Mixed Birkhoff spectra of one-dimensional Markov maps

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1. Introduction

Borel's normal number theorem states that for Lebesgue almost every real number the limiting frequency of each digit of its decimal expansion is 1/10. On the other hand, there exist plenty of real numbers with other limiting frequencies. In the 1930s and 1940s, Besicovitch [3] and Eggleston [4] have investigated these exceptional sets of numbers in terms of their Hausdorff dimension. Eggleston showed that for any $\boldsymbol{\alpha} = (\alpha_0, \dots, \alpha_9) \in (0,1)^9$ such that $\sum_{k=0}^{9} \alpha_k = 1$ the set

$$E(\boldsymbol{\alpha}) = \left\{ x = 0.x_1x_2\cdots \in [0,1] \mid \lim_{n\to\infty} \frac{1}{n} \# \{1 \leq i \leq n \mid x_i = k\} = \alpha_k \ 0 \leq k \leq 9 \right\}$$

has Hausdorff dimension

$$\dim_H E(\boldsymbol{\alpha}) = \frac{-\sum_{k=0}^{9} \alpha_k \log(\alpha_k)}{\log 10}. \tag{1.1}$$

Note that the Hausdorff dimension coincides with the dimension $\dim(\mu)$ of the Bernoulli measure μ on decimal expansions of real numbers $0.x_1x_2\dots$ such that $\mu(\{x_i = k\}) = \alpha_k$. Here, $\dim(\mu) = h(\mu)/\chi(\mu)$ is the quotient of the *Kolmogorov-Sinai entropy* $h(\mu)$ and the *Lyapunov exponent* $\chi(\mu) = \int \log |f'| d\mu$ of the measure-preserving transformation $f : [0,1] \to [0,1]$, $f(x) = 10x \mod 1$ with respect to μ.

The above formula (1.1) is a special case of a variational description of the Hausdorff dimension of Birkhoff spectra in terms of invariant probability measures. In this article we outline a generalization of this formalism for mixed Birkhoff spectra of countably many observables in the context of non-uniformly expanding one-dimensional Markov maps. As an application, we discuss the arithmetic mean spectrum and digit frequencies of backward continued fraction expansions of real numbers. A more detailed exposition of our results, together with their proofs, will appear elsewhere [11]. For related recent results we refer to [12], [5] and the references therein. For a general introduction to dimension theory in dynamical systems and multifractal analysis we refer to [16].

2. Statement of main results

A C^1 *Markov map* $f\colon \Delta \to [0,1]$ is given by a countable family $\{\Delta_a\}_{a\in S}$ of connected subsets of $[0,1]$ with pairwise disjoint interiors such that $\Delta = \bigcup_{a\in S} \Delta_a$ and $f|_{\Delta_a}$ extends to a C^1 diffeomorphism $\tilde{f}_a$ from $\overline{\Delta_a}$ onto its image, for each $a \in S$. Moreover, f has the *Markov property*, i.e., if $a, b \in S$ and $f\Delta_a \cap \Delta_b$ has non-empty interior, then $f\Delta_a \supset \Delta_b$. We note that, in the case $f(x) = 10x \mod 1$ the Markov partition is given by the first digit of the decimal expansion, and each branch of f is full.

Throughout, we will assume a strong transitivity assumption on f called *finitely irreducibility* [14] which is well known for Markov maps with countably many branches.

A $\mathcal{C}^1$ Markov map $f\colon \Delta \to [0,1]$ is *non-uniformly expanding* if $|\tilde{f}_a'(x)| > 1$ for all but finitely many $(a,x) \in S \times [0,1]$ with $x \in \overline{\Delta}_a$.

We say that f has *uniform decay of cylinders* if the length of any interval $\bigcap_{j=0}^{n-1} f^{-j}\Delta_{\omega_j}$, with $\omega_0 \cdots \omega_{n-1} \in S^n$, tends to zero uniformly as $n \to \infty$. We assume that the maximal invariant set

$$J = \bigcap_{n=0}^{\infty} f^{-n}\Delta$$

is non-empty. Denote by $\mathcal{M}(f)$ the set of f-invariant Borel probability measures on J with $\chi(\mu) < \infty$. We say that f is *saturated* if

$$\dim_H J = \sup\{\dim(\mu) : \mu \in \mathcal{M}(f)\}.$$

The finiteness parameter of f is denoted by

$$\beta_\infty = \inf\left\{\beta \in \mathbb{R}\colon \sup\{h(\mu) - \beta\chi(\mu)\colon \mu \in \mathcal{M}(f)\} < \infty\right\}.$$

We introduce the class $\mathcal{F}$ of observables $\phi\colon \Delta \to \mathbb{R}$ admitting a mild distortion bound (see [11, Section 3.3] for the details). We say f has *mild distortion* if $\log|f'| \in \mathcal{F}$. For $\boldsymbol{\phi} = (\phi_k)_{k\in\mathbb{N}} \in \mathcal{F}^{\mathbb{N}}$ and $\boldsymbol{\alpha} = (\alpha_k)_{\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ we define

$$B(\boldsymbol{\phi}, \boldsymbol{\alpha}) = \left\{x \in J\colon \lim_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} \phi_k(x) = \alpha_k \ \forall k \geq 1\right\}.$$

Theorem 2.1 (Conditional variational formula for mixed Birkhoff spectra)**.** *Let $f\colon \Delta \to [0,1]$ be a finitely irreducible non-uniformly expanding Markov map which has mild distortion and uniform decay of cylinders. Further, assume that f is saturated. Then for every $\boldsymbol{\phi} \in \mathcal{F}^{\mathbb{N}}$ and $\boldsymbol{\alpha} \in \mathbb{R}^{\mathbb{N}}$ such that $B(\boldsymbol{\phi}, \boldsymbol{\alpha}) \neq \emptyset$ we have*

$$\dim_H B(\boldsymbol{\phi}, \boldsymbol{\alpha}) = \lim_{k\to\infty}\lim_{\epsilon\to 0} \sup\left\{\dim(\mu)\colon \mu \in \mathcal{M}(f),\ \left|\int \phi_j d\mu - \alpha_j\right| < \epsilon \ \forall j \leq k\right\}.$$

If moreover each ϕ_i is bounded, then $\dim_H B(\boldsymbol{\phi}, \boldsymbol{\alpha}) \geq \beta_\infty$.

By *a frequency vector* we mean an element $\boldsymbol{\alpha} \in \mathbb{R}^{\mathbb{N}}$ such that $\alpha_i \geq 0$ holds for every $i \geq 1$ and $\sum_{i=1}^{\infty} \alpha_i \leq 1$. For each frequency vector $\boldsymbol{\alpha}$ we introduce the *Besicovitch-Eggleston set*

$$BE(\boldsymbol{\alpha}) = \left\{x \in J\colon \lim_{n\to\infty} \frac{1}{n}\#\{0 \leq j \leq n-1\colon f^j x \in \Delta_i\} = \alpha_i \ \ \forall i \geq 1\right\}.$$

Corollary 2.2 (Dimension of Besicovitch-Eggleston sets)**.** *Under the assumptions of Theorem 2.1 we have for each frequency vector $\boldsymbol{\alpha} \in \mathbb{R}^{\mathbb{N}}$ such that $BE(\boldsymbol{\alpha}) \neq \emptyset$,*

$$\dim_H BE(\boldsymbol{\alpha}) = \lim_{k\to\infty}\lim_{\epsilon\to 0} \sup\left\{\dim(\mu)\colon \mu \in \mathcal{M}(f),\ \max_{1\leq i\leq k} |\mu(\Delta_i) - \alpha_i| < \epsilon\right\} \geq \beta_\infty.$$

3. Application to backward continued fractions

Recall that each irrational number $x \in (0,1) \setminus \mathbb{Q}$ has a unique *Backward Continued Fraction (BCF)* expansion

$$x = 1 - \cfrac{1}{b_1(x) - \cfrac{1}{b_2(x) - \cfrac{1}{\ddots}}}, \tag{3.1}$$

where each digit $b_j(x)$ is an integer greater than or equal to 2. The behavior of the arithmetic mean of the BCF digits is peculiar. Aaronson [1] proved that the arithmetic mean convergences to 3 in measure as $n \to \infty$. Aaronson and Nakada [2] proved that

$$\liminf_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} b_j(x) = 2 \quad \text{and} \quad \limsup_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} b_j(x) = \infty$$

for Lebesgue a.e. $x \in (0,1) \setminus \mathbb{Q}$.

The digits in this expansion are generated by iterating the Rényi map [17]

$$R\colon x \in [0,1) \mapsto \frac{1}{1-x} - \left\lfloor \frac{1}{1-x} \right\rfloor \in [0,1).$$

This means that for all $x \in (0,1) \setminus \mathbb{Q}$,

$$b_j(x) = \left\lfloor \frac{1}{1-R^{j-1}x} \right\rfloor + 1 \quad \forall j \geq 1.$$

The graph of the Rényi map can be obtained from that of the Gauss map by reflecting the latter in the line $x = 1/2$. For this reason, (3.1) is called the Backward Continued Fraction expansion of the irrational number x. It is not difficult to verify that the Rényi map is a fully branched non-uniformly expanding Markov map having $x = 0$ as a unique parabolic fixed point.

To prove that R is saturated, we consider the induced $\mathcal{C}^1$ Markov map $\tilde{R} : (1/2,1) \to (1/2,1)$ given by $\tilde{R}(x) = R^{n(x)}(x)$, where $n(x) = \inf\{n \geq 1 \mid R^n(x) \in (1/2,1)\}$ denotes the first return time to $(1/2,1)$. One then verifies that $\tilde{R}$ is uniformly expanding, that is, $\inf |\tilde{R}'| > 1$ and that $\tilde{R}$ satisfies Rényi's condition. It is then standard to verify that $\tilde{R}$ is saturated. Finally, invoking the Kac-formula, we obtain that R is saturated.

Combining Theorem 2.1 with direct computations, we are able to establish the following.

Proposition 3.1 (Completely flat arithmetic mean spectrum of BCF expansion)**.** *For any $\alpha \in [2,\infty]$ we have*

$$\dim_H \left\{ x \in (0,1) \setminus \mathbb{Q} \colon \lim_{n\to\infty} \frac{1}{n}(b_1(x) + \cdots + b_n(x)) = \alpha \right\} = 1.$$

We conjecture the following dichotomy for the complete flatness of the spectrum.

Conjecture. *Let $\psi : \{2,3,\dots\} \to \mathbb{R}$. Then we have*

$$\dim_H \left\{ x \in (0,1) \setminus \mathbb{Q} \colon \lim_{n\to\infty} \frac{1}{n}(\psi(b_1(x)) + \cdots + \psi(b_n(x))) = \alpha \right\} = 1$$

for all $\alpha \in [\psi(2),\infty]$ if and only if $\limsup_{n\to\infty} \psi(n)/\log(n) = \infty$.

We also discuss the Besicovitch-Eggleston sets for the BCF expansions. For the Rényi map we have for any frequency vector $\boldsymbol{\alpha}$ of $\mathbb{R}^{\mathbb{N}}$,

$$BE(\boldsymbol{\alpha}) = \left\{ x \in (0,1) \setminus \mathbb{Q} \colon \lim_{n\to\infty} \frac{1}{n}\#\{1 \leq j \leq n \colon b_j(x) = i\} = \alpha_{i-1} \ \ \forall i \geq 2 \right\}.$$

Note that, for the Rényi map as well as the Gauss map, the finiteness parameter β_∞ is equal to $1/2$. The following theorem is then a consequence of Corollary 2.2.

Proposition 3.2 (Dimension of Besicovitch-Eggleston sets)**.** *For every frequency vector $\boldsymbol{\alpha}$ we have*

$$\dim_H BE(\boldsymbol{\alpha}) = \lim_{k\to\infty} \lim_{\epsilon\to 0} \max\left\{ \sup\left\{ \dim(\mu)\colon \mu \in \mathcal{M}(R),\ \max_{1\le i\le k} |\mu(\Delta_i) - \alpha_i| < \epsilon \right\}, \frac{1}{2} \right\}.$$

4. Discussion of main theorem and related results

Recently, conditional variational formulas have been presented in [5] for mixed Birkhoff spectra assuming that the Markov map is uniformly expanding and full-branched. For finitely generated parabolic iterated function systems, similar formulas have been established in [12]. Our framework of non-uniformly expanding Markov maps contains both these settings, and moreover, allows us to deal with infinitely branched Markov maps with parabolic fixed points. Such an example is given by the Rényi map.

Let us point out that infinitely branched Markov maps with parabolic fixed points may behave rather different from finitely branched ones. Namely, the set of points with zero Lyapunov exponent

$$L(0) := \left\{ x \in J \mid \liminf_{n\to\infty} \frac{1}{n} \log |f^{n\prime}(x)| = 0 \right\}$$

may intersect many level sets of the Birkhoff level sets $B(\boldsymbol{\phi}, \boldsymbol{\alpha})$. In fact, Proposition 3.1 may be strengthened as follows: For every $\alpha \ge 2$ we have

$$\dim_H \left(L(0) \cap \left\{ x \in (0,1) \setminus \mathbb{Q}\colon \lim_{n\to\infty} \frac{1}{n} (b_1(x) + \cdots + b_n(x)) = \alpha \right\} \right) = 1.$$

This indicates that a careful analysis of the set $L(0)$ is necessary.

On the other hand, for finitely branched Markov maps with one parabolic fixed point, the set $L(0)$ has non-empty intersection with only one of the Birkhoff level sets.

Let us finally comment on the two limits $\epsilon \to 0$ and $k \to \infty$ in Theorem 2.1. It is shown in [5] (see also [6]) that, for uniformly expanding Markov maps, if the potentials Φ_i are bounded, Theorem 2.1 can be stated as follows:

$$\dim_H B(\boldsymbol{\phi}, \boldsymbol{\alpha}) = \max\left\{ \sup\left\{ \dim(\mu)\colon \mu \in \mathcal{M}(f),\ \int \boldsymbol{\phi}_j d\mu = \alpha_j\ \forall j \right\}, \beta_\infty \right\}. \tag{4.1}$$

We remark that, in contrast to the uniformly expanding setting, the formula in (4.1) may fail for non-uniformly expanding Markov maps, even if the number of branches is finite. An example is given by the Lyapunov spectrum of a finitely generated non-elementary, free Fuchsian group with parabolic elements.

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