ON VALUES OF LOGARITHMIC DERIVATIVES OF L-FUNCTIONS

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ABSTRACT. This article is an extended version of a talk delivered at RIMS conference on "Problems and Prospects in Analytic Number Theory" held in November, 2020. In this note, we give a brief overview of the theme 'values of logarithmic derivatives of L-functions and zeta functions and its related topics'. We end by providing an outline of a recent work on values of logarithmic derivatives of L-functions attached to cuspidal elliptic Hecke eigenforms of integral weight.

1. INTRODUCTION

In this article, we would like to describe some recent progress towards the theme 'values of logarithmic derivatives of L-functions and zeta functions'. The purpose of next four sections is to draw an analogy and give motivation to the current investigations of the main topics discussed in Section 6. Therefore these sections are very short in nature and for most of the results in these sections we give reference to the original articles or survey articles. The next section is on an classical object 'the Euler constant'. To the best of author's knowledge, there are more than 300 articles in the literature on Euler constant and it's connection to other topics which is an area of current research as well. Hence we refer the interested reader to a survey article as well as some original research articles. In the Section 3, we describe another constant known as the Euler–Kronecker constant which is a generalization of Euler constant and it is also an active current research topic for it's connections to various other arithmetic problems. Sections 4, 5 are on the values of logarithmic derivatives of Dirichlet L-functions and the Artin L-functions respectively. In the last section, we describe briefly the distribution of logarithmic derivatives of L-functions attached to holomorphic cusp forms. Yet another motivation for this investigation is given in the beginning of the last section and we end the article by providing a brief sketch of the proof of the main results.

2. RIEMANN ZETA FUNCTION AND EULER CONSTANT

For $\Re(s) > 1$, the Riemann zeta function $\zeta(s)$ is defined by the Dirichlet series

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

Date: 27th March, 2021.

2010 *Mathematics Subject Classification.* 11F11, 11F66, 11M41.

Key words and phrases. logarithmic derivative, modular forms, Hecke eigenforms, zero-density estimate.

By work of Riemann, one knows that the function $\zeta(s)$ can be extended as a meromorphic function to whole complex plane with only a simple pole at $s = 1$ with residue 1. Therefore, at $s = 1$, the function $\zeta(s)$ has the following Laurent series expansion

$$\zeta(s) = \frac{1}{s-1} + \gamma + (s-1)h(s),$$

where $h(s)$ is a holomorphic function in a neighbourhood of $s = 1$. The constant γ is known as **Euler constant** (often known as *Euler-Mascheroni constant*). The Euler constant appears in various context in mathematics and even after three centuries of investigations, it remains mysterious in nature. For example, one of the famous unsolved problems about γ is:

Conjecture 1. *The Euler constant γ is irrational.*

The constant γ has various interpretations. The interpretation, we are interested in this article, is the following

$$\frac{\zeta'}{\zeta}(s) = -\frac{1}{s-1} + \gamma + O(s-1). \tag{1}$$

The Euler constant γ is known to be directly related to number of topics and/or problems in number theory. For example, it is known to be related to *gamma function, distribution of prime numbers and the Riemann hypothesis, Kontsevich–Zagier period, cycle structures of random permutations in the symmetric group S_N, Dirichlet divisor problem, extreme value problem of $\zeta(1+it)$*, etc. Moreover, some generalized Euler constants are known to be directly related to the Riemann hypothesis as well as transcendence problems. For more details on the Euler constant see the beautiful article of Lagarious [14] and references there. For some of the more recent works on generalized Euler constants see [6, 7, 8].

3. Dedekind zeta functions and Euler-Kronecker invariants

The main aim of this section is to discuss one of the generalizations of the Euler constant known as *Euler–Kronecker constant*. To introduce this constant, we need more notations. Let K be a number field and $\mathcal{O}_K$ be the ring of integers in K. For $\Re(s) > 1$, we define the *Dedekind zeta function* by:

$$\zeta_K(s) := \sum_{\mathfrak{a} \neq 0} \frac{1}{(N\mathfrak{a})^s},$$

where $\mathfrak{a}$ runs over all non-zero integral ideals of K. As in the case of the Riemann zeta function $\zeta(s)$, the Dedekind zeta function $\zeta_K(s)$ admits analytic continuation to whole complex plane as a meromorphic function and having a simple pole at $s = 1$. Therefore we have

$$\frac{\zeta_K'}{\zeta_K}(s) = -\frac{1}{s-1} + \gamma_K + O(s-1). \tag{2}$$

The constant γ_K is known as the *Euler–Kronecker constant (or invariant)* of K. Note that in the case when $K = \mathbb{Q}$, the function $\zeta_K(s)$ is nothing but the Riemann zeta function $\zeta(s)$, that is, $\zeta_{\mathbb{Q}}(s) = \zeta(s)$ and $\gamma_K = \gamma$.

As in the case of Euler constant γ, this constant γ_K also appears in literature frequently but its systematic study has been initiated only recently by Ihara [9]. Since then the distribution of γ_K and its relationship with other objects in mathematics have been studied by several mathematicians. Till date, it is known to be related to the 'Kronecker limit formula' [9, Section 2.2], the generalized Brauer–Siegel conjecture [4] and the Hardy–Littlewood conjecture about counts of prime k-tuples [5] among others. This is an area of current research.

4. Dirichlet L-functions

Let us now move to another set of constants which are closely related to the previous ones, namely, the constants coming from Dirichlet L-functions. Let $N \geq 1$ be a positive integer and $\chi \bmod N$ be a Dirichlet character. For $\Re(s) > 1$, define the Dirichlet L-function $L(s, \chi)$ by

$$L(s, \chi) \; := \; \sum_{m=1}^{\infty} \frac{\chi(m)}{m^s}.$$

The study of this function seems to go back to Dirichlet in 1830's when he investigated the distribution of primes in arithmetic progressions. We know that the function $L(s, \chi)$ can be extended analytically to the whole complex plane as an entire function and it does not vanish in the region $\Re(s) \geq 1$. In particular, $L(1, \chi) \neq 0$. Therefore, the value $\frac{L'}{L}(1, \chi)$ is well-defined. In this section, we mention few properties of this constant and it's connection to other topics or questions.

Let $K = \mathbb{Q}(\mu_p)$ be the cyclotomic field, where μ_p is a primitive p-th root of unity. Then one has

$$\gamma_K \; = \; \gamma \; + \; \sum_{\chi \neq \chi_0} \frac{L'}{L}(1, \chi).$$

Motivated by this connection, Y. Ihara, V. Kumar Murty and M. Shimura [10] studied the distribution of $\frac{L'}{L}(1, \chi)$. Assuming the Generalized Riemann Hypothesis (GRH), they proved that:

$$\left| \frac{L'}{L}(1, \chi) \right| \; \leq \; 2 \log\log |d_\chi| + O(1), \tag{3}$$

where d_χ is the conductor of χ. Note that their result is more general (please see [10] for more details). They also studied moments of $\frac{L'}{L}(1, \chi)$. On the other hand, Mourtada–Murty [18] showed the following omega result.

Theorem 1 (Mourtada–Murty)**.** *There are infinitely many fundamental discriminants D such that*

$$\frac{L'}{L}(1, \chi_D) \; \geq \; \log\log |D| + O(1).$$

Moreover, for large enough x*, there are* $\geq x^{\frac{1}{10}}$ *fundamental discriminants* $0 < D \leq x$ *such that*

$$\frac{L'}{L}(1, \chi_D) \ \leq \ -\log\log|D| + O(1).$$

where χ_D *is the quadratic Dirichlet character of conductor* D*.*

For further details see [18, 15].

5. Artin L-functions

Let K be a number field such that the degree of the extension $K/\mathbb{Q}$ be n and d_K be its discriminant. Also let $\widehat{K}$ be the Galois closure of $K/\mathbb{Q}$. Then we have the following decomposition of the Dedekind zeta function

$$\zeta_K(s) = \zeta(s)L(s, \rho),$$

where ρ is a $(n-1)$-dimensional complex representation of the Galois group Gal$(\widehat{K}/\mathbb{Q})$. Hence

$$\gamma_K = \gamma + \frac{L'}{L}(1, \rho).$$

Motivated by this connection, Cho and Kim [2] studied the distribution of the values $\frac{L'}{L}(1, \rho)$ in terms of degree n and discriminant d_K under the Artin conjecture, the generalized Riemann hypothesis and a certain zero-density hypothesis. Further, they showed that their conditional bounds are optimal, that is, there exist infinitely many (Galois) number fields such that for those fields the constants $\frac{L'}{L}(1, \rho)$ take maximal values. The last result is unconditional. For more details we refer the reader to [2].

6. L-functions attached to cusp forms

The purpose of this section is to introduce one more set of constants and describe their distribution what had been studied recently by the author. We need some more notions and notations! Let $k \geq 2$ be an even integer, $N \geq 1$ be an integer and $S_k(N)$ be the space of elliptic cusp forms of weight k for the group $\Gamma_0(N)$. Also assume that $S_k^{new}(N)$ denotes the subspace of newforms of $S_k(N)$. Let

$$f(z) \ := \ \sum_{n=1}^{\infty} \lambda_f(n) n^{(k-1)/2} q^n \ \in \ S_k^{new}(N)$$

be a normalized Hecke eigenform which is an eigenfunction of Fricke involution (cf. [3, p. 204]). We shall denote the set of such eigenforms by $H_k(N)$. For $f \in H_k(N)$, $\Re(s) > 1$ and primitive character $\chi \bmod M$, one defines

$$L(s, f, \chi) \ := \ \sum_{n=1}^{\infty} \frac{\lambda_f(n)\chi(n)}{n^s}.$$

As in the case of Dirichlet L-function, it is well-known (cf. [21, Theorem 3.66]) that, when $(M, N) = 1$, the function $L(s, f, \chi)$ admits analytic continuation to whole complex plane as an entire function. One also knows that $L(1, f, \chi) \neq 0$ when χ primitive. Therefore, the value $\frac{L'}{L}(1, f, \chi)$ is well defined and, in this section, we are interested in the distribution of $\frac{L'}{L}(1, f, \chi)$.

The main motivation of this study comes from two different sources. One source of motivation is the analogy with the above mentioned Euler constant, the Euler–Kronecker constants, and the values of logarithmic derivatives of Dirichlet L-functions and Artin L-functions with respect to their connection with the Euler–Kronecker constants. Another source of motivation is the values of L-functions associated to primitive cusp forms. It is known that these values encode deep arithmetic information of cusp forms. For example, by a work of Luo and Ramakrishnan [12, Theorem B], the twisted central values $L(1/2, f, \chi)$, where χ runs over the set of all quadratic characters, determine primitive cusp form f uniquely. The analogous result can also be derived from the values of L-functions at $s = 1$ (see Luo [13] for more details).

In a recent work [19], we established the following conditional estimate.

Theorem 2 (Paul, [19])**.** *We keep the above notations. Let $f \in H_k(N)$ and $\chi \bmod M$ be a primitive Dirichlet character. Assume $(M, N) = 1$ and the Riemann hypothesis holds for $L(s, f, \chi)$ and $\zeta(s)$. Then there is an absolute constant C such that*

$$\left|\frac{L'}{L}(1, f, \chi)\right| \leq 4 \log\log(M\sqrt{N}k) + C + O\left(\frac{\left(\log\log(M\sqrt{N}k)\right)^2}{\log(M\sqrt{N}k)}\right).$$

A natural question is to ask: Is the above bound optimal, i.e. does there any Omega result? Our next result answers this question positively.

Theorem 3 (Paul, [19])**.** *Let $f \in S_k(\mathrm{SL}_2(\mathbb{Z}))$ be a normalized Hecke eigenform. Then there are infinitely many fundamental discriminant D and quadratic Dirichlet character $\chi \bmod D$ such that*

$$\frac{L'}{L}(1, f, \chi) \geq \frac{1}{8} \log\log(Dk) + O(1).$$

Further, there are infinitely many fundamental discriminant D and quadratic Dirichlet character $\chi \bmod D$ such that

$$\frac{L'}{L}(1, f, \chi) \leq -\frac{1}{8} \log\log(Dk) + O(1).$$

The rest of the article is devoted to a sketch of the proofs of the above two results.

6.1. **Preliminaries and preparatory results.** As in [10], our main tool is the following explicit formula. For any $x > 1$, let us define

$$\Phi(f, \chi, x) := \frac{1}{x-1} \sum_{n \leq x} \left(\frac{x}{n} - 1\right) \chi(n) \Lambda_f(n).$$

Note that

$$\Phi(f, \chi, x) = \frac{1}{x-1}\left[x\psi_0^{(1)}(f, \chi, x) - \psi_0(f, \chi, x)\right],$$

where

$$\psi_0(f,\chi,x) := \sum_{n\le x}{}' \chi(n)\Lambda_f(n) \quad \text{and} \quad \psi_0^{(1)}(f,\chi,x) := \sum_{n\le x}{}' \frac{\chi(n)\Lambda_f(n)}{n}.$$

Here $\sum'$ means if x is a prime power then the term corresponding to $n = x$ will have weight $1/2$ and $\Lambda_f(n)$ is defined as follows:

$$\Lambda_f(n) := \begin{cases} (\alpha_p^m + \beta_p^m)\log p & \text{if } n = p^m \\ 0 & \text{otherwise.} \end{cases}$$

Recall that α_p, β_p are defined in terms of the Euler product $L(s,f,\chi)$ in the region $\Re(s) > 1$:

$$L(s,f,\chi) = \prod_p \left(1 - \frac{\lambda_f(p)\chi(p)}{p^s} + \frac{\phi_0(p)\chi(p)^2}{p^{2s}}\right)^{-1} := \prod_p \left(1 - \frac{\alpha_p\chi(p)}{p^s}\right)^{-1}\left(1 - \frac{\beta_p\chi(p)}{p^s}\right)^{-1} \tag{4}$$

where ϕ_0 is the trivial character $\bmod N$ and when $(p,N) = 1$,

$$\lambda_f(p) = \alpha_p + \beta_p \quad \text{and} \quad \alpha_p\beta_p = 1.$$

With these notations one can derive the following proposition.

Proposition 4. *Let $f \in H_k(N)$, $\chi \bmod M$ be a primitive character, $(M,N) = 1$ and $x > 2$ be a real number. Then*

$$\begin{aligned}\Phi(f,\chi,x) &= \frac{1}{x-1}\sum_\rho \frac{x^\rho}{\rho(1-\rho)} - \frac{x}{x-1}\frac{L'}{L}(1,f,\chi) - \frac{1}{x-1}\frac{L'}{L}(1,f,\overline{\chi}) \\ &\quad - \frac{2\log(M\sqrt{N}k/4\pi)}{x-1} + 2\log\left(\frac{\sqrt{x-1}}{\sqrt{x}-1}\right) - 2\sum_{\substack{1\le n\le k,\\ n \text{ odd}}} \frac{x^{-n/2}}{n} + O\left(\frac{1}{k\sqrt{x}}\right).\end{aligned}$$

If we restrict ourselves to the case of real primitive characters $\chi \bmod M$ then we have the following.

Corollary 5. *Let $f \in H_k(N)$, $\chi \bmod M$ be a real primitive character, $(M,N) = 1$ and $x > 2$ be a real number. Then*

$$\begin{aligned}\Phi(f,\chi,x) &= \frac{1}{x-1}\sum_\rho \frac{x^\rho}{\rho(1-\rho)} - \frac{x+1}{x-1}\frac{L'}{L}(1,f,\chi) - \frac{2\log(M\sqrt{N}k/4\pi)}{x-1} \\ &\quad + 2\log\left(\frac{\sqrt{x-1}}{\sqrt{x}-1}\right) - 2\sum_{\substack{1\le n\le k,\\ n \text{ odd}}} \frac{x^{-n/2}}{n} + O\left(\frac{1}{k\sqrt{x}}\right).\end{aligned}$$

We need another formula for $\frac{L'}{L}(1,f,\chi)$ whose proof will also be omitted.

Lemma 6. *Let $f \in H_k(N)$, $\chi \bmod M$ be a primitive Dirichlet character and $(M,N) = 1$. Then*

$$\frac{L'}{L}(1,f,\chi) + \frac{L'}{L}(1,f,\overline{\chi}) = \sum_\rho \frac{1}{\rho(1-\rho)} - 2\log\left(\frac{M\sqrt{N}}{2\pi}\right) - 2\frac{\Gamma'}{\Gamma}\left(\frac{k+1}{2}\right), \tag{5}$$

where the summation $\sum_\rho$ runs over all non-trivial zeros of $L(s,f,\chi)$.

6.2. **Sketch of proof of Theorem 2.** First use the Riemann hypothesis for $L(s, f, \chi)$ to get

$$\sum_{\rho} \left| \frac{x^{\rho}}{\rho(1-\rho)} \right| \leq \sqrt{x} \sum_{\rho} \frac{1}{\rho(1-\rho)} = \sqrt{x} \sum_{\rho} \frac{1}{|\rho|^2}.$$

Next use the Riemann hypothesis for $\zeta(s)$ to derive (see, [17, p. 419])

$$\sum_{n \leq x} \Lambda(n) = x + O(\sqrt{x}(\log x)^2),$$

where $\Lambda(n)$ is the von Mangoldt Lambda function. Now above two estimates along with Proposition 4 and Lemma 6 yield

$$\left| \frac{L'}{L}(1, f, \chi) \right| \leq 2 \log x + \frac{\sqrt{x}+1}{\sqrt{x}(\sqrt{x}-1)} \left| \frac{L'}{L}(1, f, \overline{\chi}) \right| + c_1 + O\left(\frac{\log(M\sqrt{N}k)}{\sqrt{x}} + \frac{(\log x)^2}{\sqrt{x}} \right). \tag{6}$$

Similarly, working with $L(s, f, \overline{\chi})$ one can derive

$$\left| \frac{L'}{L}(1, f, \overline{\chi}) \right| \leq 2 \log x + \frac{\sqrt{x}+1}{\sqrt{x}(\sqrt{x}-1)} \left| \frac{L'}{L}(1, f, \chi) \right| + c_1 + O\left(\frac{\log(M\sqrt{N}k)}{\sqrt{x}} + \frac{(\log x)^2}{\sqrt{x}} \right). \tag{7}$$

Thus from (6) and (7) we have

$$\left| \frac{L'}{L}(1, f, \chi) \right| \leq 2 \log x + c_1 + O\left(\frac{\log(M\sqrt{N}k)}{\sqrt{x}} + \frac{(\log x)^2}{\sqrt{x}} \right).$$

Choose $x = [\log(M\sqrt{N}k)]^2$ to complete the proof.

6.3. **Sketch of proof of Theorem 3.** To prove Theorem 3, we shall make use of Corollary 5. Recall from Corollary 5 that for any normalized Hecke eigenform $f \in S_k(\mathrm{SL}_2(\mathbb{Z}))$, real primitive character $\chi \bmod D$ and for $y > 2$ we have

$$\frac{L'}{L}(1, f, \chi) = \frac{1}{y+1} \sum_{\rho} \frac{y^{\rho}}{\rho(1-\rho)} - \frac{y-1}{y+1} \Phi(f, \chi, y) + O\left(\frac{\log(Dk)}{\sqrt{y}} \right). \tag{8}$$

Let us start by choosing the real primitive characters carefully. Let X be sufficiently large positive real number, $\eta > 0$ be a suitably small real number and

$$g := \left[\eta \frac{\log X}{\log \log X} \right].$$

By p_i, we denote the i-th odd prime with $p_1 = 3$. Also assume that

$$a := \prod_{i=1}^{g} p_i$$

then $a \ll X^{2\eta}$. Now by Chinese remainder theorem, we can choose $1 < b < 8a$ such that $b \equiv 1 \bmod 8$ and

$$\left(\frac{b}{p_i} \right) = \begin{cases} -1 & \text{for all } 1 \leq i \leq g \text{ with } \lambda_f(p_i) \geq 0; \\ 1 & \text{for all } 1 \leq i \leq g \text{ with } \lambda_f(p_i) < 0. \end{cases}$$

For square-free integers $D := 8an + b$, we define

$$\chi_n(m) \ := \ \left(\frac{m}{8an+b}\right)$$

which is a real primitive character $\bmod D$. Here $(-)$ denotes the Jacobi symbol. These are the real characters what we use in this subsection. Note that for odd m by the reciprocity law for Jacobi symbols we have

$$\chi_n(m) \ = \ \left(\frac{8an+b}{m}\right).$$

We now estimate the size of the term $\Phi(f, \chi, y)$ for which we use the following lemma.

Lemma 7. *Let $\Phi(f, \chi, y)$ be as above. Then we have*

$$\Phi(f, \chi, y) \ = \ \sum_{p \le y} \frac{\lambda_f(p)\chi(p)\log p}{p} \ + \ O(1).$$

Divide the sum (in the above lemma) over primes into two parts $p \le p_g$ and $p_g < p \le y$. In the sum over $p \le p_g$, we use our choice of real characters and prime number theorem (cf. [20, Theorem 2], also see [16, p.194-196]) and in the sum over $p_g < p \le y$, we apply Pólya–Vinogradov inequality. Putting all together we have the following lemma.

Lemma 8. *Keeping the notations as above and taking $y > p_g$ we have*

$$\sum_{\substack{X < n \le 2X \\ 8an+b \text{ square-free}}} \Phi(f, \chi_n, y) \ \le \ -\frac{N}{4}\log p_g \ + \ O(N) \ + \ O\left(a\sqrt{Xy}\log^2(Xy)\right),$$

where $N := \#\{X < n \le 2X \mid 8an + b$ is square-free$\}$. Note that $N \asymp X$, by [1, Lemma 3].

Next we estimate the terms involving the sum over non-trivial zeros of $L(s, f, \chi)$. Here, one of the main ingredients is the following zero-density estimate for $L(s, f, \chi)$. For a proof of Theorem 9 we refer to [19].

Theorem 9 (Paul, [19])**.** *Let $f \in S_k(\mathrm{SL}_2(\mathbb{Z}))$ be a normalized Hecke eigenform and $\chi_D \bmod D$ be a real primitive Dirichlet character. For $1/2 \le a \le 1$ and $T \ge 1$, let*

$$N(a, T, f, \chi_D) \ := \ \#\big\{\rho \in \mathbb{C} \mid L(\rho, f, \chi_D) = 0, a \le \Re(\rho) \le 1, |\Im(\rho)| \le T\big\}.$$

Then for $3/4 \le a \le 1$, $T \ge 1$, $X > 3$ and $\epsilon > 0$ we have

$$\sum_{|D| \le X} N(a, T, f, \chi_D) \ \ll_\epsilon \ (XkT)^{\frac{9-8a}{2}+\epsilon}.$$

Theorem 9 is analogous to a result of Jutila [11]. Note that when a is near to 1, we have saving in the power of (XkT), in particular, in the power of X which is crucial in our application. We apply Theorem 9 in the range $8/9 \le \Re(s) \le 1$ and in the range $1/2 \le \Re(s) \le 8/9$, we bound the sum 'trivially' to derive the following lemma.

Lemma 10. *Let the notations be as above. Then we have*

$$\sum_{\substack{X<n\leq 2X\\ 8an+b \text{ square-free}}} \frac{1}{y+1}\sum_{\rho}\frac{y^{\rho}}{\rho(1-\rho)} \ll (aXk)^{\frac{17}{18}+\epsilon}\log^2(aXk) + N\frac{\log(aXk)}{y^{\frac{1}{9}}}.$$

Put all these results together and choose the parameters carefully, we find that there is n with $X < n < 2X$ such that

$$\frac{L'}{L}(1, f, \chi_n) \geq \frac{1}{4}\log p_g + O(1).$$

Finally, choose $D := 8an + b$ and $\log(Dk) \leq (\log X)^2$ to conclude the first part of Theorem 3.

For the second part repeat the above proof with the following choice of real characters. Let X, η, g, a be as above. Choose $b \in \mathbb{N}$ with $1 < b < 8a$ such a way that $b \equiv 1 \bmod 8$ and

$$\left(\frac{b}{p_i}\right) = \begin{cases} 1 & \text{for all } 1 \leq i \leq g \text{ with } \lambda_f(p_i) \geq 0; \\ -1 & \text{for all } 1 \leq i \leq g \text{ with } \lambda_f(p_i) < 0. \end{cases}$$

Then, for square-free $D := 8an + b$, define primitive real character $\bmod D$ as

$$\chi_n(m) := \left(\frac{m}{8an+b}\right).$$

These will be the real character what will be used in this case. Then, as in Lemma 8, derive

$$\sum_{\substack{X<n\leq 2X\\ 8an+b \text{ square-free}}} \Phi(f, \chi_n, y) \geq \frac{N}{4}\log p_g + O(N) + O\left(a\sqrt{Xy}\log^2(Xy)\right)$$

for $y > p_g$. Here as above $N := \#\{X < n \leq 2X \mid 8an + b \text{ is square-free}\}$ and $N \asymp X$. For the rest of the proof follow the first part of the proof.

Acknowledgments: The author would like to thank RIMS, Kyoto and the organisers for giving an opportunity to speak in the conference. The author is partially supported by Grant-in-Aid for scientific research (JSPS KAKENHI Grant No. 19F19318). He would like to thank his academic host Professor Masanobu Kaneko for the support and continuous encouragement.

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