Algebraic independence properties of the values of Hecke-Mahler series for several irrational numbers

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1 Introduction

Hecke-Mahler series is the generating function

$$h_\omega(z) = \sum_{k=1}^{\infty} [k\omega] z^k$$

of the sequence $\{[k\omega]\}_{k\geq 1}$, where ω is a real number and $[x]$ denotes the largest integer not exceeding a real number x. Hecke [2] proved that, if ω is irrational, then $h_\omega(z)$ has the unit circle as its natural boundary, which assures that $h_\omega(z)$ is transcendental over $\mathbb{C}(z)$. Mahler [3] proved that, if ω is quadratic irrational, then the value $h_\omega(\alpha)$ is transcendental at any nonzero algebraic number α inside the unit circle. Moreover, there are several results on the algebraic independence of the values of the Hecke-Mahler series including its derivatives in the case where ω is a real quadratic irrational number.

Theorem 1 (Nishioka [5], see also Nishioka [7]). *Let ω be a real quadratic irrational number. If α is an algebraic number with $0 < |\alpha| < 1$, then the infinite set $\{h_\omega^{(l)}(\alpha) \mid l \geq 0\}$ is algebraically independent.*

Theorem 2 (Masser [4]). *Let ω be a real quadratic irrational number. Then the infinite set $\{h_\omega(\alpha) \mid \alpha \in \overline{\mathbb{Q}},\ 0 < |\alpha| < 1\}$ is algebraically independent.*

The author proved the algebraic independence of the ‘direct product’ of the infinite sets treated in the above two results.

Theorem 3 (Tanuma [9]). *Let ω be a real quadratic irrational number. Then the infinite set $\{h_\omega^{(l)}(\alpha) \mid l \geq 0,\ \alpha \in \overline{\mathbb{Q}},\ 0 < |\alpha| < 1\}$ is algebraically independent.*

On the other hand, there are several results treating the values of the Hecke-Mahler series for several quadratic irrational numbers. Nishioka proved the algebraic independence of the values of the Hecke-Mahler series for several quadratic irrational numbers generating different quadratic fields at a single algebraic number by establishing a criterion for the algebraic independence of the values of Mahler functions under different transformations.

Theorem 4 (Nishioka [6]). *Let $\omega_1, \ldots, \omega_r$ be real quadratic irrational numbers such that $\mathbb{Q}(\omega_i) \neq \mathbb{Q}(\omega_j)$ $(i \neq j)$. If α is an algebraic number with $0 < |\alpha| < 1$, then $h_{\omega_1}(\alpha), \ldots, h_{\omega_r}(\alpha)$ are algebraically independent.*

Remark 1. As a matter of fact, Nishioka [6] obtained the algebraic independence of the infinite set $\{f_d(\alpha),\ g_d(\alpha),\ h_{\omega_j}(\alpha) \mid d \geq 2,\ 1 \leq j \leq r\}$ including the values of $f_d(z) = \sum_{k=0}^{\infty} z^{d^k}$ and $g_d(z) = \prod_{k=0}^{\infty}(1 - z^{d^k})$.

The algebraic independence of the values of the Hecke-Mahler series for several quadratic irrational numbers at a single algebraic number is completely determined by Masser. We denote by $\{x\}$ the fractional part of a real number x.

Theorem 5 (Masser [4]). *Let $\omega_1, \ldots, \omega_r$ be real quadratic irrational numbers and α an algebraic number with $0 < |\alpha| < 1$. Then $h_{\omega_1}(\alpha), \ldots, h_{\omega_r}(\alpha)$ are algebraically independent if and only if $\{\pm\omega_j\}$ are distinct.*

Theorem 5 asserts that the algebraic relations among the values $h_{\omega_1}(\alpha), \ldots, h_{\omega_r}(\alpha)$ are generated only by the linear relations among the functions themselves of the forms

$$h_\omega(z) + h_{-\omega}(z) = -\frac{z}{1-z} \in \mathbb{Q}(z) \tag{1}$$

and

$$h_\omega(z) - h_{\{\omega\}}(z) = \frac{[\omega]z}{(1-z)^2} \in \mathbb{Q}(z). \tag{2}$$

For the algebraic independence of their values at several algebraic numbers, Adamczewski and Faverjon recently announced the following result.

Theorem 6 (Adamczewski and Faverjon [1]). *Let $\omega_1, \ldots, \omega_r$ be real quadratic irrational numbers such that $\mathbb{Q}(\omega_i) \neq \mathbb{Q}(\omega_j)$ $(i \neq j)$. Then the infinite set $\{h_{\omega_j}(\alpha) \mid 1 \leq j \leq r,\ \alpha \in \overline{\mathbb{Q}},\ 0 < |\alpha| < 1\}$ is algebraically independent.*

Theorem 6 is an application of their criterion for the algebraic independence of the values of Mahler functions more general than that of Nishioka. For proving Theorem 6, using their criterion, they reduced the algebraic independence of the infinite set $\{h_{\omega_j}(\alpha) \mid 1 \le j \le r,\ \alpha \in \overline{\mathbb{Q}},\ 0 < |\alpha| < 1\}$ to that of each of the infinite sets $\{h_{\omega_j}(\alpha) \mid \alpha \in \overline{\mathbb{Q}},\ 0 < |\alpha| < 1\}$ $(1 \le j \le r)$. Hence the algebraic independence of $\{h_{\omega_j}(\alpha) \mid 1 \le j \le r,\ \alpha \in \overline{\mathbb{Q}},\ 0 < |\alpha| < 1\}$ follows from Theorem 2. On the other hand, Theorem 3 shows that each of the infinite sets $\{h_{\omega_j}^{(l)}(\alpha) \mid l \ge 0,\ \alpha \in \overline{\mathbb{Q}},\ 0 < |\alpha| < 1\}$ $(1 \le j \le r)$ is algebraically independent. Therefore, using the criterion of Adamczewski and Faverjon and Theorem 3, we can obtain the following generalization of Theorem 6.

Theorem 7. *Let $\omega_1, \ldots, \omega_r$ be real quadratic irrational numbers such that $\mathbb{Q}(\omega_i) \ne \mathbb{Q}(\omega_j)$ $(i \ne j)$. Then the infinite set $\{h_{\omega_j}^{(l)}(\alpha) \mid l \ge 0,\ 1 \le j \le r,\ \alpha \in \overline{\mathbb{Q}},\ 0 < |\alpha| < 1\}$ is algebraically independent.*

In the case where $\omega_1, \ldots, \omega_r$ generate the same quadratic field, some algebraic relations occur. For example, we have

$$2h_{2\omega}(\alpha^2) - (h_\omega(\alpha) + h_\omega(-\alpha)) = 0$$

and

$$h_{4\omega}(\alpha^2) - h_{2\omega}(\alpha) + h_\omega(\alpha) - h_{\omega+1/2}(\alpha) \in \overline{\mathbb{Q}}.$$

In contrast with the case of the values at a single algebraic number, the algebraic relations among the values of $h_{\omega_1}(z), \ldots, h_{\omega_r}(z)$ at several algebraic numbers seems to be complicated. In the present article, we study the algebraic independence of the values of $h_{\omega_1}(z), \ldots, h_{\omega_r}(z)$ including their derivatives at several algebraic numbers in the case where $\mathbb{Q}(\omega_1) = \cdots = \mathbb{Q}(\omega_r)$.

2 Main result: case of same quadratic field

Let $\omega_1, \ldots, \omega_r$ be real quadratic irrational numbers such that $\mathbb{Q}(\omega_1) = \cdots = \mathbb{Q}(\omega_r)$. Then there exists a real quadratic irrational number ω such that $\mathbb{Q}(\omega) = \mathbb{Q}(\omega_1) = \cdots = \mathbb{Q}(\omega_r)$ and

$$\omega_j = p_j\omega + q_j \quad (1 \le j \le r),$$

where $p_1, \ldots, p_r$ are nonzero integers and $q_1, \ldots, q_r$ are rational numbers. By (1), we may assume that $p_1, \ldots, p_r$ are positive integers. If $(p_i, \{q_i\}) = (p_j, \{q_j\})$ for some $i \ne j$, then $\omega_i - \omega_j = q_i - q_j \in \mathbb{Z}$ and so the values $h_{\omega_i}(\alpha)$ and $h_{\omega_j}(\alpha)$ are algebraically dependent by Theorem 5. Hence we assume that the pairs $(p_1,\{q_1\}),\dots,(p_r,\{q_r\})$ are distinct. Moreover, we may assume that $0<\omega_1,\dots,\omega_r<1$ by (2).

First we consider the case where $q_1=\cdots=q_r=0$. In this case, we can obtain the following proposition from Theorem 3 by considering the generating functions of subsequences of the sequence $\{[k\omega]\}_{k\geq 1}$.

Proposition 1. *Let ω be a real quadratic irrational number and $\alpha_1,\dots,\alpha_n$ algebraic numbers with $0<|\alpha_i|<1$ $(1\leq i\leq n)$. If $\alpha_1,\dots,\alpha_n$ are pairwise multiplicatively independent, then the infinite set $\{h_{d\omega}^{(l)}(\alpha_i)\mid d\geq 1,\ l\geq 0,\ 1\leq i\leq n\}$ is algebraically independent.*

Proof. Let $h_{(d,t)}(z)=\sum_{k=0}^{\infty}[(dk+t)\omega]z^{dk+t}$ $(d\geq 1,\ 1\leq t\leq d)$. Then we have $h_\omega(z)=\sum_{t=1}^{d}h_{(d,t)}(z)$. We note that $h_{(d,d)}(z)=\sum_{k=0}^{\infty}[(dk+d)\omega]z^{dk+d}=h_{d\omega}(z^d)$. Letting ζ_d be the primitive d-th root of unity, we see that

$$h_\omega(\zeta_d^j z)=\sum_{t=1}^{d}h_{(d,t)}(\zeta_d^j z)=\sum_{t=1}^{d}(\zeta_d^t)^j h_{(d,t)}(z)\quad(0\leq j\leq d-1)$$

and so $\zeta_d^{jl}h_\omega^{(l)}(\zeta_d^j z)=\sum_{t=1}^{d}(\zeta_d^t)^j h_{(d,t)}^{(l)}(z)$ $(l\geq 0,\ 0\leq j\leq d-1)$. Hence we obtain

$$\operatorname{diag}\left(1,\zeta_d^l,\dots,\zeta_d^{(d-1)l}\right)\begin{pmatrix}h_\omega^{(l)}(z)\\ h_\omega^{(l)}(\zeta_d z)\\ \vdots\\ h_\omega^{(l)}(\zeta_d^{d-1}z)\end{pmatrix}=\begin{pmatrix}1&1&\cdots&1\\ \zeta_d&\zeta_d^2&\cdots&1\\ \vdots&\vdots&\ddots&\vdots\\ \zeta_d^{d-1}&(\zeta_d^2)^{d-1}&\cdots&1\end{pmatrix}\begin{pmatrix}h_{(d,1)}^{(l)}(z)\\ h_{(d,2)}^{(l)}(z)\\ \vdots\\ h_{(d,d)}^{(l)}(z)\end{pmatrix}$$

for any $d\geq 1$. Let $\alpha_1,\dots,\alpha_n$ be pairwise multiplicatively independent algebraic numbers with $0<|\alpha_i|<1$ $(1\leq i\leq n)$. Then the numbers $\alpha_i,\alpha_i^{1/2},-\alpha_i^{1/2},\dots,\alpha_i^{1/d},\zeta_d\alpha_i^{1/d},\dots,\zeta_d^{d-1}\alpha_i^{1/d}$ $(1\leq i\leq n)$ are all distinct. Hence by Theorem 3 the infinite set $\{h_{(d,d)}^{(l)}(\alpha_i^{1/d})\mid d\geq 1,\ l\geq 0,\ 1\leq i\leq n\}$ is algebraically independent. Since $h_{(d,d)}(z)=\sum_{k=0}^{\infty}[(dk+d)\omega]z^{dk+d}=h_{d\omega}(z^d)$, we see that $\{h_{d\omega}^{(l)}(\alpha_i)\mid d\geq 1,\ l\geq 0,\ 1\leq i\leq n\}$ is algebraically independent. □

Now we state the main result. Recall that $\omega_j=p_j\omega+q_j$ $(1\leq j\leq r)$, where ω is a real quadratic irrational number, $p_1,\dots,p_r$ are positive integers, and $q_1,\dots,q_r$ are rational numbers such that the pairs

$(p_1,\{q_1\}),\dots,(p_r,\{q_r\})$ are distinct. The main result treats the case where $q_1,\dots,q_r$ are not necessarily zero.

Theorem 8. *Let $\alpha_1,\dots,\alpha_n$ be algebraic numbers with $0<|\alpha_i|<1$ $(1\le i\le n)$ such that α_i/α_j is not a root of unity for any $i\ne j$. If $(1-\{q_j\})/p_j$ $(1\le j\le r)$ are distinct, then $\{h^{(l)}_{\omega_j}(\alpha_i)\mid l\in\mathbb{Z}_{\ge 0},\ 1\le i\le n,\ 1\le j\le r\}$ is algebraically independent.*

As a corollary, we obtain the following generalization of Proposition 1.

Corollary 1. *Let $\alpha_1,\dots,\alpha_n$ be algebraic numbers with $0<|\alpha_i|<1$. If α_i/α_j is not a root of unity for any $i\ne j$, then the infinite set $\{h^{(l)}_{d\omega}(\alpha_i)\mid d\ge 1,\ l\ge 0,\ 1\le i\le n\}$ is algebraically independent.*

In the rest of this article, we give a sketch of the proof of Theorem 8.

3 Construction of auxiliary functions

For any positive number ω, let

$$H_\omega(z_1,z_2)=\sum_{k_1=1}^{\infty}\sum_{k_2=1}^{[k_1\omega]} z_1^{k_1}z_2^{k_2}.$$

Then $H_\omega(z,1)=h_\omega(z)$. We have

$$\begin{aligned}H_\omega(z_1,z_2)+H_{1/\omega}(z_2,z_1)&=\sum_{\substack{k_1\ge 1,k_2\ge 1\\ k_2<k_1\omega}} z_1^{k_1}z_2^{k_2}+\sum_{\substack{k_1\ge 1,k_2\ge 1\\ k_1>k_2\omega}} z_1^{k_2}z_2^{k_1}\\ &=\frac{z_1}{1-z_1}\frac{z_2}{1-z_2}\end{aligned}\tag{3}$$

and

$$\begin{aligned}H_{a+\omega}(z_1,z_2)&=\sum_{k_1=1}^{\infty}\sum_{k_2=1}^{k_1a+[k_1\omega]} z_1^{k_1}z_2^{k_2}\\ &=\sum_{k_1=1}^{\infty}\left(z_1^{k_1}\sum_{k_2=1}^{ak_1} z_2^{k_2}+(z_1z_2^a)^{k_1}\sum_{k_2=1}^{[k_1\omega]} z_2^{k_2}\right)\\ &=\frac{z_1z_2}{(1-z_1)(1-z_2)}-\frac{z_1z_2^{a+1}}{(1-z_2)(1-z_1z_2^a)}+H_\omega(z_1z_2^a,z_2)\end{aligned}\tag{4}$$

for any positive integer a.

Let $T = \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix}$ be a 2×2 matrix with nonnegative integer entries and define

$$T(z_1, z_2) = (z_1^{t_{11}} z_2^{t_{12}}, z_1^{t_{21}} z_2^{t_{22}}). \tag{5}$$

Let $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $C(a) = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$ for any positive integer a. Define $\begin{pmatrix} b & c \\ d & e \end{pmatrix} \omega = (b\omega + c)/(d\omega + e)$, where b, c, d, e are nonnegative integers. Then we see that

$$H_{B\omega}(z_1, z_2) \equiv -H_\omega(B(z_1, z_2)) \pmod{\mathbb{Q}(z_1, z_2)} \tag{6}$$

and that

$$H_{C(a)\omega}(z_1, z_2) \equiv H_\omega(C(a)(z_1, z_2)) \pmod{\mathbb{Q}(z_1, z_2)} \tag{7}$$

by (3) and (4), respectively, where $B(z_1, z_2)$ and $C(a)(z_1, z_2)$ are defined by (5).

For simplicity, we assume that $\omega_1, \ldots, \omega_r$ is expanded in purely periodic continued fractions. Suppose that the positive quadratic irrational ω_j is expanded in the continued fraction as follows:

$$\omega_j = [0; a_1^{(j)}, a_2^{(j)}, \ldots],$$

where $\{a_k^{(j)}\}_{k \ge 1}$ is an purely periodic sequence of positive integers. Let ν_j be its even period. Then we see that

$$\omega_j = BC(a_1^{(j)})BC(a_2^J) \cdots BC(a_{\nu_j}^{(j)})\omega_j.$$

Let

$$T_j = C(a_{\nu_j}^{(j)})BC(a_{\nu_j - 1}^{(j)})B \cdots C(a_1^{(j)})B.$$

Using (6) and (7), we have

$$H_{\omega_j}(z_1, z_2) \equiv H_{\omega_j}(T_j(z_1, z_2)) \qquad \pmod{\mathbb{Q}(z_1, z_2)}. \tag{8}$$

Combining Lemmas 3.3 and 7.3 of [4], we construct auxiliary functions.

Lemma 1 (cf. Adamczewski and Faverjon [1])**.** *Let $\omega_1, \ldots, \omega_r$ be real quadratic irrational numbers such that $\mathbb{Q}(\omega_1) = \cdots = \mathbb{Q}(\omega_r)$. Let $\alpha_1, \ldots,$ α_n be algebraic numbers with $0 < |\alpha_i| < 1$ $(1 \le i \le n)$. Then there exist multiplicatively independent algebraic numbers $\beta_1, \ldots, \beta_m$ with $0 <$ $|\beta_j| < 1$ $(1 \le j \le m)$, roots of unity $\zeta_1, \ldots, \zeta_n$, a 2×2 matrix T, and power series $G_{ij}(\boldsymbol{z}) = G_{ij}(x_1, y_1, x_2, y_2, \ldots, x_m, y_m)$ such that*

(i) $G_{ij}(x_1, 1, x_2, 1, \dots, x_m, 1) \equiv h_{\omega_j}(\zeta_i M_i(\boldsymbol{x})) \pmod{\overline{\mathbb{Q}}(\boldsymbol{x}))}$, *where* M_i *is a monomial.*

(ii) $G_{ij}(\boldsymbol{z}_0) - h_{\omega_j}(\alpha_i) \in \overline{\mathbb{Q}}$, *where* $\boldsymbol{z}_0 = (\beta_1, 1, \beta_2, 1, \dots, \beta_m, 1)$.

(iii) $G_{ij}(\boldsymbol{z}) \equiv G_{ij}(T(x_1, y_1), T(x_2, y_2) \dots, T(x_m, y_m)) \pmod{\overline{\mathbb{Q}}(\boldsymbol{z}))}$.

Although we do not go into the detail, we remark that we can take the same transformation matrix T for H_{ω_j} $(1 \leq j \leq r)$. Since the maximum eigenvalue ρ_j of T_j is a non-trivial unit of $\mathbb{Q}(\omega_j)$, there exist positive integers a_j, b_j such that $\rho_1^{a_j} = \rho_j^{b_j}$. Let $a = \prod_j a_j$ and $T^{(1)} = T_1^a$. Then the maximum eigenvalue ρ of $T^{(1)}$ is $\rho_j^{ab_j/a_j}$. Since $\det T_j = 1$, we see that $T^{(1)}$ is conjugate to $T_j^{ab_j/a_j}$. This is the reason why we can take the same transformation matrix T. Hence we can apply Nishioka's criterion. We need the criterion of Adamczewski and Faverjon to treat the distinct transformations simultaneously.

4 Linear independence of Hecke-Mahler series

By Nishioka's criterion, it is sufficient to show that

$$\left\{ \sum_{k_1=1}^{\infty} \sum_{k_2=1}^{[k_1\omega_j]} k_1^l k_2^{l'} (\zeta_i z^{t_i})^{k_1} \,\middle|\, 0 \leq l, l' \leq L,\ 1 \leq i \leq n,\ 1 \leq j \leq r \right\}$$

is linearly independent over $\overline{\mathbb{Q}}$ modulo $\overline{\mathbb{Q}}(z)$ for any L, where $\zeta_1, \dots, \zeta_n$ are roots of unity and $t_1 < \cdots < t_n$ are positive integers. On the contrary, we assume that $\{\sum_{k_1=1}^{\infty} \sum_{k_2=1}^{[k_1\omega_j]} k_1^l k_2^{l'} (\zeta_i z^{t_i})^{k_1} \mid 0 \leq l, l' \leq L,\ 1 \leq i \leq n,\ 1 \leq j \leq r\}$ is linearly dependent over $\overline{\mathbb{Q}}$ modulo $\overline{\mathbb{Q}}(z)$. Then we see that

$$\left\{ \sum_{k=1}^{\infty} k^l [k\omega_j]^{l'} (\zeta_i z^{t_i})^k \,\middle|\, 0 \leq l \leq L,\ 1 \leq l' \leq L+1,\ 1 \leq i \leq n,\ 1 \leq j \leq r \right\}$$

is linearly dependent over $\overline{\mathbb{Q}}$ modulo $\overline{\mathbb{Q}}(z)$. Therefore there exist algebraic integers $\lambda_{ijll'}$, not all zero, such that

$$\sum_{i=1}^{n} \sum_{j=1}^{r} \sum_{l=0}^{L} \sum_{l'=1}^{L+1} \lambda_{ijll'} \sum_{k=1}^{\infty} k^l [k\omega_j]^{l'} (\zeta_i z^{t_i})^k = \sum_{k=0}^{\infty} a_k z^k \in \overline{\mathbb{Q}}(z),$$

where

$$a_k = \sum_{t_i|k} \sum_{j=1}^{r} \sum_{l=0}^{L} \sum_{l'=1}^{L+1} \lambda_{ijll'} \left(\frac{k}{t_i}\right)^l \left[\frac{k\omega_j}{t_i}\right]^{l'} \zeta_i^{k/t_i}.$$

Since $\{a_k\}_{k\geq 0}$ is a linear recurrence sequence of algebraic integers and since $a_k = O(k^{2L+1})$, we see that the characteristic roots $\xi_1, \ldots, \xi_v$ are roots of unity. Let N be a positive integer such that $\zeta_1^N = \cdots = \zeta_n^N = \xi_1^N = \cdots = \xi_v^N = 1$. Put $s = t_1 \cdots t_n N$ and $s_i = s/t_i$. For any $k \geq 0$ and for any fixed nonnegative integer h, we have

$$\begin{aligned} a_{sk+h} &= \sum_{t_i|h} \sum_{j=1}^{r} \sum_{l=0}^{L} \sum_{l'=1}^{L+1} \lambda_{ijll'} \zeta_i^{h/t_i} \left(\frac{sk+h}{t_i}\right)^l \left[\frac{(sk+h)\omega_j}{t_i}\right]^{l'} \\ &= \sum_{t_i|h} \sum_{j=1}^{r} \sum_{l=0}^{L} \sum_{l'=1}^{L+1} \lambda_{ijll'} \zeta_i^{h/t_i} \left(\frac{sk+h}{t_i}\right)^l \\ &\qquad \times \left(\frac{(sk+h)\omega_j}{t_i} - \left\{\frac{(sk+h)\omega_j}{t_i}\right\}\right)^{l'}. \end{aligned}$$

We may assume that $\lambda_{1jll'}$ $(1 \leq j \leq r, 0 \leq l \leq L, 1 \leq l' \leq L+1)$ are not all zero. We replace N with a positive integral multiple N' of N such that $N'q_j \in \mathbb{Z}$ $(1 \leq j \leq r)$. Then we have

$$a_{sk+t_1} = \sum_{j=1}^{r} \sum_{l=0}^{L} \sum_{l'=1}^{L+1} \lambda_{1jll'} \zeta_1 (s_1 k + 1)^l \big((s_1 k+1)\omega_j - \{s_1 p_j k\omega + \omega_j\}\big)^{l'}.$$

For each j, we have

$$\begin{aligned} &\sum_{l=0}^{L} \sum_{l'=1}^{L+1} \lambda_{1jll'} \zeta_1 (s_1 k + 1)^l \big((s_1 k+1)\omega_j - \{s_1 p_j k\omega + \omega_j\}\big)^{l'} \\ =& Q_{d_j j}(\{s_1 p_j k\omega + \omega_j\}) k^{d_j} + \cdots + Q_{0j}(\{s_1 p_j k\omega + \omega_j\}), \end{aligned}$$

where $Q_{d_j j}(X), \ldots, Q_{0j}(X) \in \overline{\mathbb{Q}}[X]$ with $Q_{d_j j}(X) \neq 0$. In addition, at least one of $Q_{d_j j}(X), \ldots, Q_{0j}(X)$ is not constant for any j such that $\lambda_{1jll'}$ $(0 \leq l \leq L, 1 \leq l' \leq L+1)$ are not all zero. Hence

$$\begin{aligned} a_{sk+t_1} =& \sum_{j=1}^{r} \big(Q_{d_j j}(\{s_1 p_j k\omega + \omega_j\}) k^{d_j} + \cdots + Q_{0j}(\{s_1 p_j k\omega + \omega_j\})\big) \\ =& Q_d(\{s_1 p_1 k\omega + \omega_1\}, \ldots, \{s_1 p_r k\omega + \omega_r\}) k^d + \cdots \\ &+ Q_0(\{s_1 p_1 k\omega + \omega_1\}, \ldots, \{s_1 p_r k\omega + \omega_r\}), \end{aligned}$$

where $Q_d(X_1,\dots,X_r),\dots,Q_0(X_1,\dots,X_r) \in \overline{\mathbb{Q}}[X_1,\dots,X_r]$ are not all constant polynomials of the form

$$Q_i(X_1,\dots,X_r) = Q_{i1}(X_1) + \cdots + Q_{ir}(X_r) \quad (0 \le i \le d)$$

with $Q_d(X_1,\dots,X_r) \neq 0$. On the other hand, for all sufficiently large k, we can write

$$a_{sk+t_1} = c_d k^d + \cdots + c_0,$$

where $c_d,\dots,c_0$ are algebraic numbers. Let i_0 be the largest integer such that $Q_{i_0}(X_1,\dots,X_r)$ is not constant. Here we need the following key lemma.

Lemma 2. *Let $Q_1(X),\dots,Q_r(X) \in \mathbb{C}[X]$ be not all constant and let $Q(X_1,\dots,X_r) = Q_1(X_1) + \cdots + Q_r(X_r) \in \mathbb{C}[X_1,\dots,X_r]$. Let $a_1,\dots,a_r$ be positive numbers and $b_1,\dots,b_r$ real numbers with $0 \le b_j < 1$ such that the pairs (a_j, b_j) $(1 \le j \le r)$ are distinct. If the numbers $(1-b_j)/a_j$ $(1 \le j \le r)$ are distinct, then*

$$f(\tau) = Q(\{a_1\tau + b_1\},\dots,\{a_r\tau + b_r\})$$

is not a constant function on $\mathbb{R}$.

Since $(1 - \{q_j\})/p_j$ are distinct, from Lemma 2, there exists a real number τ_0 such that

$$Q_{i_0}(\{p_1\tau_0 + q_1\},\dots,\{p_r\tau_0 + q_r\}) \neq c_{i_0}.$$

Lemma 3 (Tanaka and Tanuma [8])**.** *Let ω be a positive irrational number and let $s_1,\dots,s_r$ be positive integers. Then, for any real number τ, there exists an increasing sequence $\{k_\nu\}_{\nu\ge 0}$ of positive integers such that*

$$\lim_{\nu\to\infty} \left(\{s_1 k_\nu \omega\},\dots,\{s_r k_\nu \omega\}\right) = \left(\{s_1\tau\},\dots,\{s_r\tau\}\right),$$

where each component of the left-hand side approaches the corresponding component of the right-hand side from the right.

From Lemma 3 there exists an increasing sequence $\{k_\nu\}_{\nu\ge 0}$ of positive integers such that

$$\lim_{\nu\to\infty} (\{s_1p_1k_\nu\omega\},\dots,\{s_1p_rk_\nu\omega\}) = (\{p_1(\tau_0-\omega)\},\dots,\{p_r(\tau_0-\omega)\}),$$

where each component of the left-hand side approaches the corresponding component of the right-hand side. Then we see that

$$\begin{aligned}
&\lim_{\nu\to\infty} (\{s_1p_1k_\nu\omega + \omega_1\},\dots,\{s_1p_rk_\nu\omega + \omega_r\}) \\
=&(\{p_1(\tau_0-\omega) + p_1\omega + q_1\},\dots,\{p_r(\tau_0-\omega) + p_r\omega + q_r\}) \\
=&(\{p_1\tau_0 + q_1\},\dots,\{p_r\tau_0 + q_r\}).
\end{aligned}$$

If $i_0 = d$, then $\lim_{\nu\to\infty} a_{sk_\nu+t_1}/k_\nu^d = Q_d(\{p_1\tau_0\},\ldots,\{p_r\tau_0\}) \neq c_d$. On the other hand $\lim_{k\to\infty} a_{sk+t_1}/k^d = c_d$, which is a contradiction. Hence we see that the polynomial $Q_d(X_1,\ldots,X_r)$ is equal to the constant c_d identically. Then, since

$$\lim_{\nu\to\infty} \frac{a_{sk_\nu+t_1} - c_d k_\nu^d}{k_\nu^{d-1}} = Q_{d-1}(\{p_1\tau_0\},\ldots,\{p_r\tau_0\})$$

and

$$\lim_{k\to\infty} \frac{a_{sk+t_1} - c_d k^d}{k^{d-1}} = c_{d-1},$$

we see that $i_0 < d-1$. Continuing this process, we obtain a contradiction.

5 Remarks on Lemma 2

In the previous work, the author proved the following lemma, which plays a crucial role in the proof of Theorem 3 and is the $b_1 = \cdots = b_r = 0$ case of Lemma 2.

Lemma 4 (Tanuma [9]). *Let $Q_1(X),\ldots,Q_r(X) \in \mathbb{C}[X]$ be not all constant and let $Q(X_1,\ldots,X_r) = Q_1(X_1)+\cdots+Q_r(X_r) \in \mathbb{C}[X_1,\ldots,X_r]$. Let $a_1,\ldots,a_r$ are real numbers with $a_1 > a_2 > \cdots > a_r > 0$. Then*

$$f(\tau) = Q(\{a_1\tau\},\ldots,\{a_r\tau\})$$

is not a constant function on $\mathbb{R}$.

The number $(1-b_j)/a_j$ in Lemma 2 is the smallest positive point of discontinuity of the function $\{a_j\tau + b_j\}$. The proof of Lemma 2 focuses on the behavior of the functions $Q_1(\{a_1\tau+b_1\}),\ldots,Q_r(\{a_r\tau+b_r\})$ around these points of discontinuity. In fact, if we can find the point of discontinuity of the function $\{a_j\tau + b_j\}$ at which all other functions $\{a_1\tau+b_1\},\ldots,\{a_{j-1}\tau+b_{j-1}\},\{a_{j+1}\tau+b_{j+1}\},\ldots,\{a_r\tau+b_r\}$ are continuous, then we can obtain the same conclusion as in Lemma 2. Hence Lemma 2 is obvious if the ratio of any pair of $a_1,\ldots,a_r$ is irrational. However, in the proof of Theorem 8, we have to treat the case where $a_1,\ldots,a_r$ are all integers. In this case, if we omit the assumption that the numbers $(1-b_j)/a_j$ $(1 \le j \le r)$ are distinct, the same conclusion does not hold:

$$\{2\tau\} - \{\tau\} - \left\{\tau + \frac{1}{2}\right\} = -\frac{1}{2}.$$

The assumption that any cross terms do not appear is also essential. For example, let us consider the case where $r = 2$, $a_1 = 2$, $a_2 = 1$, and

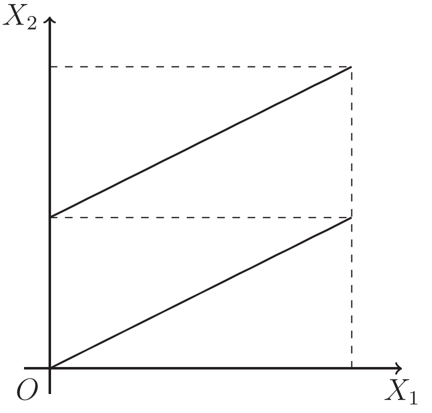

Figure 1: Line segments where the point $(\{2\tau\}, \{\tau\})$ varies

$b_1 = b_2 = 0$. The point $(\{2\tau\}, \{\tau\})$ varies on the two line segments shown in Figure 1. Then, for the polynomial $Q(X_1, X_2) = (X_1 - 2X_2)(X_1 - 2X_2 + 1) = X_1^2 - 4X_1X_2 + 4X_2^2 + X_1 - 2X_2$, we have $Q(\{2\tau\}, \{\tau\}) = 0$ for any τ.

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