Compression of $\mathrm{M}^\natural$-convex Functions — Flag Matroids and Valuated Permutohedra

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Abstract

Murota (1998) and Murota and Shioura (1999) introduced concepts of M-convex function and $\mathrm{M}^\natural$-convex function as discrete convex functions, which are generalizations of valuated matroids due to Dress and Wenzel (1992). In the present paper we consider a new operation defined by a convolution of sections of an $\mathrm{M}^\natural$-convex function that transforms the given $\mathrm{M}^\natural$-convex function to an M-convex function, which we call a *compression* of an $\mathrm{M}^\natural$-convex function. For the class of valuated generalized matroids, which are special $\mathrm{M}^\natural$-convex functions, the compression induces a *valuated permutohedron* together with a decomposition of the valuated generalized matroid into *flag-matroid strips*, each corresponding to a maximal linearity domain of the induced valuated permutohedron. We examine the details of the structure of flag-matroid strips and the induced valuated permutohedron by means of discrete convex analysis of Murota.

Keywords: Discrete convex functions, compression, flag matroids, permutohedra

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1. Introduction

Murota (1998) and Murota and Shioura (1999) introduced the concepts of M-convex function [13] and $\mathrm{M}^\natural$-convex function [17], as discrete convex functions. Their original ideas can be traced back to Dress and Wenzel's valuated matroids [4] introduced in 1992. See [14, 15, 16] for details about the theory of discrete convex analysis and its applications developed by Murota and others (also see [10, Chapter VII]).

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In the present paper we consider a new operation defined by a convolution of sections of an M$^\natural$-convex function that transforms the given M$^\natural$-convex function to an M-convex function, which we call a *compression* of an M$^\natural$-convex function. For the class of valuated generalized matroids, which is a special class of M$^\natural$-convex functions, the compression induces a valuated permutohedron together with a decomposition of the valuated generalized matroid into flag-matroid strips, each corresponding to a maximal linearity domain of the valuated permutohedron. We examine the details of the structure of flag-matroid strips and the induced valuated permutohedron. We investigate the structures of the strip decomposition of valuated generalized-matroids, special M$^\natural$-convex functions by means of discrete convex analysis of Murota ([14]). The strip decomposition of a valuated generalized-matroid uniquely determines a valuated permutohedron, identified with a special M-convex function.

The present paper is organized as follows. In Section 2 we give some definitions and preliminaries about (i) submodular/supermodular functions and related polyhedra such as base polyhedra, submodular/supermodular polyhedra, and generalized polymatroids, and (ii) M-/M$^\natural$-convex functions and L-/L$^\natural$-convex functions. In Section 3 we introduce a new operation called a *compression* of M$^\natural$-convex functions, which leads us to the concepts of flag-matroid strips and a strip decomposition of M$^\natural$-convex functions in Section 4. In Section 5 we consider valuated generalized matroids, which are special M$^\natural$-convex functions, and examine implications of our results in valuated generalized matroids. The strip decomposition of a valuated generalized matroid gives a collection of strips of flag matroids [2], each inducing a sub-permutohedra. The compression of a valuated generalized matroid induces a valuated permutohedron whose maximal linearity domains corresponds to flag-matroid strips of the valuated generalized matroid. Section 6 gives some concluding remarks.

Note: We gratefully acknowledge that Georg Loho informed us of the closely related results independently and almost at the same time obtained by Madeline Brandt, Christopher Eur, and Leon Zhang [3].

2. Definitions and Preliminaries

Let $E = [n](= \{1, \cdots, n\})$ for a positive integer $n > 1$. For any $x \in \mathbb{R}^E$ and $X \subseteq E$ define $x(X) = \sum_{e \in X} x(e)$, where $x(\emptyset) = 0$. For any subset $X \subseteq E$ its *characteristic vector* χ_X in $\mathbb{R}^E$ is defined by $\chi_X(e) = 1$ if $e \in X$ and $\chi_X(e) = 0$ if $e \in E \setminus X$. We also write χ_e instead of $\chi_{\{e\}}$ for $e \in E$.

2.1. Basics of submodular/supermodular functions

A function $f : 2^E \to \mathbb{R}$ is called a *submodular function* if it satisfies

$$f(X) + f(Y) \geq f(X \cup Y) + f(X \cap Y) \qquad (\forall X, Y \subseteq E). \tag{2.1}$$

We assume that $f(\emptyset) = 0$ for any set function $f : 2^E \to \mathbb{R}$ in the sequel. A negative of a submodular function is called a *supermodular function*. (Also see [5, 10].)

For a submodular function $f : 2^E \to \mathbb{R}$ define

$$\mathrm{P}(f) = \{x \in \mathbb{R}^E \mid \forall X \subseteq E : x(X) \leq f(X)\}, \tag{2.2}$$

which is called the *submodular polyhedron* associated with submodular function f. Also define

$$\mathrm{B}(f) = \{x \in \mathrm{P}(f) \mid x(E) = f(E)\}, \tag{2.3}$$

which is called the *base polyhedron* associated with submodular function f. As is well known (see [10]), the base polyhedron $\mathrm{B}(f)$ is always nonempty (and is a face of $\mathrm{P}(f)$).

For a supermodular function $g : 2^E \to \mathbb{R}$ we define in a dual manner the *supermodular polyhedron*

$$\mathrm{P}(g) = \{x \in \mathbb{R}^E \mid \forall X \subseteq E : x(X) \geq g(X)\} \tag{2.4}$$

and the *base polyhedron*

$$\mathrm{B}(g) = \{x \in \mathrm{P}(g) \mid x(E) = g(E)\}. \tag{2.5}$$

For a submodular function $f : 2^E \to \mathbb{R}$ define a supermodular function $f^\# : 2^E \to \mathbb{R}$ by

$$f^\#(X) = f(E) - f(E \setminus X) \qquad (\forall X \subseteq E), \tag{2.6}$$

which is called the *dual supermodular function* of f. Then we have $\mathrm{B}(f) = \mathrm{B}(f^\#)$. For more details about submodular/supermodular functions and associated polyhedra see [10, Chapter II], where submodular/supermodular functions defined on distributive lattices $\mathcal{D} \subseteq 2^E$ are also investigated and their base polyhedra are unbounded unless $\mathcal{D} = 2^E$.

Define $Q_\mathbb{Z} = Q \cap \mathbb{Z}^E$ for any set $Q \subseteq \mathbb{R}^E$. Also denote by $\mathrm{Conv}(Q)$ the convex hull of Q in $\mathbb{R}^E$. When $\mathrm{Conv}(Q_\mathbb{Z}) = Q$, we identify Q with $Q_\mathbb{Z}$.

When a submodular function $f : 2^E \to \mathbb{R}$ is integer-valued, its submodular polyhedron and base polyhedron are integral (i.e., every vertex of the polyhedra is an integral vector). Moreover, we have

$$\mathrm{Conv}(\mathrm{P}(f)_\mathbb{Z}) = \mathrm{P}(f), \qquad \mathrm{Conv}(\mathrm{B}(f)_\mathbb{Z}) = \mathrm{B}(f). \tag{2.7}$$

Most of the following arguments are valid even if we regard $\mathbb{Z}$ in place of $\mathbb{R}$ as the underlying totally ordered additive group. When f is integer-valued, we call $\mathrm{P}(f)_\mathbb{Z}$ and $\mathrm{B}(f)_\mathbb{Z}$

the submodular polyhedron and the base polyhedron, respectively, associated with f as well. (This is the approach taken in [10], indeed.)

For a submodular function $f : 2^E \to \mathbb{R}$ and a nonempty proper subset A of E define functions $f^A : 2^A \to \mathbb{R}$ and $f_A : 2^{E\setminus A} \to \mathbb{R}$ by

$$f^A(X) = f(X) \quad (\forall X \subseteq A), \quad f_A(X) = f(X \cup A) - f(A) \quad (\forall X \subseteq E \setminus A). \quad (2.8)$$

We also define $f^E = f_\emptyset = f$. We call f^A the *restriction* of f on A and f_A the *contraction* of f by A. Similarly we define the restriction and contraction for supermodular functions.

2.2. Permutohedra and sub-permutohedra

For a permutation $\pi : [n] \to [n]$ we identify π with the *permutation vector* $(\pi(1), \cdots, \pi(n))$ in $\mathbb{Z}^n$, which we denote by v_π (or simply by π when there is no possibility of confusion). Suppose that an integer-valued submodular function $f : 2^E \to \mathbb{Z}$ is given by $f(X) = \sum_{i=1}^{|X|}(n - i + 1)$ for all $X \subseteq E = [n]$. Then the base polyhedron $\mathrm{B}(f)$ has the $n!$ extreme points, each being a permutation vector identified with a permutation of $[n]$, which is called the *permutohedron* (or *permutahedron*) in $\mathbb{R}^E$.

For any permutation π of $[n]$ we have a unique complete flag

$$\mathcal{F} : F_1 \subset \cdots \subset F_n = [n] \quad (2.9)$$

such that for each $i \in [n]$ F_i is the set of the first i elements of $(\pi(1), \cdots, \pi(n))$, i.e.,

1. $|F_i| = i$ for each $i = 1, \cdots, n$ and
2. $\sum_{i=1}^{n} \chi_{F_i} = v_\pi$, where χ_{F_i} is the characteristic vector of a set $F_i \subseteq [n]$.

Denote the flag $\mathcal{F}$ in (2.9) by $\mathcal{F}^\pi : F_1^\pi \subset \cdots \subset F_n^\pi$.

Let us consider a polyhedron P satisfying the following two:

(P1) P is the convex hull of a set of some permutation vectors.

(P2) P is a base polyhedron.

We call such a polyhedron P a *sub-permutohedron*.[1] A sub-permutohedron is precisely the Coxeter matroid polytope associated with a flag matroid of complete flag (see [2] and the discussions to be made in Section 5). A recent interesting appearance of a sub-permutohedron is from the theory of Bruhat order ([1]), due to Tsukerman and Williams [21], that every Bruhat interval polytope is a sub-permutohedron.

[1] We may call the sub-permutohedron a *permutohedron*, and an ordinary permutohedron a *complete permutohedron*, but we resist the temptation.

2.3. Base polyhedra, generalized polymatroids, and strong maps

Suppose that a submodular function $f : 2^E \to \mathbb{R}$ and a supermodular function $g : 2^E \to \mathbb{R}$ with $f(\emptyset) = g(\emptyset) = 0$ satisfy the following inequalities

$$f(X) - g(Y) \geq f(X \setminus Y) - g(Y \setminus X) \quad (\forall X, Y \subseteq E). \tag{2.10}$$

Then the polyhedron

$$\mathrm{P}(f, g) = \{x \in \mathbb{R}^E \mid \forall X \subseteq E : g(X) \leq x(X) \leq f(X)\} \tag{2.11}$$

is called a *generalized polymatroid* ([7, 11]). There exists a one-to-one correspondence between base polyhedra and generalized polymatroids up to translation along a coordinate axis as follows (see [10, Figure 3.7]).

Theorem 2.1 ([9, 10]): *For the base polyhedron* $\mathrm{B}(f)$ *associated with a submodular function* $f : 2^E \to \mathbb{R}$ *the projection of* $\mathrm{B}(f)$ *along an axis* $e \in E$ *on the coordinate subspace given by the hyperplane* $x(e) = 0$ *is a generalized polymatroid* $\mathrm{P}(f', g')$ *in* $\mathbb{R}^{E'}$ *with* $E' = E - \{e\}$*, where* f' *is the restriction of* f *on* E' *and* g' *is the restriction of* $f^{\#}$ *on* E'*.*

Conversely, every generalized polymatroid in $\mathbb{R}^{E'}$ *is obtained in this way.*

When a generalized polymatroid is a convex hull of $\{0, 1\}$-valued points (vertices), then it is called a *generalized-matroid polytope*, which can be identified with the family $\mathcal{G}$ of subsets $X \subseteq E$ such that χ_X are vertices of the generalized-matroid polytope. The family $\mathcal{G}$ is called a *generalized matroid* (see [8]).

An ordered pair (f_1, f_2) of submodular functions $f_i : 2^E \to \mathbb{R}$ $(i = 1, 2)$ is called a *weak map* if we have

$$\mathrm{P}(f_2) \subseteq \mathrm{P}(f_1). \tag{2.12}$$

Moreover, the ordered pair (f_1, f_2) is called a *strong map* if we have

$$\mathrm{P}((f_2)_X) \subseteq \mathrm{P}((f_1)_X) \qquad (\forall X \subset E), \tag{2.13}$$

i.e., every ordered pair $((f_1)_X, (f_2)_X)$ of contractions of f_1 and f_2 by $X \subset E$ is a weak map. The concept of strong map was originally considered for matroids (see [19, 20, 22]), and we adapt it to any submodular functions (or submodular systems).[2]

We also have the following theorem.

Theorem 2.2: $\mathrm{P}(f, g)$ *is a generalized polymatroid if and only if* $(f, g^{\#})$ *is a strong map.*

[2] As is remarked below (2.7), we can regard $\mathbb{R}$ appearing here as the set $\mathbb{Z}$ of integers.

(Proof) Relation (2.13) is equivalent to the following inequalities

$$f_2(Z \cup X) - f_2(X) \leq f_1(Z \cup X) - f_1(X) \qquad (X \subset E,\ Z \subseteq E \setminus X). \tag{2.14}$$

Putting $W = E \setminus (Z \cup X)$, (2.14) is rewritten in terms of the dual supermodular function $f_2^{\#}$ of f_2 as

$$f_2^{\#}(Z \cup W) - f_2^{\#}(W) \leq f_1(Z \cup X) - f_1(X) \qquad (X \subset E,\ Z \subseteq E \setminus X) \tag{2.15}$$

with $W = E \setminus (Z \cup X)$. Because of the supermodularity of $f_2^{\#}$ (2.15) is equivalent to

$$f_2^{\#}(Z \cup W) - f_2^{\#}(W) \leq f_1(Z \cup X) - f_1(X) \tag{2.16}$$

for all $X, W, Z \subset E$ with $X \cap W = X \cap Z = Z \cap W = \emptyset$, which is equivalent to (2.10). □

A sequence of submodular functions $f_1, \cdots, f_p : 2^E \to \mathbb{R}$ is called a *strong map sequence* if for each $i = 1, \cdots, p-1$ the pair (f_{i+1}, f_i) is a strong map. It follows from Theorem 2.2 that

(F1) Given a strong map sequence $f_1, \cdots, f_p : 2^E \to \mathbb{R}$, we have a sequence of generalized polymatroids $\mathrm{P}(f_{i+1}, (f_i)^{\#})$ for $i = 1, \cdots, p-1$.

We also have the following.

(F2) For a generalized polymatroid $\mathrm{P}(f, g)$ and $\alpha \in \mathbb{R}$ such that $g(E) \leq \alpha \leq f(E)$ the intersection of $\mathrm{P}(f, g)$ and the hyperplane $x(E) = \alpha$ is a base polyhedron (we call such a base polyhedron a *section* of $\mathrm{P}(f, g)$ and denote it by $\mathrm{P}(f, g)_{(\alpha)}$). When $\mathrm{P}(f, g)$ is integral and α is an integer, the section $\mathrm{P}(f, g)_{(\alpha)}$ is integral. Moreover, letting $\mathrm{B}(f')$ be a section of $\mathrm{P}(f, g)$ with a submodular function f', we have a strong map sequence $g^{\#}, f', f$, i.e., $\mathrm{P}(f', g)$ and $\mathrm{P}(f, (f')^{\#})$ are generalized polymatroids.

A strong map sequence $f_1, \cdots, f_p : 2^E \to \mathbb{Z}$ with f_i $(i = 1, \cdots, p)$ being matroid rank functions is called a *flag matroid* (see [2]).

2.4. M$^\natural$-convex functions and L$^\natural$-convex functions

Let $f : \mathbb{R}^E \to \mathbb{R} \cup \{+\infty\}$ be a polyhedral convex function such that

1. its effective domain, $\mathrm{dom} f \equiv \{x \in \mathbb{R}^E \mid f(x) < +\infty\}$, is a generalized polymatroid (hence nonempty) and

2. every linearity domain of f is a generalized polymatroid,

where a *linearity domain* (or *affinity domain*) of f is $\operatorname{Arg\,min}(f-h)$ for a linear function $h(x) = \langle z, x\rangle (\equiv \sum_{e\in E} z(e)x(e))$ with some $z \in (\mathbb{R}^E)^*$. Then f is called an *M$^\natural$-convex function*[3], which is due to Murota and Shioura [17, 14] (also see [10, Section 17]). The negative of an M$^\natural$-convex function is called an *M$^\natural$-concave function* (see Figure 1).

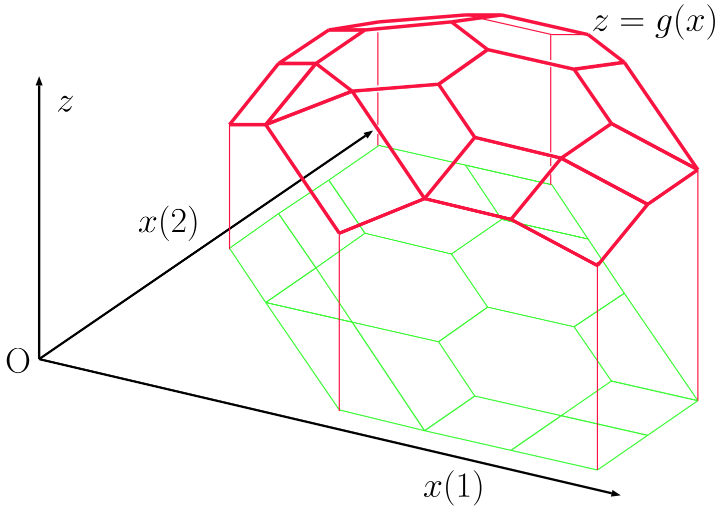

Figure 1: An M$^\natural$-concave function g ([10, Fig. 17.4]).

When $\mathrm{dom} f$ of an M$^\natural$-convex function f is a base polyhedron, f is called an *M-convex function* (see [14]).[4] Any concept related to M-/M$^\natural$-concave functions is defined in a natural way from that defined for M-/M$^\natural$-convex functions.

Now let $f : \mathbb{R}^E \to \mathbb{R} \cup \{+\infty\}$ be an M$^\natural$-convex function satisfying the following (a) and (b):

(a) The effective domain of f is an *integral* generalized polymatroid.

(b) Every linearity domain of f is also an *integral* generalized polymatroid.

Such an M$^\natural$-convex function f can be identified with f being restricted on the integer lattice $\mathbb{Z}^E$. So we can consider an M$^\natural$-convex function $f : \mathbb{Z}^E \to \mathbb{R} \cup \{+\infty\}$. An integer-valued M-concave function $g : \mathbb{Z}^E \to \mathbb{Z} \cup \{-\infty\}$ with its effective domain being a matroid base polytope coincides with a *valuated matroid* due to Dress and Wenzel [4]. Also, if an

[3] Here we employ another equivalent definition of M$^\natural$-convexity, instead of the original definition by means of the exchange axiom introduced by Murota and Shioura [17, 18, 14]. Recall that arguments here are valid if $\mathbb{R}$ is regarded as the set $\mathbb{Z}$ of integers, as well.

[4] M-convex functions were introduced earlier than M$^\natural$-convex functions by Murota (see [13, 14]). There is a one-to-one correspondence between M-convex functions and M$^\natural$-convex convex functions because of Theorem 2.1.

M$^\natural$-convex function $f : \mathbb{Z}^E \to \mathbb{R} \cup \{+\infty\}$ has an effective domain $\mathrm{dom}(f)$ whose convex hull $\mathrm{Conv}(\mathrm{dom}(f))$ is a permutohedron, we call f a *valuated permutohedron*.

For any M$^\natural$-convex function $f : \mathbb{R}^E \to \mathbb{R} \cup \{+\infty\}$ define the *Legendre-Fenchel transform* (or *convex conjugate*) of f by

$$f^\bullet(y) = \sup\{\langle y, x\rangle - f(x) \mid x \in \mathbb{R}^E\} \qquad (y \in (\mathbb{R}^E)^*), \tag{2.17}$$

where $\langle y, x\rangle = \sum_{e\in E} y(e)x(e)$. The function $f^\bullet$ is called an *L$^\natural$-convex function* ([14]), which is equivalent to submodular integrally convex function due to Favati and Tardella [6] when the underlying $\mathbb{R}$ is regarded as $\mathbb{Z}$. The original f is recovered from $f^\bullet$ by taking another Legendre-Fenchel transform as follows.

$$f(x) = \sup\{\langle y, x\rangle - f^\bullet(y) \mid y \in (\mathbb{R}^E)^*\} \qquad (x \in \mathbb{R}^E). \tag{2.18}$$

(See [14, 15].) Hence there exists a one-to-one correspondence between M$^\natural$-convex functions and L$^\natural$-convex functions. Furthermore, Murota [14] showed the integrality property that (2.17) and (2.18) with $\mathbb{R}$ being replaced by $\mathbb{Z}$ hold for any M$^\natural$-convex function $f : \mathbb{Z}^E \to \mathbb{Z} \cup \{+\infty\}$. When f is an M-convex function, its Legendre-Fenchel transform is what is called an *L-convex function* ([14]).

For any $x \in \mathbb{R}^E$ the *subdifferential* of f at x, denoted by $\partial f(x)$, is defined by

$$\partial f(x) = \{w \in (\mathbb{R}^E)^* \mid \forall z \in \mathbb{R}^E : f(z) \geq f(x) + \langle w, z - x\rangle\}. \tag{2.19}$$

The subdifferential of $f^\bullet$ is defined similarly as

$$\partial f^\bullet(w) = \{x \in \mathbb{R}^E \mid \forall y \in (\mathbb{R}^E)^* : f^\bullet(y) \geq f^\bullet(w) + \langle y - w, x\rangle\}. \tag{2.20}$$

Then we have the following.

Lemma 2.3: *We have $w \in \partial f(x)$ if and only if $x \in \partial f^\bullet(w)$.*

(Proof) Note that both statements, $w \in \partial f(x)$ and $x \in \partial f^\bullet(w)$, are equivalent to that $f(x) + f^\bullet(w) = \langle w, x\rangle$. □

Remark: When $f : \mathbb{R}^E \to \mathbb{R} \cup \{+\infty\}$ is an M$^\natural$-convex function, subdifferentials $\partial f^\bullet(w)$ for all $w \in \mathrm{dom}(f^\bullet)$ are generalized polymatroids (or M$^\natural$-convex sets). Furthermore, if f is defined on integer lattice $\mathbb{Z}^E$, then $\partial f^\bullet(w)$ for all $w \in \mathrm{dom}(f^\bullet)$ are integral generalized polymatroids (restricted on $\mathbb{Z}^E$). □

For more details about M-/M$^\natural$-convex functions and L-/L$^\natural$-convex functions see [14, 15] and [10, Chapter VII].

3. Compression of M$^\natural$-convex Functions

In this section we introduce a new transformation, called *compression*, of M$^\natural$-convex functions defined on the integer lattice $\mathbb{Z}^E$. The compression of an M$^\natural$-convex function $f : \mathbb{Z}^E \to \mathbb{R} \cup \{+\infty\}$ is a transformation of the M$^\natural$-convex function f to an M-convex function $\hat{f} : \mathbb{Z}^E \to \mathbb{R} \cup \{+\infty\}$.

Consider any M$^\natural$-convex function $f : \mathbb{Z}^E \to \mathbb{R} \cup \{+\infty\}$. We suppose the following:

- The effective domain $\mathrm{dom}(f)$ is bounded and full-dimensional. That is, $\mathrm{dom}(f)$ is a full-dimensional generalized polymatroid $\mathrm{P}(f^*, g^*)$ (with finite $f^*(E) > g^*(E)$).

For each integer α such that $f^*(E) \geq \alpha \geq g^*(E)$ let $f_{(\alpha)} : \mathbb{Z}^E \to \mathbb{R} \cup \{+\infty\}$ be the M-convex function defined by

$$f_{(\alpha)}(x) = \begin{cases} f(x) & \text{if } x(E) = \alpha \\ +\infty & \text{otherwise} \end{cases} \qquad (x \in \mathbb{Z}^E) \tag{3.1}$$

(cf. [14, 15, 16]). We call $f_{(\alpha)}$ the *α-section* of f. Then put

$$I_f = \{\alpha \in \mathbb{Z} \mid f^*(E) \geq \alpha \geq g^*(E)\} \tag{3.2}$$

and consider the *convolution* of all the sections $f_{(\alpha)}$ $(\alpha \in I_f)$, which is given by

$$\hat{f}(x) = \min\left\{ \sum_{\alpha \in I_f} f_{(\alpha)}(y_\alpha) \;\middle|\; x = \sum_{\alpha \in I_f} y_\alpha,\ \forall \alpha \in I_f : y_\alpha \in \mathbb{Z}^E \right\} \qquad (x \in \mathbb{Z}^E). \tag{3.3}$$

where note that $f_{(\alpha)}(y_\alpha) < +\infty$ only if $y_\alpha(E) = \alpha$. We call $\hat{f}$ the *compression* of f.

It is shown ([14, Theorem 6.13]) that the convolution of M-convex functions is an M-convex function. Hence we have the following theorem about the compression of an M$^\natural$-convex function f, where we give its proof for completeness.

Theorem 3.1: *For the compression $\hat{f}$ of an M$^\natural$-convex function $f : \mathbb{Z}^E \to \mathbb{R} \cup \{+\infty\}$ the Legendre-Fenchel transform of $\hat{f}$ is given by*

$$\hat{f}^\bullet = \sum_{\alpha \in I_f} f_{(\alpha)}{}^\bullet. \tag{3.4}$$

We also have

$$\mathrm{dom}(\hat{f}) = \sum_{\alpha \in I_f} \mathrm{dom}(f_{(\alpha)}), \tag{3.5}$$

where the right-hand side is the Minkowski sum of the effective domains $\mathrm{dom}(f_{(\alpha)})$ for all $\alpha \in I_f$.

(Proof) For any $w \in (\mathbb{R}^E)^*$ we have from (3.3)

$$\begin{aligned} \hat{f}^\bullet(w) &= \sup\{\langle w, x\rangle - \hat{f}(x) \mid x \in \mathbb{Z}^E\} \\ &= \sup\left\{ \sum_{\alpha\in I_f} (\langle w, y_\alpha\rangle - f_{(\alpha)}(y_\alpha)) \;\middle|\; \forall \alpha \in I_f : y_\alpha \in \mathbb{Z}^E \right\} \\ &= \sum_{\alpha\in I_f} \sup\{\langle w, x\rangle - f_{(\alpha)}(x) \mid x \in \mathbb{Z}^E\} \\ &= \sum_{\alpha\in I_f} {f_{(\alpha)}}^\bullet(w). \end{aligned} \tag{3.6}$$

This implies (3.4) and (3.5) because of the definition of the Legendre-Fenchel transform.

□

4. M$^\natural$-convex Functions and Strip Decomposition

Now, we introduce the concept of the *strip decomposition* of an M$^\natural$-convex function defined on the integer lattice $\mathbb{Z}^E$. Let $f : \mathbb{Z}^E \to \mathbb{R} \cup \{+\infty\}$ be an M$^\natural$-convex function with a full-dimensional and bounded effective domain $\mathrm{dom}(f)$.

4.1. Strip decomposition of M$^\natural$-convex functions

For the compression $\hat{f}$ of f and for any $w \in (\mathbb{R}^E)^*$ let us define $D(\hat{f}, w) \subseteq \mathbb{Z}^E$ and $D(f_{(\alpha)}, w) \subseteq \mathbb{Z}^E$ $(\alpha \in I_f)$ by

$$D(\hat{f}, w) = \mathrm{Arg\,min}\{\hat{f}(x) - \langle w, x\rangle \mid x \in \mathbb{Z}^E\} \tag{4.1}$$

$$D(f_{(\alpha)}, w) = \mathrm{Arg\,min}\{f_{(\alpha)}(x) - \langle w, x\rangle \mid x \in \mathbb{Z}^E\} \quad (\alpha \in I_f), \tag{4.2}$$

where recall (3.2) for the definition of I_f. We call $D(\hat{f}, w)$ and $D(f_{(\alpha)}, w)$ $(\alpha \in I_f)$ *linearity domains of* $\hat{f}$ and $f_{(\alpha)}$ $(\alpha \in I_f)$, respectively, *associated with* w. We see from Lemma 2.3 that for every $x \in \mathbb{Z}^E$ we have

$$x \in D(\hat{f}, w) \Longleftrightarrow w \in \partial\hat{f}(x) \Longleftrightarrow x \in \partial\hat{f}^\bullet(w). \tag{4.3}$$

This means that the linearity domains of $\hat{f}$ are exactly the subdifferentials of $\hat{f}^\bullet$.

Let $\mathbb{S}$ be the collection of all maximal linearity domains of $\hat{f}$ (or maximal subdifferentials of $\hat{f}^\bullet$). Then $\mathbb{S}$ gives a polyhedral division of $\mathrm{dom}(\hat{f})$ into generalized polymatroids. Since $\mathrm{dom}(f)$ is full-dimensional, the dimension of $\mathrm{dom}(\hat{f})$ is equal to $|E| - 1 (= n - 1)$.

For each maximal linearity domain $S \in \mathbb{S}$ of $\hat{f}$ let w be a vector in $(\mathbb{R}^E)^*$ such that $S = \partial\hat{f}^\bullet(w)$ and put $D(\alpha, S) = D(f_{(\alpha)}, w)$ $(\alpha \in I_f)$, which are independent of the choice of w satisfying $S = \partial \hat{f}^{\bullet}(w)$. Define $D(S) = \bigcup\{D(\alpha, S) \mid \alpha \in I_f\}$ and let $f^{D(S)}$ be the restriction of f on $D(S)$. Then we call $f^{D(S)} = (f_{(\alpha)}^{D(\alpha,S)} \mid \alpha \in I_f)$ a *strip of* f *associated with* $S \in \mathbb{S}$, where $f_{(\alpha)}^{D(\alpha,S)}$ is the restriction of $f_{(\alpha)}$ on $D(\alpha, S)$. (See Figure 2 for an example of a strip of an M$^\natural$-concave function.) The collection of the strips $f^{D(S)}$ for all $S \in \mathbb{S}$ is called the *strip decomposition* of f.

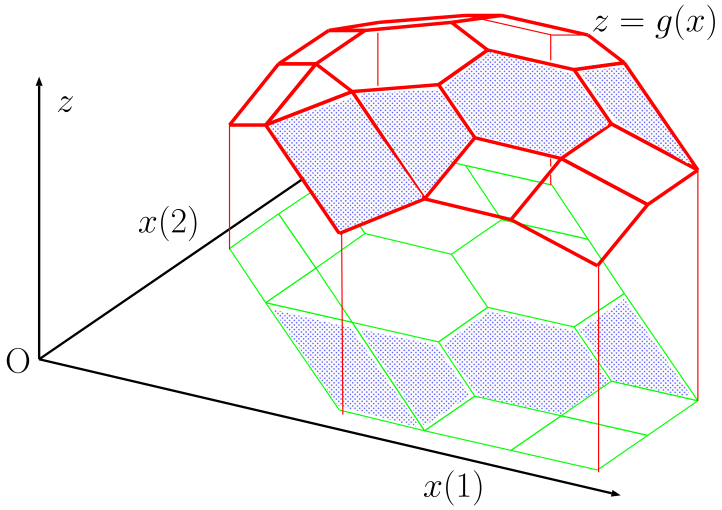

Figure 2: A strip of an M$^\natural$-concave function g indicated by shade.

4.2. Strips viewed from parametric optimization

For any strip $f^{D(S)} = (f_{(\alpha)}^{D(\alpha,S)} \mid \alpha \in I_f)$ of an M$^\natural$-convex function f associated with $S \in \mathbb{S}$ let w be a vector in $(\mathbb{R}^E)^*$ such that $S = \partial \hat{f}^{\bullet}(w)$. Then consider a parametric optimization problem $\mathbf{P}(\lambda)$ with a parameter $\lambda \in \mathbb{R}$ described as follows.

$$\mathbf{P}(\lambda): \quad \text{Minimize } f(x) - \langle w + \lambda \mathbf{1}, x\rangle \ \text{ subject to } \ x \in \mathrm{dom}(f), \tag{4.4}$$

where $\mathbf{1} = \chi_E$ is the n-dimensional vector of all ones. We then have the following theorem.

Theorem 4.1: *For $w \in (\mathbb{R}^E)^*$ chosen as above there exist a finite sequence of values $\lambda_0 = -\infty < \lambda_1 < \cdots < \lambda_p < \lambda_{p+1} = +\infty$ and that of integers $k_0 = g^*(E) < k_1 < \cdots < k_p = f^*(E)$ such that the set $X^*(\lambda)$ of optimal solutions of $\mathbf{P}(\lambda)$ for each $\lambda \in \mathbb{R}$ is given by*

$$X^*(\lambda) = \begin{cases} \bigcup_{\alpha=k_{\ell-1}}^{k_\ell} D(\alpha, S) & \text{if } \lambda = \lambda_\ell \quad (\ell = 1, \cdots, p) \\ D(k_\ell, S) & \text{if } \ \lambda \in (\lambda_\ell, \lambda_{\ell+1}) \quad (\ell = 0, \cdots, p). \end{cases} \tag{4.5}$$

(Proof) Because of the discrete structure of the M$^\natural$-convex function f and the assumption that $\mathrm{dom}(f)$ is bounded, there exists a finite sequence of values $\lambda_0 = -\infty < \lambda_1 < \cdots < \lambda_p < \lambda_{p+1} = +\infty$ such that

1. for each $i = 0, 1, \cdots, p$ Problems $\mathbf{P}(\lambda)$ for all $\lambda \in (\lambda_i, \lambda_{i+1})$ have one and the same optimal solution set, and
2. the set $X^*(\lambda_i)$ of optimal solutions of $\mathbf{P}(\lambda_i)$ for each $i = 1, \cdots, p$ consists of more than one optimal solution and we have $X^*(\lambda_i) \cap X^*(\lambda_{i+1}) = X^*(\lambda)$ for each $i = 1, \cdots, p-1$ and $\lambda \in (\lambda_i, \lambda_{i+1})$.

Hence the optimal solution sets $X^*(\lambda)$ are expressed as (4.5) for a sequence of some integers $k_0 = g^*(E) < k_1 < \cdots < k_p = f^*(E)$ that gives a division of the interval I_f. □

Here it should be noted that values λ_i $(i = 1, \cdots, p)$ depend on the choice of $w \in S$, while the vectors $w + \lambda_i \mathbf{1}$ $(i = 1, \cdots, p)$ are uniquely determined by f and $S \in \mathbb{S}$, because of the assumptions that $\mathrm{dom}(f)$ is full-dimensional and $S \in \mathbb{S}$ is a maximal linearity domain of $\hat{f}$.

5. Valuated Generalized Matroids

In this section we further investigate the structures of strips and their compressions for a class of valuated generalized matroids, which are special M$^\natural$-convex functions defined on the unit hypercube $\{0, 1\}^E$, in more details.

Let $f : \mathbb{Z}^E \to \mathbb{Z} \cup \{+\infty\}$ be an M$^\natural$-convex function such that $\mathrm{dom}(f) = \{0, 1\}^E$, which is called a *valuated generalized matroid*. For any $X \subseteq E$ we often write $f(X)$ as $f(\chi_X)$ and regard f as a function on 2^E in the sequel.

5.1. Compression of valuated generalized matroids and valuated permutohedra

For a valuated generalized matroid $f : 2^E \to \mathbb{Z}$ the compression $\hat{f}$ of f given by (3.3) becomes

$$\hat{f}(x) = \min \left\{ \sum_{\alpha \in [n]} f(Y_\alpha) \,\middle|\, x = \sum_{\alpha \in [n]} \chi_{Y_\alpha},\ \forall \alpha \in [n] : Y_\alpha \in \binom{E}{\alpha} \right\} \quad (x \in \mathbb{Z}^E), \quad (5.1)$$

where $\binom{E}{\alpha} = \{X \subseteq E \mid |X| = \alpha\}$ for $\alpha \in [n]$, and we define $\hat{f}(x) = +\infty$ if the minimum on the right-hand side does not exist for $x \in \mathbb{Z}^E$. Then the effective domain of the compression $\hat{f}$ given by (3.6) is expressed by the following Minkowski sum:

$$\mathrm{dom}(\hat{f}) = \sum_{\alpha\in[n]} \left\{ \chi_Y \,\middle|\, Y \in \binom{E}{\alpha} \right\}. \tag{5.2}$$

Recall that $E = [n]$.

Theorem 5.1: *The effective domain* $\mathrm{dom}(\hat{f})$ *of the compression* $\hat{f}$ *is a permutohedron. Hence the compression* $\hat{f}$ *is a valuated permutohedron, whose linearity domains are sub-permutohedra.*

(Proof) The right-hand side of (5.2) is the Minkowski sum of the sets of the characteristic vectors of bases of uniform matroids $U_{\alpha,n}$ of rank α for $\alpha \in [n]$. Hence it is a base polyhedron whose every extreme point (a greedy solution in the sense of Edmonds [5]) is a permutation $(\pi(1), \cdots, \pi(n)) \in \mathbb{Z}^n$ of $[n]$ and *vice versa*. It follows that (the convex hull of) $\mathrm{dom}(\hat{f})$ is a permutohedron and $\hat{f}$ is a valuated permutohedron. Moreover, for any generic $w \in (\mathbb{R}^E)^*$ and every $\alpha \in [n]$, $D(f_{(\alpha)}, w)$ in (4.2) is a singleton, $\chi_{F(\alpha)}$ say. Then, for each $\alpha = 1, \cdots, n$ we have $F(\alpha - 1) \subset F(\alpha)$ and $\chi_{F(\alpha)} - \chi_{F(\alpha-1)} = \chi_i$ for some $i \in [n]$ $(= E)$ with $F(0) = \emptyset$. Hence sets $F(\alpha)$ for $\alpha = 1, \cdots, n$ form a complete flag

$$\emptyset = F(0) \subset F(1) \subset F(2) \subset \cdots \subset F(n) = [n] \tag{5.3}$$

and it determines a permutation π of $[n]$ with the permutation vector $v_\pi = \sum_{\alpha=1}^{n} \chi_{F(\alpha)}$. It follows that every linearity domain of the compression $\hat{f}$ is a sub-permutohedron. □

5.2. Strips of valuated generalized matroids and flag matroids

For every $S \in \mathbb{S}$ we have the strip $f^{D(S)} = (f_{(\alpha)}^{D(\alpha,S)} \mid \alpha = 0, 1, \cdots, n)$ of f associated with $S \in \mathbb{S}$, which is characterized as follows. We identify χ_X with X for any $X \subseteq [n]$.

Theorem 5.2: *For each* $\alpha = 0, 1, \cdots, n$ *we have a base family* $D(\alpha, S)$ *of a matroid* $([n], \rho_\alpha^S)$ *with a rank function* ρ_α^S *satisfying* $\rho_\alpha^S([n]) = \alpha$*. Moreover, the sequence of* $(\rho_\alpha^S \mid \alpha = 0, 1, \cdots, n)$ *is that of strong maps, i.e., a flag matroid.*

(Proof) The present theorem follows from the definition of the strip $f^{D(S)} = (f_{(\alpha)}^{D(\alpha,S)} \mid \alpha = 0, 1, \cdots, n)$ and the assumption that f is a valuated generalized matroid. □

We call the flag matroid $(\rho_\alpha^S \mid \alpha = 0, 1, \cdots, n)$ the *flag matroid associated with a strip* $S \in \mathbb{S}$ (or a *flag-matroid strip*) of the valuated generalized matroid f. We see from Theorems 5.1 and 5.2 the following.

Theorem 5.3: *Every valuated generalized matroid* f *induces a valuated permutohedron* $\hat{f}$ *by its compression and each flag-matroid strip of* f *corresponds to a maximal linearity domain, a sub-permutohedron, of the induced valuated permutohedron.*

Every valuated generalized matroid is regarded as a valuated permutohedron endowed with *valuated* flag-matroid strips, one for each maximal linearity domain of it.

6. Concluding Remarks

We have introduced the concepts of strip decomposition and compression of $M^\natural$-convex functions. We have examined the structures of valuated generalized-matroids by considering the strip decomposition of a valuated generalized-matroid into flag-matroid strips. The compression of a valuated generalized matroid induces a valuated permutohedron, a special M-convex function of Murota [14]. We thus have a new transformation, which we call the compression, of a valuated generalized-matroid (an $M^\natural$-convex function) to a valuated permutohedron (a special M-convex function). Every Bruhat interval polytope is known to be a sub-permutohedron, due to Tsukerman and Williams [21]. It is interesting to investigate Bruhat interval polytopes from a point of view of the strip decomposition of valuated generalized-matroids and also from a point of view of valuated permutohedra.

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