Observability Estimates for Schrödinger Operators on Eucleadian space minus tube

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Abstract

Observability estimates on Euclidean space are considered. We prove the observability estimate for Schrödinger operators on the Eucleadian set minus tube. Our proof is stable under perturbation by bounded and smooth real-valued potential. Our assumption on the operator are saisfied by some non-elliptic operators and differential operators with unbounded coefficients. Results in this article are collaboration work by the author with Fabricio Macià and Shu Nakamura.

1 Introduction

In this article, we introduce our collaboration work by the author with Fabricio Macià and Shu Nakamura on the observability estimates for Schrödinger operators on Euclidean space minus tube.

1.1 Statement of the main result

Let $\Omega \subset \mathbb{R}^d$ be such that $\Omega = \mathbb{R}^k \times \omega$ with $1 < k < d$ and $\omega \subset \mathbb{R}^{d-k}$. We assume $\mathbb{R}^{d-k} \setminus \omega$ is compact.

Let P_1 and P_2 be diffrential operators on $\mathbb{R}^k$ and $\mathbb{R}^{d-k}$ respectively. We define diffrential operator P on $\mathbb{R}^d$ by $P = P_1 + P_2 + V(x)$.

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Assumption A. P_1, P_2 and V satisfy followings:

1. P is essentially selfadjoint on $C_0^\infty(\mathbb{R}^d)$.
2. P_1 is a differential operator with smooth real valued coefficient and essentially selfadjoint on $C_0^\infty(\mathbb{R}^k)$.
3. $P_2 = -\triangle_{\mathbb{R}^{d-k}}$.
4. $V \in L^\infty(\mathbb{R}^d : \mathbb{R})$ and $\sup_{x \in \mathbb{R}^d} |\partial_{x_2}^\alpha V(x)| < \infty$ for any $\alpha \in \mathbb{N}^d$ with $|\alpha| \leq 1$.

Example. 1. If $k = 1$, $P_1 = -\triangle_{\mathbb{R}} + x_1$ and $V = 0$, then P is Stark Hamiltonian and Assumption A is satisfied.

2. If $k = 1$, $P_1 = \triangle_{\mathbb{R}}$ and $V = 0$, then P is d'Alembert operator and Assumption A is satisfied.

Theorem 1.1. *(Observability estimates)*
Suppose Assumtion A is satisfied. For any $T > 0$, there exists $C_{\Omega,T} > 0$ such that

$$\|u\|_{L^2(\mathbb{R}^d)}^2 \leq C_{\Omega,T} \int_0^T \int_\Omega |e^{-itP} u(x)|^2 \mathrm{d}x \mathrm{d}t,$$

for any $u \in L^2(\mathbb{R}^d)$.

In [9], Lions proved that observability estimate holds if and only if corresponding exact control problem has a solution. In our setting, exact control problem is a following problem: for given $T > 0$ and $u_0, u_t \in L^2(\mathbb{R}^d)$, can we find $f \in L^2(\mathbb{R}_t \times \mathbb{R}_x^d)$ such that

$$\begin{cases} i\partial_t u(t,x) - Pu(t,x) = \mathbb{1}_\Omega(x) f(t,x), (t,x) \in \mathbb{R} \times \mathbb{R}^d, \\ u|_{t=0} = u_0 \in L^2(\mathbb{R}^d). \end{cases} \tag{1.1}$$

has a solution $u \in L^2(\mathbb{R} \times \mathbb{R}^d)$?

Also, Miller showed that this this exact control problem is equivalent to the following spectral inequality:

Theorem 1.2 (Miller [13, Corollary 2.17]). *Let A be a selfadjoint operator on $L^2(\mathbb{R}^d)$, which is the infinitesimal generator of a strongly continuous group $(e^{itA})_{t\in\mathbb{R}}$ on $L^2(\mathbb{R}^d)$. If the evolution equation (1.1) with P replace by A is* *exactly observable from a measurable subset* $\Omega \subset \mathbb{R}^d$ *at some time* $T > 0$ *then there exist some positive constants* $k > 0$ *and* $D > 0$ *such that*

$$\forall \lambda \in \mathbb{R}, \forall f \in \mathbb{1}_{\{|A-\lambda| \leq \sqrt{D}\}}\big(L^2(\mathbb{R}^d)\big), \quad \|f\|_{L^2(\mathbb{R}^d)} \leq \sqrt{k}\|f\|_{L^2(\Omega)}. \tag{1.2}$$

Conversely, when the spectral estimates (1.2) *hold for some* $k > 0$ *and* $D > 0$*, then the system (1.1) is exactly observable from* Ω *at any time*

$$T > \pi\sqrt{\frac{1+k}{D}}.$$

It is known that on compact manifolds, observability estimates are related to the property of the geodesic. We say $\Omega \subset M$ satisfies the geometric control condition (GCC) if any geodesic with length L intersects with Ω. Lebeau proved in [7] that for a compact Riemannian manifold (M, g), if $\Omega \subset M$ satisfies GCC, the observability estimate for Laplace-Beltrami operator on Ω holds.

There are two main difficulties in our setting: Ω does not satisfy the geometric control conditions, and $\mathbb{R}^d$ is not compact. The first difficulty is relaxed by assuming that Ω is a product of Euclidean space and Euclidean space minus compact set.

The second difficulty is much more severe since the proof of observability estimate in [7] uses compactness of the space. In the [7], the observability estimate in high energy regime is shown in then it is shown that low energy regime can be regarded as a minor error. In the second part, compactness plays a critical role. We use Logvinenko-Sereda theorem to avoid this difficulty. See section 3.2 for the detail.

1.2 Thick set and Logvinenko-Sereda theorem

Definition 1.1. A measureble subset $S \subset \mathbb{R}^d$ is thick if there exists a cube $K \subset \mathbb{R}^d$ with sides parallel to coordinate axes and a positive constant $0 < \gamma \leq 1$ such that

$$\forall x \in \mathbb{R}^d, \quad |(K + x) \cap S| \geq \gamma |K| > 0,$$

where $|A|$ denotes the Lebesgue measure of the measurable set A.

Theorem 1.3 (Logvinenko-Sereda [8])**.** *Let* $S, \Sigma \subset \mathbb{R}^d$ *be measurable sets with* Σ *compact. The following assertions are equivalent:*

- *The subset S is thick.*

- *There exists a positive constant $C = C(S, \Sigma) > 0$ such that for all $f \in L^2(\mathbb{R}^d)$,*

$$\int_{\mathbb{R}^d} |f(x)|^2 dx \leq C\Big(\int_S |f(x)|^2 dx + \int_{\mathbb{R}^d \setminus \Sigma} |\widehat{f}(\xi)|^2 d\xi \Big).$$

Logvinenko-Sereda Theorem states any function f in $L^2(\mathbb{R}^d)$ never concentrates in a thick subset if its energy is concentrated in a compact set. In this sense, Logvinenko-Sereda theorem can be regarded as a sort of uncertainty principle for a thick set.

Kovrjikine obtained some exact constant in Logvinenko-Sereda theorem.

Theorem 1.4. *Let $S \subset \mathbb{R}^d$ be measurable set. We say S is (γ, L)-thick set if*

$$\forall x \in \mathbb{R}^d, \quad |([0, L]^d + x) \cap S| \geq \gamma L^d,$$

with $\gamma \in (0, 1)$. There exists a constant $C > 0$ such that for any $f \in L^2(\mathbb{R}^d)$ with supp$\hat{f} \subset J$ where J is a cube with sides of length b parallel to coordinate axis,

$$\|f\|_{L^2(\mathbb{R}^d)} < c(\gamma, d, L, b)\|f\|_{L^2(S)},$$

with $c(\gamma, d, L, b) = \left(\frac{C^d}{\gamma}\right)^{Cd(Lb+1)}$.

Remark. As $b \to \infty$, $c(\gamma, d, L, b) \to \infty$. Thus it is impossible to obtain observability estimates directly from Logvinenko-Sereda theorem.

Thickness may be considered as a higher dimensional analogue of the geometric control condition. We define the κ-dimensional geometric control condition as follows:

Definition 1.2. Fix $\kappa \in \{1, \ldots, d\}$. For $\ell > 0$ and $\gamma > 0$, a set $E \subset \mathbb{R}^d$ satisfies the κ-dimensional (ℓ, γ)-GCC if for any κ-dimensional cube $Q \subset \mathbb{R}^n$ of side-length ℓ,

$$|Q \cap E|_\kappa \geq \gamma |Q|_\kappa,$$

where $|\cdot|_\kappa$ denotes the κ-dimensional Hausdorff measure. We say that E satisfies the κ-GCC if it satisfies the κ-dimensional (ℓ, γ)-GCC for some $\ell > 0$ and γ.

With this notion of the κ-dimensional GCC, a thick set is a set that satisfies the d-dimensional GCC. On the other hand, on a compact manifold, the 1-dimengional GCC and the GCC used in [7] are equivalent. However, we have to assume uniformness, in general, to obtain to obtain the 1-dimengional GCC from the usual GCC. Assume $S \subset \mathbb{R}^d$ satisfies the κ-dimensional GCC.

From Fubini theorem, S must satisfy the κ'-dimensional GCC for $\kappa' > \kappa$. Let $Q = \Pi_{\ell=1}^d (a_k, a_k + L) \subset \mathbb{R}^d$ and $\tilde{Q} = \Pi_{\ell=k+1}^d (a_k, a_k + L)$. We see

$$\begin{aligned}
&|S \cap Q| \\
&= \int_Q \mathbb{1}_S(x)\mathrm{d}x \\
&= \int_{\tilde{Q}} \int_{a_k}^{a_k+L} \cdots \int_{a_1}^{a_1+L} \mathbb{1}_S(x_1, \cdots, x_k, x')\mathrm{d}x_1 \cdots \mathrm{d}x_k \mathrm{d}x' \\
&= \int_{\tilde{Q}} |S \cap \{(x_1, x') \mid a_\ell < x_1 < a_\ell + L, \ell = 1 \cdots k\}|_k \mathrm{d}x' \\
&> \gamma L \int_{\tilde{Q}} \mathrm{d}x' \\
&= \gamma L^d.
\end{aligned}$$

Consider $P_s = (-\triangle)^s$ for $s > 0$. In [12], Martin and Pravda-Starov showed that if $S \subset \mathbb{R}^d$ satisfies observability estimates for P_s, S must be a thick set. Further, when $s > \frac{1}{2}$, it is proved that there exists $T_0 > 0$ such that observability estimates on S with time $T > 0$ holds if $T > T_0$. Huang, Wang and Wang showed the same results when $s = 1$ and $d = 1$ in [5] independently.

Also, Martin and Pravda-Starov showed that if S satisfies the κ-GCC for $\kappa \in \{1, \cdots d-1\}$, δ neighbourhood of Ω satisfies observability estimates for P_s and sufficiently large $T > 0$.

Assume Ω satisfies Assumption A. Then one can easily see ω is a thick set and Ω is also a thick set. However, Ω does not satisfy GCC. Therefore Theorem 1.1 gives an example of Ω that is thick and satisfies observability estimates for P and any $T > 0$ but does not satisfy the κ-GCC for $\kappa \in \{1, \cdots d-1\}$.

Theorem 1.1 also shows that the observability estimate is stable under perturbation by bounded and smooth potential. Furthermore, Theorem 1.1 covers some non-elliptic operators and operators with unbounded potentials (See examples after Theorem 1.1 for the detail).

2 Preliminary

2.1 Pseudodifferential operators and semiclassical defect measures

As we stated before, in [7], Lebeau first proves the semiclassical observability estimates to obtain the observability estimates. In his proof of the observability estimates, Lebeau used semiclassical defect measure.

This subsection aims to provide some basic notions of semiclassical analysis needed later. This subsection also aims to provide some concepts of semiclassical measure. You can find all the proof of the theorem in this subsection in [16].

Let $a \in C_0^\infty(T^*\mathbb{R}^d)$. We define Weyl quantization of a by

$$a^{\mathrm{w}}(x, hD_x)u(x) = \frac{1}{(2\pi h)^d} \int_{\mathbb{R}^{2d}} e^{i\frac{(x-y)\cdot\xi}{h}} a\left(\frac{x+y}{2}, \xi\right) u(y)\mathrm{d}y\mathrm{d}\xi,$$

for $u \in \mathcal{S}(\mathbb{R}^d)$. Then $a^{\mathrm{w}}(x, hD_x)$ is extended to a bounded linear operator on $L^2(\mathbb{R}^d)$. Further we obtain following theorem on the properties of $a^{\mathrm{w}}(x, hD_x)$ as a bounded operator on $L^2(\mathbb{R}^d)$.

Theorem 2.1. *(Calderon-Vaillancourt Theorem)*
For $a \in C_0^\infty(T^*\mathbb{R}^d)$, there exists $C > 0$ such that
$\|a^{\mathrm{w}}(hX, D_X)\|_{\mathcal{L}(L^2(\mathbb{R}^d))} \leq C \sup_{(x,\xi)\in\mathbb{R}^{2d}} |a(x, \xi)| + \mathcal{O}(h^{\frac{1}{2}})$ *as* $h \to 0$.

Theorem 2.2. *(Sharp Gårding inequality)*
Suppose $a \in C_0^\infty(T^*\mathbb{R}^d)$ is positive. Then there exist $C > 0$ and $h_0 > 0$ such that

$$\langle u, a^{\mathrm{w}}(hX, D_X)u\rangle_{L^2(\mathbb{R}^d)} \geq -Ch\|u\|^2_{L^2(\mathbb{R}^d)}$$

for $u \in L^2(\mathbb{R}^n)$ and $0 < h < h_0$.

From the Riesz-Markov-Kakutani theorem and Theorems 2.1 and 2.2, we obtain following theorem.

Theorem 2.3. *(Existence of semiclassical defect measure)*
Let $u_h \in L^2(\mathbb{R}^d)$ be a bounded sequence in h. There exists a sequence of

positive numbers h_m *and a positive finite Radon measure* μ *on* $T^*\mathbb{R}^d$ *such that* $h_m \to 0$ *as* $m \to \infty$ *and*

$$\langle u_{h_m}, a^{\mathrm{w}}(x, hD_x)u_{h_m}\rangle_{L^2(\mathbb{R}^d)} \to \int_{T^*\mathbb{R}^d} a\mathrm{d}\mu \text{ as } m \to \infty,$$

for all $a \in C_0^\infty(T^*\mathbb{R}^d)$.

We call this μ semiclassical defect measure of u_h. We remark that this μ depends on the choice of h_m.

Wigner first introduced the notion of the semiclassical measure in [14]. The study of the partial differential equation using defect measure appeared in [10], and Patrick Gérard refined it in [4]. You can find several proofs of the existence of semiclassical measures in [2, 3, 11, 15]. You can find a survey of this subject in [1].

2.2 Estimates on propagators

This subsection aims to provide some estimates on propagators P, which we will use in the next section. Let $\tilde{P} = P_1 + P_2$

Lemma 2.4. *For any* $\varepsilon > 0$, $\|e^{-i\varepsilon t\tilde{P}} - e^{-i\varepsilon tP}\|_{\mathcal{B}(L^2(\mathbb{R}^d))} \leq \varepsilon t\|V\|_{L^\infty(\mathbb{R}^d)}$.

Proof.

$$\begin{aligned}
&\|e^{-i\varepsilon t\tilde{P}} - e^{-i\varepsilon tP}\|_{\mathcal{B}(L^2(\mathbb{R}^d))} \\
&= \|\int_0^t e^{-i\varepsilon s\tilde{P}}\varepsilon V e^{-i\varepsilon(t-s)P}\mathrm{d}s\|_{\mathcal{B}(L^2(\mathbb{R}^d))} \\
&\leq \int_0^t \|e^{-i\varepsilon s\tilde{P}}\varepsilon V e^{-i\varepsilon(t-s)P}\|_{\mathcal{B}(L^2(\mathbb{R}^d))}\mathrm{d}s \\
&\leq \varepsilon t\|V\|_{L^\infty(\mathbb{R}^d)}.
\end{aligned}$$

□

Lemma 2.5. *Let* $\chi \in C_0^\infty(\mathbb{R})$ *and* $h > 0$. $\|[\chi(h^2P_2), e^{-ihtP}]\|_{\mathcal{B}(L^2(\mathbb{R}^d)} = t\mathcal{O}(h^2)$ *as* $h \to 0$.

Proof. From Hellfer-Sjöstrand formula,

$$\begin{aligned}
&[\chi(h^2P_2), e^{-ihtP}] \\
&= \frac{1}{2\pi i}\int \partial\chi^{(A.A)}(z)[(h^2P_2-z)^{-1}, e^{-ihtP}]\mathrm{d}z\mathrm{d}\bar{z} \\
&= \frac{1}{2\pi i}\int \partial\chi^{(A.A)}(z)\int_0^t e^{ihsP}[(h^2P_2-z)^{-1}, hP]e^{-ih(t-s)P}\mathrm{d}s\mathrm{d}z\mathrm{d}\bar{z},
\end{aligned}$$

where $\chi^{(A.A)}$ is almost analytic extension of χ.

$$\begin{aligned}
&[(h^2P_2-z)^{-1}, hP] \\
&= (h^2P_2-z)^{-1}[h^2P_2, hP](h^2P_2-z)^{-1} \\
&= (h^2P_2-z)^{-1}[h^2P_2, hV](h^2P_2-z)^{-1} \\
&= h^2(h^2P_2-z)^{-1}c^w(x, hD_x)(h^2P_2-z)^{-1},
\end{aligned}$$

where $c(x,\xi) = \xi\cdot\partial_x V$.

Then there exists $C > 0$ and $h_o > 0$ such that for for any $h \in (0, h_0)$ $\|(h^2P_2-z)^{-1}c^w(x,hD_x)(h^2P_2-z)^{-1}\|_{\mathcal{B}(L^2(\mathbb{R}^d))} \le C|\mathrm{Im}z|^{-2}$ for some $C > 0$. Since $\chi^{(A.A)}$ is almost analytic extension of $\chi \in C_0^\infty(\mathbb{R}^d)$, $|\partial\chi^{(A.A)}(z)||\mathrm{Im}z|^{-2}$ is integrable on $\mathbb{C}$, which concludes the proof. □

3 Proof of the main theorem

By modifying the arguement in [7], one can prove observebility estimate from following two energy localized estimates:

Theorem 3.1. *Let $\tilde{\chi} \in C^\infty(\mathbb{R}^d; [0,1])$ be such that $\mathbb{1}_\Omega\tilde{\chi} = \mathbb{1}_\Omega$. If Assumption (A) is satisfied, for any $T > 0$, there exists $C_{\omega,T}, h_0 > 0$ such that*

$$\begin{aligned}
&\|\chi(h^2P_2)u\|_{L^2(\mathbb{R}^d)} \\
&\le C_{\omega,T}\left(\int_0^T \|\chi(h^2P_2)\tilde{\chi}e^{-itP}u(x)\|^2_{L^2(\mathbb{R}^d)}\mathrm{d}x\mathrm{d}t + h^2\|u\|^2_{L^2(\mathbb{R}^d)}\right),
\end{aligned}$$

for any $u \in L^2(\mathbb{R}^d)$ and $0 < h < h_0$.

Theorem 3.2. *(Observability for low energy)*
Let $\chi \in C_0^\infty(\mathbb{R})$. Then there exists $C_{\omega,\chi} > 0$ such that for any $T > 0$,

$$\begin{aligned}
&\|\chi(P_2)u\|^2_{L^2(\mathbb{R}^d)} \\
&\le \frac{C_{\omega,\chi}}{T}\int_0^T\int_\Omega |e^{-itP}u(x)|^2\mathrm{d}x\mathrm{d}t + C_{\omega,\chi}T^2\|V\|_{L^\infty(\mathbb{R}^d)}\|u\|^2_{L^2(\mathbb{R}^d)}
\end{aligned}$$

for any $u \in L^2(\mathbb{R}^d)$.

3.1 Semiclassical observability estimates

This subsection is devoted to proving semiclassical observability estimates. We follow the argument in [7]. However, we have to change the discussion a bit due to non-compactness.

Theorem 3.3. *(semiclassical observability estimates)*
Let $\chi \in C_0^\infty(\mathbb{R})$ *be such that* $0 \notin \mathrm{supp}\chi$. *For any* $T > 0$, *there exists* $C_{\omega,T} > 0$ *and* $h_0 > 0$ *such that*

$$\begin{aligned}&\|\chi(h^2P_2)u\|^2_{L^2(\mathbb{R}^{d-k})}\\ &\le C_{\omega,T}\left(\int_0^T\int_\omega |e^{-ithP_2}\chi(h^2P_2)u(x)|^2\mathrm{d}x\mathrm{d}t + h^2\|u\|^2_{L^2(\mathbb{R}^{d-k})}\right)\end{aligned}$$

for any $u \in L^2(\mathbb{R}^{d-k})$ *and* $0 < h < h_0$.

We prove this theorem by contradiction. Assume the assertion does not hold. Then there exists $u_\ell \in L^2(\mathbb{R}^{d-k})$ and $h_\ell > 0$ such that

1. $h_\ell \to 0$ as $\ell \to \infty$.
2. $\|\chi(h_\ell^2P_2)u_\ell\|_{L^2(\mathbb{R}^n)} = 1$.
3. $\int_0^T \|e^{-ith_\ell P_2}\chi(h_\ell^2P_2)u_\ell\|^2_{L^2(\omega)}\mathrm{d}t \to 0$ as $\ell \to \infty$.

From second and third condition, we obtain $v_\ell(t) = \Pi e^{-it_0h_\ell P_2}\chi(h_\ell^2P_2)u_\ell$ satisfies $\lim_{\ell\to\infty}\|v(t)\|_{L^2(\mathbb{R}^n)} = 1$ for any $t \in (0,T)$. We then apply the semiclassical defect measure argument in [7] as u_{h_ℓ} is supported in a compact set in a semiclassical sense.

Since $L^2(\mathbb{R}^d) = L^2(\mathbb{R}^k) \otimes L^2(\mathbb{R}^{d-k})$, one can extend Theorem 3.1 to semiclassical estimate for $P_1 + P_2$ on $\mathbb{R}^d$. From Lemma 2.1, we obtain following proposition.

Proposition 3.4. *(semiclassical observability estimates on* $\mathbb{R}^d$*)*
Let $\chi \in C_0^\infty(\mathbb{R})$ *be such that* $0 \notin \mathrm{supp}\chi$. *For any* $T > 0$, *there exists* $C_{\omega,T} > 0$ *and* $h_0 > 0$ *such that*

$$\begin{aligned}&\|\chi(h^2P_2)u\|^2_{L^2(\mathbb{R}^d)}\\ &\le C_{\omega,T}\left(\int_0^T\int_\Omega |\chi(h^2P_2)e^{-ithP}u(x)|^2\mathrm{d}x\mathrm{d}t + h^2\|u\|^2_{L^2(\mathbb{R}^d)}\right)\end{aligned}$$

for any $u \in L^2(\mathbb{R}^d)$ *and* $0 < h < h_0$.

From lemma 2.3 and the arguement in [7], one can replace e^{-ithP} by e^{-itP} to obtain Theorem 3.1.

3.2 Low energy observability estimates

From Logvinenko-Sereda theorem in [8], there exists a $C > 0$

$$\|\chi(P_2)u\|^2_{L^2(\mathbb{R}^{d-k})} \leq C\|e^{-itP_2}\chi(P_2)u\|^2_{L^2(\omega)}$$

for any $u \in L^2(\mathbb{R}^{d-k})$ since Ω is a thick set. By replacing u by $e^{-itP_2}u$ and integrate above inequality on $(0,T)$ to obtain following lemma:

Lemma 3.5. *Let* $\chi \in C_0^\infty(\mathbb{R})$. *Then there exists* $C_{\omega,\chi} > 0$ *such that for any* $T > 0$,

$$\|\chi(P_2)u\|^2_{L^2(\mathbb{R}^{d-k})} \leq \frac{C_{\omega,\chi}}{T}\int_0^T\int_\omega |e^{-itP_2}u(x)|^2 \mathrm{d}x\mathrm{d}t \tag{3.1}$$

for any $u \in L^2(\mathbb{R}^{d-k})$.

Similarly to the semiclassical case, we can extend this estimate to the estimates on Ω to obtain Theorem 3.2.

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