Overview of the parabolic positive representations of $\mathcal{U}_q(\mathfrak{g}_\mathbb{R})$

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Abstract

We give a simplified overview of the construction of a new family of irreducible representations for split real quantum groups $\mathcal{U}_q(\mathfrak{g}_\mathbb{R})$ known as the *parabolic positive representations* by truncating the standard positive representations. This corresponds to quantizing the parabolic induction representation.

1 Positive representations of $\mathcal{U}_q(\mathfrak{g}_\mathbb{R})$

Let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbb{C}$ with root index I and Cartan matrix $A = (a_{ij})_{i,j\in I}$. For simplicity, we will focus on the simply-laced case, but the main results of this paper hold for all semisimple Lie types. We first define the Drinfeld's double of the quantum Borel subalgebra using generators and relations as follows:

Definition 1.1. The *Drinfeld's double* $\mathcal{D}_q(\mathfrak{g})$ is the algebra over $\mathbb{C}(q)$ generated by the Chevalley generators $\{\mathbf{E}_i, \mathbf{F}_i, \mathbf{K}_i^{\pm}, \mathbf{K}_i'^{\pm}\}_{i\in I}$ such that it satisfies

$$\begin{aligned} \mathbf{K}_i\mathbf{E}_j &= q^{a_{ij}}\mathbf{E}_j\mathbf{K}_i, & \mathbf{K}_i\mathbf{F}_j &= q^{-a_{ij}}\mathbf{F}_j\mathbf{K}_i,\\ \mathbf{K}_i'\mathbf{E}_j &= q^{-a_{ij}}\mathbf{E}_j\mathbf{K}_i', & \mathbf{K}_i'\mathbf{F}_j &= q^{a_{ij}}\mathbf{F}_j\mathbf{K}_i',\\ [\mathbf{E}_i, \mathbf{F}_j] &= \delta_{ij}\frac{\mathbf{K}_i - \mathbf{K}_i'}{q-q^{-1}}, & \mathbf{K}_i\mathbf{K}_i' &= \mathbf{K}_i'\mathbf{K}_i, \end{aligned}$$

together with the standard quantum Serre relations for $\mathbf{E}_i$ and $\mathbf{F}_i$.

Definition 1.2. The *quantum group* $\mathcal{U}_q(\mathfrak{g})$ is defined as the quotient

$$\mathcal{U}_q(\mathfrak{g}) := \mathcal{D}_q(\mathfrak{g})/\langle \mathbf{K}_i\mathbf{K}_i' = 1\rangle_{i\in I}. \tag{1.1}$$

Both $\mathcal{D}_q(\mathfrak{g})$ and $\mathcal{U}_q(\mathfrak{g})$ have a Hopf algebra structure which will not be needed.

Definition 1.3. Let $q = e^{\pi\sqrt{-1}b^2}$ with $b \in (0,1)$ so that $|q| = 1$. The *split real quantum group* $\mathcal{U}_q(\mathfrak{g}_\mathbb{R})$ is defined to be the real form of $\mathcal{U}_q(\mathfrak{g})$ induced by the star structure

$$\mathbf{E}_i^* = \mathbf{E}_i, \qquad \mathbf{F}_i^* = \mathbf{F}_i, \qquad \mathbf{K}_i^* = \mathbf{K}_i, \tag{1.2}$$

where $q^* := \overline{q} = q^{-1}$ acts as complex conjugate.

The theory of *positive representations* is initiated in [4] to study the representations of split real quantum group $\mathcal{U}_q(\mathfrak{g}_\mathbb{R})$ and its modular double $\mathcal{U}_{q\widetilde{q}}(\mathfrak{g}_\mathbb{R})$ [2], such that the generators are represented by positive self-adjoint operators on Hilbert spaces. This is a generalization of a special class of representations $\mathcal{P}_\lambda$ for $\mathcal{U}_{q\widetilde{q}}(\mathfrak{sl}(2,\mathbb{R}))$ discovered by Teschner et al. [19, 20] in his study of quantum Liouville theory, a certain non-compact conformal field theory. The positive representations for arbitrary semisimple types $\mathcal{U}_{q\widetilde{q}}(\mathfrak{g}_\mathbb{R})$ has been constructed in [7, 8], and recently a geometric interpretation is given in [5].

In the case of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$, it is shown in [1, 6, 20] that various properties hold in parallel to the compact case, such as (1) closure under taking tensor product, (2) existence of a braiding structure, and (3) Peter-Weyl type decomposition of the regular representations. However in the split real case, the usual expressions involving direct sums of finite dimensional irreducible representations are replaced by an appropriate direct integral of continuous slices of this class of infinite dimensional representations $\mathcal{P}_\lambda$ with certain Plancherel measure. These properties are now also known to hold at least for type A_n quantum groups due to [9, 14, 23, 24], related to open Toda-Coxeter integrable systems.

Recall that we set $q = e^{\pi\sqrt{-1}b^2}$ with $b \in (0,1)$ in the split real case. We define the rescaled generators by

$$\mathbf{e}_k := \left(\frac{\sqrt{-1}}{q-q^{-1}}\right)^{-1} E_k, \qquad \mathbf{f}_k := \left(\frac{\sqrt{-1}}{q-q^{-1}}\right)^{-1} F_k. \tag{1.3}$$

Note that $\dfrac{\sqrt{-1}}{q-q^{-1}} = (2\sin \pi b^2)^{-1} > 0$.

The following Theorem summarizes the main features of positive representations constructed in [4, 7, 8]:

Theorem 1.4. *There exists a family of irreducible representations $\mathcal{P}_\lambda$ of $\mathcal{U}_{q\tilde{q}}(\mathfrak{g}_\mathbb{R})$ parametrized by the $\mathbb{R}_{\geq 0}$-span of the cone of dominant weights $\lambda \in P_\mathbb{R}^+ \subset \mathfrak{h}_\mathbb{R}^*$, or equivalently by $\lambda := (\lambda_1, ..., \lambda_n) \in \mathbb{R}_{\geq 0}^n$ where $n = rank(\mathfrak{g})$, such that*

- *The generators $\mathbf{e}_i, \mathbf{f}_i, \mathbf{K}_i$ are represented by positive essentially self-adjoint operators acting on $L^2(\mathbb{R}^N)$, where $N = l(w_0)$ is the length of the longest element $w_0 \in W$ of the Weyl group.*
- *$\mathbf{e}_i, \mathbf{f}_i, \mathbf{K}_i$ are expressed in terms of Laurent polynomials of the Weyl algebra $\{e^{\pi b x_k}, e^{2\pi b p_k}\}_{k=1}^N$, where the momentum and position operators act as self-adjoint operators on $L^2(\mathbb{R}^N)$ satisfying $[x_k, p_k] = \frac{1}{2\pi i}$.*
- *It is compatible with the modular double structure by interchanging b with b^{-1}.*
- *One can recover any finite dimensional irreducible representations of $\mathcal{U}_q(\mathfrak{g})$ by appropriate analytic continuation on the parameters $\lambda \in P_\mathbb{R}^+$.*

By replacing the Weyl algebra with appropriate quantum variables, we can express the representation explicitly over a quiver diagram. Let us first recall the notion of quantum torus algebra.

2 Embedding to quantum torus algebra

Let $\mathbf{Q} = (Q, Q_0, B)$ be a *cluster seed*, where Q is a finite set, $Q_0 \subset Q$ is the *frozen* subset, and $B = (\epsilon_{ij})_{i,j\in Q}$ a skew-symmetric $\frac{1}{2}\mathbb{Z}$-valued *exchange matrix.*

Definition 2.1. The *quantum torus algebra* $\mathcal{X}_q^\mathbf{Q}$ is the algebra generated by $\{X_i\}_{i\in Q}$ over $\mathbb{C}[q]$ such that

$$X_i X_j = q^{-2\epsilon_{ij}} X_j X_i \tag{2.1}$$

where X_i are called the quantum cluster variables.

Alternatively, we can let $\Lambda_\mathbf{Q}$ be a $\frac{1}{2}\mathbb{Z}$-lattice with basis $\{e_i\}_{i\in Q}$ and a skew-symmetric form $(e_i, e_j) := \epsilon_{ij}$, then $\mathcal{X}_q^\mathbf{Q}$ is the algebra generated by $\{X_\lambda\}_{\lambda\in\Lambda_\mathbf{Q}}$ over $\mathbb{C}[q^{\frac{1}{2}}]$ such that

$$X_{\lambda+\mu} = q^{(\lambda,\mu)} X_\lambda X_\mu. \tag{2.2}$$

By abuse of notation, we write $X_i := X_{e_i}$ and $X_{i_1,i_2,...,i_k} := X_{e_{i_1}+e_{i_2}+\cdots+e_{i_k}}$.

A cluster seed can be represented by a quiver with vertex set Q, and ϵ_{ij} arrows between node i and j. We use dashed arrows if $\epsilon_{ij} = \frac{1}{2}$, which only occurs between frozen vertices (denoted by *square* nodes on the quiver diagram).

Definition 2.2. Let $q = e^{\pi\sqrt{-1}b^2}$ where $b \in (0,1)$. A *polarization* π_λ of $\mathcal{X}_q^\mathbf{Q}$ on a Hilbert space $L^2(\mathbb{R}^M)$ is an assignment

$$\pi_\lambda : X_i \mapsto e^{2\pi b L_i}, \qquad i \in Q \tag{2.3}$$

where $L_i := L_i(u_k, p_k, \lambda_k)$ is a linear combination of position and momentum operators, and real parameters $\lambda = (\lambda_k)$ such that

$$[L_i, L_j] = \frac{\epsilon_{ij}}{2\pi i} \tag{2.4}$$

and the center of $\mathcal{X}_q^{\mathbf{Q}}$ acts by scalar. Each generator X_i acts as a positive essentially self-adjoint operator on $\mathcal{H}$ and together gives an *integrable representation* of $\mathcal{X}_q^{\mathbf{Q}}$ on $\mathcal{H}$ in the sense of [21], i.e. the relation (2.1) is interpreted as a collection of relations of *bounded* operators by functional calculus:

$$X_i^{\sqrt{-1}s} X_j^{\sqrt{-1}t} = q^{2\epsilon_{ij}st} X_j^{\sqrt{-1}t} X_i^{\sqrt{-1}s}, \qquad \forall s, t \in \mathbb{R}. \tag{2.5}$$

Note that different polarization that fixes the action of the center are unitary equivalent. Furthermore, there exists *quantum cluster mutations* $\mu_k^q : \mathbf{T}_q^{\mathbf{Q}'} \longrightarrow \mathbf{T}_q^{\mathbf{Q}}$ between the field of fractions of the quantum torus algebra which induces unitary equivalence, but we will not need the formula here.

We have the notion of *amalgamation* of two quantum torus algebras which glues the corresponding quivers $\widetilde{\mathbf{Q}} := \mathbf{Q} * \mathbf{Q}'$ along some frozen vertices, and naturally identifies the algebra

$$\mathcal{X}_q^{\widetilde{\mathbf{Q}}} \subset \mathcal{X}_q^{\mathbf{Q}} \otimes \mathcal{X}_q^{\mathbf{Q}'}. \tag{2.6}$$

Finally we define the *basic quiver*, which was first constructed in [11, 17] for all Lie types. We briefly describe the construction following the systematic approach by [5].

Definition 2.3. Let $i, k \in I$. The *elementary quiver* $\overline{\mathbf{J}}_k(i)$ has vertices $Q = Q_0 = (I \setminus \{i\}) \cup \{i_l\} \cup \{i_r\} \cup \{k_e\}$ with adjacency matrix

$$c_{i_l,j} = c_{j,i_r} := \frac{a_{ij}}{2}, \qquad c_{i,i_r} = c_{i_r,k_e} = c_{k_e,i_l} := 1. \tag{2.7}$$

We denote by $\mathbf{J}(i)$ the subquiver without the extra vertex $\{k_e\}$.

Let $\mathbf{i} = (i_1, ..., i_m)$ be a reduced word, and $\{\beta_j\}_{j=1}^m$ a list of positive roots by

$$\beta_j := s_{i_m} s_{i_{m-1}} \cdots s_{i_{j+1}}(\alpha_{i_j}), \qquad \alpha_i \in \Delta_+. \tag{2.8}$$

The quiver $\mathbf{H}(\mathbf{i})$ has vertices $Q = Q_0 = I$ with adjacency matrix

$$c_{ij} := \begin{cases} \operatorname{sgn}(r-s)\frac{a_{ij}}{2} & \beta_s = \alpha_i \text{ and } \beta_r = \alpha_j, \\ 0 & \text{otherwise.} \end{cases} \tag{2.9}$$

The *basic quiver* $\mathbf{Q}(\mathbf{i})$ is defined by

$$\mathbf{Q} := \mathbf{J}_{\mathbf{i}}^{\#}(i_1) * \mathbf{J}_{\mathbf{i}}^{\#}(i_2) * \cdots * \mathbf{J}_{\mathbf{i}}^{\#}(i_m) * \mathbf{H}(\mathbf{i}), \tag{2.10}$$

$$\mathbf{J}_{\mathbf{i}}^{\#}(i_j) := \begin{cases} \overline{\mathbf{J}}_k(i_j) & \text{if } \beta_j = \alpha_k, \\ \mathbf{J}(i_j) & \text{otherwise,} \end{cases} \tag{2.11}$$

where the amalgamations glue side-by-side the right nodes of $\mathbf{J}_{\mathbf{i}}^{\#}(i_k)$ to the left nodes of $\mathbf{J}_{\mathbf{i}}^{\#}(i_{k+1})$ with the same label.

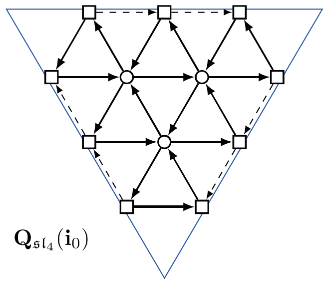
Finally the *symplectic double quiver* $\mathbf{D}(\mathbf{i})$ is defined by the amalgamation

$$\mathbf{D}(\mathbf{i}) := \mathbf{Q}(\mathbf{i}^{op}) * \mathbf{Q}(\mathbf{i}) \tag{2.12}$$

where $\mathbf{i}^{op}$ is the word with opposite order.

Remark 2.4. When $\mathbf{i} = \mathbf{i}_0$ is the longest word, $\mathbf{Q}(\mathbf{i}_0)$ is naturally associated to a triangle, and $\mathbf{D}(\mathbf{i}_0)$ is associated to the triangulation of a punctured disk with two marked points.

Example 2.5. For $\mathfrak{g} = \mathfrak{sl}_4$ and $\mathbf{i}_0 = (3, 2, 1, 3, 2, 3)$, the basic quiver coincides with the n-triangulation of Fock-Goncharov [3].



We can now state the other characterization of positive representations in terms of quantum group embedding.

Theorem 2.6. *[11, 22] For any longest word* $\mathbf{i}_0$*, there exists a quiver* $\mathbf{D}(\mathbf{i}_0)$ *such that we have an embedding of the Drinfeld's double*

$$\mathcal{D}_q(\mathfrak{g}) \hookrightarrow \mathcal{X}_q^{\mathbf{D}(\mathbf{i}_0)}$$

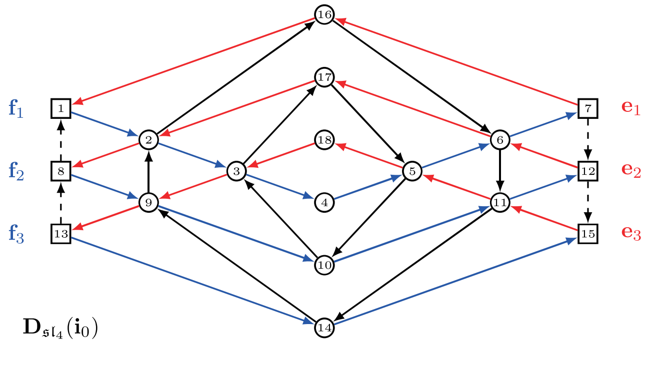
where $\mathbf{K}_i\mathbf{K}_i'$ *lies in the center of* $\mathcal{X}_q^{\mathbf{D}(\mathbf{i}_0)}$*. In particular we have an embedding*

$$\mathcal{U}_q(\mathfrak{g}) \hookrightarrow \mathcal{X}_q^{\mathbf{D}(\mathbf{i}_0)}/\langle \mathbf{K}_i\mathbf{K}_i' = 1\rangle.$$

There exists a polarization π_λ *of* $\mathcal{X}_q^{\mathbf{D}(\mathbf{i}_0)}$ *where* $\pi_\lambda(\mathbf{K}_i\mathbf{K}_i') = 1$ *and the other* n *central characters act by* $e^{2\pi b\lambda_i} \in \mathbb{R}_{>0}$*, such that the composition with the embedding coincides with the expression of the positive representations* $\mathcal{P}_\lambda$*.*

Remark 2.7. The embedding for the lower Borel algebra $\langle \mathbf{f}_i, \mathbf{K}_i'\rangle$ coincides with the *Feigin's homomorphism*, and the explicit expression is very simple. On the other hand, the expressions for $\langle \mathbf{e}_i, \mathbf{K}_i\rangle$ is obtained by repeatedly applying quantum cluster mutations and the expression depends on the choice of $\mathbf{i}_0$ and can be very complicated. However, the representations corresponding to different choice of $\mathbf{i}_0$ are unitarily equivalent.

Example 2.8. The symplectic double quiver $\mathbf{D}_{\mathfrak{sl}_4}(\mathbf{i}_0)$ of the previous example is presented as follows. The embedding of $\mathcal{U}_q(\mathfrak{sl}_4)$ can be expressed as *paths* on the quiver, denoting a telescopic sums (omitting the last term). Each Cartan generator $\mathbf{K}_i, \mathbf{K}_i'$ is given by a single monomial as the product of the nodes along the path.



For example,

$$\begin{aligned}
\mathbf{f}_1 &= X_1 + X_{1,2} + X_{1,2,3} + X_{1,2,3,4} + X_{1,2,3,4,5} + X_{1,2,3,4,5,6},\\
\mathbf{K}_1' &= X_{1,2,3,4,5,6,7},\\
\mathbf{e}_2 &= X_{12} + X_{12,6} + X_{12,6,17} + X_{12,6,17,2},\\
\mathbf{K}_2 &= X_{12,6,17,2,8}.
\end{aligned}$$

For another example of other type, see Example 3.1.

A polarization of this quantum torus algebra recovers the positive representations. We notice that the explicit expressions of the quantum group generators are given by polynomials in the quantum cluster variables.

3 Parabolic Positive Representations

3.1 Main Theorem

Let $J \subset I$ and $W_J \subset W$ be the corresponding *parabolic subgroup* generated by the simple reflections $\{s_j\}_{j\in J}$. Let $\mathbf{i}_J, \mathbf{i}_0$ be the longest word of the longest element $w_J \in W_J, w_0 \in W$, such that $w_0 = w_J\overline{w}$ for some $\overline{w} \in W$. Then we can write the longest word as

$$\mathbf{i}_0 = (\mathbf{i}_J, \bar{\mathbf{i}}) \tag{3.1}$$

for some reduced word $\bar{\mathbf{i}}$ of $\overline{w}$. We observe that

$$\mathbf{Q}(\mathbf{i}_0) = \mathbf{Q}(\mathbf{i}_J) * \mathbf{Q}(\bar{\mathbf{i}}). \tag{3.2}$$

We can now state the new construction of a representation of $\mathcal{U}_q(\mathfrak{g}_\mathbb{R})$:

Main Theorem. *There is a homomorphism $\mathcal{D}_q(\mathfrak{g}) \longrightarrow \mathcal{X}_q^{\mathbf{D}(\bar{\mathbf{i}})}$ such that the image of the generators are universally Laurent polynomials (in the sense of quantum cluster mutations).*

Furthermore, a polarization of $\mathcal{X}_q^{\mathbf{D}(\bar{\mathbf{i}})}$ induces a family of irreducible representations $\mathcal{P}_\lambda^J$ of $\mathcal{U}_q(\mathfrak{g}_\mathbb{R})$ and its modular double $\mathcal{U}_{q\tilde{q}}(\mathfrak{g}_\mathbb{R})$, parametrized by $\lambda \in \mathbb{R}^{|I\setminus J|}$ as positive essentially self-adjoint operators on $L^2(\mathbb{R}^{|\bar{\mathbf{i}}|})$.

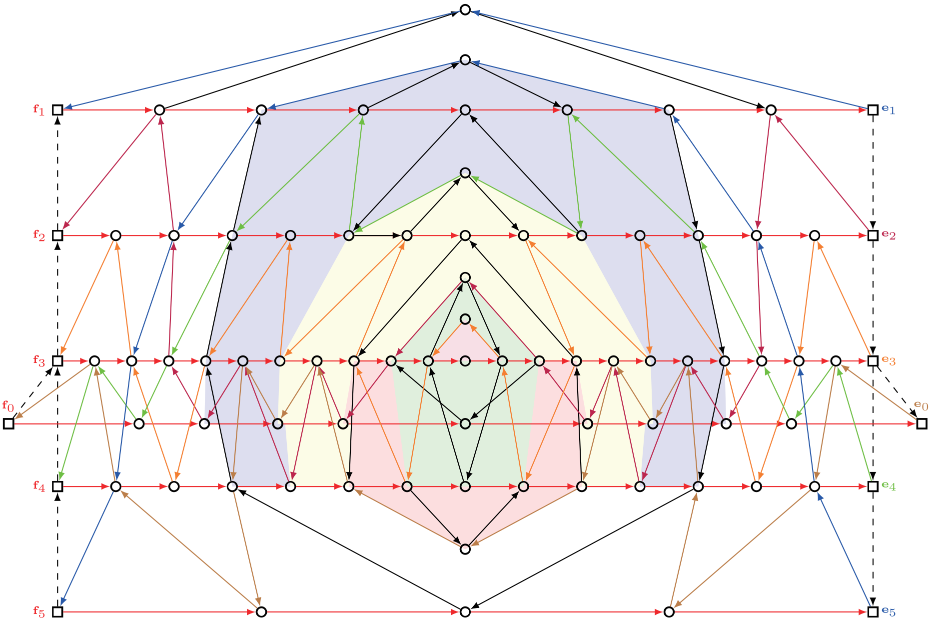
Example 3.1. Consider type E_6 and the longest word decomposition

$$\mathbf{i}_0 = (3\ 43\ 034\ 230432\ 12340321\ 5432103243054321)$$

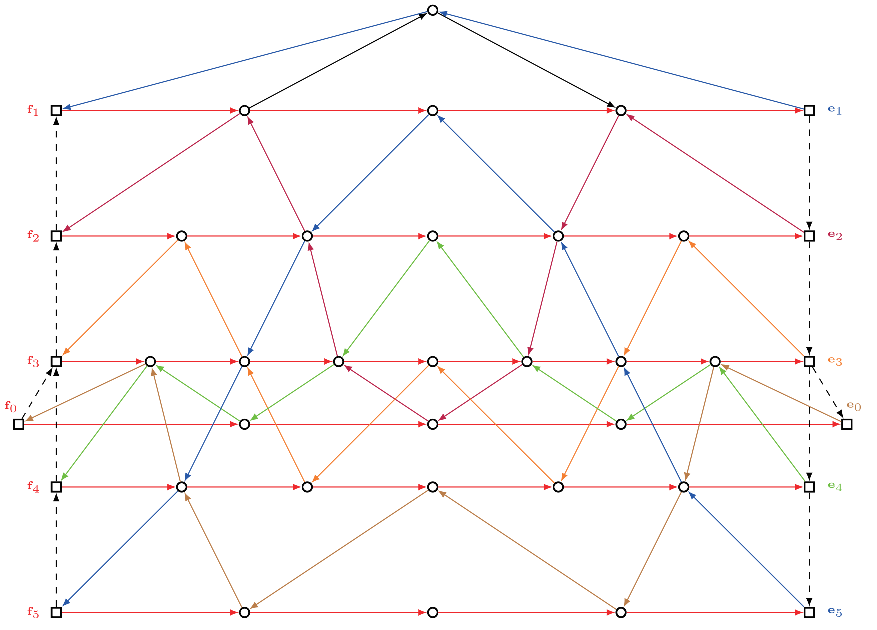
corresponding to the chain of parabolic subgroups of the respective root index

$$A_1 \subset A_2 \subset A_3 \subset D_4 \subset D_5 \subset E_6.$$

We shaded the portion of $\mathbf{D}(\mathbf{i}_J)$ corresponding to the parabolic subgroups below



Correspondingly, for example, the parabolic positive representations of $\mathcal{U}_q(\mathfrak{g}_\mathbb{R})$ for $D_5 \subset E_6$ can be realized by modding out the shaded portion, and we obtain the following paths on the reduced quiver $\mathbf{D}(\bar{\mathbf{i}})$.



3.2 Sketch of Proof

The idea of the proof of the Main Theorem is to observe that one only needs to focus on half of the quantum group embedding to construct a homomorphic image of $\mathcal{U}_q(\mathfrak{g})$.

Definition 3.2. The *generalized Heisenberg double* $\mathcal{H}^{\pm}_{q,\omega}(\mathfrak{g})$ is the associative algebra generated by $\langle \mathbf{e}_i^{\pm}, \mathbf{f}_i^{\pm}, \mathbf{K}_i^{\pm}, \mathbf{K'}_i^{\pm} \rangle_{i\in I}$ satisfying

$$\frac{[\mathbf{e}_i^+, \mathbf{f}_j^+]}{q-q^{-1}} = \delta_{ij}\mathbf{K'}_i^{+} + \omega_{ij}\mathbf{K}_i^{+}, \qquad \frac{[\mathbf{e}_i^-, \mathbf{f}_j^-]}{q-q^{-1}} = \delta_{ij}\mathbf{K}_i{}^{-} - \omega_{ij}\mathbf{K'}_i{}^{-}, \tag{3.3}$$

and other standard quantum group relations, where $\omega_{ij} \in \mathbb{C}$.

Proposition 3.3. *If $\mathcal{H}^{\pm}_{q,\omega}(\mathfrak{g})$ are commuting copies, then*

$$\mathbf{e}_i := \mathbf{e}_i^+ + \mathbf{K}_i^+\mathbf{e}_i^-, \qquad \mathbf{f}_i := \mathbf{f}_i^- + \mathbf{K'}_i{}^{-}\mathbf{f}_i^+,$$
$$\mathbf{K}_i := \mathbf{K}_i^+\mathbf{K}_i^-, \qquad \mathbf{K}'_i := \mathbf{K'}_i{}^{+}\mathbf{K'}_i{}^{-},$$

gives a homomorphic image of $\mathcal{U}_q(\mathfrak{g})$.

Proof. The nontrivial part is to check the Serre relations, which follows since the decomposition has the same algebraic relations as the coproduct Δ. □

The usual Heisenberg double [16] corresponds to the case when $\omega_{ij} \equiv 0$.

Proposition 3.4. *The embedding $\mathcal{D}_q(\mathfrak{g}) \hookrightarrow \mathcal{X}_q^{\mathbf{D}(\mathbf{i}_0)} \subset \mathcal{X}_q^{\mathbf{Q}(\mathbf{i}_0^{op})} \otimes \mathcal{X}_q^{\mathbf{Q}(\mathbf{i})}$ decomposes as in Proposition 3.3 where $\langle \mathbf{e}_i^{\pm}, \mathbf{f}_i^{\pm}, \mathbf{K}_i^{\pm}, \mathbf{K'}_i^{\pm} \rangle$ generates $\mathcal{H}^{\pm}_{q,0}$ with*

$$\mathcal{H}^+_{q,0}(\mathfrak{g}) \hookrightarrow 1 \otimes \mathcal{X}_q^{\mathbf{Q}(\mathbf{i}_0)}, \qquad \mathcal{H}^-_{q,0}(\mathfrak{g}) \hookrightarrow \mathcal{X}_q^{\mathbf{Q}(\mathbf{i}_0^{op})} \otimes 1.$$

Therefore it suffices to study one half $\mathbf{Q}(\mathbf{i}_0) \subset \mathbf{D}(\mathbf{i}_0)$ of the quiver.

Definition 3.5. Let $J \subset I$. The *double Dynkin involution* of $i \in I$ is the unique index $i^{**} \in I$ such that

$$w_0 s_i = s_{i^*} w_0 = s_{i^*} w_J \overline{w} = w_J s_{i^{**}} \overline{w}. \tag{3.4}$$

Equivalently

$$\Longleftrightarrow i^{**} := (i^{*_W})^{*_{W_J}},$$

where $*_W$ is the standard Dynkin involution, and $i^{*_{W_J}} = i$ if $i \notin J$.

The proof of the Main Theorem now follows immediately from the following *Decomposition Lemma*, which is the technical heart of the result.

Lemma 3.6. *The embedding $\mathcal{H}_q^+(\mathfrak{g}) \hookrightarrow \mathcal{X}_q^{\mathbf{Q}(\mathbf{i}_0)} \subset \mathcal{X}_q^{\mathbf{Q}(\mathbf{i}_J)} \otimes \mathcal{X}_q^{\mathbf{Q}(\bar{\mathbf{i}})}$ can be decomposed into the form*

$$\mathbf{e}_i^+ = \overline{\mathbf{e}_i} + \overline{\mathbf{K}_i}\mathbf{e}_{i^{**}}^J, \qquad \mathbf{f}_i^+ = \mathbf{f}_i^J + {\mathbf{K}'_i}^J\overline{\mathbf{f}_i},$$
$$\mathbf{K}_i^+ = \mathbf{K}_{i^{**}}^J\overline{\mathbf{K}_i}, \qquad {\mathbf{K}'_i}^+ = {\mathbf{K}'_i}^J\overline{\mathbf{K}'_i},$$

where $\mathbf{e}_i^J = \mathbf{f}_i^J := 0$ and $\mathbf{K}_i^J = {\mathbf{K}'_i}^J := 1$ if $i \notin J$, such that

- *$\mathbf{X}_i^J \in \mathcal{X}_q^{\mathbf{Q}(\mathbf{i}_J)} \otimes 1$ and $\overline{\mathbf{X}_i} \in 1 \otimes \mathcal{X}_q^{\mathbf{Q}(\bar{\mathbf{i}})}$ for $\mathbf{X} = \mathbf{e}, \mathbf{f}, \mathbf{K}, \mathbf{K}'$,*
- *$\{\mathbf{e}_i^J, \mathbf{f}_i^J, \mathbf{K}_i^J, {\mathbf{K}'_i}^J\} \simeq \mathcal{H}_q^+(\mathfrak{g}_J)$ in $\mathcal{X}_q^{\mathbf{Q}(\mathbf{i}_J)}$ where $\mathfrak{g}_J \subset \mathfrak{g}$,*
- *we have*
$$\langle \overline{\mathbf{e}_i}, \overline{\mathbf{f}_i}, \overline{\mathbf{K}_i}, \overline{\mathbf{K}'_i} \rangle \simeq \mathcal{H}_{q,\omega}^+(\mathfrak{g})$$
on $\mathcal{X}_q^{\mathbf{Q}(\bar{\mathbf{i}})}$ for some $\omega_{ij} \in \{0,1\}$.

Proof. The decomposition of $\mathbf{f}_i, \mathbf{K}'_i$ follows from explicit calculation using Feigin's embedding. The decomposition of $\mathbf{e}_i, \mathbf{K}_i$ requires combinatorics of Coxeter moves, namely, if $\mathbf{i}$ is the reduced word of $w \in W$ and $l(s_i w s_j) = l(w)$, then there is a sequence of Coxeter moves that brings

$$\mathbf{i} = (i, ...,) \mapsto \mathbf{i}' = (..., j)$$

where the sequence of Coxeter moves splits into 2 stages, the second of which increase in indices consecutively from first to last letter. Together with the explicit construction of the representation of $\mathbf{e}_i$ by quantum cluster mutations give the required decomposition. □

4 Applications

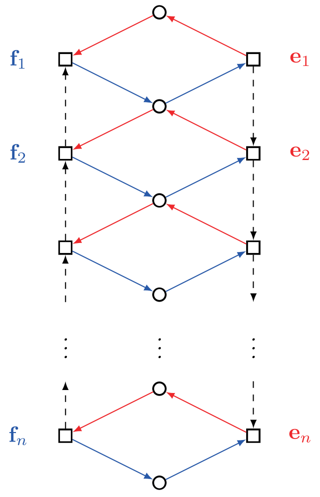
4.1 Minimal Positive Representations

Let us consider type A_n and take the subset $J = \{1, 2, ..., n-1\} \subset I$. Then the longest word can be decomposed as

$$\mathbf{i}_0 = (\mathbf{i}_J \bar{\mathbf{i}}) \tag{4.1}$$

where $\bar{\mathbf{i}} = (n\ \cdots\ 3\ 2\ 1)$ has length n. Hence according to the Main Theorem, we obtain the parabolic positive representation of $\mathcal{U}_q(\mathfrak{sl}(n+1, \mathbb{R}))$ on the space $L^2(\mathbb{R}^n)$, which we call the **minimal positive representation**, and can be explicitly realized on the quiver diagram as follows.

Theorem 4.1. *[12] The polarization of the quiver $\mathbf{D}(\mathbf{i})$ for $\mathbf{i} = (n, ..., 3, 2, 1)$ gives a family of irreducible representations $\mathcal{P}_\lambda^J$ of $\mathcal{U}_q(\mathfrak{sl}(n+1, \mathbb{R}))$ acting on $L^2(\mathbb{R}^n)$ as positive essentially self-adjoint operators, parametrized by $\lambda \in \mathbb{R}$.*



The representation has the following properties:

- *The non-simple generators* $\mathbf{e}_\alpha, \mathbf{f}_\alpha$, $\alpha \in \Phi_+$ *defined by Lusztig's braid group action, are non-degenerate, and hence the universal* $\mathcal{R}$ *operator*

$$\mathcal{R} = \mathcal{K} \prod_{\alpha\in\Phi_+} g_b(\mathbf{e}_\alpha \otimes \mathbf{f}_\alpha)$$

  is well-defined. Here $\mathcal{K}$ *is the Cartan part and* g_b *is the non-compact quantum dilogarithm function.*

- *The Casimir elements* $\mathbf{C}_k$ *acts by real-valued scalar, and lie outside the* positive spectrum *[10] of the standard positive representations.*

4.2 Evaluation Modules

Interestingly, this construction gives another model for the **evaluation module** of the affine $\mathcal{U}_q(\widehat{sl}_{n+1})$.

More precisely, by wrapping the quiver around and gluing the top to the bottom, we obtain naturally a quiver $\widehat{\mathbf{D}(\mathbf{i})}$ that describes a positive representation of the split real form of $\mathcal{U}_q(\widehat{\mathfrak{sl}}_{n+1})$.

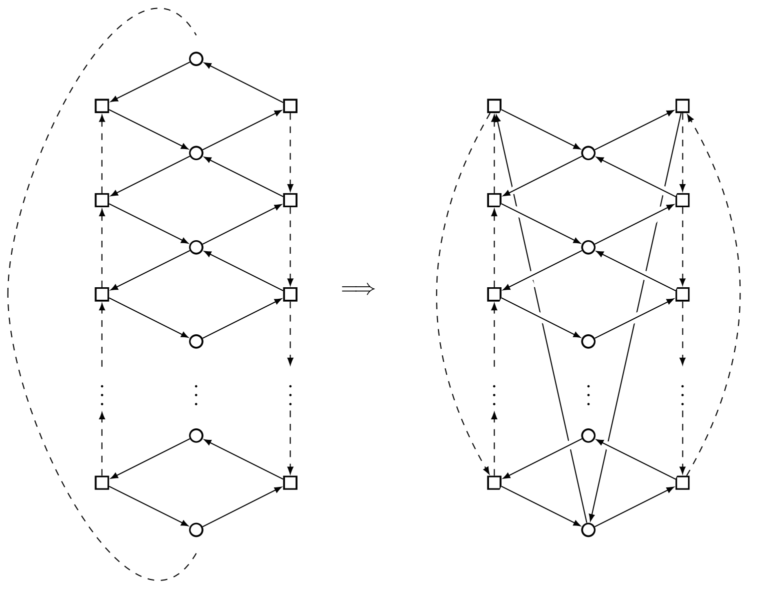
Theorem 4.2. *The positive representation of* $\mathcal{U}_q(\widehat{\mathfrak{sl}}_{n+1})$ *defined by the polarization of* $\widehat{\mathbf{D}(\mathbf{i})}$ *is unitarily equivalent to the evaluation module [15]* $\mathcal{P}_\lambda^\mu$

$$\mathcal{U}_q(\widehat{\mathfrak{sl}}_{n+1}) \longrightarrow \mathcal{U}_q(\mathfrak{sl}_{n+1})$$

of the minimal positive representations $\mathcal{P}_\lambda^J$ *of* $\mathcal{U}_q(\mathfrak{sl}_{n+1})$*, where the evaluation parameter* $\mu \in \mathbb{R}$ *is given by the action of the central element*

$$e^{\pi b\mu} := \pi(D_0^{\frac{1}{n+1}} D_1).$$

Here D_0 *is the product of all cluster variables of the middle vertices, and* D_1 *is the product of all vertices on the right column.*



$\mathbf{D}(\mathbf{i})$ $\widehat{\mathbf{D}(\mathbf{i})}$

4.3 Further Discussions

The nice properties of the minimal positive representations suggest that one can investigate further the structure of parabolic positive representations, including:

- Tensor product decomposition of $\mathcal{P}_\lambda^J \otimes \mathcal{P}_{\lambda'}^J$, since we have seen that the $\mathcal{R}$ matrix is well-defined, and one can also study the spectrum of the Casimir operators. Following the strategy in [13], one may also ask whether the minimal positive representations admit a semiclassical limit to a tensor product decomposition without multiplicities.
- To understand the geometric meaning of the cluster structure of $\mathbf{D}(\bar{\mathbf{i}})$ in terms of the **partial configuration space** $Conf_{\overline{w}}^e(\mathcal{A})$ described in [5].
- We observe that the positive representations in the parabolic case are not only universally Laurent polynomials, but in fact they are always honest polynomials in the cluster variables in any cluster seed. So it will be natural to give a combinatorial description of the explicit embeddings.
- One may also seek generalization to other modules of affine quantum groups $\mathcal{U}_q(\widehat{\mathfrak{g}}_{\mathbb{R}})$ by utilizing some special form of parabolic positive representations as above, or constructing certain "building blocks" such that one can construct a representation for any quantum Kac-Moody algebra.

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