Stationary scattering theory for repulsive Hamiltonians

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1 Introduction

We discuss stationary scattering theory for repulsive Hamiltonians with *short-range* perturbations. For such Hamiltonians, it is well-known that their spectrum is purely absolutely continuous in $\mathbb{R}$ and time-dependent wave operators exist and are complete, see [1]. However stationary scattering theory for repulsive Hamiltonians is not well studied even for short-range case, as far as the speaker knows.

In the present paper, we deal with several topics on stationary scattering theory. The first one is existence and completeness of stationary wave operators. To construct stationary wave operators we need radiation condition bounds for limiting resolvents stated as Corollary 2.5 below. The second one is unitarity of scattering matrix. The last one is a characterization of asymptotic behaviors of generalized eigenfunctions with minimal growth order at infinity. We characterize their leading term by outgoing/incoming spherical waves. We note these topics are not dealt with in [1].

1.1 Basic setting

We consider the following repulsive Hamiltonians.

$$H = \tfrac{1}{2}p^2 - \tfrac{1}{2}|x|^{2\alpha} + q(x) \quad \text{on } L^2(\mathbb{R}^d),$$

where $d \in \mathbb{N} = \{1, 2, \ldots\}$, $\alpha \in (0, 1)$, $p = -\mathrm{i}\partial$ and q is a real-valued function.

Let us impose on q more precise condition. We choose and fix a cut-off function $\chi \in C^\infty(\mathbb{R})$ which satisfies

$$\chi = \chi(s) = \begin{cases} 1 & s \le 1, \\ 0 & s \ge 2, \end{cases} \qquad \tfrac{\mathrm{d}}{\mathrm{d}s}\chi = \chi' \le 0. \tag{1.1}$$

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By using the function χ we introduce the function $r \in C^\infty(\mathbb{R}^d)$ by

$$r = r(x) = \chi(|x|) + \bigl(1 - \chi(|x|)\bigr)|x|.$$

Now we define our *escape function* $f \in C^\infty(\mathbb{R}^d)$ as

$$f = f(x) = \frac{r^{1-\alpha} - 1}{1 - \alpha} + 1.$$

Such a choice of f is based on the scattering trajectory of classical particles subject to the repulsive electric field, see [4, 5]. We note $f \geq 1$ on $\mathbb{R}^d$.

Condition 1.1. The perturbation q is a real-valued function belonging to $C^1(\mathbb{R}^d)$. Moreover there exist $\rho, C_k > 0$ for $k = 0, 1$ such that

$$|\partial^k q| \leq C_k f^{-1-k-\rho}.$$

Under Condition 1.1 it follows by the Faris–Lavine theorem (cf. [8]) that H is essentially self-adjoint on $C_0^\infty(\mathbb{R}^d)$. We denote its self-adjoint extension by the same letter for simplicity.

Next we introduce the Agmon–Hörmander spaces associated with f. We let $F(S)$ be the sharp characteristic function of a general subset $S \subseteq \mathbb{R}^d$, and set

$$F_n = F\bigl(\bigl\{x \in \mathbb{R}^d \,\big|\, 2^n \leq f(x) < 2^{n+1}\bigr\}\bigr) \quad \text{for } n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}.$$

Then define the Agmon–Hörmander spaces $\mathcal{B}$, $\mathcal{B}^*$ and $\mathcal{B}_0^*$ as

$$\mathcal{B} = \Bigl\{\psi \in L^2_{\mathrm{loc}}(\mathbb{R}^d) \,\Big|\, \|\psi\|_{\mathcal{B}} := \sum_{n\in\mathbb{N}_0} 2^{n/2}\|F_n\psi\|_{L^2} < \infty\Bigr\},$$
$$\mathcal{B}^* = \Bigl\{\psi \in L^2_{\mathrm{loc}}(\mathbb{R}^d) \,\Big|\, \|\psi\|_{\mathcal{B}^*} := \sup_{n\in\mathbb{N}_0} 2^{-n/2}\|F_n\psi\|_{L^2} < \infty\Bigr\},$$
$$\mathcal{B}_0^* = \Bigl\{\psi \in \mathcal{B}^* \,\Big|\, \lim_{n\to\infty} 2^{-n/2}\|F_n\psi\|_{L^2} = 0\Bigr\}.$$

$\mathcal{B}$ is a Banach space with respect to the norm $\|\cdot\|_{\mathcal{B}}$, and $\mathcal{B}^*$ and $\mathcal{B}_0^*$ are Banach spaces with respect to the same norm $\|\cdot\|_{\mathcal{B}^*}$. We introduce the *$f$-weighted L^2-space of order $s \in \mathbb{R}$* as

$$L^2_s = f^{-s}L^2,$$

and then, for any $s > 1/2$ the following inclusion relations hold:

$$L^2_s \subsetneq \mathcal{B} \subsetneq L^2_{1/2} \subsetneq L^2 \subsetneq L^2_{-1/2} \subsetneq \mathcal{B}_0^* \subsetneq \mathcal{B}^* \subsetneq L^2_{-s}. \tag{1.2}$$

We introduce differential operators ∂^r and ∂^f as

$$\partial^r = (\partial r)\partial, \quad \partial^f = (\partial f)\partial = r^{-\alpha}(\partial r)\partial,$$

respectively, and then we define *conjugate operator* A as

$$A = \mathrm{i}[H, f] = \mathrm{Re}\, p^f = \tfrac{1}{2}\left((p^f)^* + p^f\right), \quad p^f = -\mathrm{i}\partial^f. \tag{1.3}$$

We note that A is different from the usual one used in Mourre theory, cf. [1, 7]. In fact the commutator $\mathrm{i}[H, A]$ has only weaker positivity which decays at infinity, see [4]. However by using A of (1.3) we can prove strong results, e.g. Theorem 2.1 below. We also note that A is self-adjoint with maximal domain $\mathcal{D}(A) = \{\psi \in L^2 \,|\, A\psi \in L^2\}$ and has expressions

$$A = p^f - \tfrac{\mathrm{i}}{2}(\Delta f) = (p^f)^* + \tfrac{\mathrm{i}}{2}(\Delta f).$$

We denote the resolvent of H for $z \in \mathbb{C} \setminus \mathbb{R}$ by $R(z)$, i.e.

$$R(z) = (H - z)^{-1}.$$

2 Results on spectral theory

From this section, we always assume Condition 1.1. In this section we state several results on generalized eigenfunction and resolvents of H. We establish stationary scattering theory based on these results, which are interesting in their own right. We note we have obtained similar results with *long-range* perturbations in [4, 5].

First we state the absence of $\mathcal{B}_0^*$-eigenfunctions, which is called Rellich's theorem.

Theorem 2.1. *Let $\lambda \in \mathbb{R}$. Suppose a function $\phi \in \mathcal{B}_0^*$ satisfies*

$$(H - \lambda)\phi = 0$$

in the distributional sense. Then $\phi = 0$ on $\mathbb{R}^d$.

We set

$$\ell_{jk} = |\partial f|^2 \delta_{jk} - (\partial_j f)(\partial_k f),$$

where δ_{jk} is Kronecker's delta. For any compact interval $I \subseteq \mathbb{R}$ we introduce

$$I_\pm = \{z = \lambda \pm \mathrm{i}\Gamma \,|\, \lambda \in I,\ \Gamma \in (0, 1)\},$$

respectively. We also use the notation $\langle T \rangle_\psi = \langle \psi, T\psi \rangle$ for a general linear operator T. Then the following limiting absorption principle bounds hold.

Theorem 2.2. *Let $I \subset \mathbb{R}$ be a compact interval. Then there exists $C > 0$ such that for any $\psi \in \mathcal{B}$ and $z \in I_\pm$*

$$\|R(z)\psi\|_{\mathcal{B}^*} + \|p^f R(z)\psi\|_{\mathcal{B}^*} + \langle p_j f^{-1} \ell_{jk} p_k \rangle_{R(z)\psi}^{1/2} \le C\|\psi\|_{\mathcal{B}},$$

respectively.

As a corollary of Theorems 2.1 and 2.2 we obtain, noting (1.2), that the spectrum of H is purely absolutely continuous in $\mathbb{R}$, i.e.

$$\sigma(H) = \sigma_{\mathrm{ac}}(H) = \mathbb{R}.$$

Using the function χ of (1.1), we define *smooth cut-off functions* $\chi_m, \bar\chi_m, \chi_{m,n} \in C^\infty(\mathbb{R}^d)$ for $m, n \in \mathbb{N}_0$ as

$$\chi_m = \chi(f/2^m), \quad \bar\chi_m = 1 - \chi_m, \quad \chi_{m,n} = \bar\chi_m \chi_n.$$

We choose and fix large $m \in \mathbb{N}$ so that on $\operatorname{supp} \bar\chi_m$

$$2(\operatorname{Re} z) - 2q_0 + r^{2\alpha} > 1, \quad r = |x|,$$

where $z \in I_\pm$ and $q_0 = q + \frac{1}{8} r^{2\alpha}(\Delta f)^2 + \frac{\alpha}{2} r^{\alpha-1}(\Delta f) + \frac{1}{4} r^{2\alpha}(\partial^f \Delta f) - \frac{\alpha}{2} r^{-2}$. Then we set an asymptotic complex phase a by

$$a = a_z = \bar\chi_m \Big[r^{-\alpha}\sqrt{2(z - q_0) + r^{2\alpha}} \pm \mathrm{i}\alpha r^{-\alpha-1} \mp \mathrm{i}\alpha \tfrac{z - q_0}{2(z-q_0)+r^{2\alpha}} r^{-\alpha-1} \Big]$$

for $z \in I_\pm$. Here we choose the branch of square root as $\operatorname{Re}\sqrt{s} > 0$ for $s \in \mathbb{C} \setminus (-\infty, 0]$. We let

$$\beta_c = \min\left\{\rho + \tfrac{1}{1-\alpha},\ 1 + \tfrac{2\alpha}{1-\alpha}\right\}.$$

Then we have radiation condition bounds for complex spectral parameters.

Theorem 2.3. *Let $I \subset \mathbb{R}$ be a compact interval. For all $\beta \in [0, \beta_c)$, there exists $C > 0$ such that for any $\psi \in f^{-\beta}\mathcal{B}$ and $z \in I_\pm$*

$$\|f^\beta (A \mp a) R(z)\psi\|_{\mathcal{B}^*} + \langle p_j f^{2\beta-1} \ell_{jk} p_k \rangle^{1/2}_{R(z)\psi} \le C \|f^\beta \psi\|_{\mathcal{B}},$$

respectively.

The following three corollaries are applications of Theorem 2.2 and Theorem 2.3. The first one is the limiting absorption principle.

Corollary 2.4. *Let $I \subset \mathbb{R}$ be a compact interval. For any $s > 1/2$ and $\omega \in (0, \beta_c) \cap (0, \min\{s - 1/2, 1\}]$ there exists $C > 0$ such that for any $z, z' \in I_+$ or $z, z' \in I_-$*

$$\|R(z) - R(z')\|_{\mathcal{L}(L^2_s, L^2_{-s})} \le C|z - z'|^\omega,$$
$$\|r^{-\alpha} p\{R(z) - R(z')\}\|_{\mathcal{L}(L^2_s, L^2_{-s})} \le C|z - z'|^\omega.$$

In particular, for any $\lambda \in \mathbb{R}$, there exist uniform limits

$$\lim_{\Gamma \to +0} R(\lambda \pm \mathrm{i}\Gamma), \quad \lim_{\Gamma \to +0} r^{-\alpha} p R(\lambda \pm \mathrm{i}\Gamma),$$

in the norm topology of $\mathcal{L}(L^2_s, L^2_{-s})$. We denote these limits by $R(\lambda \pm \mathrm{i}0), r^{-\alpha}pR(\lambda \pm \mathrm{i}0)$, respectively. These limiting resolvents belong to $\mathcal{L}(\mathcal{B}, \mathcal{B}^*)$.

The second one is the radiation condition bounds for real spectral parameters, which follows from Theorem 2.3 and Corollary 2.4. We set

$$a_\pm := \lim_{I_\pm \ni z \to \lambda \pm \mathrm{i}0} a_z, \quad \lambda \in I.$$

Corollary 2.5. *Let $I \subset \mathbb{R}$ be a compact interval and $\lambda \in I$. Then for all $\beta \in [0, \beta_c)$, there exists $C > 0$ such that for any $\psi \in f^{-\beta}\mathcal{B}$*

$$\|f^\beta (A \mp a_\pm) R(\lambda \pm \mathrm{i}0)\psi\|_{\mathcal{B}^*} + \langle p_j f^{2\beta-1} \ell_{jk} p_k \rangle^{1/2}_{R(\lambda\pm\mathrm{i}0)\psi} \le C \|f^\beta \psi\|_{\mathcal{B}},$$

respectively.

The last one is Sommerfeld's uniqueness theorem.

Corollary 2.6. *Let $\lambda \in \mathbb{R}, \phi \in f^\beta \mathcal{B}^*$ and $\psi \in f^{-\beta}\mathcal{B}$ with $\beta \in [0, \beta_c)$. Then $\phi = R(\lambda \pm \mathrm{i}0)\psi$ hold if and only if both of the following conditions hold:*

(i) $(H - \lambda)\phi = \psi$ *in the distributional sense.*

(ii) $(A \mp a_\pm)\phi \in f^{-\beta}\mathcal{B}_0^*$,

respectively.

3 Results on stationary scattering theory

We can obtain several topics on stationary scattering theory by using the results of Section 2, especially Corollaries 2.5 and 2.6. We can prove them by similar approaches and schemes to [2, 3, 6]. However we omit the details.

Let us introduce the function θ_λ for $\lambda \in \mathbb{R}$ by

$$\theta_\lambda(x) = \frac{r^{1+\alpha}}{1+\alpha} + \lambda f.$$

Note that the function θ_λ is an approximate solution to the eikonal equation

$$\tfrac{1}{2}\left|\tfrac{\partial \theta_\lambda}{\partial x}(x)\right|^2 - \tfrac{1}{2}|x|^{2\alpha} + q - \lambda = 0,$$

in the sense that for $1/3 < \alpha < 1$ the quantity of the left-hand side tends to 0 faster than f^{-1} as $f \to \infty$. More precisely, the function θ_λ satisfies

$$\tfrac{1}{2}\left|\tfrac{\partial \theta_\lambda}{\partial x}(x)\right|^2 - \tfrac{1}{2}|x|^{2\alpha} + q - \lambda = \mathcal{O}(f^{-1-\min\{\rho,(3\alpha-1)/(1-\alpha)\}}). \tag{3.1}$$

Remark 3.1. We constructed θ_λ by the following simple approximation.

$$\theta_\lambda(x) \sim \int (r^{2\alpha} + 2\lambda)^{1/2} (\partial r)\, \mathrm{d}x \sim \int \big(r^\alpha + \lambda r^{-\alpha} + \mathcal{O}(r^{-3\alpha})\big)(\partial r)\, \mathrm{d}x.$$

Thus by adding some lower order terms to $\theta_\lambda(x)$, we can improve the order of the right-hand side of (3.1). In particular, for all $\alpha \in (0,1)$, there exists a smooth function θ_λ such that for some $\tilde\rho > 0$

$$\tfrac{1}{2}\left|\tfrac{\partial \theta}{\partial x}(\lambda, x)\right|^2 - \tfrac{1}{2}|x|^{2\alpha} + q - \lambda = \mathcal{O}(f^{-1-\tilde\rho}).$$

In the following we consider the case of $1/3 < \alpha < 1$ for simplicity. However our results hold for all $\alpha \in (0,1)$ by retaking θ_λ appropriately as stated in the above remark.

3.1 stationary wave operator

We set $c_\pm = (2\pi)^{1/2}\exp\{\pm\frac{\pi i}{4}(\frac{d+\alpha-1}{1+\alpha})\}$ and $J = r^{-(d+\alpha-1)}$, and then, introduce the operators $\mathcal{F}^\pm(\lambda, f)$ which map from $C_0^\infty(\mathbb{R}^d)$ to $L^2(\mathbb{S}^{d-1})$ by

$$\big(\mathcal{F}^\pm(\lambda, f)\psi\big)(\omega) = \pm(2\pi\mathrm{i})^{-1}c_\pm[J^{-1/2}e^{\mp\mathrm{i}\theta_\lambda}R(\lambda\pm\mathrm{i}0)\psi](\cdot\,\omega),$$

respectively, where $\psi \in C_0^\infty(\mathbb{R}^d)$ and $\omega \in \mathbb{S}^{d-1}$.

Theorem 3.2. *The operators $\mathcal{F}^\pm(\lambda, f)$ extend to bounded operators from $\mathcal{B}$ into $L^2(\mathbb{S}^{d-1})$. In particular, for any $\psi \in C_0^\infty(\mathbb{R}^d)$ there exist limits*

$$\lim_{f\to\infty}\mathcal{F}^\pm(\lambda, f)\psi \equiv \mathcal{F}^\pm(\lambda)\psi \quad \text{in} \quad L^2(\mathbb{S}^{d-1}).$$

Moreover it holds that for any $\psi \in \mathcal{B}$

$$\tfrac{1}{2\pi i}\langle\psi,\, R(\lambda+\mathrm{i}0)\psi - R(\lambda-\mathrm{i}0)\psi\rangle = \|\mathcal{F}^\pm(\lambda)\psi\|^2_{L^2(\mathbb{S}^{d-1})}.$$

In addition, the operators $\mathcal{F}^\pm(\lambda)$ are continuous in $\lambda \in \mathbb{R}$.

We introduce the spaces

$$\mathcal{H} = L^2(\mathbb{R}^d), \quad \widetilde{\mathcal{H}} = L^2\big(\mathbb{R}, \mathrm{d}\lambda; L^2(\mathbb{S}^{d-1})\big),$$

and define the operators $\mathcal{F}^\pm : \mathcal{B} \to C\big(\mathbb{R}; L^2(\mathbb{S}^{d-1})\big)$ as

$$\big(\mathcal{F}^\pm\psi\big)(\lambda) = \mathcal{F}^\pm(\lambda)\psi, \quad \psi \in \mathcal{B},$$

respectively. Let M_λ be the multiplication operator by λ on $\widetilde{\mathcal{H}}$.

Theorem 3.3. *The operators $\mathcal{F}^\pm$ are extended to unitary operators $\mathcal{H} \to \widetilde{\mathcal{H}}$, and satisfy*

$$\mathcal{F}^\pm H = M_\lambda\mathcal{F}^\pm.$$

In particular, H and M_λ are unitary equivalent.

We note we use a density of $\mathcal{F}^\pm(\lambda)$ stated as (3.4) below to prove $\operatorname{Ran}\mathcal{F}^\pm = \widetilde{\mathcal{H}}$.

The operators $\mathcal{F}^\pm : \mathcal{H} \to \widetilde{\mathcal{H}}$ are called stationary wave operators, and their existence and completeness follows from Theorem 3.3.

3.2 Characterization of generalized eigenfunctions

Let us introduce the functions $\phi_\lambda^\pm[v]$ for $v \in L^2(\mathbb{S}^{d-1})$ by

$$\phi_\lambda^\pm[v](f, \omega) = c_\pm^{-1}J^{1/2}e^{\pm\mathrm{i}\theta_\lambda}v(\pm\omega), \tag{3.2}$$

respectively. We may call these functions outgoing/incoming approximate generalized eigenfunctions. In fact for $v \in C^\infty(\mathbb{S}^{d-1})$ we can see by straightforward calculations that

$$\psi^\pm_\lambda[v] := (H-\lambda)\phi^\pm_\lambda[v] \in \mathcal{B}. \tag{3.3}$$

The adjoints of $\mathcal{F}^\pm(\lambda)$:

$$\mathcal{F}^\pm(\lambda)^* \in \mathcal{L}(L^2(\mathbb{S}^{d-1}), \mathcal{B}^*),$$

which are called the *stationary wave matrices*, are characterized by $\phi^\pm_\lambda$ and $\psi^\pm_\lambda$ as follows.

Proposition 3.4. *Let $v \in C^\infty(\mathbb{S}^{d-1})$, and let $\phi^\pm_\lambda[v]$ and $\psi^\pm_\lambda[v]$ be given by* (3.2) *and* (3.3)*, respectively. Then*

$$\mathcal{F}^\pm(\lambda)^* v = \phi^\pm_\lambda[v] - R(\lambda \mp \mathrm{i}0)\psi^\pm_\lambda[v] \ \ (\in \mathcal{B}^*),$$

respectively.

$\mathcal{F}^\pm(\lambda)^*$ are also called *eigenoperators*. In fact, by Proposition 3.4 and a density argument we obtain

$$(H-\lambda)\mathcal{F}^\pm(\lambda)^* v = 0 \quad \text{for any } v \in L^2(\mathbb{S}^{d-1}).$$

By Corollary 2.6, we have

$$\phi^\pm_\lambda[v] - R(\lambda \pm \mathrm{i}0)\psi^\pm_\lambda[v] = 0 \quad \text{for } \ v \in C^\infty(\mathbb{S}^{d-1}).$$

We can deduce from this equality that

$$v = \pm 2\pi\mathrm{i}\mathcal{F}^\pm(\lambda)\psi^\pm_\lambda[v] \quad \text{for } \ v \in C^\infty(\mathbb{S}^{d-1}).$$

This implies

$$C^\infty(\mathbb{S}^{d-1}) \subseteq \operatorname{Ran}\mathcal{F}^\pm(\lambda) \subseteq L^2(\mathbb{S}^{d-1}). \tag{3.4}$$

Therefore we can define the *scattering matrix* $S(\lambda)$ as satisfying for $\psi \in \mathcal{B}$

$$\mathcal{F}^+(\lambda)\psi = S(\lambda)\mathcal{F}^-(\lambda)\psi. \tag{3.5}$$

Then by Theorem 3.2 we obtain the following proposition.

Proposition 3.5. *$S(\lambda)$ defined by* (3.5) *is extended to a unitary operator on $L^2(\mathbb{S}^{d-1})$ and is strongly continuous in $\lambda \in \mathbb{R}$.*

Finally, we obtain a characterization of the $\mathcal{B}^*$-eigenfunctions in terms of $\phi^\pm_\lambda$ similar to [6]. Let us introduce the set of minimal generalized eigenfunctions.

$$\mathcal{E}_\lambda := \{\phi \in \mathcal{B}^* \,|\, (H-\lambda)\phi = 0 \ \text{in the distributional sense.}\}.$$

Theorem 3.6. *For any fixed $\lambda \in \mathbb{R}$ the following assertions hold.*

(i) For any one of $\xi_\pm \in L^2(\mathbb{S}^{d-1})$ or $\phi \in \mathcal{E}_\lambda$ the two other quantities in $\{\xi_+, \xi_-, \phi\}$ uniquely exist such that

$$\phi - \phi^+_\lambda[\xi_+] - \phi^-_\lambda[\xi_-] \in \mathcal{B}^*_0. \tag{3.6}$$

(ii) For the quantities $\{\xi_+, \xi_-, \phi\}$ *satisfying* (3.6)*, the following relations hold.*

$$\phi = \mathcal{F}^{\pm}(\lambda)^*\xi_{\pm}, \qquad \xi_+ = S(\lambda)\xi_-,$$
$$\xi_{\pm} = \frac{1}{2}c_{\pm} \lim_{R\to\infty} \frac{1}{R}\int_R^{2R} J^{-1/2}e^{\mp i\theta_\lambda}\big(1 \pm p^f\big)\phi\,\mathrm{d}f,$$

In particular the wave matrices $\mathcal{F}^{\pm}(\lambda)^*$ *give one-to-one correspondences between the spaces* $L^2(\mathbb{S}^{d-1})$ *and* $\mathcal{E}_\lambda$*.*

(iii) The operators $\mathcal{F}^{\pm}(\lambda) : \mathcal{B} \to L^2(\mathbb{S}^{d-1})$ *are surjections.*

It is guaranteed by Theorem 3.6 that $\mathcal{E}_\lambda$ has many elements. In particular, Theorem 2.1 is sharp, cf. (1.2).

Acknowledgement

This work was supported by the Research Institute for Mathematical Science, an International Joint Usage/Research Center located in Kyoto University.

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