PARABOLIC TRANSMISSION EIGENVALUES-FREE REGIONS FOR MAXWELL EQUATIONS

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1. Introduction

Let $E, \hat{E}, H, \hat{H} \in H^1(\Omega : \mathbb{C}^3)$ be vector-valued functions in open bounded domain $\Omega \subset \mathbb{R}^3$ with C^∞ boundary Γ satisfying the system

$$\begin{cases} \operatorname{curl} E = \mathbf{i}\lambda\mu H, \\ \operatorname{curl} H = -\mathbf{i}\lambda\gamma E, \end{cases} \quad \begin{cases} \operatorname{curl} \hat{E} = \mathbf{i}\lambda\hat{\mu}\hat{H}, \\ \operatorname{curl} \hat{H} = -\mathbf{i}\lambda\hat{\gamma}\hat{E}, \end{cases} \quad x \in \Omega, \tag{1.1}$$

with boundary conditions

$$\nu \wedge E = \nu \wedge \hat{E}, \ \nu \wedge H = \nu \wedge \hat{H}, \ x \in \Gamma. \tag{1.2}$$

Here $\nu(x)$ is the exterior unit normal vector of Γ at $x \in \Gamma$, and $\gamma(x), \hat{\gamma}(x), \mu(x), \hat{\mu}(x)$ are positive smooth functions. The values $\lambda \in \mathbb{C} \setminus \{0\}$ for which (1.1)-(1.2) has a non-trivial solution $(E, \hat{E}, H, \hat{H}) \neq 0$ are called *interior transmission* eigenvalues (ITE). These eigenvalues play a crucial role in the linear sampling and factorization methods in inverse scattering. We refer to [2] for the results of these topics and related references.

The (ITE) for the wave equation are studied very intensively in the last 20 years in many works. The main problems were the discreteness of the (ITE) in $\mathbb{C}$ and their location in the complex plan. A more difficult problem is the existence of (ITE) and the Weyl asymptotic of the counting function $N(r) = \sharp\{\lambda_j \in \mathbb{C} : |\lambda_j| \leq r\}$ as $r \to \infty$ (see [8]). For the location of the transmission eigenvalues the reader may see [10], [11], [12] and the reference cited there. The analysis of the (ITE) for Maxwell equations attired the attention of many researchers. Under different assumptions the discreteness of the (ITE) has been established in [4], [1], [7], [3]. The location of the (ITE) has been examined in [3] and under the hypothesis

$$d(x) = \gamma(x)\hat{\mu}(x) - \hat{\gamma}(x)\mu(x) \neq 0, \ \gamma(x) \neq \hat{\gamma}(x), \ \mu(x) \neq \hat{\mu}(x), \ \forall x \in \Gamma,$$

the authors prove that for any $\epsilon > 0$ there are no (ITE) in the domain

$$\{\lambda \in \mathbb{C} : |\operatorname{Im} \lambda| \geq \epsilon |\operatorname{Re} \lambda|, \ |\operatorname{Re} \lambda| \geq C_\epsilon > 0\}.$$

As $\epsilon \searrow 0$, one could have $C_\epsilon \nearrow \infty$ and the region, where $|\operatorname{Re} \lambda|$ is bounded, is not covered in [3]. As in the case of wave equation [10], we expect that if $(\mu(x) - \hat{\mu}(x))d(x) < 0, \ x \in \Gamma$, then there exist an infinite number of (ITE) with $|\operatorname{Re} \lambda| \leq 1$ converging to the imaginary axis. On the other hand, in the case studied in [3] the celebrated complementing condition of Agmon, Douglas and Nirenberg is satisfied.

Here we discuss transmission eigenvalues-free regions in $\mathbb{C}$ in the **isotropic case** with analogy with the results in [10]. Notice that in this case the complementing condition mentioned above is not satisfied. In a recent paper under the condition

$$\mu(x) = \hat{\mu}(x),\ \gamma(x) \neq \hat{\gamma}(x),\ \forall x \in \Gamma \tag{1.3}$$

G. Vodev proved in [13] that there are no transmission eigenvalues in the region

$$\big\{\lambda \in \mathbb{C}:\ |\operatorname{Im}\lambda| \geq C_1(1 + |\operatorname{Re}\lambda|)^{5/7}\big\},\ C_1 > 0 \tag{1.4}$$

with a constant $C_1 > 0$ independent of λ. Under stronger assumptions we improve [9] the above result in the following

Theorem 1. *Assume the conditions:*

$$\mu(x) = \hat{\mu}(x),\ \partial_\nu \mu(x) = \partial_\nu \hat{\mu}(x),\ \gamma(x) \neq \hat{\gamma}(x),\ \forall x \in \Gamma, \tag{1.5}$$

$$\operatorname{grad}(\log\gamma)(x) = \operatorname{grad}(\log\hat{\gamma})(x),\ \forall x \in \Gamma. \tag{1.6}$$

Then there exists a constant $C_1 > 0$ independent of λ such that there are no transmission eigenvalues in the region

$$\Lambda = \big\{\lambda \in \mathbb{C}:\ |\operatorname{Im}\lambda| \geq C_1(1 + |\operatorname{Re}\lambda|)^{3/5}\big\}. \tag{1.7}$$

The condition (1.6) is related to our argument and it is not clear if it is possible to obtain the eigenvalues-free region (1.7) without it. On the other hand, it is natural to conjecture that under the assumption (1.5) one has an eigenvalues-free region $\{\lambda \in \mathbb{C} : |\operatorname{Im}\lambda| \geq C_0 > 0\}$ with a suitable constant $C_0 > 0$ (see [11] for the results concerning the wave equation).

2. Preliminaries

From (1.1) one deduces

$$\operatorname{curl}\operatorname{curl} E = \lambda^2\gamma\mu E + \frac{\operatorname{grad}(\mu)}{\mu} \wedge \operatorname{curl} E.$$

On the other hand, $\operatorname{div}(\gamma E) = \operatorname{div}(\hat{\gamma}\hat{E}) = 0$ and

$$\operatorname{div} E + \langle \frac{\operatorname{grad}(\gamma)}{\gamma}, E\rangle = 0,\ \operatorname{div}\hat{E} + \langle \frac{\operatorname{grad}(\hat{\gamma})}{\hat{\gamma}}, \hat{E}\rangle = 0.$$

Therefore, E satisfies the system

$$L(E) = -\Delta E - \lambda^2\gamma\mu E - \operatorname{grad}\langle \operatorname{grad}(\log\gamma), E\rangle - \operatorname{grad}(\log\mu) \wedge \operatorname{curl} E = 0,$$

while $\hat{E}$ satisfies the system

$$\hat{L}(\hat{E}) = -\Delta\hat{E} - \lambda^2\hat{\gamma}\hat{\mu}\hat{E} - \operatorname{grad}\langle \operatorname{grad}(\log\hat{\gamma}), \hat{E}\rangle - \operatorname{grad}(\log\hat{\mu}) \wedge \operatorname{curl}\hat{E} = 0.$$

For $E = (E_1, E_2, E_3)$ consider the operator

$$\operatorname{grad}(\langle \operatorname{grad}(\log\gamma), E\rangle) = \operatorname{grad}\Big(\sum_k \partial_{x_k}(\log\gamma)E_k\Big) = \Big(\sum_k \partial_{x_k}(\log\gamma)\partial_{x_j}E_k\Big)_{j=1,2,3}$$
$$+\Big(\sum_k \partial^2_{x_j x_k}(\log\gamma)E_k\Big)_{j=1,2,3}$$

and set

$$m_{1,j,k}(x,\partial_x;\gamma)=\partial_{x_k}(\log\gamma)\partial_{x_j},\ \mathcal{M}_1(x,\partial_x;\gamma)=\Big\{m_{1,j,k}(x,\partial_x;\gamma)\Big\}_{j,k=1,2,3},$$

$$m_{0,j,k}(x,\partial_x;\gamma)=\partial^2_{x_jx_k}(\log\gamma),\ \mathcal{M}_0(x;\gamma)=\Big\{m_{0,j,k}(x,\partial_x;\gamma)\Big\}_{j,k=1,2,3}.$$

In the same way we get

$$\mathrm{grad}\,(\log\mu)\wedge\mathrm{curl}\,E=\mathcal{M}_1(x,\partial_x;\mu)E.$$

We can write the equation for E close to the boundary as

$$-\Delta E-(\mathcal{M}_1(x,\partial_x;\gamma)+\mathcal{M}_1(x,\partial_x;\mu))E+\mathcal{M}_0(x;\gamma)E-\lambda^2\gamma\mu E=0$$

and similarly, one has

$$-\Delta\hat{E}-(\mathcal{M}_1(x,\partial_x;\hat{\gamma})+\mathcal{M}_1(x,\partial_x;\hat{\mu}))\hat{E}+\mathcal{M}_0(x;\hat{\gamma})\hat{E}-\lambda^2\hat{\gamma}\hat{\mu}\hat{E}=0.$$

Here $\mathcal{M}_1(x,\partial_x;\gamma)$, $\mathcal{M}_1(x,\partial_x;\mu)$ are first order matrix-valued differential operators, while $\mathcal{M}_0(x;\gamma)$ is a smooth matrix. We use similar notations for the equation for $\hat{E}$.

Introduce the sets

$$\begin{aligned}Z_1 &:= \{z\in\mathbb{C}:\ \mathrm{Re}\,z=1,\quad 0<|\mathrm{Im}\,z|\le 1\},\\ Z_2 &:= \{z\in\mathbb{C}:\mathrm{Re}\,z=-1,\quad 0\le|\mathrm{Im}\,z|\le 1\},\\ Z_3 &:= \{z\in\mathbb{C}:|\mathrm{Re}\,z|\le 1,\quad |\mathrm{Im}\,z|=1\}.\end{aligned}$$

Set $\lambda^2=\frac{z}{h^2}$, $z\in Z_1\cup Z_2\cup Z_3$, $0<h\le 1$. Then we obtain the system

$$\begin{cases}-h^2\Delta E-h(\mathcal{M}_1(x,h\partial_x;\gamma)+\mathcal{M}_1(x,h\partial_x;\mu))E+h^2\mathcal{M}_0(x;\gamma)E-z\gamma\mu E=0,\\ -h^2\Delta\hat{E}-h(\mathcal{M}_1(x,h\partial_x;\hat{\gamma})+\mathcal{M}_1(x,h\partial_x;\hat{\mu}))\hat{E}+h^2\mathcal{M}_0(x;\hat{\gamma})\hat{E}-z\hat{\gamma}\hat{\mu}\hat{E}=0,\end{cases}\quad x\in\omega \tag{2.1}$$

with boundary conditions

$$E|_{tan}=\hat{E}|_{tan},\ \ \nu\wedge\Big(\frac{1}{\mu}\mathrm{curl}\,E\Big)=\nu\wedge\Big(\frac{1}{\hat{\mu}}\mathrm{curl}\,\hat{E}\Big),\ \ x\in\Gamma. \tag{2.2}$$

Here $\omega\subset\Omega$ is a small neighborhood of the boundary Γ. Let $D_{x_j}=-\mathbf{i}\partial_{x_j}$, $j=1,2,3$, $\mathrm{grad}\,f=\{D_{x_j}f\}_{j=1,2,3}$. Consider the geodesic normal coordinates $(y_1,y')\in\mathbb{R}^3$ on a neighborhood of a point $x_0\in\Gamma$ determined as follows. For a point $x\in\omega$, $y'(x)$ is the closest point in Γ and $y_1=\mathrm{dist}\,(x,\Gamma)$. Let $\nu(x)$ be the unit normal in the direction of increasing y_1 to the surface $y_1=$ constant through x. Thus $\nu(x)$ is an extension of the unit normal vector to a unit vector field. The boundary Γ becomes $y_1=0$ and

$$x\ =\ \alpha(y_1,y')\ =\ \beta(y')\ +\ y_1\nu(y').$$

Therefore, setting $D_\nu=-\mathbf{i}\partial_\nu$, one has

$$\nu\wedge\frac{1}{\mathbf{i}\mu}\mathrm{curl}\,E\big|_{tan}=-\frac{1}{\mu}(D_\nu E_{tan})|_{tan}+(\mathrm{grad}\,(\frac{1}{\mu}E_{nor})|_{tan}-\mathbf{i}g_0(\frac{1}{\mu}E|_{tan})|_{tan},$$

where $E_{nor} := \langle E, \nu\rangle,\ \hat{E}_{nor} := \langle \hat{E}, \nu\rangle$. From (2.2) we get for $x \in \Gamma$ the boundary conditions

$$\begin{cases} E_{tan} - \hat{E}_{tan} = 0, \\ \frac{1}{\mu}(D_\nu E_{tan})|_{tan} - \frac{1}{\mu}(\mathrm{grad}\, E_{nor})|_{tan} - \frac{1}{\hat{\mu}}(D_\nu \hat{E}_{tan})|_{tan} + \frac{1}{\hat{\mu}}(\mathrm{grad}\, \hat{E}_{nor})|_{tan} \\ = -\mathbf{i}\Big(\frac{1}{\hat{\mu}} - \frac{1}{\mu}\Big)\langle E_{tan}, \partial_x \nu\rangle|_{tan}, \\ \mathrm{div}\, E + \langle \mathrm{grad}\,(\log\gamma), E\rangle = 0, \\ \mathrm{div}\, \hat{E} + \langle \mathrm{grad}\,(\log\hat{\gamma}), \hat{E}\rangle = 0, \end{cases} \tag{2.3}$$

Next for a vector-valued function $u = (u_1, u_2, u_3) \in \mathbb{C}^3$ we have the equality

$$\begin{aligned} \mathrm{div}\, u := \frac{1}{\mathbf{i}}\mathrm{div}\, u(\alpha(y_1, y')) &= \langle D_{y_1} u(\alpha(y_1, y')), \nu(y')\rangle \\ &+ \sum_{k=1}^3 \sum_{j=2}^3 \frac{\partial y_j}{\partial x_k} D_{y_j} u_k(\alpha(y_1, y')) \end{aligned} \tag{2.4}$$

$$= D_{y_1}\Big(u_{nor}(t, y')\Big) + \sum_{j=2}^3 \Big\langle D_{y_j} u_{tan}(\alpha(y_1, y')), \frac{\partial y_j}{\partial x}\Big\rangle - \mathbf{i} u_{nor}\mathrm{div}\,\nu,$$

where $\langle u(\alpha(y_1, y')), \nu(y')\rangle := u_{nor}(y_1, y')$.

Below we introduce some notations for h-pseudo-differential operators (see [6] for more details). Let X be a C^∞ smooth compact manifold without boundary with dimension $d \geq 2$. Let (x, ξ) be the coordinates in $T^*(X)$ and let $a(x, \xi, h) \in C^\infty(T^*(X))$. Given $\ell \in \mathbb{R}, \delta > 0$ and a function, one denotes by S_δ^ℓ the set of symbols so that

$$|\partial_x^\alpha \partial_\xi^\beta a(x, \xi, h)| \leq C_{\alpha,\beta} h^{\ell - \delta(|\alpha| + |\beta|)},\ \forall\alpha, \forall\beta, \quad (x, \xi) \in T^*(X).$$

The $h-$pseudo-differential operator with symbol $a(x, \xi, h)$ is defined by

$$(Op_h(a)f)(x) := \ (2\pi h)^{-d} \int_{T^*X} e^{\mathbf{i}\langle y - x, \xi\rangle/h} a(x, \xi, h) f(y) dy d\xi.$$

Also, as in [10], we introduce for $k \in \mathbb{R},\ 0 \leq \delta \leq 1/2$, the class of symbols $\mathcal{S}_\delta^k$ such that

$$|\partial_\alpha \partial_\beta a(x, \xi)| \leq C_{\alpha,\beta} h^{-\delta(|\alpha| + |\beta|)} \langle\xi\rangle^{k - |\beta|},\ \forall\alpha,\ \forall\beta.$$

We will recall some results for the Dirichlet-to-Neumann map (see [10], [5]). Setting $n = \gamma > 0$, consider the operator

$$\mathcal{P}(\mathcal{M}_1, \mathcal{M}_0; h)F = (-h^2\Delta_x - h\mathcal{M}_1(x, h\partial_x) + h^2\mathcal{M}_0(x))F.$$

In the geodesic normal coordinates (y_1, y') close the boundary it has the form

$$\mathcal{P}(\mathcal{M}_1, \mathcal{M}_0; h) = h^2 D_{y_1}^2 + r(y, hD_{y'}) + h\langle q_1(y), hD_y\rangle - ih\tilde{\mathcal{M}}_1(y, hD_y) + h^2\tilde{\mathcal{M}}_0(y)$$

with $r(y, \eta') = \langle R(y)\eta', \eta'\rangle,\ q_1(y) \in C^\infty$. Here

$$R(y) = \Big\{\sum_{k=1}^3 \frac{\partial y_m}{\partial x_k}\frac{\partial y_j}{\partial x_k}\Big\}_{m,j=2}^3 = \Big\{\Big\langle\frac{\partial y_m}{\partial x}, \frac{\partial y_j}{\partial x}\Big\rangle\Big\}_{m,j=2}^3$$

is a symmetric (2×2) matrix and $r(0, y', \eta') = r_0(y', \eta')$, where $r_0(y', \eta')$ is the principal symbol of the Laplace-Beltrami operator $-h^2\Delta_\Gamma$ on Γ equipped with the Riemannian metric induced by the Euclidean one in $\mathbb{R}^3$. We use the notation $\mathcal{P}(\mathcal{M}_1, \mathcal{M}_0; h)$ to precise the dependence on the matrix-valued operators $\mathcal{M}_1(y, hD_y)$, $\mathcal{M}_0(y)$. Clearly, if $\gamma, \hat{\gamma}, \mu, \hat{\mu}$ are constants, we have $\mathcal{M}_1 = \mathcal{M}_0 = 0$ and one obtains the vector-valued Laplacian $-h^2\Delta$. For $z \in Z_1 \cup Z_2 \cup Z_3$ introduce $\rho_n(y', \eta', z) = \sqrt{\mu n z - r_0(y', \eta')} \in C^\infty(T^*\Gamma)$ as the root of the equation

$$\rho^2 + r_0(y', \eta') - \mu n z = 0$$

with $\operatorname{Im} \rho_n(y', \eta', z) > 0$. For $z \in Z_1 \cup Z_3$ we have (see [10])

$$\operatorname{Im} \rho_n(y', \eta', z) \ \geq \ \frac{|\operatorname{Im} z|}{C}, \ |\rho_n| \geq C\sqrt{|\operatorname{Im} z|}, \tag{2.5}$$

while for $r_0 \geq 2\mu n$ we have $C_1\sqrt{r_0 + 1} \geq 2\operatorname{Im} \rho_n \geq |\rho_n| \geq C_2\sqrt{r_0 + 1}$. For $z \in Z_2$ the last inequalities hold if $r_0 \geq 0$. Similarly, let $\rho_{\hat{n}}(y', \eta', z)$ be the root of the equation $\rho^2 + r_0(y', \eta') - \hat{\mu}\hat{n}z = 0$ with $\hat{n} = \hat{\gamma}$, $\operatorname{Im} \rho_{\hat{n}}(y', \eta', z) > 0$. Let $u \in \mathbb{C}^3$ be the solution of the Dirichlet problem

$$\begin{aligned}(\mathcal{P}(\mathcal{M}_1, \mathcal{M}_0; h) - \mu\gamma z)u = (-h^2\Delta_x - h\mathcal{M}_1(x, h\partial_x) + h^2\mathcal{M}_0(x) - \mu\gamma z)u = 0, \text{ in } \Omega, \\ u = g \text{ on } \Gamma.\end{aligned} \tag{2.6}$$

Consider the semi-classical Sobolev spaces $H_h^k(\Gamma)$ with norm $\|(1 - h^2\Delta)^{s/2}u\|_{L^2(\Gamma)}$ and introduce the semi-classical Dirichlet-to-Neumann map

$$\mathcal{N}(nz, h) : H_h^s(\Gamma) \ni g \longrightarrow hD_\nu u|_\Gamma \in H_h^{s-1}(\Gamma).$$

G. Vodev [10], [12] established for bounded domains $\Omega \subset \mathbb{R}^d$, $d \geq 2$, with C^∞ boundary and *scalar solutions* u of the problem $(-h^2\Delta - \mu n z)u = 0, u|_\Gamma = g$ the following approximation of the Dirichlet-to-Neumann map.

Theorem 2 ([12]). *For every* $0 < \epsilon \ll 1$ *there exists* $0 < h_0(\epsilon) \ll 1$ *such that for* $z \in Z_1(1/2 - \epsilon) := \{z \in Z_1, \ |\operatorname{Im} z| \geq h^{\frac{1}{2}-\epsilon}\}$ *and* $0 < h \leq h_0(\epsilon)$ *we have*

$$\|hD_\nu u|_\Gamma - Op_h(\rho_n + hb)g\|_{H_h^1(\Gamma)} \leq \frac{Ch}{|\operatorname{Im} z|^{3/2}}\|g\|_{L^2(\Gamma)}, \tag{2.7}$$

where $b \in \mathcal{S}_0^0(\Gamma)$ *does not depend on* $(\mu n)|_\Gamma$. *Moreover,* (2.7) *holds for* $z \in Z_2 \cup Z_3$ *with* $|\operatorname{Im} z|$ *replaced by* 1.

The result of Theorem 2 holds for vector-valued solutions G of the equation $(-h^2\Delta - \mu n z)G = 0$ (see [5]). For a more general operator $\mathcal{P}(\mathcal{M}_1, \mathcal{M}_0; h)$ with matrix-valued lower order terms given above some modifications in the construction of the parametrix are necessary (see [9]). For the problem (2.6) with $g = E_0 = E|_\Gamma$, $E_{nor}|_\Gamma = \langle E_0, \nu\rangle$, with a matrix-valued symbol $\mathbf{b} \in \mathcal{S}_0^0(\Gamma)$ which depends on μ, n we obtain an analog of (2.7)

$$\begin{aligned}\Big\|h(D_\nu E)|_\Gamma - Op_h(\rho_n + h\mathbf{b})E_0\Big\|_{H_h^1(\Gamma)} \leq \frac{Ch}{|\operatorname{Im} z|^{3/2}}\|E_0\|_{L^2(\Gamma)}, \\ \Big\|h\langle D_\nu E, \nu\rangle|_\Gamma - \langle Op_h(\rho_n + h\mathbf{b})E_0, \nu\rangle\Big\|_{H_h^1(\Gamma)} \leq \frac{Ch}{|\operatorname{Im} z|^{3/2}}\|E_0\|_{L^2(\Gamma)}\end{aligned} \tag{2.8}$$

with the same improvement for $z \in Z_2 \cup Z_3$. For $\hat{E}$ one has the same result with symbols $\rho_{\hat{n}}, \hat{\mathbf{b}}$ which depend on $\hat{\mu}, \hat{\gamma}$. The important point is that exploiting the assumptions (1.5), (1.6), we show that $\mathbf{b} = \hat{\mathbf{b}}$.

Obviously, $\langle D_\nu G, \nu\rangle|_\Gamma = D_\nu(\langle G, \nu\rangle)|_\Gamma$ so we have an approximation for the derivative $D_\nu(\langle G, \nu\rangle)|_\Gamma$. Notice that for $z \in Z_1(1/2-\epsilon)$ we have $\rho_n \in \mathcal{S}^1_{1/2-\epsilon}$, while for $z \in Z_2 \cup Z_3$ one has $\rho_n \in \mathcal{S}^1_0$. More precisely, the following properties have been proved in [10]. Let $\phi \in C_0^\infty(\mathbb{R})$ be such that $\phi(t) = 1$ for $|t| \leq 1$, $\phi(t) = 0$ for $|t| \geq 2$. Set

$$\eta(x', \xi') = \phi(\delta_0 r_0(x', \xi')),\ \mathcal{M}_1 := Z_{1,0} \times \operatorname{supp} \eta,$$

$$\mathcal{M}_2 := \Big(Z_1 \times \ \operatorname{supp}\,(1-\eta)\Big) \cup \Big(Z_2 \times T^*(\Gamma)\Big) \cup \Big(Z_3 \times T^*(\Gamma)\Big),$$

where $0 < 2\delta_0 \leq \mu n$. Then (see [10])

$$|\partial_{x'}^\alpha \partial_{\xi'}^\beta \rho_n| \leq C_{\alpha,\beta} |\operatorname{Im} z|^{1/2-|\alpha|-|\beta|} \text{ for } (z, x', \xi') \in \mathcal{M}_1,\ |\alpha| + |\beta| \geq 1, \tag{2.9}$$

$$|\partial_{x'}^\alpha \partial_{\xi'}^\beta \rho_n| \leq C_{\alpha,\beta} |\rho|^{1-|\beta|} \text{ for } (z, x', \xi') \in \mathcal{M}_2. \tag{2.10}$$

The commutator $[Op_h(\rho_n), \nu(x)]$ is a pseudo-differential operator with symbol in $\mathcal{S}^0_\delta$. Clearly,

$$\langle Op_h(\rho_n + h\mathbf{b})E_0, \nu\rangle = \langle Op_h(\rho_n + h\mathbf{b})(E_{tan}|_\Gamma + E_{nor}|_\Gamma \nu), \nu\rangle.$$

To simplify the terms involving $\mathbf{b}$, let $\mathbf{b}$ have the form $\mathbf{b} = \{b_{k,j}\}_{k,j=1}^3$. Then

$$\langle Op_h(\mathbf{b})E_{nor}|_\Gamma \nu, \nu\rangle = \sum_{k,j=1}^3 \nu_k [Op_h(b_{k,j}), \nu_j](E_{nor}|_\Gamma) + \sum_{k,j=1}^3 \nu_k \nu_j Op_h(b_{k,j}) E_{nor}|_\Gamma$$

$$= Op_h(b) E_{nor}|_\Gamma + Op_h(b_{-1}) E_{nor}|_\Gamma,$$

where b, b_{-1} are *scalar symbols* in $\mathcal{S}^0_0, \mathcal{S}^{-1}_0$, respectively, and we deduce

$$\begin{aligned} \Big\| hD_\nu(E_{nor})|_\Gamma - Op_h(\rho_n + hb)(E_{nor}|_\Gamma) - \langle [Op_h(\rho_n), \nu]|_{tan}, E_{tan}|_\Gamma\rangle \\ -h\langle Op_h(\mathbf{b})E_{tan}, \nu\rangle\Big\|_{H^1_h(\Gamma)} \leq C\frac{h}{|\operatorname{Im} z|^{3/2}} \|E_0\|_{L^2(\Gamma)}. \end{aligned} \tag{2.11}$$

The same result holds for $D_\nu \hat{E}_{nor}|_\Gamma$ with ρ_n replaced by of $\rho_{\hat{n}}$. For the components E_{tan}, $\hat{E}_{tan}$ we obtain

$$\begin{aligned} h(D_\nu E_{\tan})|_{tan} - h(D_\nu \hat{E}_{tan})|_{tan} = \Big(\Big(Op_h(\rho_n) - Op_h(\rho_{\hat{n}})\Big) E_{tan}|_\Gamma\Big)\Big|_{tan} \\ + \Big([Op_h(\rho_n), \nu] E_{nor}|_\Gamma\Big)\Big|_{tan} - \Big([Op_h(\rho_{\hat{n}}), \nu]\hat{E}_{nor}|_\Gamma\Big)\Big|_{tan} \\ + h\Big((Op_h(\mathbf{b})(E_{nor} - \hat{E}_{nor})\nu|_\Gamma\Big)\Big|_{tan} - T_1 \end{aligned} \tag{2.12}$$

with

$$\|T_1\|_{H^1_h(\Gamma)} \leq C\frac{h}{|\operatorname{Im} z|^{3/2}}(\|E_0\|_{L^2(\Gamma)} + \|\hat{E}_0\|_{L^2(\Gamma)})\Big). \tag{2.13}$$

3. Pseudo-differential system on the boundary

We return to the system (2.3) and we suppose the conditions (1.5), (1.6) fulfilled. Set

$$X = E_{tan}|_\Gamma,\ Y = E_{nor}|_\Gamma,\ \hat{Y} = \hat{E}_{nor}|_\Gamma,\ Z = Y - \hat{Y},\ \ X = (X_1, X_2, X_3),\ R = \rho_n - \rho_{\hat{n}}.$$

According to (1.6), the matrix-valued operators $\mathcal{M}_1(x, \partial_x; \gamma), \mathcal{M}_1(x, \partial_x; \hat{\gamma})$ coincide for $x \in \Gamma$.

On the other hand, the condition (1.5) implies that for $x \in \Gamma$ we have $\mathcal{M}(x, \partial_x; \mu) = \mathcal{M}(x, \partial_x; \hat{\mu})$, $\forall x \in \Gamma$. Since the matrix-valued symbols $\mathbf{b}$, $\hat{\mathbf{b}}$ involved in the approximation of $hD_\nu E|_\Gamma$ and $hD_\nu \hat{E}|_\Gamma$ coincide, we have

$$h\Big(Op_h(\mathbf{b})E_{nor}\nu|_\Gamma\Big)\Big|_{\tan} - h\Big(Op_h(\mathbf{b})\hat{E}_{nor}\nu|_\Gamma\Big)\Big|_{tan} = h\{Op_h(g_k)Z\}_{k=1,2,3}$$

with scalar symbols $g_k \in \mathcal{S}^0_0$. For $x \in \Gamma$ we write

$$\langle \text{grad}\,(\log\gamma), E\rangle = \langle \text{grad}\,(\log\gamma)|_{tan}, E|_{tan}\rangle + \gamma_\nu E_{nor},\ \gamma_\nu = \frac{\partial \log(\gamma)}{\partial\nu}.$$

In the system (2.3) some terms cancel and the system modulo terms whose $H^1_h(\Gamma)$ norms are estimated by

$$\frac{Ch}{|\operatorname{Im} z|^{3/2}}\Big(\|E_0\|_{L^2(\Gamma)} + \|\hat{E}_0\|_{L^2(\Gamma)}\Big) \tag{3.1}$$

becomes

$$\begin{cases} Op_h(R)X_k - (b_k - hf_{n,k} - hg_k)Z = B_k - hOp_h(f_{n,k} - f_{\hat{n},k})\hat{Y} = \tilde{B}_k,\ k = 1, 2, 3, \\ \sum_{j=1}^3 Op_h(b_j + hf_{n,j})X_j + h\langle Op_h(\mathbf{b})X, \nu\rangle + h\langle \text{grad}\,(\log\gamma)|_{\tan}, X\rangle \\ +Op_h(\rho_n + hb - \mathbf{i}h\text{div}\,\nu)Z + Op_h(\rho_n + hb - \mathbf{i}h\text{div}\,\nu)\hat{Y} + h\gamma_\nu Y = D_1, \\ \sum_{j=1}^3 Op_h(b_j + hf_{\hat{n},j})X_j + h\langle Op_h(\mathbf{b})X, \nu\rangle + h\langle \text{grad}\,(\log\gamma)|_{tan}, X\rangle \\ +Op_h(\rho_{\hat{n}} + hb - \mathbf{i}h\text{div}\,\nu)\hat{Y} + h\gamma_\nu\hat{Y} = D_2. \end{cases} \tag{3.2}$$

Here

$$b_k = \sum_{j=2,3}\frac{\partial y_j}{\partial x_k}D_{y_j},\ f_{n,k} = \mathbf{i}\sum_{j=2,3}D_{\xi'_j}(\rho_n)D_{x_j}\nu_k,\ f_{\hat{n},k} = \mathbf{i}\sum_{j=2,3}D_{\xi'_j}(\rho_{\hat{n}})D_{x_j}\nu_k,\ k = 1, 2, 3,$$

$$f_{n,k} - f_{\hat{n},k} = \mathbf{i}\sum_{j=2,3}D_{\xi'_j}(\rho_n - \rho_{\hat{n}})D_{x_j}\nu_k, k = 1, 2, 3,$$

and $\|\tilde{B}_k\|_{H^1_h(\Gamma)}, \|D_j\|_{H^1_h(\Gamma)}$, $k = 1, 2, 3,\ j = 1, 2$, are estimated by (3.1). In fact the terms $h\|Op_h(f_{n,k} - f_{\hat{n},k})X_j\|_{H^1_h(\Gamma)}$, $k = 1, 2, 3$ can be estimated since

$$f_{n,k} - f_{\hat{n},j} = \mathbf{i}\sum_{j=2,3}D_{\xi'_j}(\rho_n - \rho_{\hat{n}})D_{x_j}\nu_k,\ R = \rho_n - \rho_{\hat{n}} = \frac{\mu z(\gamma - \hat{\gamma})}{\rho_n + \rho_{\hat{n}}} \in \mathcal{S}^{-1}_{1/2-\epsilon}$$

and $|\partial_{\xi'}(\rho_n - \rho_{\hat{n}})| \le C|\text{Im}\, z|^{-1/2}(1 + |\xi'|)^{-2}$.

In the following we denote by C some positive constants which may change from line to line. Taking the difference of the last two equations in (3.2), one obtains the system

$$\begin{cases} Op_h(R)X_k - (b_k - hf_{n,k} - hg_k)Z = \tilde{B}_k,\ k = 1,2,3, \\ Op_h(\rho_n + h(b+\gamma_\nu) - \mathbf{i}h\mathrm{div}\,\nu)Z + Op_h(R)\hat{Y} \\ = D_1 - D_2 - h\sum_{k=1}^3 Op_h(f_{n,k} - f_{\hat{n},k})X_k = \tilde{D}_1, \\ \sum_{k=1}^3 Op_h(b_k + hf_{\hat{n},k})X_k + Op_h(\rho_{\hat{n}} + h(b+\gamma_\nu) - \mathbf{i}h\mathrm{div}\,\nu)\hat{Y} = D_2. \end{cases} \tag{3.3}$$

Notice that the norm $\|\tilde{D}_1\|_{H_h^1(\Gamma)}$ has also an estimate by (3.1). The determinant of the symbol Q on the left hand side of (3.3) becomes

$$\det Q =$$

$$\det \begin{pmatrix} R & 0 & 0 & -(b_1 - hf_{n,1} - hg_1) & 0 \\ 0 & R & 0 & -(b_2 - hf_{n,2} - hg_2) & 0 \\ 0 & 0 & R & -(b_3 - hf_{n,3} - hg_3) & 0 \\ 0 & 0 & 0 & \rho_n + h(b+\gamma_\nu) - \mathbf{i}h\mathrm{div}\,\nu & R \\ b_1 + hf_{\hat{n},1} & b_2 + hf_{\hat{n},2} & b_3 + hf_{\hat{n},3} & 0 & \rho_{\hat{n}} + h(b+\gamma_\nu) - \mathbf{i}h\mathrm{div}\,\nu \end{pmatrix}.$$

Set $\tilde{b} = b + \gamma_\nu$. A simple calculus yields

$$\det Q = R^3(\rho_n + h\tilde{b} - \mathbf{i}h\mathrm{div}\,\nu)(\rho_{\hat{n}} + h\tilde{b} - \mathbf{i}h\mathrm{div}\,\nu) - R^3\sum_{k=1}^3 (b_k - hf_{n,k} - hg_k)(b_k + hf_{\hat{n},k})$$

$$= R^2(R\rho_n\rho_{\hat{n}} - R\sum_{j=1}^3 b_k^2) + hR^2(\tilde{b} - \mathbf{i}\mathrm{div}\,\nu)\mu z(\gamma - \hat{\gamma}) + h^2R^3(\tilde{b} - \mathbf{i}\mathrm{div}\,\nu)^2$$

$$+hR^3\sum_{k=1}^3 b_k((f_{n,k} - f_{\hat{n},k}) + g_k) + h^2R^3q_3,$$

where $q_1 = \mu z(\gamma - \hat{\gamma})(\tilde{b} - \mathbf{i}\mathrm{div}\,\nu)$, $q_2 = (\tilde{b} - \mathbf{i}\mathrm{div}\,\nu)^2 \in \mathcal{S}_0^0$ are zero order symbols and

$$\sum_{k=1}^3 b_k(f_{n,k} - f_{\hat{n},k}),\ q_3 \in \mathcal{S}^0_{1/2-\epsilon}.$$

On the other hand, $\sum_{j=1}^3 b_j^2 = r_0(y', \eta')$, since

$$\sum_{k=1}^3 \sum_{m,j=2}^3 \frac{\partial y_m}{\partial x_k}\frac{\partial y_j}{\partial x_k}\eta_m\eta_j = r_0(y', \eta')$$

and

$$\begin{aligned}(\rho_n - \rho_{\hat{n}})\rho_n\rho_{\hat{n}} &= \rho_{\hat{n}}(\gamma\mu z - r_0) - \rho_n(\hat{\gamma}\hat{\mu}z - r_0) \\ &= \mu(\gamma\rho_{\hat{n}} - \hat{\gamma}\rho_n)z + Rr_0.\end{aligned}$$

According to the above calculus, one has

$$\det Q = R^2\Big[\mu z(\gamma\rho_{\hat{n}} - \hat{\gamma}\rho_n) + h\Big(q_1 + R\sum_{k=1}^3 b_k((f_{n,k} - f_{\hat{n},k}) + g_k)\Big)$$

$$+h^2R(q_2+q_3)\Big] = R^2q(x',\xi';h).$$

Here

$$q(x',\xi';h) := R_2 + h(a_1 + hRa_2),\ R_2 := \mu z(\gamma\rho_{\hat{n}} - \hat{\gamma}\rho_n)$$

with zero order symbols a_1, a_2. Concerning R_2, we obtain

$$R_2 = \mu z\frac{\gamma^2\rho_{\hat{n}}^2 - \hat{\gamma}^2\rho_n^2}{\gamma\rho_{\hat{n}} + \hat{\gamma}\rho_n} = \mu z\frac{\gamma^2(\hat{\gamma}\hat{\mu}z - r_0) - \hat{\gamma}^2(\gamma\mu z - r_0)}{\gamma\rho_{\hat{n}} + \hat{\gamma}\rho_n}$$

$$= \mu z(\gamma - \hat{\gamma})\frac{\gamma\hat{\gamma}\mu z - (\gamma + \hat{\gamma})r_0}{\gamma\rho_{\hat{n}} + \hat{\gamma}\rho_n}.$$

Consequently, since $\gamma \neq \hat{\gamma}$, for $z \in Z_1 \cup Z_2 \cup Z_3$ the symbol

$$p(x',\xi';z) := \mu z(\gamma - \hat{\gamma})\Big(\gamma\hat{\gamma}\mu z - (\gamma + \hat{\gamma})r_0\Big)$$

is not vanishing. Moreover,

$$|R_2(x',\xi';z)| \geq C_2\langle\xi'\rangle \text{ for } |\xi'| \geq A \gg 1.$$

Since $\frac{h}{|\operatorname{Im} z|} \leq h^{1/2}$ for $z \in Z_1(1/2 - \epsilon)$ and $\gamma \neq \hat{\gamma}$, one deduces that for small h we have

$$|q(x',\xi';h)| \geq c_2|\operatorname{Im} z|,\ c_2 > 0 \text{ for } |\xi'| \leq A,\ z \in Z_1(1/2 - \epsilon),$$

while $|q(x',\xi';h)| \geq c_2 > 0$ for $|\xi'| \leq A,\ z \in Z_2 \cup Z_3$.

Set $\mu_1 = \mu\gamma\hat{\gamma}$. As in [10], we conclude that for $(z, x', \xi') \in \mathcal{M}_1$, $|\mu_1 \operatorname{Re} z - (\gamma + \hat{\gamma})r_0| \leq \delta'$, small h and $\operatorname{Im} z \neq 0$ we have

$$|\partial_{x'}^\alpha \partial_{\xi'}^\beta q^{-1}(x',\xi';h)| \leq C_{\alpha,\beta}|\operatorname{Im} z|^{-1-|\alpha|-|\beta|},\ |\alpha| + |\beta| \geq 1, \tag{3.4}$$

while for $(z, x', \xi') \in \mathcal{M}_1$, $|\mu_1 \operatorname{Re} z - (\gamma + \hat{\gamma})r_0| \geq \delta'$, small h and $\operatorname{Im} z \neq 0$ we have

$$|\partial_{x'}^\alpha \partial_{\xi'}^\beta q^{-1}(x',\xi';h)| \leq C_{\alpha,\beta}|\operatorname{Im} z|^{-1/2-|\alpha|-|\beta|},\quad |\alpha| + |\beta| \geq 1. \tag{3.5}$$

For $z \in Z_2 \cup Z_3$ these estimates hold with $|\operatorname{Im} z|$ replaced by 1. Moreover,

$$|\partial_{x'}^\alpha \partial_{\xi'}^\beta q^{-1}(x',\xi';h)| \leq C_{\alpha,\beta}\langle\xi'\rangle^{-1-|\beta|} \tag{3.6}$$

for $|\xi'| \gg 1$.

Now we pass to the analysis of the inverse matrix Q^{-1}. Let

$$Q = \{\alpha_{i,j}\}_{i,j=1}^5,\ ({}^{co}Q)^{tr} = \{\beta_{i,j}\}_{i,j=1}^5,\ Q^{-1} = \{d_{i,j}\}_{i,j=1}^5$$

with $d_{i,j} = \frac{\beta_{i,j}}{R^2q}$. A straightforward calculus yields

$$\beta_{i,j} = R^2b_ib_j + R^2(hL_{i,j} + h^2M_{i,j}),\ i,j = 1,2,3,\ i \neq j,$$
$$\beta_{i,i} = R^2(\rho_n\rho_{\hat{n}} - r_0 + b_i^2) + R^2(hL_{i,i} + h^2M_{i,i}),\ i = 1,2,3,$$
$$\beta_{i,4} = R^2\rho_{\hat{n}}b_i + R^2(hL_{i,4} + h^2M_{i,4}),\ \beta_{4,i} = R^3(b_i + hf_{\hat{n},i})\,, i = 1,2,3,$$
$$\beta_{4,4} = R^3(\rho_{\hat{n}} + hM_{4,4}),\ \beta_{4,5} = -R^4,$$
$$\beta_{i,5} = -R^3(b_i - hf_{n,i}),\ \beta_{5,i} = -R^2(\rho_{\hat{n}}b_i + hL_{5,i} + h^2M_{5,i}),\ i = 1,2,3,$$
$$\beta_{5,4} = R^2(r_0 + hL_{5,4} + h^2M_{5,4}),\ \beta_{5,5} = R^3(\rho_n + hM_{5,5}).$$

Here $L_{i,j}$ are symbols of first order operators, while $M_{i,j}$ are symbols of zero order operators. Therefore, we may cancel R^2 in $d_{i,j} = \frac{\beta_{i,j}}{R^2q}$ and $d_{i,j}$ become symbols of pseudo-differential operators of order $\kappa \leq 1$. In particular, $d_{i,5}, i = 1, ..., 5$, contain the factor R and they are symbols of operators of order -1. For the analysis in the next section it is important to estimate the derivatives $\partial_{x'}^{\alpha}\partial_{\xi'}^{\beta} d_{i,j}$. To do this, taking into account the form of q, we write

$$d_{i,j} = \frac{\gamma_{i,j}(\gamma\rho_{\hat{n}} + \hat{\gamma}\rho_n)}{\mu z(\gamma - \hat{\gamma})(\gamma\hat{\gamma}\mu z - (\gamma + \hat{\gamma})r_0)}(1 + (ha_1 + h^2 Ra_2)R_2^{-1})^{-1},$$

where $\gamma_{i,j} = \beta_{i,j} R^{-2}$. Therefore by using the estimates (2.5), (2.6), (3.4)-(3.6) , one gets

$$|\partial_{x'}^{\alpha}\partial_{\xi'}^{\beta} d_{i,j}| \leq C_{i,j,\alpha,\beta} |\operatorname{Im} z|^{-1-|\alpha|-|\beta|} \langle \xi' \rangle^{m-|\beta|},\ \forall \alpha,\ \forall \beta \tag{3.7}$$

with $m = 0$ or $m = 1$.

4. Location of transmission eigenvalues

We use the notation of the previous section and we assume the conditions on $\mu, \hat{\mu}, \gamma, \hat{\gamma}$ introduced in the beginning of the previous section fulfilled. Set

$$V = (X_1, X_2, X_3, Z, \hat{Y}), W = (\tilde{B}_1, \tilde{B}_2, \tilde{B}_3, \tilde{D}_1, D_2).$$

As we have mentioned in the previous section, we have the system $Op_h(Q)V = W$. We have for $z \in Z_1(1/2 - \epsilon)$

$$\|Op_h(Q^{-1})W\|_{L^2(\Gamma)} \leq C|\operatorname{Im} z|^{-1} \|W\|_{H_h^1(\Gamma)} \leq C \frac{h}{|\operatorname{Im} z|^{5/2}} \|V\|_{L^2(\Gamma)}, \tag{4.1}$$

where for $z \in Z_2 \cup Z_3$ the factor $|\operatorname{Im} z|$ must be replaced by 1. Here we have used the estimate by (3.1) of the $H_h^1(\Gamma)$ norms of the components of W. Also dealing with $d_{i,j}$, the factor q^{-1} has been estimated by $c_2 |\operatorname{Im} z|^{-1}$ for $z \in Z_{1,\epsilon}$ and by c_2 for $z \in Z_2 \cup Z_3$. Next

$$V = \Big(Id - Op_h(Q^{-1})Op_h(Q)\Big)V + Op_h(Q^{-1})W.$$

The principal symbol of the product $Op_h(Q^{-1})Op_h(Q)$ is Id and we must study the symbol

$$\zeta = \sum_{j=1}^{N-1} \sum_{|\alpha|=j} \frac{(\mathbf{i}h)^j}{j!} D_{\xi'}^{\alpha}(Q^{-1}) D_{x'}^{\alpha} Q + R_N = \sum_{j=1}^{N-1} (\mathbf{i}h)^j \zeta_j + R_N.$$

It is important to examine the term ζ_1 since $\zeta_j,\ j \geq 2$, are symbols of lower order operators. Our purpose is to bound $\|Op_h(\zeta_1)V\|_{L^2(\Gamma)}$ by $\|V\|_{L^2(\Gamma)}$ multiplied by a small factor. The difficulty here is caused by the symbols $\partial_{x_k} Q$, since some components $\partial_{x_k} \alpha_{i,j}$ of the matrix $\partial_{x_k} Q$ are symbols of first order operators. More precisely, these terms are

$$\partial_{x_k}(\alpha_{5,j}),\ j = 1, 2, 3, 5,\ \partial_{x_k}(\alpha_{i,4}),\ i = 1, 2, 3, 4.$$

The other components in $\partial_{x_k} Q$ are equal to $\partial_{x_k} R$ or to 0. Recall that we have

$$|\partial_{x'}^{\alpha}\partial_{\xi'}^{\beta} R| \leq C_{\alpha,\beta} |\operatorname{Im} z|^{1/2-|\alpha|-|\beta|},\ |\alpha| + |\beta| \geq 1 \tag{4.2}$$

for $(z, x', \xi') \in \mathcal{M}_1$ and

$$|\partial_{x'}^{\alpha}\partial_{\xi'}^{\beta} R| \leq C_{\alpha,\beta} \langle \xi' \rangle^{-1-|\beta|} \tag{4.3}$$

for $(z, x', \xi') \in \mathcal{M}_2$. Thus we obtain

$$h \Big\| Op_h\Big(D_{\xi_k}(d_{i,j}) D_{x_k}(\alpha_{j,m})\Big) V \Big\|_{L^2(\Gamma)} \leq C_{i,j,m} \frac{h}{|\operatorname{Im} z|^{5/2}} \|V\|_{L^2(\Gamma)} \tag{4.4}$$

for the components of $\partial_{\xi_k}(Q^{-1})\partial_{x_k}Q$ containing $\partial_{x_k}R$.

Now consider the symbols

$$e_{i,j} = \sum_k \partial_{\xi_k}(d_{i,5})\partial_{x_k}(\alpha_{5,j}).$$

As we have mentioned above, $d_{i,5}$ are of order -1. Then applying the estimates (3.4)-(3.6), (4.2), (4.3), we deduce

$$|\partial_{x'}^{\alpha}\partial_{\xi'}^{\beta}e_{i,j}| \leq C_{i,j,\alpha,\beta}|\operatorname{Im} z|^{-5/2-|\alpha|-|\beta|}$$

for $(z, x', \xi') \in \mathcal{M}_1$ and

$$|\partial_{x'}^{\alpha}\partial_{\xi'}^{\beta}e_{i,j}| \leq C_{k,l,\alpha,\beta}\langle\xi'\rangle^{-1-|\beta|}$$

for $(z, x', \xi') \in \mathcal{M}_2$. For $j = 1, 2, 3$ we have a better estimate since $\alpha_{5,j} = b_j + hf_{\hat{n},j}$. Consequently, by using once more (2.2), these operators will produce again terms having the estimate (4.4).

It remains to handle the terms

$$p_{i,4} = \sum_k \sum_{m=1}^{4} \partial_{\xi_k}(d_{i,m})\partial_{x_k}(\alpha_{m,4}),\ i = 1, ..., 5$$

for which no simplifications are possible, hence these symbols are related to first order operators. Here $\alpha_{m,4}$, $m = 1, 2, 3$, have classical principal symbols and this simplifies the expression of their derivatives. The same is true for the principal part of the symbols $d_{i,4}$. Thus for $(x', \xi', z) \in \mathcal{M}_1$ we deduce the estimates

$$|\partial_{x'}^{\alpha}\partial_{\xi'}^{\beta}p_{i,4}| \leq C_{i,\alpha,\beta}|\operatorname{Im} z|^{-2-|\alpha|-|\beta|}, \tag{4.5}$$

while for $(x', \xi', z) \in \mathcal{M}_2$ we have

$$|\partial_{x'}^{\alpha}\partial_{\xi'}^{\beta}p_{i,4}| \leq C_{\alpha,\beta}\langle\xi'\rangle.$$

The problem is reduced to obtain a bound

$$h\|Op_h(p_{i,4})Z\|_{L^2(\Gamma)} \leq C\Big(\frac{h}{|\operatorname{Im} z|^{5/2}} + \frac{h^2}{|\operatorname{Im} z|^4}\Big)\|V\|_{L^2(\Gamma)},\ i = 1, ..., 5.$$

We prove the above estimate exploiting the system (3.3). Combining the estimates obtained above, we conclude that for $z \in Z_1(1/2 - \epsilon)$ we have

$$\|V\|_{L^2(\Gamma)} \leq C\Big(\frac{h}{|\operatorname{Im} z|^{5/2}} + \frac{h^2}{|\operatorname{Im} z|^4}\Big)\|V\|_{L^2(\Gamma)}.$$

For $z \in Z_2 \cup Z_3$ one obtains the same estimate with $|\operatorname{Im} z|$ replaced by 1. Consequently, for $|\operatorname{Im} z| \geq C_1 h^{2/5} > C_1 h^{1/2+\epsilon}$ with a large constant $C_1 > 0$ as well as for $z \in Z_2 \cup Z_3$ with $0 < h \leq \delta_0$ and $\delta_0 > 0$ small enough one deduces $\|V\|_{L^2(\Gamma)} = 0$. This implies $E_{|\Gamma} = \hat{E}_{|\Gamma} = 0$ which leads to $E = \hat{E} = H = \hat{H} = 0$ and this completes the proof of Theorem 1.

References

[1] F. Cakoni, H. Haddar, and S. Meng, *Boundary integral equations for the transmission eigenvalue problem for Maxwell′ s equations*, J. Integral Equations Appl. **27** (3) (2015), 375-406.
[2] F. Cakoni, D. Colton, and H. Haddar, *Inverse scattering theory and transmission eigenvalues*, CBMS-NSF Regional Conference Series in Applied Mathematics, **88**, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2016.
[3] F. Cakoni and H-M. Hguyen, *On the discreteness of the transmission eigenvalues for the Maxwell equations*, SIAM J. of Math. Analysis, to appear (arXiv: 2004.14670).
[4] L. Chesnel, *Transmission eigenvalue problem for Maxwell′ s equations: the T -coercivity as an alternative approach*, Inverse Problems **28** (6) (2012), 065005.
[5] F. Colombini, V. Petkov and J. Rauch, *Eigenvalues of the Maxwell's equations with dissipative boundary conditions*, Asymptotic Analysis, **99** (2016), 105-124.
[6] M. Dimassi and J. Sjöstrand, *Spectral asymptotics in semi-classical limits*, London Mathematical Society, Lecture Notes Series, **268**, Cambridge University Press, 1999.
[7] H. Haddar and S. Meng, *The spectral analysis of the interior transmission eigenvalue problem for Maxwell′ s equations*, J. Math. Pures Appl. **120** (9) (2018), 1-32.
[8] V. Petkov and G. Vodev, *Asymptotics of the number of the interior transmission eigenvalues*, J. Spectral theory, **7** (1) (2017), 1- 31.
[9] V. Petkov, *Parabolic transmission eigenvalues-free regions for Maxwell equations*, in preparation.
[10] G. Vodev, *Transmission eigenvalue-free regions*, Commun. Math. Phys. **336** (2015), 1141-1166.
[11] G. Vodev, *High-frequency approximation of the interior Dirichlet-to-Neumann map and applications to the transmission eigenvalues*, Anal. & PDE. **11** (1) (2018), 213-236.
[12] G. Vodev, *Parabolic transmission eigenvalues-free regions in the degenerate isotropic case*, Asymptotic Analysis **106** (2018), 147-168.
[13] G. Vodev, *Semiclassical parametrix for the Maxwell equation and applications to the electromagnetic transmission eigenvalues*, arXix: Math.AP: 2102.08662.

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