Feynman propagator on the Minkowski spacetime

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Abstract

In this short note, we explain the construction of the Feynman/anti-Feynman propagators by Gérard and Wrochna and show that they coincide with limits of the resolvent via elementary calculus.

1 Introduction

Feynman/anti-Feynman propagators on Lorentzian manifolds are fundamental object in quantum field theory. In [5], it is proved that Feynman/anti-Feynman parametrix (inverses up to smooth kernels) exist under a non-trapping condition. Recently, on general globally hyperbolic spacetimes, Feynman/anti-Feynman propagators are constructed by scattering or spectral techniques on L^2-based spaces ([2], [3], [6], [7]). In particular, Dereziński and Siemssen propose a method of construction for Feynman/anti-Feynman propagators based on the essential self-adjointness of wave operators:

Conjecture 1.1. *[4, Conjecture 8.3] For a large class of asymptotically stationary spacetimes (M, g), the wave operator P is essentially self-adjoint on $C_c^\infty(M)$ and the Feynman propagator defined in [4] coincides with a limit of its resolvent at the real line.*

In this note, we consider the wave operator

$$P_0 = \partial_{x_1}^2 - \sum_{j=2}^{n+1} \partial_{x_j}^2 = \partial_t^2 - \Delta_y \quad \text{on} \quad \mathbb{R}_x^{n+1} = \mathbb{R}_t \times \mathbb{R}_y^n.$$

Moreover, we denote its symbol by p_0:

$$p_0(\xi) = -\xi_1^2 + \sum_{j=2}^{n+1} \xi_j^2.$$

The purpose of this note is to explain how to construct the Feynman-propagator in [6] and [7] for our model operator P_0 and to show the result in [11] by a more elementary method. Since P_0 is constant coefficient and static, all the arguments are simpler and more elementary than on curved spacetimes.

First, we refer to the essential self-adjointness for P_0. Although the proof for essential self-adjointness of wave operators on curved spacetimes is not trivial (see [12] or [9]), the essential self-adjointness of P_0 is easily proved in this case:

Lemma 1.2. *P_0 is essentially self-adjoint on $C_c^\infty(\mathbb{R}^{n+1})$.*

Proof. It suffices to prove that

$$(P_0 \pm i)u = 0, \quad u \in L^2(\mathbb{R}^{n+1})$$

implies $u = 0$. By the Fourier transform, this is equivalent to the fact that $(p_0(\xi) \pm i)\hat{u}(\xi) = 0$ and $u \in L^2(\mathbb{R}^{n+1})$ imply $u = 0$. Since $p_0(\xi)$ is real-valued and $\pm i$ is purely imaginary, dividing $(p_0(\xi) \pm i)\hat{u}(\xi) = 0$ by $(p_0(\xi) \pm i)$, we obtain $\hat{u} = 0$. This implies $u = 0$. □

We denote the unique self-adjoint extension of P_0 by the same symbol P_0. The main result of this note is the following theorem.

Theorem 1.3. *For $s > \frac{1}{2}$, the limits*

$$R_\pm := \lim_{\varepsilon \to 0, \varepsilon > 0} (P_0 + m_0^2 \mp i\varepsilon)^{-1}$$

exist in $B(\langle x\rangle^s L^2(\mathbb{R}^{n+1}), \langle x\rangle^{-s} L^2(\mathbb{R}^{n+1}))$. Moreover, R_+ coincides with the anti-Feynman propagator defined in [6] and [7].

Remark 1.4. The convention of the Feynman/anti-Feynman propagators in [6] and [7] are opposite to in physics.

This result is also proved in [2] for more general static spacetimes by using the theory of dissipative operators. In this paper, we use the Fourier analysis and the spectral theory instead.

One of important properties of Feynman/anti-Feynman propagators is the following asymptotics: For $m_0 > 0$ and $(P_0 + m_0^2)u = f$ with $f \in \mathcal{S}(\mathbb{R}^{n+1})$,

$$u \text{ is Feynman} \Leftrightarrow u(t) \sim e^{\pm it\sqrt{-\Delta_y + m_0^2}} b_{\pm,\pm} \quad \text{as} \quad t \to \pm\infty,$$

$$u \text{ is anti-Feynman} \Leftrightarrow u(t) \sim e^{\mp it\sqrt{-\Delta_y + m_0^2}} b_{\mp,\pm} \quad \text{as} \quad t \to \pm\infty.$$

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2 Limiting absorption principle

In this section, we prove the existence of the powers of the outgoing/incoming resolvent for P_0 away from the zero energy. The main result of this section is the following theorem.

Theorem 2.1. *Let $k \geq 1$ be an integer, $I \Subset \mathbb{R} \setminus \{0\}$ be a bounded interval and $s > k/2$. Then*

$$\sup_{z \in I_\pm} \|\langle x\rangle^{-s} (P_0 - z)^{-k} \langle x\rangle^{-s}\|_{B(L^2(\mathbb{R}^{n+1}))} < \infty, \tag{2.1}$$

where $I_\pm = \{z \in \mathbb{C} \,|\, \mathrm{Re}\, z \in I, \pm \mathrm{Im}\, z > 0\}$. Moreover, the limits

$$\langle x\rangle^{-s} (P_0 - \lambda \pm i0)^{-k} \langle x\rangle^{-s} = \lim_{\varepsilon \to +0} \langle x\rangle^{-s} (P_0 - \lambda \pm i\varepsilon)^{-k} \langle x\rangle^{-s} \tag{2.2}$$

exist uniformly in $\lambda \in I$.

Remark 2.2. Similar estimates hold on an ultrastatic Lorentizan manifold $M = \mathbb{R} \times Y$, where Y is compact [1, Theorem C.5].

Remark 2.3. The case $k = 1$ is proved in [10].

Proof. Set $\tilde{x} = (-x_1, x_2..., x_{n+1})$ and consider the differential operator

$$A = \tilde{x} \cdot D_x (I - \Delta)^{-1} + (I - \Delta)^{-1} D_x \cdot \tilde{x}$$

with the domain $C_c^\infty(\mathbb{R}^{n+1})$. By using Nelson's commutator theorem with a conjugate operator $-\Delta + |x|^2 + 1$, it turns out that A is essentially self-adjoint on $C_c^\infty(\mathbb{R}^{n+1})$. We denote its unique self-adjoint extension by the same symbol A. A simple calculation gives

$$[P_0, iA] = -2\Delta(I - \Delta)^{-1} = \mathcal{F}^{-1}\left(\frac{2|\xi|^2}{1 + |\xi|^2}\mathcal{F}\right).$$

In the following, we see that $[P, iA]$ satisfy the Mourre estimate except at 0, that is,

$$E_I(P_0)[P_0, iA]E_I(P_0) \geq \frac{4a}{1 + a} E_I(P_0), \tag{2.3}$$

where $E_I(P_0)$ is the spectral projection of P_0 to I and χ_I is the characteristic function of $I \subset \mathbb{R}$. Fix $I \Subset \mathbb{R} \setminus \{0\}$. We set $a = \inf\{|\lambda| \mid \lambda \in I\} > 0$. Then for $\xi \in p_0^{-1}(I)$, we have

$$|\xi|^2 = \sum_{j=1}^{k} |\xi_j|^2 + \sum_{j=k+1}^{d} |\xi_j|^2 \geq a.$$

Hence we obtain

$$\chi_I(p_0(\xi)) \frac{4|\xi|^2}{1 + |\xi|^2} \chi_I(p_0(\xi)) \geq \frac{4a}{1 + a} \chi_I(p_0(\xi))$$

where χ_I is the characteristic function for I. This proves (2.3). Moreover since $\mathrm{ad}_A^k P_0$ are bounded operators for all $k \geq 1$, it follows that $P_0 \in C^k(A)$ for all $k \geq 1$. By the results in [8], we obtain (2.1) and (2.2).

$\square$

3 Feynman propagator

In this section, we construct the Feynman/anti-Feynman propagators along the strategy in [6] and [7].

3.1 Diagonalized operator

In this subsection, we shall construct the Feynman/anti-Feynman propagators for the diagonalized operator P^{ad} defined below.

Notation

For $m_0 > 0$, we set

$$H^{ad} = \begin{pmatrix} \sqrt{-\Delta + m_0^2} & 0 \\ 0 & -\sqrt{-\Delta + m_0^2} \end{pmatrix}, \ P^{ad} = D_t - H^{ad}$$

$$\mathcal{U}^{ad}(t) = e^{itH^{ad}} = \begin{pmatrix} e^{it\sqrt{-\Delta+m_0^2}} & 0 \\ 0 & e^{-it\sqrt{-\Delta+m_0^2}} \end{pmatrix}, \quad \pi^+ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \pi^- = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Moreover, for $\frac{1}{2} < \nu < 1$, we denote

$$\mathcal{H}^m = H^m \oplus H^m, \ \mathcal{Y}^{ad,m} = \langle t \rangle^{-\nu} L^2(\mathbb{R}; \mathcal{H}^m),$$
$$\mathcal{X}^{ad,m} = \{u \in C(\mathbb{R}; \mathcal{H}^m) \mid P^{ad}u \in \mathcal{H}^m\}, \ \|u\|_m^2 = \|\rho_0^{ad} u\|_{\mathcal{H}^m}^2 + \|P^{ad}u\|_{\mathcal{Y}^{ad,m}}^2.$$

For $u = (u_0, u_1) \in C(\mathbb{R}; \mathcal{D}'(\mathbb{R}^n) \oplus \mathcal{D}'(\mathbb{R}^n))$, we denote

$$\rho_t^{ad} u = (u_0(t), u_1(t)) \in \mathcal{D}'(\mathbb{R}^n) \oplus \mathcal{D}'(\mathbb{R}^n).$$

Construction of propagators

Proposition 3.1. *(i) There exist bounded linear operators $G_\pm^{ad} : L^1(\mathbb{R}; \mathcal{H}^m) \to C(\mathbb{R}; \mathcal{H}^m)$ such that*

$$P^{ad} G_\pm^{ad} = Id_{L^1(\mathbb{R};\mathcal{H}^m)}$$

and for $f \in L^1(\mathbb{R}; \mathcal{H}^m)$,

$$G_+^{ad} f(t) \to 0 \quad as \quad t \to -\infty, \quad G_-^{ad} f(t) \to 0 \quad as \quad t \to +\infty.$$

The operator $G_\pm^{ad}$ are called retarded/advanced propagators for P^{ad} respectively.
(ii) There exist bounded linear operators $G_{F/AF}^{ad} : L^1(\mathbb{R}; \mathcal{H}^m) \to C(\mathbb{R}; \mathcal{H}^m)$ such that

$$P^{ad} G_{F/AF}^{ad} = Id_{L^1(\mathbb{R};\mathcal{H}^m)}$$

and for $f \in L^1(\mathbb{R}; \mathcal{H}^m)$,

$$\pi_\pm G_{F/AF}^{ad} f(t) \to 0 \quad as \quad t \to -\infty, \quad \pi_\mp G_{F/AF}^{ad} f(t) \to 0 \quad as \quad t \to +\infty.$$

The operator $G_{F/AF}^{ad}$ are called Feynman/anti-Feynman propagators for P^{ad} respectively.

Proof. Set

$$(G_\pm^{ad} f)(t) = i \int_{\mp\infty}^t \mathcal{U}^{ad}(t-s) f(s) ds \quad \text{for} \quad f = (f_0, f_1) \in L^1(\mathbb{R}_t; \mathcal{H}^m), \tag{3.1}$$

$$(G_{F/AF}^{ad} f)(t) = (i \int_{\mp\infty}^t e^{i(t-s)\sqrt{-\Delta+m^2}} f_0(s) ds, i \int_{\pm\infty}^t e^{-i(t-s)\sqrt{-\Delta+m^2}} f_1(s) ds) \tag{3.2}$$

Since

$$\sup_{t \in \mathbb{R}} \|\mathcal{U}^{ad}(t)\|_{B(\mathcal{H}^m)} < \infty,$$

we have $G_\pm^{ad}, G_{F/AF}^{ad} \in B(L^1(\mathbb{R}; \mathcal{H}^m), C(\mathbb{R}; \mathcal{H}^m))$. The rest of the properties can be easily checked.

□

Propagators as inverses between some function spaces

We shall show that $G^{ad}_{\pm}$ and $G^{ad}_{F/AF}$ can be realized as inverses between some Banach spaces. Such observation is useful for constructing Feynman/anti-Feynman propagators on curved spacetimes. First, we define the boundary data at $t = \pm\infty$.

Definition 1. We define

$$\rho^{ad}_{out/in} = s - \lim_{t\to\pm\infty} \mathcal{U}^{ad}(-t)\rho^{ad}_t,$$
$$\rho^{ad}_{out/in}u = \lim_{t\to\pm\infty}(e^{-it\sqrt{-\Delta+m^2}}u_0(t), e^{it\sqrt{-\Delta+m^2}}u_1(t)).$$

We also define the Feynman/anti Feynman scattering data:

$$\rho^{ad}_{F/AF} := \pi^{\pm}\rho^{ad}_{out} + \pi^{\mp}\rho^{ad}_{in}.$$

The next lemma assures that the boundary maps are well-defined for the function belonging to $\mathcal{X}^{ad,m}$.

Lemma 3.2. *We have*

$$\rho^{ad}_{out/in}, \rho^{ad}_{F/AF} \in B(\mathcal{X}^{ad,m}, \mathcal{H}^m).$$

Proof. It suffices to prove $\rho^{ad}_{out/in} \in B(\mathcal{X}^{ad,m}, \mathcal{H}^m)$. Let $u = (u_0, u_1) \in \mathcal{X}^{ad,m}$ and $f = (f_0, f_1) := Pu \in \mathcal{Y}^{ad,m} \subset L^1(\mathbb{R}; \mathcal{H}^m)$. By the Duhamel formula, we have

$$\rho^{ad}_t u = u(t) = e^{itH^{ad}}u(0) + i\int_0^t e^{i(t-s)H^{ad}}f(s)ds. \tag{3.3}$$

This implies

$$\rho^{ad}_{out/in}u = \lim_{t\to\pm\infty} e^{-itH^{ad}}\rho^{ad}_t u = u(0) + i\int_0^{\pm\infty} e^{-isH^{ad}}f(s)ds. \tag{3.4}$$

This representation gives $\rho^{ad}_{out/in} \in B(\mathcal{X}^{ad,m}, \mathcal{H}^m)$. □

Now we introduce the out/in and Feynman/anti-Feynman function spaces.

Definition 2. Set

$$\mathcal{X}^{ad,m}_{out/in} = \{u \in \mathcal{X}^{ad,m} \mid \rho^{ad}_{in/out}u = 0\}, \quad \mathcal{X}^{ad,m}_{F/AF} = \{u \in \mathcal{X}^{ad,m} \mid \rho^{ad}_{AF/F}u = 0\}.$$

Proposition 3.3. (i) *The operator*

$$P^{ad} : \mathcal{X}^{ad,m}_{out/in} \to \mathcal{Y}^{ad,m}$$

is invertible and its inverse is equal to $G^{ad}_{\pm}$.
(ii) *The operator*

$$P^{ad} : \mathcal{X}^{ad,m}_{F/AF} \to \mathcal{Y}^{ad,m}$$

is invertible and its inverse is equal to $G^{ad}_{F/AF}$.

Proof. Let $u \in \mathcal{X}^{ad,m}$ and set $f = P^{ad}u$. The relation (3.4) imply that $\rho^{ad}_{in/out}u = 0$ is equivalent to

$$u(0) = -\int_{\mp\infty}^{0} e^{-isH^{ad}} f(s)ds.$$

By (3.3) and (3.1), the above identity is also equivalent to $u = G^{ad}_{\pm} f$. This proves (i).

Next, we show (ii). For $u = (u_0, u_1) \in \mathcal{X}^{ad,m}$ with $f = (f_0, f_1) = P^{ad}u \in \mathcal{Y}^{ad,m}$, we have $\rho^{ad}_{AF/F}u = 0$ is equivalent to

$$u_0(0) = i\int_{\mp\infty}^{0} e^{-is\sqrt{-\Delta+m^2}} f_0(s)ds \in H^m,\ u_1(0) = i\int_{\pm\infty}^{0} e^{is\sqrt{-\Delta+m^2}} f_1(s)ds \in H^m.$$

By (3.3) and and (3.2), it is also equivalent to $u = G^{ad}_{F/AF} f$. $\square$

3.2 Propagators for P_0

Now we construct Feynman/anti-Feynman propagators for P_0.

Notation

Let $m_0 > 0$. We set

$$H = \begin{pmatrix} 0 & 1 \\ A^2 & 0 \end{pmatrix}, \quad P = \partial_t^2 - \Delta_y + m_0^2 = P_0 + m_0^2,$$
$$T = -\frac{i}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ A & A \end{pmatrix} A^{-\frac{1}{2}}, \quad T^{-1} = \frac{i}{\sqrt{2}} A^{-\frac{1}{2}} \begin{pmatrix} A & 1 \\ -A & 1 \end{pmatrix}, \quad A = \sqrt{-\Delta_y + m_0^2}.$$

We denote

$$\mathcal{U}(t) = e^{itH},$$

where we note H is not self-adjoint and hence $\mathcal{U}(t)$ is not unitary. Moreover, for $u \in C^1(\mathbb{R}; \mathcal{D}'(\mathbb{R}^n))$, we set

$$\rho_t u = (u(t), D_t u(t)), \quad (\rho u)(t) = \rho_t u, \quad \pi_j(u_0, u_1) = u_j.$$

Then we have

$$P = \pi_1 \circ (D_t - H) \circ \rho.$$

For $\frac{1}{2} < \nu < 1$, we define function spaces $\mathcal{X}^m, \mathcal{Y}^m$ by

$$\mathcal{Y}^m = \langle t\rangle^{-\nu} L^2(\mathbb{R}; H^m),\ \mathcal{X}^m = \{u \in C(\mathbb{R}; H^{m+1}) \cap C^1(\mathbb{R}; H^m) \mid Pu \in \mathcal{Y}^m\},$$
$$\|u\|^2_{\mathcal{X}^m} := \|u\|^2_m := \|u(0)\|^2_{H^{m+1}} + \|\partial_t u(0)\|^2_{H^m} + \|Pu\|^2_{\mathcal{Y}^m}.$$

Moreover, we set

$$c^{\pm}_{free} = \frac{1}{2}\begin{pmatrix} 1 & \pm A^{-1} \\ \pm A & 1 \end{pmatrix}$$

Representation of the solution to $Pu = f$

Lemma 3.4. *Suppose that $u \in C(\mathbb{R}; H^m)$ satisfies $Pu \in L^1(\mathbb{R}; H^m)$. Then we have*

$$\begin{aligned} u(t) =& \frac{1}{2}A^{-1}e^{itA}(Au(0) + D_t u(0) - i\int_0^t e^{-isA}f(s)ds) \\ &+ \frac{1}{2}A^{-1}e^{-itA}(Au(0) - D_t u(0) + i\int_0^t e^{isA}f(s)ds) \end{aligned} \tag{3.5}$$

Proof. By the Duhamel formula, we have

$$u(t) =(\cos tA)u(0) + \frac{\sin tA}{A}(\partial_t u)(0) + \int_0^t \frac{\sin(t-s)A}{A}f(s)ds.$$

Rewriting this formula, we obtain (3.5). $\square$

Now we set

$$b_\pm = b_\pm(t) = \frac{1}{2}(Au(0) \pm D_t u(0) \mp i\int_0^t e^{-isA}f(s)ds). \tag{3.6}$$

Next lemma shows that $c^\pm_{free}$ is the spectral projections of H in the energy space.

Lemma 3.5. *For $u \in \mathcal{X}^m$, we have*

$$c^\pm_{free}\begin{pmatrix} u(t) \\ D_t u(t) \end{pmatrix} = \begin{pmatrix} A^{-1}e^{\pm itA}b_\pm \\ \pm e^{\pm itA}b_\pm \end{pmatrix}$$

Proof. We observe $u(t) = A^{-1}e^{itA}b_+ + A^{-1}e^{-itA}b_-$ and $D_t u(t) = e^{itA}b_+ - e^{-itA}b_-$. Thus,

$$\begin{aligned} c^\pm_{free}\begin{pmatrix} u(t) \\ D_t u(t) \end{pmatrix} =& \frac{1}{2}\begin{pmatrix} 1 & \pm A^{-1} \\ \pm A & 1 \end{pmatrix}\begin{pmatrix} A^{-1}e^{itA}b_+ + A^{-1}e^{-itA}b_- \\ e^{itA}b_+ - e^{-itA}b_- \end{pmatrix} \\ =& \begin{pmatrix} A^{-1}e^{\pm itA}b_\pm \\ \pm e^{\pm itA}b_\pm \end{pmatrix}. \end{aligned}$$

$\square$

Construction of propagators

Proposition 3.6. (i) *There exist bounded linear operators $G_\pm : L^1(\mathbb{R}; H^m) \to C(\mathbb{R}; H^{m+1})$ such that*

$$PG_\pm = Id_{L^1(\mathbb{R};H^m)}$$

and for $f \in L^1(\mathbb{R}; H^m)$,

$$G_\pm f(t) \to 0 \quad as \quad t \to \mp\infty.$$

The operator $G_\pm$ are called retarded/advanced propagators for P respectively.
(ii) *There exist bounded linear operators $G_{F/AF} : L^1(\mathbb{R}; H^m) \to C(\mathbb{R}; H^{m+1})$ such that*

$$PG_{F/AF} = Id_{L^1(\mathbb{R};H^m)}$$

and for $f \in L^1(\mathbb{R}; H^m)$,

$$c^\pm_{free}\rho G_{F/AF}f(t) \to 0 \quad as \quad t \to -\infty, \quad c^\mp_{free}\rho G_{F/AF}f(t) \to 0 \quad as \quad t \to +\infty.$$

The operator $G_{F/AF}$ are called Feynman/anti-Feynman propagators for P respectively.

Proof. For $f(t) \in L^1(\mathbb{R}_t; H^m)$, we set

$$\begin{aligned}(G_\pm f)(t) &= -\frac{i}{2}\int_{\mp\infty}^t A^{-1}(e^{i(t-s)A} - e^{-i(t-s)A})f(s)ds,\\ (G_{F/AF}f)(t) &= \frac{-i}{2}A^{-1}\int_{\mp\infty}^t e^{i(t-s)A}f(s)ds + \frac{i}{2}A^{-1}\int_{\pm\infty}^t e^{-i(t-s)A}f(s)ds.\end{aligned}$$

(i) and (ii) are easily proved.

□

Remark 3.7. We can write

$$G_\pm f(t) = -i\pi_0\int_{\mp\infty}^t \mathcal{U}(t-s)\begin{pmatrix}0\\ f(s)\end{pmatrix}ds.$$

Propagators as inverses

Similar to the last section, we shall show that $G_\pm$ and $G_{F/AF}$ are realized as inverses between some Banach spaces. We introduce boundary maps at $t = \pm\infty$.

Definition 3. We define

$$\rho_{out/in} = s - \lim_{t\to\pm\infty} T^{-1}\mathcal{U}(-t)\rho_t$$

We also define the Feynman/anti Feynman scattering data:

$$\rho_{F/AF} := \pi^\pm\rho_{out} + \pi^\mp\rho_{in},$$

where $\pi^\pm$ is defined in the last section.

Now we set

$$b_{+,\pm} = Au(0) + D_tu(0) - i\int_0^{\pm\infty} e^{-isA}f(s)ds,\ b_{-,\pm} = Au(0) - D_tu(0) + i\int_0^{\pm\infty} e^{isA}f(s)ds.$$

Lemma 3.8. *We have*

$$\begin{aligned}\rho_{out/in}u &= \frac{i}{\sqrt{2}}A^{-\frac12}\lim_{t\to\pm\infty}\begin{pmatrix}e^{-itA}(Au(t) + D_tu(t))\\ e^{itA}(-Au(t) + D_tu(t))\end{pmatrix} = i\sqrt{2}A^{-\frac12}\begin{pmatrix}b_{+,\pm}\\ -b_{-,\pm}\end{pmatrix}\\ \rho_{F/AF}u &= i\sqrt{2}A^{-\frac12}\begin{pmatrix}b_{+,\pm}\\ -b_{-,\mp}\end{pmatrix}\end{aligned}$$

Moreover,

$$\rho_{out/in}, \rho_{F/AF} \in B(\mathcal{X}^m, H^{m+\frac12}\oplus H^{m+\frac12}).$$

Proof. The first formula follows from the following calculation:

$$\begin{aligned}\rho_{out/in}u &= \lim_{t\to\pm\infty} T^{-1}\begin{pmatrix}\cos tAu(t) - iA^{-1}\sin tA(D_tu)(t)\\ -iA\sin tAu(t) + \cos tA(D_tu)(t)\end{pmatrix},\\ &\pm Au(t) + D_tu(t) = \pm 2b_\pm(t),\end{aligned}$$

where $b_\pm$ are defined in (3.6). The mapping properties of $\rho_{out/in}, \rho_{F/AF}$ can be proved by using $\|e^{itA}\|_{H^m\to H^m} = 1$.

□

Corollary 3.9. *For $u \in \mathcal{X}^m$, we have*

$$\rho_{out}u = 0 \Leftrightarrow b_{+,-} = b_{-,-} = 0, \quad \rho_{in}u = 0 \Leftrightarrow b_{+,+} = b_{-,+} = 0,$$
$$\rho_F u = 0 \Leftrightarrow b_{+,+} = b_{-,-} = 0, \quad \rho_{AF}u = 0 \Leftrightarrow b_{+,-} = b_{-,+} = 0,$$

Definition 4. Set

$$\mathcal{X}^m_{out/in} = \{u \in \mathcal{X}^m \mid \rho_{in/out}u = 0\}, \quad \mathcal{X}^m_{F/AF} = \{u \in \mathcal{X}^m \mid \rho_{AF/F}u = 0\}.$$

Proposition 3.10. (i) *The operator*

$$P : \mathcal{X}^m_{out/in} \to \mathcal{Y}^m$$

is invertible and its inverse is equal to $G_\pm$.
(ii) *The operator*

$$P : \mathcal{X}^m_{F/AF} \to \mathcal{Y}^m$$

is invertible and its inverse is equal to $G_{F/AF}$.

Proof. This proposition follows from the representation (3.5) and Corollary 3.9. □

Connection to the diagonalized operator

Lemma 3.11. *We have*

$$\mathcal{U}(t) = T\mathcal{U}^{ad}(t)T^{-1}$$

Proof. We recall

$$H_{ad} = \begin{pmatrix} A & 0 \\ 0 & -A \end{pmatrix}, \quad \mathcal{U}^{ad}(t) = e^{itH^{ad}} = \begin{pmatrix} e^{itA} & 0 \\ 0 & e^{-itA} \end{pmatrix}.$$

Thus we have

$$\begin{aligned} T\mathcal{U}^{ad}(t)T^{-1} =& \frac{1}{2}\begin{pmatrix} 1 & -1 \\ A & A \end{pmatrix}\begin{pmatrix} A^{-1}e^{itA} & 0 \\ 0 & A^{-1}e^{-itA} \end{pmatrix}\begin{pmatrix} A & 1 \\ -A & 1 \end{pmatrix} \\ =& \frac{1}{2}\begin{pmatrix} A^{-1}e^{itA} & -A^{-1}e^{-itA} \\ e^{itA} & e^{-itA} \end{pmatrix}\begin{pmatrix} A & 1 \\ -A & 1 \end{pmatrix} \\ =& \frac{1}{2}\begin{pmatrix} e^{itA} + e^{-itA} & A^{-1}(e^{itA} - e^{-itA}) \\ A(e^{itA} - e^{-itA}) & e^{itA} + e^{-itA} \end{pmatrix} \\ =& \begin{pmatrix} \cos tA & iA^{-1}\sin tA \\ iA\sin tA & \cos tA \end{pmatrix} \end{aligned}$$

On the other hand, we note

$$\begin{aligned} H^{2k} =& A^{2k}I, \quad H^{2k+1} = A^{2k}H = A^{2k+1}\begin{pmatrix} 0 & A^{-1} \\ A & 0 \end{pmatrix}, \\ \mathcal{U}(t) =& \cos tH + i\sin tH = \begin{pmatrix} \cos tA & iA^{-1}\sin tA \\ iA\sin tA & \cos tA \end{pmatrix}. \end{aligned}$$

□

4 Resolvent and propagators

In this section, we shall show that the outgoing resolvent for P_0 coincides with the anti-Feynman propagator constructed in the last section. For $\varepsilon > 0$, set

$$A = \sqrt{-\Delta_y + m_0^2}, \quad A(\varepsilon) = \sqrt{-\Delta_y + m_0^2 - i\varepsilon},$$

where we choose the branch of $A(\varepsilon)$ as Im $\sqrt{-\Delta_y + m_0^2 - i\varepsilon} \leq 0$. Explicitly, we can write

$$\begin{aligned} A(\varepsilon) =& \sqrt{\frac{A^2}{2} + \frac{\sqrt{A^4+\varepsilon}}{2}} - i\sqrt{-\frac{A^2}{2} + \frac{\sqrt{A^4+\varepsilon}}{2}} \\ =& A_1(\varepsilon) - iA_2(\varepsilon) \end{aligned}$$

with $A_2(\varepsilon) \geq 0$. Mimicking the definition of the anti-Feynman propagator, we define

$$(G_{AF}(\varepsilon)f)(t) = \frac{-i}{2}A(\varepsilon)^{-1}\int_{\infty}^{t} e^{i(t-s)A(\varepsilon)}f(s)ds + \frac{i}{2}A(\varepsilon)^{-1}\int_{-\infty}^{t} e^{-i(t-s)A(\varepsilon)}f(s)ds.$$

Proposition 4.1. *For $\varepsilon > 0$, we denote the resolvent of P_0 by $R(\varepsilon)$: $R(\varepsilon) = (P_0 + m_0^2 - i\varepsilon)^{-1}$. Then we have $G_{AF}(\varepsilon) = R(\varepsilon)$ on $L^2(\mathbb{R}^{n+1})$.*

Proof. Since P_0 is essentially self-adjoint on $C_c^\infty(\mathbb{R}^{n+1})$, it suffices to prove

$$f = G_{AF}(\varepsilon)(P + m_0^2 - i\varepsilon)f, \quad f = (P + m_0^2 - i\varepsilon)G_{AF}(\varepsilon)f, \quad f \in C_c^\infty(\mathbb{R}^{n+1})$$

and $G_{AF}(\varepsilon) \in B(L^2(\mathbb{R}^{n+1}))$. The above two identities can be proved by a simple calculation and by the integration by parts. Hence we shall prove $G_{AF}(\varepsilon) \in B(L^2(\mathbb{R}^{n+1}))$. We denote $A_\eta = \sqrt{|\eta|^2 + m_0^2}$ and

$$\widehat{G_{AF}(\varepsilon)} = \mathcal{F}_{y\to\eta}G_{AF}(\varepsilon)\mathcal{F}_{\eta\to y}, \; A(\varepsilon,\eta) = \sqrt{|\eta|^2 + m_0^2 - i\varepsilon} = A_1(\varepsilon,\eta) - iA_2(\varepsilon,\eta).$$

By Plancherel's theorem, it suffices to prove $\widehat{G_{AF}(\varepsilon)} \in B(L^2(\mathbb{R}^{n+1}))$. We observe

$$A_2(\varepsilon,\eta) = \frac{\frac{\varepsilon}{4}}{\sqrt{\frac{A_\eta^2}{2} + \frac{\sqrt{A_\eta^4+\varepsilon}}{2}}} \geq \frac{\varepsilon}{4}A_\eta^{-1}. \tag{4.1}$$

For $f \in L^2(\mathbb{R}^{n+1})$, set $u = \widehat{G_{AF}(\varepsilon)}f$. Then we have

$$\begin{aligned} u(A_\eta t) =& \frac{-i}{2}A(\varepsilon,\eta)^{-1}\int_{\infty}^{A_\eta t} e^{i(A_\eta t-s)A(\varepsilon,\eta)}f(s)ds \\ &+ \frac{i}{2}A(\varepsilon,\eta)^{-1}\int_{-\infty}^{A_\eta t} e^{-i(A_\eta t-s)A(\varepsilon,\eta)}f(s)ds \\ =& \frac{-i}{2}\int_{\infty}^{t} e^{i(t-s)A_\eta A(\varepsilon,\eta)}f(A_\eta s)ds + \frac{i}{2}\int_{-\infty}^{t} e^{-i(t-s)A_\eta A(\varepsilon,\eta)}f(A_\eta s)ds. \end{aligned}$$

Since A_1 is real-valued, the inequality (4.1) implies

$$\begin{aligned}\|u(A_\eta t)\|_{L^2(\mathbb{R}_t)} \leq & \frac{1}{2}\| \int_\infty^t e^{\frac{\varepsilon(t-s)}{4}} f(A_\eta s)ds\|_{L^2(\mathbb{R}_t)} + \frac{1}{2}\| \int_\infty^t e^{\frac{-\varepsilon(t-s)}{4}} f(A_\eta s)ds\|_{L^2(\mathbb{R}_t)} \\ \leq & C_\varepsilon A_\eta^{-\frac{1}{2}} \|f\|_{L^2(\mathbb{R}_t)}.\end{aligned}$$

Thus we obtain $\|u\|_{L^2(\mathbb{R}_t)} \leq C_\varepsilon \|f\|_{L^2(\mathbb{R})}$ and hence $\|u\|_{L^2(\mathbb{R}^{n+1})} \leq C_\varepsilon \|f\|_{L^2(\mathbb{R}^{n+1})}$.

□

Lemma 4.2. *For $s > \frac{1}{2}$ and $f \in \langle t \rangle^{-s} L^2(\mathbb{R}^{n+1}) \subset L^1(\mathbb{R}; L^2(\mathbb{R}^n))$, we have $G_{AF}(\varepsilon)f \to G_{AF}f$ in $\langle t \rangle^s L^2(\mathbb{R}^{n+1})$ as $\varepsilon \to 0$.*

Proof. By the spectral theorem, for $t \geq 0$, we have

$$e^{-itA(\varepsilon)} \to e^{-itA}, \quad A(\varepsilon)^{-1} \to A^{-1}$$

strongly in $L^2(\mathbb{R}^n_y)$. This implies $G_{AF}(\varepsilon)f \to G_{AF}f$ in $\langle t \rangle^s L^2(\mathbb{R}^{n+1})$.

□

Now Theorem 1.3 can be proved by the two result above.

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