

CAMPANA POINTS, HEIGHT ZETA FUNCTIONS, AND LOG MANIN'S CONJECTURE

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ABSTRACT. This is a report of the author's talk at RIMS workshop 2020 Problems and Prospects in Analytic Number Theory held online on Zoom. We discuss a recent formulation of log Manin's conjecture for klt Campana points and an approach to this conjecture using the height zeta function method.

1. INTRODUCTION

One of fundamental tools in diophantine geometry is the notion of height functions and this height function measures the geometric and arithmetic complexities of rational points on an algebraic variety. These are crucial to various finiteness results in diophantine geometry such as Mordell-Weil theorem, Siegel's theorem, Mordell-Faltings' theorem, and so on. One of basic properties of height functions is the Northcott property which claims that for a height function associated to an ample divisor, the set of rational points whose height is less than T is finite. Thus one may consider the counting function of rational points of bounded height, and one natural question is the asymptotic formula for such a counting function when T goes to infinity.

Around the late 1980's, Yuri Manin and his collaborators proposed a general framework to understand this asymptotic formula in terms of geometric and arithmetic invariants of the underlying projective variety, and this leads to Manin's conjecture whose formulation is developed in a series of papers [FMT89], [BM90], [Pey95], [BT98a], [Pey03], [Pey17], and [LST18]. One of fertile testing grounds for this conjecture is a class of equivariant compactifications of homogeneous spaces, and there are mainly two methods available, i.e., the method of mixing and the height zeta function method.

Mixing is a concept from ergodic theory, and this idea has been successfully used to prove equidistribution of rational points on homogeneous spaces acted by semi-simple groups ([GMO08] and [GO11]). The height zeta function method can be applied to a variety of equivariant compactifications of connected algebraic groups including, but not limited to, generalized flag varieties ([FMT89]), toric varieties ([BT96] and [BT98a]), equivariant compactifications of vector groups ([CLT02]), wonderful compactifications of semi-simple groups of adjoint type ([STBT07]), and biequivariant compactifications of unipotent groups ([ST16]).

The height zeta function method also has its advantage to studying the counting problem of integral points associated to a reduced boundary divisor, and this has been implemented for equivariant compactifications of vector groups ([CLT12]), toric varieties ([CLT10b]), wonderful compactifications of semi-simple groups of adjoint type ([TBT13] and [Cho19]), and biequivariant compactifications of the Heisenberg group ([Xia20]). These results suggest that there should be an analogous formulation of log Manin's conjecture for integral points,

however certain subtleties of geometric and arithmetic nature prevent a general formulation of such a conjecture.

Campana and subsequently Abramovich proposed the notion of Campana points in [Cam05] and [Abr09] and this notion interpolates between rational points and integral points. The counting problem of Campana points has been originally featured in [VV11], [BVV12], and [VV12]. Recently many mathematicians started to look at this problem and develop a series of results, attested by [BY20], [PSTVA20], [PS20], [Xia20], and [Str20]. In [PSTVA20], Pieropan, Smeets, Várilly-Alvarado, and the author initiated a systematic study of the counting problem for Campana points, and formulated a log Manin's conjecture for klt Campana points. Then we confirmed this conjecture for equivariant compactifications of vector groups using the height zeta function method for vector groups which is developed by Chambert-Loir and Tschinkel in [CLT02] and [CLT12].

In this survey paper, we discuss the formulation of log Manin's conjecture for klt Campana points and applications of the height zeta function method to study this problem for various equivariant compactifications of connected algebraic groups.

Here is a plan of this paper: In Section 2, we review the notion of height functions. In Section 3, we introduce two definitions of (weak) Campana points. In Section 4, we discuss a formulation of log Manin's conjecture for klt Campana points. Finally in Section 5, we discuss the height zeta function method and its applications to equivariant compactifications of algebraic groups.

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2. HEIGHT FUNCTIONS

In this section we review the notion of height functions and their basic properties. The main references are [HS00] and [CLT10a], and they include different treatments of height functions. In [HS00] height functions are introduced using the machinery of Weil height machine and some basic properties of height functions such as the Northcott property are proved. In [CLT10a], adelic metrizations are used to define height functions, and this definition is frequently used in the literature in Manin's conjecture. It is well-known that two definitions are essentially equivalent. See [HS00] for more details. In this paper, we employ the definition of height functions using adelic metrizations described in [CLT10a].

Let us fix our notation: let F be any number field and \mathcal{O}_F be its ring of integers. We denote the set of places of F by Ω_F , the set of archimedean places by Ω_F^∞ , and the set of non-archimedean places by $\Omega_F^{\leq\infty}$. For any finite set $S \subset \Omega_F$ containing Ω_F^∞ , $\mathcal{O}_{F,S}$ denotes

the ring of S -integers. For each $v \in \Omega_F$, we denote the completion of F with respect to v by F_v . When v is non-archimedean, we denote the ring of integers for F_v by \mathcal{O}_v with maximal ideal \mathfrak{m}_v and residue field k_v of size q_v . We denote the adèle ring of F by \mathbb{A}_F .

For each $v \in \Omega_F$, F_v is a locally compact subgroup and it comes with a self-Haar measure $dx_v = \mu_v$ which is normalized in a way Tate did in [Tat67]. We define the absolute value $|\cdot|_v$ on F_v by requiring

$$\mu_v(xB) = |x|_v \cdot \mu_v(B).$$

This normalization satisfies the product formula, i.e., for any $x \in F^\times$, we have

$$\prod_{v \in \Omega_F} |x|_v = 1$$

See [CLT10a] for more details.

Let F be a number field and $v \in \Omega_F$ be a place of F . Let U be an open set of F_v^n in the analytic topology. A complex valued function on U is smooth if it is C^∞ when v is archimedean and it is locally constant when v is non-archimedean. This notion is local and extends to any v -adic analytic manifold.

Let X be a smooth variety defined over F_v and L be a line bundle on X . For each local point $x \in X(F_v)$, we denote the fiber of L at x by L_x .

Definition 2.1. A smooth metric on L is a collection of metrics $\|\cdot\| : L_x(F_v) \rightarrow \mathbb{R}_{\geq 0}$ for all $x \in X(F_v)$ such that

- for $\ell \in L_x(F_v) \setminus \{0\}$, $\|\ell\| > 0$;
- for any $a \in F_v$, $x \in X(F_v)$, and $\ell \in L_x(F_v)$, $\|a\ell\| = |a|_v \|\ell\|$, and;
- for any open subset $U \subset X(F_v)$ and any non-vanishing section $f \in \Gamma(U, L)$, the function $x \mapsto \|f(x)\|$ is smooth.

An integral model of a projective variety can be used to define a metric on it:

Example 2.2. Let X be a smooth projective variety defined over F_v and L be a line bundle on X where v is non-archimedean. Suppose that we have a flat projective \mathcal{O}_v -scheme \mathcal{X} and a line bundle \mathcal{L} on \mathcal{X} extending X and L . Let $x \in X(F_v) = \mathcal{X}(\mathcal{O}_v)$. Then we define a smooth metric on L by insisting that for any $\ell \in L_x(F_v)$

$$\|\ell\| \leq 1 \iff \ell \in \mathcal{L}_x(\mathcal{O}_v).$$

This metric is called as the induced metric by an integral model $(\mathcal{X}, \mathcal{L})$.

Next we define adelic metrizations on a smooth projective variety defined over a number field F .

Definition 2.3. Let X be a smooth projective variety defined over a number field F and L be a line bundle on X . An adelic metrization on L is a collection of v -adic smooth metrics $\{\|\cdot\|_v\}_{v \in \Omega_F}$ on X such that there exist a finite set S of places including Ω_F^∞ , a flat $\mathcal{O}_{F,S}$ -projective model \mathcal{X} , and a line bundle \mathcal{L} on \mathcal{X} extending X and L such that for any $v \notin S$, the metric $\|\cdot\|_v$ is induced by $(\mathcal{X}, \mathcal{L})$. Note that two integral models are isomorphic outside of finitely many places so that two adelic metrizations differ only at finitely many places.

Finally we define the notion of height functions:

Definition 2.4. Let X be a smooth projective variety defined over a number field F and $\mathcal{L} = (L, \{\|\cdot\|_v\})$ be an adelicly metrized line bundle on X . For each rational point $x \in X(F)$, choose $\ell \in L_x(F)$ and we define the height function $H_{\mathcal{L}} : X(F) \rightarrow \mathbb{R}_{\geq 0}$ by

$$H_{\mathcal{L}}(x) = \prod_{v \in \Omega_F} \|\ell\|_v^{-1}.$$

This is well-defined due to the product formula mentioned above.

Here is an example of height functions for the projective space:

Example 2.5. Let $X = \mathbb{P}^n$ and $L = \mathcal{O}_X(1)$. We consider the standard integral model $\mathcal{X} = \mathbb{P}_{\mathcal{O}_F}^n$. For each non-archimedean place $v \in \Omega_F$, we let $\|\cdot\|_v$ be the metric at v induced by \mathcal{X} . For any archimedean place v , we define a smooth metric at v by insisting

$$\|\ell(x)\|_v = \frac{|\ell(x)|_v}{\sqrt{\sum_{i=0}^n |x_i|_v^2}},$$

where $x = (x_0 : \cdots : x_n) \in X(F)$ and $\ell \in H^0(X, \mathcal{O}_X(1))$. Then it is an easy exercise to prove that the height function associated to L with this adelic metrization is given by

$$H(x) = \prod_{v \in \Omega_F^{\infty}} \max\{|x_0|_v, \dots, |x_n|_v\} \prod_{v \in \Omega_F^{\infty}} \sqrt{|x_0|_v^2 + \cdots + |x_n|_v^2}.$$

When $F = \mathbb{Q}$, we may assume that x_i 's are integers and $\gcd(x_0, \dots, x_n) = 1$. In this situation, the above formula reduces to

$$H(x) = \sqrt{|x_0|_{\infty}^2 + \cdots + |x_n|_{\infty}^2}.$$

Let us mention a few basic properties of height functions:

Proposition 2.6. *Let X be a smooth projective variety defined over a number field F and $\mathcal{L} = (L, \{\|\cdot\|_v\})$ be an adelicly metrized line bundle on X . Then the following statements are true:*

- *Let \mathcal{L}' be another adelicly metrized line bundle associated to L . Then there exist positive constants $C_1 \leq C_2$ such that for any $x \in X(F)$, we have*

$$C_1 H_{\mathcal{L}'}(x) \leq H_{\mathcal{L}}(x) \leq C_2 H_{\mathcal{L}'}(x);$$

- *Let B be the base locus of the complete linear series $|L|$. Then there exists a positive constant $C > 0$ such that for any $x \in (X \setminus B)(F)$, we have*

$$H_{\mathcal{L}}(x) \geq C;$$

- *When L is ample, for any real number $T > 0$ the set*

$$\{x \in X(F) \mid H_{\mathcal{L}}(x) \leq T\}$$

is a finite set.

The last property is called as the Northcott property which is fundamental in diophantine geometry and it is also foundational for Manin's conjecture. For more details, see [HS00].

3. CAMPANA POINTS

In this section, we review the notion of Campana points. Campana points were originally considered by Campana for curves in [Cam05], and its higher dimensional analogue was explored by Abramovich in [Abr09]. One may consider Campana points as integral points on Campana orbifolds developed by again Campana himself:

Definition 3.1. Let F be an arbitrary field and X be a smooth projective variety defined over F . Let $D_\epsilon = \sum_{\alpha \in \mathcal{A}} \epsilon_\alpha D_\alpha$ be an effective \mathbb{Q} -divisor on X with D_α 's irreducible and distinct. We say (X, D_ϵ) is a Campana orbifold if the following statements are true:

- For any $\alpha \in \mathcal{A}$, a non-negative rational number ϵ_α takes the form of

$$1 - \frac{1}{m_\alpha},$$

where m_α is a positive integer or $+\infty$;

- the reduced divisor $D = \sum_{\alpha \in \mathcal{A}} D_\alpha$ is a strict normal crossings divisor.

We say a Campana orbifold (X, D_ϵ) is Fano if $-(K_X + D_\epsilon)$ is ample.

Let (X, D_ϵ) be a Campana orbifold. Then (X, D_ϵ) is a divisorial log terminal (dlt for short) pair in the sense of birational geometry. When $\epsilon_\alpha < 1$ for any α , (X, D_ϵ) is a kawamata log terminal (klt for short) pair. See [KM98] for the definitions and their basic properties. We say a Campana orbifold (X, D_ϵ) is klt if $\epsilon_\alpha < 1$ for every $\alpha \in \mathcal{A}$.

To define the notion of Campana points, one needs to fix an integral model of a Campana orbifold. Let (X, D_ϵ) be a Campana orbifold defined over a number field F with $D_\epsilon = \sum_{\alpha \in \mathcal{A}} \epsilon_\alpha D_\alpha$. Let S be a finite set of places including all archimedean places. A good integral model of (X, D_ϵ) away from S is a flat projective $\mathcal{O}_{F,S}$ -scheme \mathcal{X} such that \mathcal{X} is extending X and \mathcal{X} is regular. Let \mathcal{D}_α be the Zariski closure of D_α in \mathcal{X} and let $\mathcal{D}_\epsilon = \sum_{\alpha \in \mathcal{A}} \epsilon_\alpha \mathcal{D}_\alpha$.

Let us fix a good integral model of a Campana orbifold (X, D_ϵ) as above. Let $\mathcal{A}_\epsilon = \{\alpha \in \mathcal{A} \mid \epsilon_\alpha \neq 0\}$. We set $X^\circ = X \setminus \cup_{\alpha \in \mathcal{A}_\epsilon} D_\alpha$. Let $P \in X^\circ(F)$ be a rational point and $v \notin S$ be a non-archimedean place of F . Then we may consider P as an \mathcal{O}_v -point $\mathcal{P}_v \in \mathcal{X}(\mathcal{O}_v)$ by valuative criterion for properness. Since $\mathcal{P}_v \notin \mathcal{D}_\alpha$ for any $\alpha \in \mathcal{A}_\epsilon$, the pullback of \mathcal{D}_α via \mathcal{P}_v defines an ideal in \mathcal{O}_v . We denote its colength by $n_v(\mathcal{D}_\alpha, P)$. When $P \in \mathcal{D}_\alpha$ for some $\alpha \in \mathcal{A}_\epsilon$, we formally set $n_v(\mathcal{D}_\alpha, P) = +\infty$. The total intersection number is given by

$$n_v(\mathcal{D}_\epsilon, P) = \sum_{\alpha \in \mathcal{A}_\epsilon} \epsilon_\alpha n_v(\mathcal{D}_\alpha, P).$$

Now we are ready to define two notions of Campana points:

Definition 3.2. We say $P \in X(F)$ is a weak Campana $\mathcal{O}_{F,S}$ -point on $(\mathcal{X}, \mathcal{D}_\epsilon)$ if the following statements are true:

- we have $P \in (\mathcal{X} \setminus \cup_{\alpha=1} \mathcal{D}_\alpha)(\mathcal{O}_{F,S})$, and;
- for $v \notin S$, if $n_v(\mathcal{D}_\epsilon, P) > 0$, then

$$n_v(\mathcal{D}_\epsilon, P) \leq \left(\sum_{\alpha \in \mathcal{A}_\epsilon} n_v(\mathcal{D}_\alpha, P) \right) - 1.$$

We denote the set of weak Campana $\mathcal{O}_{F,S}$ -points by $(\mathcal{X}, \mathcal{D}_\epsilon)_w(\mathcal{O}_{F,S})$.

Definition 3.3. We say $P \in X(F)$ is a Campana $\mathcal{O}_{F,S}$ -point on $(\mathcal{X}, \mathcal{D}_\epsilon)$ if the following statements are true:

- we have $P \in (\mathcal{X} \setminus \cup_{\epsilon_\alpha=1} \mathcal{D}_\alpha)(\mathcal{O}_{F,S})$, and;
- for $v \notin S$ and for all $\alpha \in \mathcal{A}_\epsilon$ with $\epsilon_\alpha < 1$ and $n_v(\mathcal{D}_\alpha, P) > 0$, we have

$$n_v(\mathcal{D}_\alpha, P) \geq m_\alpha,$$

where $\epsilon_\alpha = 1 - \frac{1}{m_\alpha}$.

A Campana $\mathcal{O}_{F,S}$ -point is klt when the underlying Campana orbifold is a klt pair.

We denote the set of Campana $\mathcal{O}_{F,S}$ -points by $(\mathcal{X}, \mathcal{D}_\epsilon)(\mathcal{O}_{F,S})$. Then we have the following inclusions:

$$\mathcal{X}^\circ(\mathcal{O}_{F,S}) \subset (\mathcal{X}, \mathcal{D}_\epsilon)(\mathcal{O}_{F,S}) \subset (\mathcal{X}, \mathcal{D}_\epsilon)_w(\mathcal{O}_{F,S}) \subset X(F),$$

where $\mathcal{X}^\circ = \mathcal{X} \setminus (\cup_{\alpha \in \mathcal{A}_\epsilon} \mathcal{D}_\alpha)$. When $\epsilon_\alpha = 0$ for all $\alpha \in \mathcal{A}$, the rightmost two inclusions are equalities. When $\epsilon_\alpha = 1$ for all $\alpha \in \mathcal{A}_\epsilon$, the leftmost two inclusions are equalities.

Here is an example of klt Campana points:

Example 3.4. For simplicity, let us assume that $F = \mathbb{Q}$ and $S = \{\infty\}$. Let $X = \mathbb{P}^m$ and $H = V(x_0)$ be a hyperplane. Let m be a positive integer and $\epsilon = 1 - 1/m$. We define

$$D_\epsilon = \epsilon H.$$

We consider the standard integral model of X . Then a rational point $x = (x_0 : \dots : x_n) \in X(\mathbb{Q})$ with $x_i \in \mathbb{Z}$ and $\gcd(x_0, \dots, x_n) = 1$ is a Campana \mathbb{Z} -point if $x_0 = 0$ or $x_0 \neq 0$ and the following statement is true: for any prime number p we have

$$p \mid x_0 \implies p^m \mid x_0.$$

Any non-zero integer with this property is said to be m -full. When $m = 2$, it is said to be squarefull.

4. LOG MANIN'S CONJECTURE

Let X be a smooth projective variety defined over a number field F and $\mathcal{L} = (L, \{\|\cdot\|_v\})$ be an adelicly metrized line bundle on X . We consider the associated height function

$$H_{\mathcal{L}} : X(F) \rightarrow \mathbb{R}_{>0}.$$

When L is ample, this height function satisfies the Northcott property so that for any subset $Q \subset X(F)$ and any positive real number $T > 0$ one may define the counting function

$$N(Q, \mathcal{L}, T) = \#\{P \in Q \mid H_{\mathcal{L}}(P) \leq T\}.$$

Manin's conjecture predicts the asymptotic formula of the above function for an appropriate Q , and a natural question is to extend this conjecture to integral points and Campana points. In [PSTVA20], Pieropan, Smeets, Várilly-Alvarado and the author formulated this log version of Manin's conjecture when the underlying Campana orbifold is a klt log Fano pair. In this section, we review a general formulation of this log Manin's conjecture.

4.1. Two birational invariants. Let X be a smooth projective variety defined over a field F . Let D_1, D_2 are \mathbb{Q} -divisors on X . We say D_1 and D_2 are numerically equivalent if for any curve $C \subset X$, we have $D_1 \cdot C = D_2 \cdot C$. In this case we write $D_1 \equiv D_2$. We define the space of \mathbb{Q} -divisors up to numerical equivalence as

$$N^1(X)_{\mathbb{Q}} = \{D : \mathbb{Q}\text{-divisors}\} / \equiv .$$

We set $N^1(X) := N^1(X)_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{R}$. Then we define the cone of pseudo-effective divisors by

$$\overline{\text{Eff}}^1(X) := \overline{\text{the cone of effective } \mathbb{Q}\text{-divisors}} \subset N^1(X).$$

Now we are ready to introduce two birational invariants which play central roles in Manin's conjecture:

Definition 4.1. Let (X, D_{ϵ}) be a klt Campana orbifold defined over a field F and L be an ample \mathbb{Q} -divisor on X . We define the Fujita invariant or a -invariant by

$$a(X, D_{\epsilon}, L) := \inf\{t \in \mathbb{R} \mid tL + K_X + D_{\epsilon} \in \overline{\text{Eff}}^1(X)\}.$$

Next assume that $a(X, D_{\epsilon}, L) > 0$. Then we define the b -invariant by

$$b(F, X, D_{\epsilon}, L) := \text{codimension of the minimal face of } \overline{\text{Eff}}^1(X) \\ \text{containing } a(X, D_{\epsilon}, L)L + K_X + D_{\epsilon}.$$

It is explained in [PSTVA20, Section 3.6.2] that these invariants are birational invariants.

Example 4.2. Let (X, D_{ϵ}) be a klt Fano orbifold defined over a field F and $L = -(K_X + D_{\epsilon})$. Then we have

$$a(X, D_{\epsilon}, L) = 1, \quad b(F, X, D_{\epsilon}, L) = \rho(X) = \dim N^1(X).$$

4.2. Thin exceptional sets. The notion of thin sets has been explored by Serre to study Galois inverse problem, and it is also fundamental to Manin's conjecture. Let us give the definition of thin sets for Campana points:

Definition 4.3. Let (X, D_{ϵ}) be a klt Campana orbifold defined over a number field F . Let S be a finite set of places of F including Ω_F^{∞} and we fix a good integral model away from S $\mathcal{X} \rightarrow \text{Spec } \mathcal{O}_{F,S}$.

A type I thin set is a set of the form

$$V(F) \cap (\mathcal{X}, \mathcal{D}_{\epsilon})(\mathcal{O}_{F,S}),$$

where $V \subset X$ is a proper closed subset of X .

A type II thin set is a set of the form

$$f(Y(F)) \cap (\mathcal{X}, \mathcal{D}_{\epsilon})(\mathcal{O}_{F,S}),$$

where $f : Y \rightarrow X$ is a dominant generically finite morphism of degree ≥ 2 defined over F with Y integral.

A thin set is any subset of a finite union of type I and type II thin sets.

Here is an example of thin sets:

Example 4.4. Let $X = \mathbb{P}^1$ with $D_{\epsilon} = 0$ defined over a number field F . We consider the morphism

$$f : \mathbb{P}^1 \rightarrow \mathbb{P}^1, (x_0 : x_1) \mapsto (x_0^d : x_1^d)$$

with $d \geq 2$. Then $f(X(F)) \subset X(F)$ is a thin set.

4.3. Log Manin’s conjecture for klt Campana points. Finally we state log Manin’s conjecture for klt Campana points:

Conjecture 4.5 (Log Manin’s conjecture for klt Campana points). Let (X, D_ϵ) be a klt Fano orbifold defined over a number field F and $\mathcal{L} = (L, \{\|\cdot\|_v\})$ be an adelically metrized ample line bundle. Assume that $(\mathcal{X}, \mathcal{D}_\epsilon)(\mathcal{O}_{F,S})$ is not thin. Then there exists a thin set $Z \subset (\mathcal{X}, \mathcal{D}_\epsilon)(\mathcal{O}_{F,S})$ such that

$$N((\mathcal{X}, \mathcal{D}_\epsilon)(\mathcal{O}_{F,S}) \setminus Z, \mathcal{L}, T) \sim c(F, \mathcal{X}, \mathcal{D}_\epsilon, \mathcal{L}, Z) T^{a(X, D_\epsilon, L)} (\log T)^{b(F, X, D_\epsilon, L) - 1},$$

as $T \rightarrow \infty$. Here the leading constant $c(F, \mathcal{X}, \mathcal{D}_\epsilon, \mathcal{L}, Z)$ is analogous to Peyre’s constant developed in [Pey95] and [BT98a] and its definition is given in [PSTVA20, Section 3.3].

Remark 4.6. For a smooth geometrically rationally connected projective variety X defined over a number field F , it is expected that $X(F)$ is not thin as soon as there is a rational point. Indeed, Colliot-Thélène’s conjecture predicts that the set of rational points is dense in the Brauer-Manin set, and this implies that X satisfies weak weak approximation. It is known that weak weak approximation property implies non-thinness of the set of rational points. The corresponding statement for klt Campana points, i.e., weak weak approximation for klt Campana sets implies non-thinness of the set of klt Campana points is established in [NS20]. So it is natural to expect that the assumption of Conjecture 4.5 is true as long as there is a klt Campana point.

Remark 4.7. It is well-documented in the case of rational points that in Conjecture 4.5 it is important to remove the contribution of a thin set Z from the counting function. There is a series of papers ([LT17], [Sen21], and [LST18]) studying birational geometry of thin exceptional subsets for rational points. In [LST18], Lehmann, Sengupta, and the author proposed a conjectural description of thin exceptional subsets and proved that it is indeed a thin set using the minimal model program and the boundedness of singular Fano varieties. It would be interesting to perform a similar study for klt Campana points.

Conjecture 4.5 is known in the following cases:

- projective space with a boundary being the union of hyperplanes ([VV11], [VV12], [BVV12], and [BY20]);
- equivariant compactifications of vector groups ([PSTVA20]);
- toric varieties defined over \mathbb{Q} ([PS20]) and;
- biequivariant compactifications of the Heisenberg group ([Xia20]).

One can also consider a similar counting problem for weak Campana points, however this problem is much harder than Conjecture 4.5. At the moment of writing this paper, we do not know how one should formulate a log Manin’s conjecture for weak Campana points, but [Str20] takes the first step towards to this problem.

5. HEIGHT ZETA FUNCTIONS

Let F be a number field and G be a connected linear algebraic group defined over F . Let X be a smooth projective equivariant compactification of G , i.e., X contains G as a Zariski open subset, and the right action of G extends to X . In this situation, the boundary

$$D = X \setminus G = \bigcup_{\alpha \in \mathcal{A}} D_\alpha$$

is a divisor where each D_α is an irreducible component. After applying an equivariant resolution, we may assume that $D = \sum_{\alpha \in \mathcal{A}} D_\alpha$ is a divisor with strict normal crossings. We also fix an adelic metrization for $\mathcal{O}(D_\alpha)$ for every $\alpha \in \mathcal{A}$.

For each $\alpha \in \mathcal{A}$, we choose m_α which is a positive integer or $+\infty$, and set $\epsilon_\alpha = 1 - \frac{1}{m_\alpha}$. We consider

$$D_\epsilon = \sum_{\alpha \in \mathcal{A}} \epsilon_\alpha D_\alpha,$$

and (X, D_ϵ) is a Campana orbifold. Let us fix a finite set S of places including Ω_F^∞ and a good integral model \mathcal{X} away from S extending X . When $-(K_X + D_\epsilon)$ is ample (or more generally big), it is natural to consider the counting problem of $\mathcal{O}_{F,S}$ -Campana points on $(\mathcal{X}, D_\epsilon)$. There is a general approach to this problem which is called as the height zeta function method.

Let $\text{Pic}(X)^G$ be the Picard group of G -linearized line bundles on X up to isomorphisms. (If the reader is not familiar with G -linearizations, she/he may ignore this term for now.) After tensoring by \mathbb{Q} , boundary components D_α form a basis for $\text{Pic}(X)_{\mathbb{Q}}^G$. We choose a section

$$f_\alpha \in H^0(X, \mathcal{O}(D_\alpha)),$$

corresponding to D_α . Then we define a local height pairing: for any place $v \in \Omega_F$,

$$H_v : G(F_v) \times \text{Pic}(X)_{\mathbb{C}}^G \rightarrow \mathbb{C}^\times, \left(g_v, \sum_{\alpha \in \mathcal{A}} s_\alpha D_\alpha \right) \mapsto \prod_{\alpha \in \mathcal{A}} \|f_\alpha(g_v)\|_v^{-s_\alpha}.$$

Using this local height pairing, we define the global height pairing as the Euler product:

$$H := \prod_{v \in \Omega_F} H_v : G(\mathbb{A}_F) \times \text{Pic}(X)_{\mathbb{C}}^G \rightarrow \mathbb{C}^\times.$$

Applying the definition of Campana points to local points, for each $v \notin S$, one can define the Campana set

$$(\mathcal{X}, D_\epsilon)(\mathcal{O}_v) \subset X(F_v).$$

We set

$$G(F_v)_\epsilon = G(F_v) \cap (\mathcal{X}, D_\epsilon)(\mathcal{O}_v),$$

and let $\delta_{\epsilon,v}(g_v)$ be the characteristic function of $G(F_v)_\epsilon$ on $G(F_v)$. When $v \in S$, we set $\delta_{\epsilon,v} \equiv 1$ and define δ_ϵ as the Euler product:

$$\delta_\epsilon = \prod_{v \in \Omega_F} \delta_{\epsilon,v} : G(\mathbb{A}_F) \rightarrow \mathbb{R}_{\geq 0}.$$

For $g \in G(\mathbb{A}_F)$ and $\mathbf{s} \in \text{Pic}(X)_{\mathbb{C}}^G$, we define the height zeta function by

$$Z(g, \mathbf{s}) := \sum_{\gamma \in G(F)} H(\gamma g, \mathbf{s})^{-1} \delta_\epsilon(\gamma g)$$

When $\Re(\mathbf{s})$ is sufficiently large, this function converges to a continuous function in $g \in G(F) \backslash G(\mathbb{A}_F)$ and a holomorphic function in $\mathbf{s} \in \text{Pic}(X)_{\mathbb{C}}^G$.

A relation of this height zeta function to log Manin's conjecture is given by Tauberian theorem. Indeed, if one can prove that for an ample (or big) line bundle L , $Z(\text{id}, sL)$ admits

a meromorphic continuation to a half plane $\Re(s) \geq a$ with a unique pole at $s = a$ of order b with $a > 0$ a positive real number, then one can conclude

$$N(G(F)_\epsilon, \mathcal{L}, T) \sim cT^a(\log T)^{b-1},$$

where c is a positive constant related to the leading constant of $Z(\text{id}, sL)$ at $s = a$. Thus our goal is reduced to obtain a meromorphic continuation of $Z(\text{id}, \mathbf{s})$.

To this end, for $\mathbf{s} \gg 0$, one can prove that

$$Z(g, \mathbf{s}) \in L^2(G(F) \backslash G(\mathbb{A}_F)),$$

thus one may apply spectral decomposition of this Hilbert space to $Z(g, \mathbf{s})$ and use this spectral decomposition to obtain a meromorphic continuation.

This program has been pioneered mainly by Tschinkel and his collaborators, and has been carried out in the following cases:

- rational points on toric varieties ([BT96], [BT98b]);
- rational points on equivariant compactifications of vector groups ([CLT02]);
- rational points on wonderful compactifications of semi-simple groups of adjoint type ([STBT07]);
- rational points on biequivariant compactifications of unipotent groups ([ST16]);
- integral points on equivariant compactifications of vector groups ([CLT12]);
- integral points on toric varieties ([CLT10b]);
- integral points on wonderful compactifications of semi-simple groups of adjoint type ([TBT13] and [Cho19]);
- Campana points on equivariant compactifications of vector groups ([PSTVA20]);
- Campana points on biequivariant compactifications of the Heisenberg group ([Xia20]), and;
- weak Campana points on certain toric varieties ([Str20]).

It would be interesting to explore Campana points on other algebraic groups. In particular, the treatment of integral points on toric varieties ([CLT10b]) is known to be incomplete, and there is some technical issue on this paper. It would be interesting to apply the height zeta function method to klt Campana points on toric varieties and see whether we have a similar issue.

Finally for the readers who are interested in working examples of this program, we recommend them to consult [PSTVA20, Interlude I].

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