Algebraic independence of the partial derivatives of certain functions with many variables

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1 Main results

This article is based on a joint work with Professor Taka-aki Tanaka. We denote by $\mathbb{Z}_{\geq 0}$ the set of nonnegative integers, by $\overline{\mathbb{Q}}$ the field of algebraic numbers, and by $\overline{\mathbb{Q}}^{\times}$ the set of nonzero algebraic numbers. Let $\{R_k\}_{k\geq 0}$ be a linear recurrence of nonnegative integers satisfying

$$R_{k+n} = c_1 R_{k+n-1} + \cdots + c_n R_k \quad (k \geq 0), \tag{1}$$

where $n \geq 2$, $R_0, \ldots, R_{n-1}$ are not all zero, and $c_1, \ldots, c_n$ are nonnegative integers with $c_n \neq 0$. Define the polynomial associated with (1) by

$$\Phi(X) := X^n - c_1 X^{n-1} - \cdots - c_n.$$

Throughout this article, we assume the following three conditions (R1)–(R3) on $\{R_k\}_{k\geq 0}$:

(R1) $\Phi(\pm 1) \neq 0$.

(R2) The ratio of any pair of distinct roots of $\Phi(X)$ is not a root of unity.

(R3) $\{R_k\}_{k\geq 0}$ is not a geometric progression.

We note that if $\{R_k\}_{k\geq 0}$ satisfies the conditions (R1) and (R2), then $R_k = c\rho^k + o(\rho^k)$, where $c > 0$ and $\rho > 1$ (cf. Tanaka [4, Remark 4]). Let $a_1, \ldots, a_s$ be multiplicatively independent algebraic numbers with $0 < |a_i| < 1$ $(1 \leq i \leq s)$ and $y_1, \ldots, y_s$ complex variables. We write $\boldsymbol{y} := (y_1, \ldots, y_s)$. For each $1 \leq i \leq s$, we define

$$G_i(y_i) := \prod_{k=0}^{\infty} \left(1 - a_i^{R_k} y_i\right), \quad H_i(y_i) := \sum_{k=0}^{\infty} \frac{a_i^{R_k}}{1 - a_i^{R_k} y_i} \tag{2}$$

and for each algebraic number β, we define

$$N_{i,\beta} \coloneqq \#\{k \geq 0 \mid a_i^{-R_k} = \beta\} = \operatorname*{ord}_{y_i=\beta} G_i(y_i). \tag{3}$$

Moreover, for each $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_s) \in \overline{\mathbb{Q}}^s$, we denote

$$\mathcal{M}_{\boldsymbol{\beta}} \coloneqq \{\boldsymbol{m} = (m_1, \ldots, m_s) \in \mathbb{Z}_{\geq 0}^s \mid m_i \geq N_{i,\beta_i} \text{ for all } 1 \leq i \leq s\}.$$

Let

$$G(\boldsymbol{y}) \coloneqq \prod_{i=1}^{s} G_i(y_i), \quad H(\boldsymbol{y}) \coloneqq \sum_{k=0}^{\infty} \prod_{i=1}^{s} \frac{a_i^{R_k}}{1 - a_i^{R_k} y_i}, \quad \Theta(\boldsymbol{y}) \coloneqq G(\boldsymbol{y}) H(\boldsymbol{y}). \tag{4}$$

For an analytic function $f(\boldsymbol{y})$ and a vector $\boldsymbol{m} = (m_1, \ldots, m_s) \in \mathbb{Z}_{\geq 0}^s$, we denote

$$f^{(\boldsymbol{m})}(\boldsymbol{y}) \coloneqq \frac{\partial^{m_1 + \cdots + m_s} f}{\partial y_1^{m_1} \cdots \partial y_s^{m_s}}(\boldsymbol{y}).$$

Main Theorem. *Suppose that* $\{R_k\}_{k\geq 0}$ *satisfies the conditions* (R1)–(R3)*. Then the infinite set*

$$\left\{\Theta^{(\boldsymbol{m})}(\boldsymbol{\beta}) \;\middle|\; \boldsymbol{\beta} \in \overline{\mathbb{Q}}^s,\ \boldsymbol{m} \in \mathcal{M}_{\boldsymbol{\beta}}\right\}$$

is algebraically independent.

As a corollary to this theorem, we obtain an explicit example of an entire function with arbitrary number s of variables having the property that the values and the partial derivatives of any order at any distinct algebraic points are algebraically independent. Let $\mathbb{Z}_{>0}$ be the set of positive integers. Suppose that $\{R_k\}_{k\geq 0}$ satisfies the conditions (R1)–(R3). Assume in addition that $\{R_k\}_{k\geq 0}$ is strictly increasing. Then $N_{i,\beta} \leq 1$ for all $\beta \in \overline{\mathbb{Q}}$ and so $\mathbb{Z}_{>0}^s$ is a subset of $\mathcal{M}_{\boldsymbol{\beta}}$ for all $\boldsymbol{\beta} \in \overline{\mathbb{Q}}^s$. Hence $\left\{\Theta^{(\boldsymbol{m})}(\boldsymbol{\beta}) \;\middle|\; \boldsymbol{\beta} \in \overline{\mathbb{Q}}^s,\ \boldsymbol{m} \in \mathbb{Z}_{>0}^s\right\}$ is an infinite subset of $\left\{\Theta^{(\boldsymbol{m})}(\boldsymbol{\beta}) \;\middle|\; \boldsymbol{\beta} \in \overline{\mathbb{Q}}^s,\ \boldsymbol{m} \in \mathcal{M}_{\boldsymbol{\beta}}\right\}$. Therefore the main theorem implies that the infinite set $\left\{\Theta^{(\boldsymbol{m})}(\boldsymbol{\beta}) \;\middle|\; \boldsymbol{\beta} \in \overline{\mathbb{Q}}^s,\ \boldsymbol{m} \in \mathbb{Z}_{>0}^s\right\}$ is algebraically independent. Letting

$$\Xi(\boldsymbol{y}) \coloneqq \frac{\partial^s \Theta}{\partial y_1 \cdots \partial y_s}(\boldsymbol{y}) = G(\boldsymbol{y}) \sum_{\substack{k_1, \ldots, k_s, l \geq 0, \\ k_1, \ldots, k_s \neq l}} \prod_{i=1}^{s} \frac{-a_i^{R_{k_i} + R_l}}{(1 - a_i^{R_{k_i}} y_i)(1 - a_i^{R_l} y_i)}, \tag{5}$$

we obtain the following

Corollary 1. *Suppose that* $\{R_k\}_{k\geq 0}$ *satisfies the the conditions* (R1)–(R3)*. Assume in addition that* $\{R_k\}_{k\geq 0}$ *is strictly increasing. Then the infinite set*

$$\left\{\Xi^{(\boldsymbol{m})}(\boldsymbol{\beta}) \;\middle|\; \boldsymbol{\beta} \in \overline{\mathbb{Q}}^s,\ \boldsymbol{m} \in \mathbb{Z}_{\geq 0}^s\right\}$$

is algebraically independent.

Example 1. Let $p_1, \dots, p_s$ be distinct rational primes and $\{F_k\}_{k\geq 0}$ the Fibonacci numbers defined by

$$F_0 = 0, \quad F_1 = 1, \quad F_{k+2} = F_{k+1} + F_k \quad (k \geq 0).$$

Putting $a_i := p_i^{-1}$ $(1 \leq i \leq s)$ and regarding $\{F_{k+2}\}_{k\geq 0}$ as $\{R_k\}_{k\geq 0}$, we define the function $\Xi(\boldsymbol{y})$ by (5), namely,

$$\Xi(\boldsymbol{y}) = \prod_{i=1}^{s} \prod_{k=2}^{\infty} \left(1 - p_i^{-F_k} y_i\right) \sum_{\substack{k_1,\dots,k_s,l\geq 2, \\ k_1,\dots,k_s \neq l}} \prod_{i=1}^{s} \frac{-p_i^{-F_{k_i}-F_l}}{(1 - p_i^{-F_{k_i}} y_i)(1 - p_i^{-F_l} y_i)}.$$

Then by Corollary 1 the infinite set

$$\left\{\Xi^{(\boldsymbol{m})}(\boldsymbol{\beta}) \;\middle|\; \boldsymbol{\beta} \in \overline{\mathbb{Q}}^s, \ \boldsymbol{m} \in \mathbb{Z}_{\geq 0}^s\right\}$$

is algebraically independent.

In the case of $s = 1$, the main theorem is deduced from the following previous result of the author, which extends Tanaka's previous result [6] asserting the algebraic independency of the infinite set

$$\left\{G^{(m)}(\beta) \;\middle|\; \beta \in \overline{\mathbb{Q}}^{\times} \setminus \{a^{-R_k}\}_{k\geq 0}, \ m \geq 0\right\}.$$

Proposition 1 (A special case of Theorem 1.7 of Ide [1])**.** *Suppose that* $\{R_k\}_{k\geq 0}$ *satisfies the the conditions* (R1)–(R3)*. Then, if* $s = 1$*, then the infinite set*

$$\left\{G^{(m)}(\beta) \;\middle|\; \beta \in \overline{\mathbb{Q}}^{\times}, \ m \geq N_\beta\right\} \bigcup \left\{G^{(m)}(0) \;\middle|\; m \geq 1\right\}$$
$$\left(= \left\{-\Theta^{(m)}(\beta) \;\middle|\; \beta \in \overline{\mathbb{Q}}, \ m \geq N_\beta\right\} \bigcup \left\{G^{(N_\beta)}(\beta) \;\middle|\; \beta \in \overline{\mathbb{Q}}^{\times}\right\}\right)$$

is algebraically independent, where $N_\beta := N_{1,\beta}$ *for each* $\beta \in \overline{\mathbb{Q}}$*.*

For obtaining the entire main theorem, we actually show the following, which includes the main theorem for the case of $s \geq 2$.

Theorem 1. *Suppose that* $\{R_k\}_{k\geq 0}$ *satisfies the conditions* (R1)–(R3)*. Assume in addition that* $s \geq 2$*. Then the infinite set*

$$\begin{aligned}
&\left\{\Theta^{(\boldsymbol{m})}(\boldsymbol{\beta}) \;\middle|\; \boldsymbol{\beta} \in \overline{\mathbb{Q}}^s, \ \boldsymbol{m} \in \mathcal{M}_{\boldsymbol{\beta}}\right\} \\
&\bigcup \left\{G_i^{(m)}(\beta) \;\middle|\; 1 \leq i \leq s, \ \beta \in \overline{\mathbb{Q}}^{\times}, \ m \geq N_{i,\beta}\right\} \\
&\bigcup \left\{G_i^{(m)}(0) \;\middle|\; 1 \leq i \leq s, \ m \geq 1\right\}
\end{aligned}$$

is algebraically independent.

Theorem 1 is deduced from the following theorem together with the lemmas and the theorem stated in the next section.

Theorem 2. *Suppose that* $\{R_k\}_{k\geq 0}$ *satisfies the conditions* (R1)–(R3)*. Assume in addition that* $s \geq 2$*. Then the infinite set*

$$\begin{aligned}&\left\{H^{(\boldsymbol{m})}(\boldsymbol{\beta}) \;\middle|\; \boldsymbol{\beta} \in \mathcal{B}^s,\ \boldsymbol{m} \in \mathbb{Z}_{\geq 0}^s\right\}\\ &\bigcup \{G_i(\beta) \mid 1 \leq i \leq s,\ \beta \in \mathcal{B} \setminus \{0\}\}\\ &\bigcup \left\{H_i^{(m)}(\beta) \;\middle|\; 1 \leq i \leq s,\ \beta \in \mathcal{B},\ m \geq 0\right\}\end{aligned}$$

is algebraically independent, where

$$\mathcal{B} := \overline{\mathbb{Q}} \setminus \bigcup_{i=1}^{s} \{a_i^{-R_k} \mid k \geq 0\} = \{\beta \in \overline{\mathbb{Q}} \mid N_{i,\beta} = 0 \text{ for all } 1 \leq i \leq s\}.$$

The proof of Theorem 2 is based on Mahler's method (cf. [2, 3]) and consists of the following four steps: First, we construct an sn-dimensional algebraic point $\boldsymbol{\alpha}$ and Mahler functions $h_{\boldsymbol{\beta},\boldsymbol{m}}(\boldsymbol{z})$, $g_{i,\beta}(\boldsymbol{z})$, $h_{i,\beta,m}(\boldsymbol{z})$ of sn variables $\boldsymbol{z} = (z_{11}, \ldots, z_{sn})$ such that $H^{(\boldsymbol{m})}(\boldsymbol{\beta}) = h_{\boldsymbol{\beta},\boldsymbol{m}}(\boldsymbol{\alpha})$, $G_i(\beta) = g_{i,\beta}(\boldsymbol{\alpha})$, $H_i^{(m)}(\beta) = h_{i,\beta,m}(\boldsymbol{\alpha})$, where n is the length of the recurrence formula (1). Secondly, using Kubota's criterion for the algebraic independence of the values of Mahler functions, we reduce the algebraic independence of the values $h_{\boldsymbol{\beta},\boldsymbol{m}}(\boldsymbol{\alpha})$, $g_{i,\beta}(\boldsymbol{\alpha})$, $h_{i,\beta,m}(\boldsymbol{\alpha})$ to that of the functions $h_{\boldsymbol{\beta},\boldsymbol{m}}(\boldsymbol{z})$, $g_{i,\beta}(\boldsymbol{z})$, $h_{i,\beta,m}(\boldsymbol{z})$ themselves over the rational function field $\overline{\mathbb{Q}}(\boldsymbol{z})$. Thirdly, using Kubota's criterion for the algebraic independence of Mahler functions themselves, we reduce the algebraic independence of the functions above to their linear independence and their multiplicative independence. (For these two criteria for the algebraic independence, see Kubota [2].) Finally, multiplexing Tanaka's result [5] on the rational function solutions of certain functional equations, we prove the linear independence and the multiplicative independence mentioned above. We omit further details of the proof of Theorem 2 in this article.

2 Proof of Theorem 1

For each $1 \leq i \leq s$, let $\{a_k^{(i)}\}_{k\geq 0}$ be a sequence of algebraic numbers satisfying

$$\sum_{k=0}^{\infty} |a_k^{(i)}| < \infty$$

and let

$$g_i(y_i) := \prod_{k=0}^{\infty} \left(1 - a_k^{(i)} y_i\right), \quad h_i(y_i) := \sum_{k=0}^{\infty} \frac{a_k^{(i)}}{1 - a_k^{(i)} y_i}. \tag{6}$$

Define

$$g(\boldsymbol{y}) \coloneqq \prod_{i=1}^{s} g_i(y_i), \quad h(\boldsymbol{y}) \coloneqq \sum_{k=0}^{\infty} \prod_{i=1}^{s} \frac{a_k^{(i)}}{1 - a_k^{(i)} y_i}, \quad \theta(\boldsymbol{y}) \coloneqq g(\boldsymbol{y}) h(\boldsymbol{y}). \tag{7}$$

First we show the following lemmas, which assert that the functions in (6) and (7) satisfy 'invertible' algebraic relations.

Lemma 1. *Let $m \in \mathbb{Z}_{\geq 0}$ and let $X_0, \ldots, X_{m-1}$ and $Y_1, \ldots, Y_m$ be variables. Then the following equations hold.*

(i) *For any $m \geq 0$,*

$$g_i^{(m)}(y_i) = g_i(y_i) A_m \left(h_i(y_i), \ldots, h_i^{(m-1)}(y_i) \right) \qquad (1 \leq i \leq s), \tag{8}$$

where $A_0 \coloneqq 1$ and $A_m(X_0, \ldots, X_{m-1}) \in \mathbb{Z}[X_0, \ldots, X_{m-1}]$.

(ii) *For any $m \geq 0$,*

$$h_i^{(m)}(y_i) = -\frac{g_i^{(m+1)}(y_i)}{g_i(y_i)} + B_m \left(-\frac{g_i'(y_i)}{g_i(y_i)}, \ldots, -\frac{g_i^{(m)}(y_i)}{g_i(y_i)} \right) \qquad (1 \leq i \leq s), \tag{9}$$

where $B_0 \coloneqq 0$ and $B_m(Y_1, \ldots, Y_m) \in \mathbb{Z}[Y_1, \ldots, Y_m]$.

Proof. Since $g_i'(y_i) = -g_i(y_i) h_i(y_i)$, we see inductively that, for any $m \geq 1$,

$$g_i^{(m)}(y_i) = -g_i(y_i) h_i^{(m-1)}(y_i) + g_i(y_i) A_m^* \left(h_i(y_i), \ldots, h_i^{(m-2)}(y_i) \right), \tag{10}$$

where $A_1^* \coloneqq 0$ and $A_m^*(X_0, \ldots, X_{m-2}) \in \mathbb{Z}[X_0, \ldots, X_{m-2}]$ $(m \geq 2)$. Letting

$$A_m(X_0, \ldots, X_{m-1}) \coloneqq \begin{cases} 1 \quad (m = 0), \\ -X_{m-1} + A_m^*(X_0, \ldots, X_{m-2}) \quad (m \geq 1), \end{cases}$$

we get (8). By (10) we have

$$h_i^{(m)}(y_i) = -\frac{g_i^{(m+1)}(y_i)}{g_i(y_i)} + A_{m+1}^* \left(h_i(y_i), \ldots, h_i^{(m-1)}(y_i) \right)$$

for any $m \geq 0$. Therefore, defining

$$B_m(Y_1, \ldots, Y_m) \coloneqq A_{m+1}^*(Y_1, Y_2 + B_1(Y_1), \ldots, Y_m + B_{m-1}(Y_1, \ldots, Y_{m-1}))$$

inductively on $m \geq 0$, we obtain (9). □

Lemma 2. *For any* $\boldsymbol{m} = (m_1, \ldots, m_s) \in \mathbb{Z}_{\geq 0}^s$*, let* $X_{i,\mu}$ $(1 \leq i \leq s,\ 0 \leq \mu \leq m_i - 1)$*,* $Y_{i,\mu}$ $(1 \leq i \leq s,\ 1 \leq \mu \leq m_i)$*, and* $Z_{\boldsymbol{\mu}}$ $(\boldsymbol{\mu} = (\mu_1, \ldots, \mu_s),\ 0 \leq \mu_i \leq m_i,\ \boldsymbol{\mu} \neq \boldsymbol{m})$ *be variables. Set* $\boldsymbol{X_m} := (X_{i,\mu})_{i,\mu}$*,* $\boldsymbol{Y_m} := (Y_{i,\mu})_{i,\mu}$*, and* $\boldsymbol{Z_m} := (Z_{\boldsymbol{\mu}})_{\boldsymbol{\mu}}$*. Then the following hold.*

(i)

$$\theta^{(\boldsymbol{m})}(\boldsymbol{y}) = g(\boldsymbol{y})h^{(\boldsymbol{m})}(\boldsymbol{y}) + g(\boldsymbol{y})C_{\boldsymbol{m}}(\boldsymbol{X_m}, \boldsymbol{Z_m}) \Big|_{X_{i,\mu}=h_i^{(\mu)}(y_i),\ Z_{\boldsymbol{\mu}}=h^{(\boldsymbol{\mu})}(\boldsymbol{y})}, \tag{11}$$

where $C_{\boldsymbol{m}}(\boldsymbol{X_m}, \boldsymbol{Z_m}) \in \mathbb{Z}[\boldsymbol{X_m}, \boldsymbol{Z_m}]$*.*

(ii)

$$h^{(\boldsymbol{m})}(\boldsymbol{y}) = \frac{\theta^{(\boldsymbol{m})}(\boldsymbol{y})}{g(\boldsymbol{y})} + D_{\boldsymbol{m}}(\boldsymbol{Y_m}, \boldsymbol{Z_m}) \Big|_{Y_{i,\mu}=-g_i^{(\mu)}(y_i)/g_i(y_i),\ Z_{\boldsymbol{\mu}}=\theta^{(\boldsymbol{\mu})}(\boldsymbol{y})/g(\boldsymbol{y})}, \tag{12}$$

where $D_{\boldsymbol{m}}(\boldsymbol{Y_m}, \boldsymbol{Z_m}) \in \mathbb{Z}[\boldsymbol{Y_m}, \boldsymbol{Z_m}]$*.*

Proof. Since $\theta(\boldsymbol{y}) = g(\boldsymbol{y})h(\boldsymbol{y})$, using (8), we obtain

$$\begin{aligned}
\theta^{(\boldsymbol{m})}(\boldsymbol{y}) = & \sum_{\substack{\boldsymbol{\mu}=(\mu_1,\ldots,\mu_s),\\ 0\leq\mu_i\leq m_i\ (1\leq i\leq s)}} \left(\prod_{i=1}^{s} \binom{m_i}{\mu_i} g_i^{(\mu_i)}(y_i)\right) h^{(\boldsymbol{m}-\boldsymbol{\mu})}(\boldsymbol{y}) \\
&= g(\boldsymbol{y})h^{(\boldsymbol{m})}(\boldsymbol{y}) \\
&\quad + g(\boldsymbol{y}) \sum_{\substack{\boldsymbol{\mu}=(\mu_1,\ldots,\mu_s),\\ 0\leq\mu_i\leq m_i\ (1\leq i\leq s),\\ \boldsymbol{\mu}\neq\boldsymbol{0}}} \left(\prod_{i=1}^{s} \binom{m_i}{\mu_i} A_{\mu_i}\left(h_i(y_i), \ldots, h_i^{(\mu_i-1)}(y_i)\right)\right) h^{(\boldsymbol{m}-\boldsymbol{\mu})}(\boldsymbol{y})
\end{aligned}$$

for any $\boldsymbol{m} \in \mathbb{Z}_{\geq 0}^s$, which implies (11). Thereby we have

$$h^{(\boldsymbol{m})}(\boldsymbol{y}) = \frac{\theta^{(\boldsymbol{m})}(\boldsymbol{y})}{g(\boldsymbol{y})} - C_{\boldsymbol{m}}(\boldsymbol{X_m}, \boldsymbol{Z_m}) \Big|_{X_{i,\mu}=h_i^{(\mu)}(y_i),\ Z_{\boldsymbol{\mu}}=h^{(\boldsymbol{\mu})}(\boldsymbol{y})}.$$

Hence, noting (9) and defining

$$\begin{aligned}
&D_{\boldsymbol{m}}(\boldsymbol{Y_m}, \boldsymbol{Z_m}) \\
&:= -\, C_{\boldsymbol{m}}(\boldsymbol{X_m}, \boldsymbol{Z'_m}) \Big|_{X_{i,\mu}=Y_{i,\mu+1}+B_\mu(Y_{i,1},\ldots,Y_{i,\mu}),\ Z'_{\boldsymbol{\mu}}=Z_{\boldsymbol{\mu}}+D_{\boldsymbol{\mu}}(\boldsymbol{Y_\mu},\boldsymbol{Z_\mu})}
\end{aligned}$$

inductively with respect to the lexicographical order of $\mathbb{Z}_{\geq 0}^s$, where $Z'_{\boldsymbol{\mu}}$ $(\boldsymbol{\mu} = (\mu_1, \ldots, \mu_s),\ 0 \leq \mu_i \leq m_i,\ \boldsymbol{\mu} \neq \boldsymbol{m})$ are variables and $\boldsymbol{Z'_m} = (Z'_{\boldsymbol{\mu}})_{\boldsymbol{\mu}}$, we obtain (12). □

Next we show the existence of invertible linear relations between the values of the above functions and those of 'shifted' functions defined below. Let $\beta_0 := 0$ and let $\beta_1, \dots, \beta_J$ be any nonzero distinct algebraic numbers. Similarly to the numbers N_{i,β_j} $(1 \le i \le s,\ 0 \le j \le J)$ defined by (3), we define the numbers $n_{i,j}$ $(1 \le i \le s,\ 0 \le j \le J)$ by

$$n_{i,j} := \#\{k \ge 0 \mid a_k^{(i)} \ne 0,\ (a_k^{(i)})^{-1} = \beta_j\} = \operatorname*{ord}_{y_i=\beta_j} g_i(y_i) \quad (1 \le i \le s,\ 0 \le j \le J).$$

For each $\boldsymbol{j} = (j_1, \dots, j_s) \in \{0, \dots, J\}^s$, let

$$\boldsymbol{\beta_j} := (\beta_{j_1}, \dots, \beta_{j_s}), \quad \boldsymbol{n_j} := (n_{1,j_1}, \dots, n_{s,j_s}).$$

Since $a_k^{(i)} \to 0$ as k tends to infinity for all $1 \le i \le s$, there exists a sufficiently large integer k_0 such that $1 - a_k^{(i)}\beta_j \ne 0$ $(1 \le i \le s,\ 1 \le j \le J)$ for all $k \ge k_0$. Let $\widetilde{a}_k^{(i)} := a_{k+k_0}^{(i)}$ $(k \ge 0)$. Let $\widetilde{g}_i(y_i)$, $\widetilde{h}_i(y_i)$ $(1 \le i \le s)$ and $\widetilde{g}(\boldsymbol{y})$, $\widetilde{h}(\boldsymbol{y})$, $\widetilde{\theta}(\boldsymbol{y})$ be the functions given respectively by (6) and (7) with the sequences $\{\widetilde{a}_k^{(i)}\}_{k\ge0}$ $(1 \le i \le s)$ in place of $\{a_k^{(i)}\}_{k\ge0}$ $(1 \le i \le s)$. Let M be any nonnegative integer and define the finite sets S_l and T_l $(l = 1, 2, 3)$ of the values by

$$\begin{aligned}
S_1 &:= \left\{ g_i^{(m+n_{i,j})}(\beta_j) \,\middle|\, 1 \le i \le s,\ 1 \le j \le J,\ 0 \le m \le M+1 \right\}, \\
S_2 &:= \left\{ g_i^{(m)}(0) \,\middle|\, 1 \le i \le s,\ 1 \le m \le M+1 \right\}, \\
S_3 &:= \left\{ \theta^{(\boldsymbol{m}+\boldsymbol{n_j})}(\boldsymbol{\beta_j}) \mid \boldsymbol{j} \in \{0, \dots, J\}^s,\ \boldsymbol{m} \in \{0, \dots, M\}^s \right\},
\end{aligned}$$

and

$$\begin{aligned}
T_1 &:= \left\{ \widetilde{g}_i^{(m)}(\beta_j) \,\middle|\, 1 \le i \le s,\ 1 \le j \le J,\ 0 \le m \le M+1 \right\}, \\
T_2 &:= \left\{ \widetilde{g}_i^{(m)}(0) \,\middle|\, 1 \le i \le s,\ 1 \le m \le M+1 \right\}, \\
T_3 &:= \left\{ \widetilde{\theta}^{(\boldsymbol{m})}(\boldsymbol{\beta_j}) \,\middle|\, \boldsymbol{j} \in \{0, \dots, J\}^s,\ \boldsymbol{m} \in \{0, \dots, M\}^s \right\}.
\end{aligned}$$

Let $N_l := \#S_l (= \#T_l)$ $(l = 1, 2, 3)$. We denote by $\mathcal{L}_N$ the set of the $N \times N$ lower triangular matrices with entries in $\overline{\mathbb{Q}}$ whose diagonal entries are nonzero. We note that $\mathcal{L}_N$ is a subset of $GL_N(\overline{\mathbb{Q}})$. For any finite set A, let $\boldsymbol{A}^*$ be a column vector whose components are given by a permutation of the elements of A. The following theorem plays a crucial role in the proof of Theorem 1.

Theorem 3. *There exist $\boldsymbol{S}_l^*$ and $\boldsymbol{T}_l^*$ $(l = 1, 2, 3)$ corresponding respectively to the sets S_l and T_l $(l = 1, 2, 3)$ such that the following hold, so that $\overline{\mathbb{Q}}[S_1 \cup S_2 \cup S_3] = \overline{\mathbb{Q}}[T_1 \cup T_2 \cup T_3]$.*

(i) $\boldsymbol{S}_1^* = L_1 \boldsymbol{T}_1^*$, *where* $L_1 \in \mathcal{L}_{N_1}$.

(ii) $\boldsymbol{S}_2^* \equiv L_2 \boldsymbol{T}_2^* \pmod{\overline{\mathbb{Q}}^{N_2}}$, *where* $L_2 \in \mathcal{L}_{N_2}$.

(iii) $\boldsymbol{S}_3^* \equiv L_3 \boldsymbol{T}_3^* \pmod{\overline{\mathbb{Q}}[T_1 \cup T_2]^{N_3}}$, *where* $L_3 \in \mathcal{L}_{N_3}$.

Proof. First, we prove (i) of the theorem. For this purpose, we fix $1 \le i \le s$ and $1 \le j \le J$ and represent $g_i^{(m+n_{i,j})}(\beta_j)$ $(0 \le m \le M+1)$ as linear combinations of $\widetilde{g}_i^{(m)}(\beta_j)$ $(0 \le m \le M+1)$. We define

$$P(y_i) \coloneqq (1-\beta_j^{-1} y_i)^{n_{i,j}} \in \overline{\mathbb{Q}}[y_i], \quad Q(y_i) \coloneqq \prod_{\substack{k=0 \\ a_k^{(i)} \neq \beta_j^{-1}}}^{k_0-1} (1-a_k^{(i)} y_i) \in \overline{\mathbb{Q}}[y_i].$$

Since

$$g_i(y_i) = \prod_{k=0}^{k_0-1} (1-a_k^{(i)} y_i) \times \prod_{k=k_0}^{\infty} (1-a_k^{(i)} y_i) = P(y_i) Q(y_i) \widetilde{g}_i(y_i),$$

we see that, for $0 \le m \le M+1$,

$$g_i^{(m+n_{i,j})}(\beta_j) = \sum_{h=0}^{m} \binom{m+n_{i,j}}{n_{i,j} \quad m-h \quad h} p Q^{(m-h)}(\beta_j) \widetilde{g}_i^{(h)}(\beta_j),$$

where $p \coloneqq P^{(n_{i,j})}(y_i) \in \overline{\mathbb{Q}}^{\times}$. Hence we have

$$\begin{pmatrix} g_i^{(n_{i,j})}(\beta_j) \\ g_i^{(1+n_{i,j})}(\beta_j) \\ \vdots \\ g_i^{(M+1+n_{i,j})}(\beta_j) \end{pmatrix} = \begin{pmatrix} pq & & & 0 \\ & \binom{1+n_{i,j}}{n_{i,j}} pq & & \\ & & \ddots & \\ * & & & \binom{M+1+n_{i,j}}{n_{i,j}} pq \end{pmatrix} \begin{pmatrix} \widetilde{g}_i(\beta_j) \\ \widetilde{g}_i'(\beta_j) \\ \vdots \\ \widetilde{g}_i^{(M+1)}(\beta_j) \end{pmatrix},$$

where $q \coloneqq Q(\beta_j) \in \overline{\mathbb{Q}}^{\times}$, which implies (i) of the theorem. In the same way, we obtain (ii), noting that $\widetilde{g}_i(0) = 1 \in \overline{\mathbb{Q}}$ $(1 \le i \le s)$.

In the remaining part of the proof we show (iii). Since

$$g(\boldsymbol{y}) = \left(\prod_{i=1}^{s} \prod_{k=0}^{k_0-1} \left(1-a_k^{(i)} y_i\right) \right) \widetilde{g}(\boldsymbol{y})$$

and since

$$h(\boldsymbol{y}) = \widetilde{h}(\boldsymbol{y}) + \sum_{k=0}^{k_0-1} \prod_{i=1}^{s} \frac{a_k^{(i)}}{1-a_k^{(i)} y_i},$$

we obtain a decomposition

$$\theta(\boldsymbol{y}) = \theta_1(\boldsymbol{y}) + \theta_2(\boldsymbol{y}), \tag{13}$$

where

$$\theta_1(\boldsymbol{y}) \coloneqq \left(\prod_{i=1}^{s} \prod_{k=0}^{k_0-1} \left(1 - a_k^{(i)} y_i\right) \right) \widetilde{\theta}(\boldsymbol{y})$$

and

$$\theta_2(\boldsymbol{y}) \coloneqq \left(\sum_{k=0}^{k_0-1} \prod_{i=1}^{s} a_k^{(i)} \prod_{\substack{k'=0 \\ k' \neq k}}^{k_0-1} (1 - a_{k'}^{(i)} y_i) \right) \widetilde{g}(\boldsymbol{y}).$$

Fix $\boldsymbol{j} = (j_1, \ldots, j_s) \in \{0, \ldots, J\}^s$. In order to prove (iii), we represent $\theta_1^{(\boldsymbol{m}+\boldsymbol{n_j})}(\boldsymbol{\beta_j})$ $(\boldsymbol{m} \in \{0, \ldots, M\}^s)$ as linear combinations of $\widetilde{\theta}^{(\boldsymbol{m})}(\boldsymbol{\beta_j})$ $(\boldsymbol{m} \in \{0, \ldots, M\}^s)$ and show that $\theta_2^{(\boldsymbol{m}+\boldsymbol{n_j})}(\boldsymbol{\beta_j})$ $(\boldsymbol{m} \in \{0, \ldots, M\}^s)$ are elements of $\overline{\mathbb{Q}}[T_1 \cup T_2]$. Let

$$I_1 \coloneqq \{i \in \{1, \ldots, s\} \mid j_i \neq 0\}, \quad I_2 \coloneqq \{i \in \{1, \ldots, s\} \mid j_i = 0\}.$$

We define

$$P_i(y_i) \coloneqq (1 - \beta_{j_i}^{-1} y_i)^{n_{i,j_i}} \in \overline{\mathbb{Q}}[y_i], \quad Q_i(y_i) \coloneqq \prod_{\substack{k=0 \\ a_k^{(i)} \neq \beta_{j_i}^{-1}}}^{k_0-1} (1 - a_k^{(i)} y_i) \in \overline{\mathbb{Q}}[y_i]$$

for each $i \in I_1$ and define

$$R_i(y_i) \coloneqq \prod_{k=0}^{k_0-1} (1 - a_k^{(i)} y_i) \in \overline{\mathbb{Q}}[y_i]$$

for each $i \in I_2$. Then we have

$$\theta_1(\boldsymbol{y}) = \left(\prod_{i \in I_1} P_i(y_i) Q_i(y_i) \right) \left(\prod_{i \in I_2} R_i(y_i) \right) \widetilde{\theta}(\boldsymbol{y}).$$

Hence, for $\boldsymbol{m} = (m_1, \ldots, m_s) \in \{0, \ldots, M\}^s$, we have

$$\theta_1^{(\boldsymbol{m}+\boldsymbol{n_j})}(\boldsymbol{\beta_j}) = \sum_{\substack{\boldsymbol{h}=(h_1,\ldots,h_s), \\ 0 \le h_i \le m_i \ (1 \le i \le s)}} \left(\prod_{i \in I_1} \binom{m_i + n_{i,j_i}}{n_{i,j_i}\ \ m_i - h_i\ \ h_i} p_i Q_i^{(m_i-h_i)}(\beta_{j_i}) \right)$$
$$\times \left(\prod_{i \in I_2} \binom{m_i}{h_i} R_i^{(m_i-h_i)}(0) \right) \widetilde{\theta}^{(\boldsymbol{h})}(\boldsymbol{\beta_j}). \tag{14}$$

We note that the coefficient of $\widetilde{\theta}^{(\boldsymbol{m})}(\boldsymbol{\beta_j})$ in the right-hand side of (14) is a nonzero algebraic number

$$\prod_{i\in I_1}\binom{m_i+n_{i,j_i}}{m_i}p_iq_i,$$

where $p_i := P_i^{(n_{i,j_i})}(y_i) \in \overline{\mathbb{Q}}^\times$ and $q_i := Q_i(\beta_{j_i}) \in \overline{\mathbb{Q}}^\times$ for $i \in I_1$.

Next, we define

$$U_i(y_i) := (1-\beta_{j_i}^{-1}y_i)^{\max\{n_{i,j_i}-1,0\}} \in \overline{\mathbb{Q}}[y_i]$$

for each $i \in I_1$ and let

$$V(\boldsymbol{y}) := \left(\sum_{k=0}^{k_0-1}\prod_{i=1}^{s}a_k^{(i)}\prod_{\substack{k'=0\\k'\neq k}}^{k_0-1}(1-a_{k'}^{(i)}y_i)\right)\prod_{i\in I_1}U_i(y_i)^{-1} \in \overline{\mathbb{Q}}[\boldsymbol{y}].$$

Then we have

$$\theta_2(\boldsymbol{y}) = \left(\prod_{i\in I_1}U_i(y_i)\right)V(\boldsymbol{y})\widetilde{g}(\boldsymbol{y}).$$

Hence, for $\boldsymbol{m} = (m_1,\dots,m_s) \in \{0,\dots,M\}^s$, we have

$$\begin{aligned}\theta_2^{(\boldsymbol{m}+\boldsymbol{n_j})}(\boldsymbol{\beta_j}) = &\sum_{\substack{\boldsymbol{h}=(h_1,\dots,h_s),\\0\leq h_i\leq m_i'\ (1\leq i\leq s)}}\left(\prod_{i\in I_1}\binom{m_i+n_{i,j_i}}{\max\{n_{i,j_i}-1,0\}\ m_i'-h_i\ h_i}u_i\widetilde{g}_i^{(h_i)}(\beta_{j_i})\right)\\ &\times\left(\prod_{i\in I_2}\binom{m_i}{h_i}\widetilde{g}_i^{(h_i)}(0)\right)V^{(\boldsymbol{m}'-\boldsymbol{h})}(\boldsymbol{\beta_j})\\ &\in \overline{\mathbb{Q}}[T_1\cup T_2],\end{aligned} \tag{15}$$

where $m_i' := m_i + \min\{1, n_{i,j_i}\} = m_i + n_{i,j_i} - \max\{n_{i,j_i}-1, 0\}$ $(1 \leq i \leq s)$, $\boldsymbol{m}' := (m_1',\dots,m_s')$, and $u_i := U_i^{(\max\{n_{i,j_i}-1,0\})}(y_i) \in \overline{\mathbb{Q}}^\times$ $(i \in I_1)$. We consider the lexicographical order of $\{0,\dots,M\}^s$ and let $\boldsymbol{\theta_j}$ and $\widetilde{\boldsymbol{\theta}}_{\boldsymbol{j}}$ be column vectors whose components are given by permuting the elements of the sets $\{\theta^{(\boldsymbol{m}+\boldsymbol{n_j})}(\boldsymbol{\beta_j}) \mid \boldsymbol{m} \in \{0,\dots,M\}^s\}$ and $\{\widetilde{\theta}^{(\boldsymbol{m})}(\boldsymbol{\beta_j}) \mid \boldsymbol{m} \in \{0,\dots,M\}^s\}$ in ascending order of $\boldsymbol{m}$, respectively. Then, from (13), (14), and (15), we see that there exists an element $L_{\boldsymbol{j}}$ of $\mathcal{L}_{(M+1)^s}$ such that $\boldsymbol{\theta_j} \equiv L_{\boldsymbol{j}}\widetilde{\boldsymbol{\theta}}_{\boldsymbol{j}} \pmod{\overline{\mathbb{Q}}[T_1\cup T_2]^{(M+1)^s}}$, which implies (iii) of the theorem. □

Remark 1. In the last part of the proof above, we can explicitly represent the coefficient matrix $L_{\boldsymbol{j}} \in \mathcal{L}_{(M+1)^s}$ as a Kronecker product of s elements of $\mathcal{L}_{M+1}$ as follows. For each $i \in \{1,\dots,s\}$, let $L_{\boldsymbol{j},i} = (l^{(i)}_{mh})_{m,h}$ be the element of $\mathcal{L}_{M+1}$ defined by

$$l^{(i)}_{mh} := \begin{cases} \binom{m-1+n_{i,j_i}}{n_{i,j_i} \quad m-h \quad h-1} p_i Q_i^{(m-h)}(\beta_{j_i}) \quad (i \in I_1,\ 1 \le h \le m \le M+1), \\ \binom{m-1}{h-1} R_i^{(m-h)}(0) \quad (i \in I_2,\ 1 \le h \le m \le M+1), \\ 0 \quad (1 \le m < h \le M+1), \end{cases}$$

namely,

$$L_{\boldsymbol{j},i} = \begin{cases} \begin{pmatrix} p_i q_i & & & 0 \\ & \binom{1+n_{i,j_i}}{n_{i,j_i}} p_i q_i & & \\ & & \ddots & \\ * & & & \binom{M+n_{i,j_i}}{n_{i,j_i}} p_i q_i \end{pmatrix} \in \mathcal{L}_{M+1} \quad (i \in I_1), \\ \begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ * & & & 1 \end{pmatrix} \in \mathcal{L}_{M+1} \quad (i \in I_2). \end{cases}$$

Then, noting that the components of $\boldsymbol{\theta_j}$ and $\widetilde{\boldsymbol{\theta}}_{\boldsymbol{j}}$ are arranged in ascending lexicographical order of $\boldsymbol{m} \in \{0,\dots,M\}^s$, we see that

$$L_{\boldsymbol{j}} = L_{\boldsymbol{j},1} \otimes L_{\boldsymbol{j},2} \otimes \cdots \otimes L_{\boldsymbol{j},s} \in \mathcal{L}_{(M+1)^s},$$

where $\otimes$ denotes the Kronecker product.

Proof of Theorem 1. Let $\beta_0 := 0$ and let $\beta_1,\dots,\beta_J$ be any nonzero distinct algebraic numbers. For the simplicity we denote $N_{i,j} := N_{i,\beta_j}$ $(1 \le i \le s,\ 0 \le j \le J)$. For any $\boldsymbol{j} = (j_1,\dots,j_s) \in \{0,\dots,J\}^s$, let $\boldsymbol{N_j} := (N_{1,j_1},\dots,N_{s,j_s})$. In order to prove Theorem 1, it is enough to prove that, for any sufficiently large nonnegative integer M, the finite set

$$\begin{aligned} S := &\left\{\Theta^{(\boldsymbol{m}+\boldsymbol{N_j})}(\boldsymbol{\beta_j}) \;\middle|\; \boldsymbol{j} \in \{0,\dots,J\}^s,\ \boldsymbol{m} \in \{0,\dots,M\}^s\right\} \\ &\bigcup \left\{G_i^{(m+N_{i,j})}(\beta_j) \;\middle|\; 1 \le i \le s,\ 1 \le j \le J,\ 0 \le m \le M+1\right\} \\ &\bigcup \left\{G_i^{(m)}(0) \;\middle|\; 1 \le i \le s,\ 1 \le m \le M+1\right\} \end{aligned}$$

is algebraically independent. Since $R_k \to \infty$ as k tends to infinity, there exists a sufficiently large integer k_0 such that $1 - a_i^{R_k}\beta_j \neq 0$ $(1 \le i \le s,\ 1 \le j \le J)$ for all

$k \geq k_0$. Let $\widetilde{R}_k := R_{k+k_0}$ $(k \geq 0)$. We note that the linear recurrence $\{\widetilde{R}_k\}_{k\geq 0}$ also satisfies the conditions (R1)–(R3) stated in Section 1. Let $\widetilde{G}_i(y_i)$, $\widetilde{H}_i(y_i)$ $(1 \leq i \leq s)$ and $\widetilde{G}(\boldsymbol{y})$, $\widetilde{H}(\boldsymbol{y})$, $\widetilde{\Theta}(\boldsymbol{y})$ be the functions given respectively by (2) and (4) with $\{\widetilde{R}_k\}_{k\geq 0}$ in place of $\{R_k\}_{k\geq 0}$. Let

$$\begin{aligned} T := &\left\{\widetilde{\Theta}^{(\boldsymbol{m})}(\boldsymbol{\beta_j}) \;\middle|\; \boldsymbol{j} \in \{0,\dots,J\}^s,\ \boldsymbol{m} \in \{0,\dots,M\}^s\right\} \\ &\bigcup \left\{\widetilde{G}_i^{(m)}(\beta_j) \;\middle|\; 1 \leq i \leq s,\ 1 \leq j \leq J,\ 0 \leq m \leq M+1\right\} \\ &\bigcup \left\{\widetilde{G}_i^{(m)}(0) \;\middle|\; 1 \leq i \leq s,\ 1 \leq m \leq M+1\right\} \end{aligned}$$

and

$$\begin{aligned} U := &\left\{\widetilde{H}^{(\boldsymbol{m})}(\boldsymbol{\beta_j}) \;\middle|\; \boldsymbol{j} \in \{0,\dots,J\}^s,\ \boldsymbol{m} \in \{0,\dots,M\}^s\right\} \\ &\bigcup \left\{\widetilde{G}_i(\beta_j) \;\middle|\; 1 \leq i \leq s,\ 1 \leq j \leq J\right\} \\ &\bigcup \left\{\widetilde{H}_i^{(m)}(\beta_j) \;\middle|\; 1 \leq i \leq s,\ 0 \leq j \leq J,\ 0 \leq m \leq M\right\}. \end{aligned}$$

By Theorem 3, we see that $\overline{\mathbb{Q}}[S] = \overline{\mathbb{Q}}[T]$. Moreover, $\overline{\mathbb{Q}}(T) = \overline{\mathbb{Q}}(U)$ since

$$\begin{aligned} &\mathbb{Z}\left[\left\{\frac{\widetilde{\Theta}^{(\boldsymbol{m})}(\boldsymbol{y})}{\widetilde{G}(\boldsymbol{y})} \;\middle|\; \boldsymbol{m} \in \{0,\dots,M\}^s\right\} \bigcup \left\{\frac{\widetilde{G}_i^{(m)}(y_i)}{\widetilde{G}_i(y_i)} \;\middle|\; 1 \leq i \leq s,\ 1 \leq m \leq M+1\right\}\right] \\ &= \mathbb{Z}\left[\left\{\widetilde{H}^{(\boldsymbol{m})}(\boldsymbol{y}) \;\middle|\; \boldsymbol{m} \in \{0,\dots,M\}^s\right\} \bigcup \left\{\widetilde{H}_i^{(m)}(y_i) \;\middle|\; 1 \leq i \leq s,\ 0 \leq m \leq M\right\}\right] \end{aligned}$$

by Lemmas 1 and 2 and since $\widetilde{G}_i(0) = 1$, $\widetilde{G}_i(\beta_j) \neq 0$ $(1 \leq i \leq s,\ 1 \leq j \leq J)$. Noting that $\#S = \#T = \#U$, we see that the algebraic independency of S is equivalent to that of U. This concludes the proof since Theorem 2 for the linear recurrence $\{\widetilde{R}_k\}_{k\geq 0}$ asserts that U is algebraically independent. □

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