$\mathcal{M}_n$ is connected

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Abstract

We consider the sets of zeros of some families of power series. We prove that the sets of zeros in the unit disk are connected. Furthermore, we apply this result to the study of the connectedness locus $\mathcal{M}_n$ for fractal n-gons. We prove that for each n, $\mathcal{M}_n$ is connected.

1 Introduction

1.1 Background

In 1985, Barnsley and Harrington ([3]) introduced a parameter set $\mathcal{M}_2$ for the iterated function systems $\{\lambda z+1, \lambda z-1\}$ on $\mathbb{C}$, where $0<|\lambda|<1$, as an analog of the Mandelbrot set for quadratic maps. The parameter set $\mathcal{M}_2$ is defined as the connectedness locus for a pair of linear maps, that is,

$$\mathcal{M}_2 = \{\lambda \in \mathbb{D}^{\times} \mid A_2(\lambda) \text{ is connected}\},$$

where $\mathbb{D}^{\times} := \{\lambda \in \mathbb{C} \mid 0 < |\lambda| < 1\}$ and the set $A_2(\lambda)$ is the attractor of the iterated function system $\{\lambda z+1, \lambda z-1\}$. For the general theory of the iterated function system, see [8]. $\mathcal{M}_2$ looks like a "ring" around the set of parameters λ for which $A_2(\lambda)$ is a Cantor set and has "whiskers" (see Figure 1). In fact, Barnsley and Harrington ([3]) proved that there is a neighborhood of the set $\{0.5, -0.5\}$ in which $\mathcal{M}_2$ is contained in $\mathbb{R}$. Furthermore, they conjectured that there is a non-trivial hole in $\mathcal{M}_2$.

Bousch ([4], [5]) proved that $\mathcal{M}_2$ is connected and locally connected. This is interesting since for the case of quadratic maps, the local connectedness of the Mandelbrot set is still an open problem. In [4] and [5], Bousch showed that $\mathcal{M}_2$ is equal to the set of zeros of power series with coefficients $0, 1$, and -1. He also studied the set of zeros of power series with coefficients 1 and -1, which is a subset of $\mathcal{M}_2$. At the same time, Odlyzko and Poonen ([12]) studied the set of zeros of power series with coefficients 1 and 0, and they proved the set of zeros is path-connected.

In 2002, Bandt ([1]) gave an algorithm to study geometric structure of $\mathcal{M}_2$, and managed to prove the existence of a non-trivial hole in $\mathcal{M}_2$ rigorously. Thus he positively answered the conjecture of Barnsley and Harrington ([3]). He also conjectured that the interior of $\mathcal{M}_2$ is dense away from $\mathcal{M}_2 \cap \mathbb{R}$, that is, $\mathrm{cl}\big(\mathrm{int}(\mathcal{M}_2)\big) \cup (\mathcal{M}_2 \cap \mathbb{R}) = \mathcal{M}_2$. Here, for a set $A \subset \mathbb{C}$, we denote by $\mathrm{cl}(A)$ and $\mathrm{int}(A)$ the closure of A and the interior of A with respect to the Euclidean topology on $\mathbb{C}$ respectively. Several authors made partial progress on Bandt's conjecture (see [13], [14] and [15]).

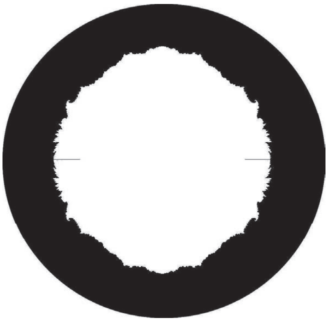

Figure 1. $\mathcal{M}_2$

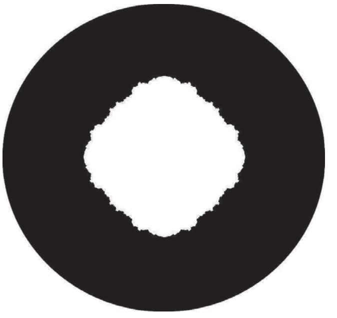

Figure 2. $\mathcal{M}_4$

In 2008, Bandt and Hung ([2]) introduced self-similar sets parametrized by $\lambda \in \mathbb{D}^\times$ which are called "*fractal n-gons*", where $\mathbb{D}^\times := \{\lambda \in \mathbb{C} \mid 0 < |\lambda| < 1\}$ and $n \in \mathbb{N}_{\geq 2}$. We give the rigorous definition of "*fractal n-gons*" in the next sub-section (see Definition 1.1). They studied the connectedness locus for "*fractal n-gons*", that is,

$$\mathcal{M}_n = \{\lambda \in \mathbb{D}^\times \mid A_n(\lambda) \text{ is connected}\},$$

where $A_n(\lambda)$ is the "*fractal n-gon*" corresponding to the parameter λ (see Figure 2). Note that "*fractal* 2*-gons*" are attractors of the iterated function systems $\{\lambda z + 1, \lambda z - 1\}$ and $\mathcal{M}_2$ is the connectedness locus for "*fractal* 2*-gons*". Bandt and Hung ([2]) discovered many remarkable properties about $\mathcal{M}_n$, including the following result. For each $n = 3$ or ≥ 5, $\mathcal{M}_n$ is regular-closed, that is, $\mathrm{cl}\big(\mathrm{int}(\mathcal{M}_n)\big) = \mathcal{M}_n$.

In 2016, Calegari, Koch and Walker ([7]) introduced new methods for constructing interior points and positively answered Bandt's conjecture, that is, $\mathrm{cl}\big(\mathrm{int}(\mathcal{M}_2)\big) \cup (\mathcal{M}_2 \cap \mathbb{R}) = \mathcal{M}_2$. Himeki and Ishii [10] proved $\mathcal{M}_4$ is regular-closed. Thus the problems about the regular-closedness of $\mathcal{M}_n$ have been completely solved. Furthermore, Calegari and Walker ([6]) characterized the extreme points in "*fractal n-gons*" and gave an alternative proof of [10, Proposition 2.1], which we need to prove the regular-closedness of $\mathcal{M}_4$.

Many authors have investigated $\mathcal{M}_n$ and discovered many remarkable properties about $\mathcal{M}_n$. However, many problems about $\mathcal{M}_n$ still remain unsolved. One of the problems is the connectedness of $\mathcal{M}_n$. Himeki [9] proved that $\mathcal{M}_3$ is connected by using the methods of Bousch ([4]). In this paper, we study the connectedness of the sets of zeros of some families of power series by extending the methods of Bousch ([4]) and by giving a new framework (see Definition 1.2, Definition 1.3, and Main result B). Furthermore, we apply this result to the study of the connectedness of $\mathcal{M}_n$ (see Main result A).

1.2 Main results

Below we fix $n \in \mathbb{N}_{\geq 2}$. We give the rigorous definition of "*fractal n-gons*" as the following.

Definition 1.1 (Fractal n-gons)**.** Let $\mathbb{D}^\times := \{\lambda \in \mathbb{C} \mid 0 < |\lambda| < 1\}$. Let $\lambda \in \mathbb{D}^\times$. We set $\xi_n = \exp(2\pi\sqrt{-1}/n)$. For each $i \in \{0, 1, ..., n-1\}$, we define $\phi_i^{n,\lambda} : \mathbb{C} \to \mathbb{C}$ by $\phi_i^{n,\lambda}(z) = \lambda z + {\xi_n}^i$. Then there uniquely exists a non-empty compact subset $A_n(\lambda)$ such that

$$\bigcup_{i=0}^{n-1} \phi_i^{n,\lambda}(A_n(\lambda)) = A_n(\lambda)$$

(See [8], [11]). We call $A_n(\lambda)$ a fractal n-gon corresponding to the parameter λ.

For each n, we define the connectedness locus $\mathcal{M}_n$ for fractal n-gons as the following.

$$\mathcal{M}_n = \{\lambda \in \mathbb{D}^\times \mid A_n(\lambda) \text{ is connected}\}.$$

We give one of the main results in this paper as the following.

Main result A. For each n, $\mathcal{M}_n$ is connected.

In [4], Bousch showed that $\mathcal{M}_2$ is equal to the set of zeros of power series with coefficients $0, 1$, and -1. Similarly, we can identify $\mathcal{M}_n$ with the set of zeros of some power series (see [2, Remark 3]). However, in the proof of the connectedness of $\mathcal{M}_n$ for general $n \in \mathbb{N}_{\geq 2}$, since the set Ω_n of coefficients of the power series, which corresponds to $\mathcal{M}_n$, is complicated for general $n \in \mathbb{N}_{\geq 2}$ (see Definition 4.3) in contrast to $\mathcal{M}_2$, we cannot use the methods to prove the connectedness of $\mathcal{M}_2$ and $\mathcal{M}_3$ which are given in [4] and [9]. Hence we study the connectedness of the sets of zeros of some power series by extending the methods of Bousch ([4]) and by using some new ideas and techniques. We need the following setting to prove Main result A, which is one of the new ideas in this paper.

Definition 1.2. Let G be a subset of $\mathbb{C}$. We say that G satisfies the condition $(*)$ if G satisfies all of the following conditions (i), (ii), and (iii).

(i) $1 \in G$.

(ii) For all $a, b \in G$ with $a \neq b$, there exist $b_1, b_2, ..., b_m \in G$ with $b_1 = a$ and $b_m = b$ such that for all $c \in G$, there exist $d_1, d_2, ..., d_{m-1} \in G$ such that

$$(b_2 - b_1)c + d_1 \in G, (b_3 - b_2)c + d_2 \in G, ..., (b_m - b_{m-1})c + d_{m-1} \in G.$$

(iii) G is compact.

Definition 1.3. Let G be a subset of $\mathbb{C}$ such that G satisfies the condition $(*)$. Let $N \in \mathbb{N}_{\geq 2}$. Let $\mathbb{D}$ be the unit disk. We set

$$\begin{aligned}
P^G &= \{1 + \sum_{i=1}^{\infty} a_i z^i \mid a_i \in G\}, \\
X^G &= \{z \in \mathbb{D} \mid \text{there exists } f \in P^G \text{ such that } f(z) = 0\}, \\
Q_N^G &= \{1 + \sum_{i=1}^{N-1} a_i z^i \mid a_i \in G\}, \\
Y_N^G &= \{z \in \mathbb{C} \mid \text{there exists } f \in Q_N^G \text{ such that } f(z) = 0\}, \\
Y^G &= \bigcup_{N \geq 2} Y_N^G.
\end{aligned}$$

Then the following theorem holds, which we need to prove Main result A.

Main result B (Theorem 3.3)**.** Let G be a subset of $\mathbb{C}$ such that G satisfies the condition $(*)$. Suppose that there exists a real number R with $0 < R < 1$ such that $\{z \in \mathbb{C} \mid R < |z| < 1\} \subset X^G$. Then X^G is connected.

1.3 Strategy for the proof of Main result A

We briefly describe our strategy for the proof of Main result A. In Sections 2 and 3, we prove Main result B by extending the methods of Bousch ([4]) and by using some new ideas. We set $I := \{0, 1, ..., n-1\}$ and $\Omega_n := \{({\xi_n}^j - {\xi_n}^k)/(1-\xi_n) \mid j, k \in I\}$. Then we have that $\mathcal{M}_n = X^{\Omega_n}$ and $\{z \in \mathbb{C} \mid 1/\sqrt{n} < |z| < 1\} \subset \mathcal{M}_n$ (see [2, Remark 3] and [2, Proposition 3]). It is highly non-trivial that Ω_n satisfies the condition $(*)$ and in order to prove that, we need Lemmas 4.1 and 4.2, which are the key lemmas in the paper. In Section 4, by using Lemmas 4.1 and 4.2, we prove that Ω_n satisfies the condition $(*)$, and hence we get Main result A as a corollary of Main result B.

Acknowledgement.

The author would like to express his gratitude Professor Hiroki Sumi for his valuable comments. This study is supported by JSPS KAKENHI Grant Number JP 19J21038.

2 Preliminaries

In this paper, for a set $A \subset \mathbb{C}$, we denote by $\mathrm{cl}(A)$ the closure of A with respect to the Euclidean topology on $\mathbb{C}$. We denote by ∂A the topological boundary of A with respect to the Euclidean topology on $\mathbb{C}$. For $a \in \mathbb{C}$, we denote by $|a|$ the Euclidean norm of a. For $x \in \mathbb{C}$ and $r > 0$, we set $B(x, r) := \{z \in \mathbb{C} \mid |x - z| < r\}$.

Lemma 2.1. Let G be a subset of $\mathbb{C}$ such that G satisfies the condition $(*)$. Then $X^G = \mathrm{cl}(Y^G) \cap \mathbb{D}$.

Proof. $(\subset)$Take $z_0 \in X^G$. Then there exists $\{a_i\}_{i=1}^{\infty} \subset G$ such that $1 + \sum_{i=1}^{\infty} a_i z_0^i = 0$. Fix $\epsilon > 0$ with $B(z_0, \epsilon) \subset \mathbb{D}$. Then there exist $N \in \mathbb{N}$ and $z' \in B(z_0, \epsilon)$ such that $1 + \sum_{i=1}^{N-1} a_i z' = 0$ by theorem of Rouché. Hence $z_0 \in \mathrm{cl}(Y^G) \cap \mathbb{D}$.

$(\supset)$Since P^G is a normal family on $\mathbb{D}$, X^G is relatively closed in $\mathbb{D}$. Hence it suffices to prove that $X^G \supset Y^G \cap \mathbb{D}$. Take $z_0 \in Y^G \cap \mathbb{D}$. Then there exists $\{a_i\}_{i=1}^{N-1} \subset G$ such that $1 + \sum_{i=1}^{N-1} a_i z_0^i = 0$. We set $\tilde{f}(z) := f(z) \times \sum_{j=0}^{\infty} z^{jN} \in P^G$. Then $\tilde{f}(z_0) = 0$.

Thus we have proved our lemma. □

Below we fix a set $G \subset \mathbb{C}$ which satisfies the condition $(*)$.

Definition 2.2. Let $N \in \mathbb{N}_{\geq 2}$. We set $L := \sup\{|a|, |ab|, |(a-b)c| \mid a, b, c \in G\} (< \infty)$. Then we define the sets of functions W^G and W_N^G as the following.

$$W^G := \{1 + \sum_{i=1}^{\infty} a_i z^i \mid |a_i| \leq L\},$$

$$W_N^G := \{1 + \sum_{i=1}^{N-1} a_i z^i \mid |a_i| \leq L\}.$$

Remark 2.3. $Q_N^G \subset W_N^G \subset W^G$ and $P^G \subset W^G$.

Let $N \in \mathbb{N}_{\geq 2}$. We identify $(1, a_1, a_2, ...)$ with the power series $1 + \sum_{i=1}^{\infty} a_i z^i$. We identify $(1, a_1, ..., a_{N-1})$ or $(1, a_1, ..., a_{N-1}, 0, 0, ...)$ with the polynomial $1 + \sum_{i=1}^{N-1} a_i z^i$. Let $f = (1, a_1, a_2, ...)$ and $g = (1, b_1, b_2, ...)$. We set $\mathrm{Val}(f, g) := \inf\{i \in \mathbb{N} \mid a_i - b_i \neq 0\}$. If $f = g$, we set $\mathrm{Val}(f, g) = \infty$. Let $N \in \mathbb{N}_{\geq 2}$. We define the map $C_N : W^G \to W_N^G$ by

$$C_N\big((1, a_1, a_2, ...)\big) := (1, a_1, ..., a_{N-1}).$$

We use the following lemma.

Lemma 2.4. [4, Lemme 2] Let $R > 0$ and $\epsilon > 0$ with $R + \epsilon < 1$. Then there exists $N_{R,\epsilon} \in \mathbb{N}_{\geq 2}$ such that for all $(f, s) \in F := \{(f, s) \in W^G \times \mathrm{cl}(B(0, R)) \mid f(s) = 0\}$ and for all $g \in W^G$ with $\mathrm{Val}(f, g) \geq N_{R,\epsilon}$, there exists $s' \in B(s, \epsilon)$ such that $g(s') = 0$.

Definition 2.5. Let $N \in \mathbb{N}_{\geq 2}$. Let $A, B \in Q_N^G$ with $A \neq B$.

Let $R := \{p_0, q_0, p_1, q_1, ..., p_{m-1}, q_{m-1}, p_m\}$ be a sequence of functions on $\mathbb{D}$. We say that R is a sequence of functions which joins A to B if R satisfies the following:

(1) for each i, $p_i \in Q_N^G$;

(2) for each i, $q_i \in W^G$;

(3) for each i, there exists a holomorphic function f_i on $\mathbb{D}$ such that $q_i(z) = f(z) \cdot p_i(z)$ for all $z \in \mathbb{D}$;

(4) for each i, $C_N(q_i) = p_{i+1}$;

(5) $p_0 = A, p_m = B$.

We prove the following lemma by extending [4, Lemme 3].

Lemma 2.6. Let $N \in \mathbb{N}_{\geq 2}$. Let $A, B \in Q_N^G$ with $A \neq B$. Then there exists a sequence of functions $p_0, q_0, p_1, q_1, ..., p_{m-1}, q_{m-1}, p_m$ which joins A to B.

Proof. This is done by induction with respect to $\mathrm{Val}(A, B) \in \{1, ..., N-1\}$. We prove that the statement holds in the case $\mathrm{Val}(A, B) = N - 1$. We set

$$\begin{aligned} A &:= (1, a_1, ..., a_{N-2}, a), \\ B &:= (1, a_1, ..., a_{N-2}, b), \end{aligned}$$

where $a \neq b$. We set

$$\begin{aligned} q_0^0 :=& \{1 + (b - a)z^{N-1}\}A \\ =& (1, a_1, ..., a_{N-2}, a) + \\ & (\underbrace{0, 0, ..., 0}_{N-1}, (b - a), (b - a)a_1, ..., (b - a)a_{N-2}, (b - a)a) \\ =& (1, a_1, ..., a_{N-2}, b, (b - a)a_1, ..., (b - a)a_{N-2}, (b - a)a) \in W^G, \\ p_1^0 :=& C_N(q_0^0) \\ =& (1, a_1, ..., a_{N-2}, b) = B \in Q_N^G. \end{aligned}$$

Hence we find a sequence $\{A, q_0^0, B\}$ of functions which joins A to B.

Fix $j \in \{1, ..., N-2\}$. Suppose that the statement holds in the case $\mathrm{Val}(A, B) > j$. We prove that the statement holds in the case $\mathrm{Val}(A, B) = j$. We set

$$\begin{aligned} A &:= (1, a_1, ..., a_{j-1}, a, * \cdots *), \\ B &:= (1, a_1, ..., a_{j-1}, b, * \cdots *), \end{aligned}$$

where $a \neq b$. Since G satisfies the condition $(*)$, there exist $(a =)b_1, b_2, ..., b_m (= b) \in G$ which satisfies Definition 1.2 (ii). Let k, l be natural numbers such that $N - 1 = jk + l$ and $0 \leq l \leq j - 1$. By the assumption of G, for each $i \in \{1, 2, ..., j\}$ and $m \in \{1, ..., k-1\}$, there exists $c_i^m \in G$ such that

$$\begin{aligned} &(b_2 - b_1)a_i + c_i^1 \in G, \\ &(b_2 - b_1)a + c_j^1 \in G, \\ &(b_2 - b_1)c_i^m + c_i^{m+1} \in G, \end{aligned}$$

where c_i^1 depends on b_1, b_2, a_i, and c_j^1 depends on $b_1(=a), b_2$, and c_i^{m+1} depends on b_1, b_2, c_i^m. We set

$$A_1 := (1, a_1, ..., a_{j-1}, a, c_1^1, ..., c_j^1, c_1^2, ..., c_j^2, ..., c_1^k, ..., c_l^k) \in Q_N^G.$$

Since $\mathrm{Val}(A, A_1) > j$, by induction hypothesis, there exists a sequence R_1 of functions which joins A to A_1. We set

$$\begin{aligned} q_1 :=& \{1 + (b_2 - b_1)z^j\}A_1 \\ =&(1, a_1, ..., a_{j-1}, a, c_1^1, ..., c_j^1, c_1^2, ..., c_j^2, ..., c_1^k, ..., c_l^k)+ \\ &(\underbrace{0, 0, ..., 0}_{j}, (b_2 - b_1), (b_2 - b_1)a_1, ..., (b_2 - b_1)a, (b_2 - b_1)c_1^1, ..., (b_2 - b_1)c_j^1, ...) \\ =&(1, a_1, ..., a_{j-1}, b_2, (b_2 - b_1)a_1 + c_1^1, ..., (b_2 - b_1)c_l^{k-1} + c_l^k, (b_2 - b_1)c_{l+1}^{k-1}, ..., (b_2 - b_1)c_l^k) \\ &\in W^G. \end{aligned}$$

Here, recall that $b_1 = a$. We set

$$\begin{aligned} p_2 :&= C_N(q_1) \\ &= (1, a_1, ..., a_{j-1}, b_2, (b_2 - b_1)a_1 + c_1^1, ..., (b_2 - b_1)c_l^{k-1} + c_l^k) \in Q_N^G. \end{aligned}$$

By the assumption of G, for each $i \in \{1, 2, ..., j\}$ and $m \in \{1, ..., k-1\}$, there exists $d_i^m \in G$ such that

$$\begin{aligned} &(b_3 - b_2)a_i + d_i^1 \in G, \\ &(b_3 - b_2)b_2 + d_j^1 \in G, \\ &(b_3 - b_2)d_i^m + d_i^{m+1} \in G, \end{aligned}$$

where d_i^1 depends on b_2, b_3, a_i, and d_j^1 depends on b_2, b_3, and d_i^{m+1} depends on b_2, b_3, d_i^m. We set

$$A_2 := (1, a_1, ..., a_{j-1}, b_2, d_1^1, ..., d_j^1, d_1^2, ..., d_j^2, ..., d_1^k, ..., d_l^k) \in Q_N^G.$$

Since $\mathrm{Val}(p_2, A_2) > j$, by induction hypothesis, there exists a sequence R_2 of functions which joins p_2 to A_2. We set

$$\begin{aligned} q_2 &:= \{1 + (b_3 - b_2)z^j\}A_2 \in W^G, \\ p_3 &:= C_N(q_2) \in Q_N^G. \end{aligned}$$

If we continue this process, we find sequences $R_1, R_2, ..., R_{m-1}$ of functions, functions $q_1, q_2, ..., q_{m-1} \in W^G$ and a function $p_m \in Q_N^G$, where R_1 joins A to A_1, R_i joins p_i to A_i for each $i \in \{2, ..., m-1\}$. Here,

$$\begin{aligned} &R_1 := \{A, q_0^1, p_1^1, q_1^1, ..., A_1\}, \\ &R_2 := \{p_2, q_0^2, p_1^2, q_1^2, ..., A_2\}, \\ &\cdots, \\ &R_{m-1} := \{p_{m-1}, q_0^{m-1}, p_1^{m-1}, q_1^{m-1}, ..., A_{m-1}\}. \end{aligned}$$

Then we find a sequence $\{A, q_0^1, p_1^1, q_1^1, ..., A_1, q_1, p_2, q_0^2, p_1^2, q_1^2, ..., A_2, ..., p_{m-1}, q_0^{m-1}, p_1^{m-1}, q_1^{m-1}, ..., A_{m-1}, q_{m-1}, p_m\}$ of functions which joins A to p_m, where p_m has the following form.

$$p_m = (1, a_1, ..., a_{j-1}, b, * \cdots *).$$

Since $\mathrm{Val}(p_m, B) > j$, by induction hypothesis, there exists a sequence of functions $R_m = \{p_m, q_0^m, p_1^m, q_1^m, ..., B\}$ which joins p_m to B. Hence we find a sequence $\{A, q_0^1, p_1^1, q_1^1, ..., A_1, q_1, p_2, q_0^2, p_1^2, q_1^2, ..., A_2, ..., p_{m-1}, q_0^{m-1}, p_1^{m-1}, q_1^{m-1}, ..., A_{m-1}, q_{m-1}, p_m, q_0^m, p_1^m, q_1^m, ..., B\}$ of functions which joins A to B. Thus we have proved our lemma.

□

3 Proof of Main result B

Definition 3.1 (ϵ-connected)**.** Let $A \subset \mathbb{C}$. Let $\epsilon > 0$. Let $x, y \in A$ and $\{e_0, ..., e_k\} \subset A$. We say that $\{e_0, ..., e_k\}$ is ϵ-chain for (x, y) if $x = e_0, y = e_k$ and for each $i \in \{0, ..., k-1\}, |e_i - e_{i+1}| \leq \epsilon$.

We say that A is ϵ-connected if for all $x, y \in A$, there exists an ϵ-chain for (x, y).

Remark 3.2. If $A \subset \mathbb{C}$ is compact, A is connected if and only if for any arbitrary small $\epsilon > 0$, A is ϵ-connected.

The following theorem is Main result B.

Theorem 3.3. *Let G be a subset of $\mathbb{C}$ such that G satisfies the condition $(*)$. Suppose that there exists a real number R with $0 < R < 1$ such that $\{z \in \mathbb{C} \mid R < |z| < 1\} \subset X^G$. Then X^G is connected.*

Proof. We set $M_R := \{z \in \mathbb{C} \mid R < |z| < 1\}$. Since $M_R \subset X^G$, it suffices to prove that $X^G \cup \partial\mathbb{D}$ is connected. By Lemma 2.1, $X^G \cup \partial\mathbb{D}$ is compact. Hence we prove that $X^G \cup \partial\mathbb{D}$ is ϵ-connected for an arbitrary small $\epsilon > 0$.

Fix $\epsilon > 0$ with $R + \epsilon < 1$. Take $s \in X^G$. We prove that there exist $s' \in M_R$ and an ϵ-chain for (s, s'). Since $s \in X^G$, there exists $f \in P^G$ such that $f(z) = 0$. Let $N_{R,\epsilon}$ be a natural number defined by Lemma 2.4. We set $A := C_{N_{R,\epsilon}}(f) \in Q^G_{N_{R,\epsilon}}$. Since $\mathrm{Val}(f, A) \geq N_{R,\epsilon}$, there exists $s_0 \in B(s, \epsilon)$ such that $A(s_0) = 0$. If $s_0 \in M_R$, our theorem holds. If $s_0 \notin M_R$, that is, $s_0 \in \mathrm{cl}(B(0, R))$, we set

$$B(z) := 1 + z + z^2 + \cdots + z^{N_{R,\epsilon}-1} = \frac{1 - z^{N_{R,\epsilon}}}{1 - z} \in Q^G_{N_{R,\epsilon}}.$$

By Lemma 2.6, there exists a sequence of functions $p_0, q_0, p_1, q_1, ..., p_{m-1}, q_{m-1}, p_m$ which joins A to B. Since $q_0(s_0) = 0$ and $\mathrm{Val}(q_0, p_1) \geq N_{R,\epsilon}$, there exists $s_1 \in B(s_0, \epsilon)$ such that $p_1(s_1) = 0$ by Lemma 2.4. If $s_1 \in M_R$, our theorem holds. If $s_1 \in \mathrm{cl}(B(0, R))$, since $q_1(s_1) = 0$ and $\mathrm{Val}(q_1, p_2) \geq N_{R,\epsilon}$, there exists $s_2 \in B(s_1, \epsilon)$ such that $p_2(s_2) = 0$ by Lemma 2.4. If we continue this process, there exists $i \in \{1, ..., m\}$ such that $s_i \in M_R$ and $p_i(s_i) = 0$.

If this is not true, there exists $s_m \in \mathbb{D}$ such that $p_m(s_m) = B(s_m) = 0$. But this contradicts that B does not have any roots in $\mathbb{D}$.

Since $A, p_j \in Q^G_{N_{R,\epsilon}}$ for each $j \in \{1, ..., i\}$, we have that $s_0, s_j \in X^G$ by Lemma 2.1. We set $s' := s_i$. Then $\{s, s_0, s_1, ..., s_i (= s')\}$ is ϵ-chain for (s, s').

Hence we have proved our theorem. □

4 Application (proof of Main result A)

We use the following lemmas, which are key lemmas in this paper.

Lemma 4.1. Let n be an odd number. Let q, r be integers such that $2 \leq q \leq (n-1)/2$ and $0 \leq r \leq (n-1)/2$. We set

$$j = \begin{cases} q+r-1 & (1 \leq q+r-1 \leq (n-1)/2) \\ n-(q+r-1) & ((n+1)/2 \leq q+r-1 \leq n-2) \end{cases}$$

and

$$k = r - q + 1.$$

Then

$$\Big(\sin\frac{q\pi}{n} - \sin\frac{(q-2)\pi}{n}\Big)\sin\frac{r\pi}{n} - \Big(\sin\frac{j\pi}{n}\Big)\Big(\sin\frac{\pi}{n}\Big) - \Big(\sin\frac{k\pi}{n}\Big)\Big(\sin\frac{\pi}{n}\Big) = 0.$$

Proof. (Case 1: $j = q+r-1$ and $k = r-q+1$)

$$\begin{aligned}
\Big(\sin\frac{j\pi}{n}\Big)\Big(\sin\frac{\pi}{n}\Big) + \Big(\sin\frac{k\pi}{n}\Big)\Big(\sin\frac{\pi}{n}\Big) &= -\frac{1}{2}\Big(\cos\frac{(j+1)\pi}{n} - \cos\frac{(j-1)\pi}{n}\Big) \\
&\quad -\frac{1}{2}\Big(\cos\frac{(k+1)\pi}{n} - \cos\frac{(k-1)\pi}{n}\Big) \\
&= -\frac{1}{2}\Big(\cos\frac{(q+r)\pi}{n} - \cos\frac{(q+r-2)\pi}{n}\Big) \\
&\quad -\frac{1}{2}\Big(\cos\frac{(r-q+2)\pi}{n} - \cos\frac{(r-q)\pi}{n}\Big) \\
&= -\frac{1}{2}\Big(\cos\frac{(q+r)\pi}{n} - \cos\frac{(q+r-2)\pi}{n}\Big) \\
&\quad -\frac{1}{2}\Big(\cos\frac{(q-r-2)\pi}{n} - \cos\frac{(q-r)\pi}{n}\Big) \\
&= -\frac{1}{2}\Big(\cos\frac{(q+r)\pi}{n} - \cos\frac{(q-r)\pi}{n}\Big) \\
&\quad +\frac{1}{2}\Big(\cos\frac{(q+r-2)\pi}{n} - \cos\frac{(q-r-2)\pi}{n}\Big) \\
&= \Big(\sin\frac{q\pi}{n} - \sin\frac{(q-2)\pi}{n}\Big)\sin\frac{r\pi}{n}
\end{aligned}$$

(Case 2: $j = n-(q+r-1)$ and $k = r-q+1$)

$$\begin{aligned}
\Big(\sin\frac{j\pi}{n}\Big)\Big(\sin\frac{\pi}{n}\Big) + \Big(\sin\frac{k\pi}{n}\Big)\Big(\sin\frac{\pi}{n}\Big) &= \Big(\sin\frac{(n-(q+r-1))\pi}{n}\Big)\Big(\sin\frac{\pi}{n}\Big) \\
&\quad + \Big(\sin\frac{(r-q+1)\pi}{n}\Big)\Big(\sin\frac{\pi}{n}\Big) \\
&= \Big(\sin\frac{(q+r-1)\pi}{n}\Big)\Big(\sin\frac{\pi}{n}\Big) \\
&\quad + \Big(\sin\frac{(r-q+1)\pi}{n}\Big)\Big(\sin\frac{\pi}{n}\Big)
\end{aligned}$$

By Case 1,

$$\Big(\sin\frac{(q+r-1)\pi}{n}\Big)\Big(\sin\frac{\pi}{n}\Big) + \Big(\sin\frac{(r-q+1)\pi}{n}\Big)\Big(\sin\frac{\pi}{n}\Big) = \Big(\sin\frac{q\pi}{n} - \sin\frac{(q-2)\pi}{n}\Big)\sin\frac{r\pi}{n}$$

$\square$

Lemma 4.2. Let n be an even number. Let q, r be integers such that $2 \leq q \leq n/2$ and $0 \leq r \leq n/2$. We set

$$j = \begin{cases} q + r - 1 & (1 \leq q + r - 1 \leq n/2 - 1) \\ n - (q + r - 1) & (n/2 \leq q + r - 1 \leq n - 1) \end{cases}$$

and

$$k = r - q + 1.$$

Then

$$\Big(\sin \frac{q\pi}{n} - \sin \frac{(q-2)\pi}{n} \Big) \sin \frac{r\pi}{n} - \Big(\sin \frac{j\pi}{n} \Big) \Big(\sin \frac{\pi}{n} \Big) - \Big(\sin \frac{k\pi}{n} \Big) \Big(\sin \frac{\pi}{n} \Big) = 0.$$

Proof. We can prove Lemma 4.2 as in the proof of Lemma 4.1. □

We define the set of coefficients Ω_n which corresponds to $\mathcal{M}_n$ as the following.

Definition 4.3. We set $I_n := \{0, 1, ..., n-1\}$. We define

$$\Omega_n := \{ ({\xi_n}^j - {\xi_n}^k)/(1 - \xi_n) \mid j, k \in I_n \}.$$

Here, recall that $\xi_n = \exp(2\pi\sqrt{-1}/n)$.

Remark 4.4. For each $a \in \Omega_n$, we have that $-a \in \Omega_n$.

The following two lemmas can be found in [2].

Lemma 4.5. [2, Remark 3]

$$\mathcal{M}_n = X^{\Omega_n}.$$

Lemma 4.6. [2, Proposition 3]

$$\Big\{ z \in \mathbb{C} \mid \frac{1}{\sqrt{n}} < |z| < 1 \Big\} \subset \mathcal{M}_n.$$

By using Lemmas 4.1 and 4.2, Ω_n satisfies the condition $(*)$. By Theorem 3.3, Lemmas 4.5 and 4.6, $\mathcal{M}_n$ is connected (Main result A holds).

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