On the inverse scattering in Stark effect

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1 Introduction

The standard two-body Schrödinger operator added a constant electric field $0 \neq E \in \mathbb{R}^n$, as a self-adjoint operator acting on $L^2(\mathbb{R}^n)$

$$-\hbar^2\Delta/(2m) - qE \cdot x \tag{1.1}$$

is called by the free Stark Hamiltonian. Here, $x \in \mathbb{R}^n$ is the position of the particle and $\Delta = \sum_{j=1}^n \partial_{x_j}^2$ is the Laplacian. We also denote the Plank constant, mass and charge of the particle by $\hbar = h/(2\pi)$, $m > 0$ and $0 \neq q \in \mathbb{R}$ respectively. However, in the following, by the suitable scale conversion and coordinate rotation, we let these physical constants be $E = e_1 = (1, 0, \ldots, 0)$ and $\hbar = m = q = 1$ without loss of generality, and employ the next free Hamiltonian

$$H_0^{\mathrm{S}} = |p|^2/2 - x_1, \tag{1.2}$$

where p is the momentum operator $-\mathrm{i}\nabla = -\sqrt{-1}(\partial_{x_1}, \ldots, \partial_{x_n})$.

Throughout this report, we assume that the space dimension $n \geqslant 2$, and denote the pairwise interaction potential by V. Under Assumption 1.1, the full Hamiltonian

$$H^{\mathrm{S}} = H_0^{\mathrm{S}} + V \tag{1.3}$$

is also realized as self-adjoint by virtue of the Kato-Rellich theorem.

In this report, we will introduce the result of the inverse scattering [I]. By applying the time-dependent method invented by Enss-Weder [EW], we can prove that the scattering operator which is defined by the wave operators determines potential V uniquely. In particular, by comparison with the previous researches ([We], [N1], [AM] and [AFI]), we can allow that the potential function V belong to the very broad classes.

The assumptions for the potential V are quite important in scattering theory. We state the details of these assumptions below, roughly speaking, V is the multiplication operator of the real-valued function $V(x)$ which is represented by $V = V^{\mathrm{vs}} + V^{\mathrm{s}} + V^{\mathrm{l}} \in \mathscr{V}^{\mathrm{vs}} + \mathscr{V}^{\mathrm{s}} + \mathscr{V}_{\mathrm{G}}^{\mathrm{l}} \cup \mathscr{V}_{\mathrm{D}}^{\mathrm{l}}$, and its value vanishes at large distance. We use the following notations. The Kitada bracket of x has the definition, $\langle x \rangle = \sqrt{1 + |x|^2}$. $F(\cdots)$ is the characteristic function of the set $\{\cdots\}$, and $\|\cdot\|$ denotes the operator norm in $L^2(\mathbb{R}^n)$ or the usual $L^2(\mathbb{R}^n)$-norm.

Assumption 1.1. *$V^{\mathrm{vs}} \in \mathscr{V}^{\mathrm{vs}}$ is decomposed into*

$$V^{\mathrm{vs}}(x) = V_1^{\mathrm{vs}}(x) + V_2^{\mathrm{vs}}(x), \tag{1.4}$$

where a singular part V_1^{vs} is $|p|^2/2$-bounded with its relative bound less than 1, $x_1 V_1^{\mathrm{vs}}$ is $|p|^2/2$-bounded, a regular part V_2^{vs} is bounded, and V^{vs} satisfies

$$\int_0^\infty \|V^{\mathrm{vs}}(x)\langle p\rangle^{-2} F(|x| \geqslant R)\| dR < \infty. \tag{1.5}$$

$V^{\mathrm{s}} \in \mathscr{V}^{\mathrm{s}}$ belongs to $C^1(\mathbb{R}^n)$ and satisfies

$$|V^{\mathrm{s}}(x)| \leqslant C\langle x\rangle^{-\gamma}, \quad |\partial_x^\beta V^{\mathrm{s}}(x)| \leqslant C_\beta \langle x\rangle^{-1-\alpha} \tag{1.6}$$

for the multi-index β with $|\beta| = 1$, where $1/2 < \gamma \leqslant 1$ and $0 < \alpha \leqslant \gamma$. $V^{\mathrm{l}} \in \mathscr{V}_{\mathrm{G}}^{\mathrm{l}}$ belongs to $C^2(\mathbb{R}^n)$ and satisfies

$$|\partial_x^\beta V^{\mathrm{l}}(x)| \leqslant C_\beta \langle x\rangle^{-\gamma_{\mathrm{G}} - \kappa|\beta|} \tag{1.7}$$

for $|\beta| \leqslant 2$, where $0 < \gamma_{\mathrm{G}} \leqslant 1/2$ and $1 - \gamma_{\mathrm{G}} < \kappa \leqslant 1$. Finally, $V^{\mathrm{l}} \in \mathscr{V}_{\mathrm{D}}^{\mathrm{l}}$ belongs to $C^2(\mathbb{R}^n)$ and satisfies

$$|\partial_x^\beta V^{\mathrm{l}}(x)| \leqslant C_\beta \langle x\rangle^{-\gamma_{\mathrm{D}} - |\beta|/2} \tag{1.8}$$

for $|\beta| \leqslant 2$, where $3/8 < \gamma_{\mathrm{D}} \leqslant 1/2$.

2 Short-range interactions

We first consider the short-range case, that is, $V^{\mathrm{l}} \equiv 0$. We see that the wave operators defined by the following strong limits

$$W^{\pm} = \operatorname*{s-lim}_{t \to \pm\infty} e^{\mathrm{i}tH^{\mathrm{S}}} e^{-\mathrm{i}tH_0^{\mathrm{S}}} \tag{2.1}$$

exist. By using these wave operators $W^{\pm}$, the scattering operator $S = S(V)$ is defined by

$$S = (W^+)^* W^-. \tag{2.2}$$

The first theorem of this paper is the following.

Theorem 2.1. *If $S(V_1) = S(V_2)$ for $V_1, V_2 \in \mathscr{V}^{\mathrm{vs}} + \mathscr{V}^{\mathrm{s}}$, then $V_1 = V_2$ holds.*

The Enss–Weder time-dependent method was developed in [EW] and, by applying its method, Weder [We] first proved this theorem for $\gamma > 3/4$. However, the borderline between the short-range and long-range is 1/2. Nicoleau [N1] proved this theorem for $V \in C^\infty(\mathbb{R}^n)$ which satisfied

$$|\partial_x^\beta V(x)| \leqslant C_\beta \langle x\rangle^{-\gamma - |\beta|} \tag{2.3}$$

under $\gamma > 1/2$ with the additional condition $n \geqslant 3$. Thereafter, these results were improved by Adachi–Maehara [AM] given $1/2 < \alpha \leqslant \gamma$. The behavior of the short-range part under their assumptions was

$$V^{\mathrm{s}}(x) = O(|x|^{-1/2-\epsilon}), \quad \nabla_x V^{\mathrm{s}}(x) = O(|x|^{-3/2-\epsilon}) \tag{2.4}$$

with small $\epsilon > 0$. In this sense, a possibility in which the condition regarding the size of α could be relaxed was left because the classical trajectory in the Stark effect is $x(t) = O(t^2)$ as $t \to \infty$. Adachi, Fujiwara, and Ishida [AFI] considered the time-dependent electric fields

$$H_0^{\mathrm{S}}(t) = |p|^2/2 - E(t) \cdot x, \quad E(t) = E_0(1 + |t|)^{-\mu}, \tag{2.5}$$

where $0 \leqslant \mu < 1$ and $0 \neq E_0 \in \mathbb{R}^n$, and proved this theorem under $\tilde{\alpha}_\mu < \alpha \leqslant \gamma$ with $1/(2-\mu) < \gamma \leqslant 1$ and

$$\tilde{\alpha}_\mu = \begin{cases} \dfrac{7 - 3\mu - \sqrt{(1-\mu)(17-9\mu)}}{4(2-\mu)} & \text{if } 0 \leqslant \mu \leqslant 1/2, \\ \dfrac{1+\mu}{2(2-\mu)} & \text{if } 1/2 < \mu < 1. \end{cases} \tag{2.6}$$

The smallest $\tilde{\alpha}_\mu$ is when $\mu = 0$, and in this case, (2.5) corresponds to the constant electric field (1.2). Therefore, the result by [AFI] is one of the improvements of [AM] because

$$\tilde{\alpha}_0 = (7 - \sqrt{17})/8 < 1/2. \tag{2.7}$$

Theorem 2.1 is a further improvement of [AM] and [AFI]. We prove that this $\tilde{\alpha}_0$ is allow to be equal to zero. This means that the tail of the first-order differential of the short-range part behaves as

$$\nabla_x V^{\mathrm{s}}(x) = O(|x|^{-1-\epsilon}). \tag{2.8}$$

Therefore, from the physical aspect and the motion of the classical trajectory, our assumptions are quite natural, and relaxing the condition on α is one of the main motivations of this study.

The following reconstruction theorem yields the proof of Theorem 2.1.

Theorem 2.2. *Let $\omega \in \mathbb{R}^n$ be given such that $|\omega| = 1$ and $|\omega \cdot e_1| < 1$. Put $v = |v|\omega$. Suppose $\Phi_0, \Psi_0 \in L^2(\mathbb{R}^n)$ such that their Fourier transforms $\mathscr{F}\Phi_0, \mathscr{F}\Psi_0 \in C_0^\infty(\mathbb{R}^n)$ with $\operatorname{supp}\mathscr{F}\Phi_0, \operatorname{supp}\mathscr{F}\Psi_0 \subset \{\xi \in \mathbb{R}^n \mid |\xi| < \eta\}$ for the given $\eta > 0$. Put $\Phi_v = e^{\mathrm{i}v\cdot x}\Phi_0, \Psi_v = e^{\mathrm{i}v\cdot x}\Psi_0$. Then*

$$\begin{aligned} |v|(\mathrm{i}[S, p_j]\Phi_v, \Psi_v) = \int_{-\infty}^{\infty} \Big\{ &(V^{\mathrm{vs}}(x + \omega t)p_j\Phi_0, \Psi_0) - (V^{\mathrm{vs}}(x + \omega t)\Phi_0, p_j\Psi_0) \\ &+ \mathrm{i}((\partial_{x_j} V^{\mathrm{s}})(x + \omega t)\Phi_0, \Psi_0)\Big\} dt + o(1) \end{aligned} \tag{2.9}$$

holds as $|v| \to \infty$ for $V^{\mathrm{vs}} \in \mathscr{V}^{\mathrm{vs}}$ and $V^{\mathrm{s}} \in \mathscr{V}^{\mathrm{s}}$, where $(\cdot, \cdot)$ is the scalar product of $L^2(\mathbb{R}^n)$ and p_j is the jth component of p.

The propagation estimate for the regular part V^{s} is one of the main techniques in this report, and is also one of the improvements on previous work.

Proposition 2.3. *Let v and Φ_v be as in Theorem 2.2. Then*

$$\int_{-\infty}^{\infty} \|\{V^{\mathrm{s}}(x) - V^{\mathrm{s}}(vt + e_1 t^2/2)\} e^{-\mathrm{i}tH_0^{\mathrm{S}}} \Phi_v\| dt = O(|v|^{-1}) \tag{2.10}$$

holds as $|v| \to \infty$ for $V^{\mathrm{s}} \in \mathscr{V}^{\mathrm{s}}$.

In [AM, Lemma 2.2], the right-hand side of (2.10) was $O(|v|^{-\alpha})$ for $1/2 < \alpha < 1$. This order was improved in [AFI, Lemma 3.4] by giving $O(|v|^{\Theta_0(\alpha)+\epsilon})$ with any small $\epsilon > 0$ and

$$\Theta_0(\alpha) = -\alpha - \frac{\alpha(1-\alpha)}{2-\alpha}. \tag{2.11}$$

The number $(7 - \sqrt{17})/8$ in (2.7) comes from the inequality $\Theta_0(\alpha) < -1/2$, which is required to prove the reconstruction theorem. As mentioned before, not only was the time-independent case (1.2) treated by [AFI], but also the time-dependent case (2.5). For more details, see [AFI, Lemma 3.4]. Our key ideas for further improvements are the efficient use of the well-known propagation estimate for the free Schrödinger dynamics

$$\|x e^{-\mathrm{i}t|p|^2/2} \Phi_0\| = O(|t|) \tag{2.12}$$

as $|t| \to \infty$ and the Hölder inequality.

3 Long-range interactions

We next consider the long-range case, that is, $V^{\mathrm{l}} \not\equiv 0$. For $V^{\mathrm{l}} \in \mathscr{V}^{\mathrm{l}}_{\mathrm{G}}$, we find the existence of the Graf-type (or Zorbas-type) modified wave operators which were proposed in Graf [G] and Zorbas [Z]

$$W_{\mathrm{G}}^{\pm} = \operatorname*{s-lim}_{t \to \pm\infty} e^{\mathrm{i}tH^{\mathrm{S}}} e^{-\mathrm{i}tH_0^{\mathrm{S}}} e^{-\mathrm{i}\int_0^t V^{\mathrm{l}}(e_1\tau^2/2)d\tau}, \tag{3.1}$$

and the Dollard-type modified wave operators introduced by Jensen and Yajima [JY] (see also White [Wh] and Adachi [A])

$$W_{\mathrm{D}}^{\pm} = \operatorname*{s-lim}_{t \to \pm\infty} e^{\mathrm{i}tH^{\mathrm{S}}} e^{-\mathrm{i}tH_0^{\mathrm{S}}} e^{-\mathrm{i}\int_0^t V^{\mathrm{l}}(p\tau + e_1\tau^2/2)d\tau} \tag{3.2}$$

by virtue of the condition $\gamma_{\mathrm{G}} + \kappa > 1$. We find also the existence of (3.2), even if $V^{\mathrm{l}} \in \mathscr{V}^{\mathrm{l}}_{\mathrm{D}}$. Then, for $V^{\mathrm{l}} \in \mathscr{V}^{\mathrm{l}}_{\mathrm{G}} \cup \mathscr{V}^{\mathrm{l}}_{\mathrm{D}}$, the Dollard-type modified scattering operator $S_{\mathrm{D}} = S_{\mathrm{D}}(V^{\mathrm{l}}; V^{\mathrm{vs}} + V^{\mathrm{s}})$ is defined by

$$S_{\mathrm{D}} = (W_{\mathrm{D}}^{+})^* W_{\mathrm{D}}^{-}. \tag{3.3}$$

The second theorem of this paper is the following.

Theorem 3.1. *Let a* $V^{\mathrm{l}} \in \mathscr{V}_{\mathrm{G}}^{\mathrm{l}} \cup \mathscr{V}_{\mathrm{D}}^{\mathrm{l}}$ *be given. If* $S_{\mathrm{D}}(V^{\mathrm{l}}; V_1) = S_{\mathrm{D}}(V^{\mathrm{l}}; V_2)$ *for* $V_1, V_2 \in \mathscr{V}^{\mathrm{vs}} + \mathscr{V}^{\mathrm{s}}$, *then* $V_1 = V_2$ *holds. Moreover, any one of the Dollard-type modified scattering operators* S_{D} *determines uniquely the total potential* V.

When $V^{\mathrm{l}} \in \mathscr{V}_{\mathrm{G}}^{\mathrm{l}}$, a similar result to Theorem 3.1 was obtained in [AM] (Note that the notation of γ_{G} was denoted by γ_{D} in [AM]), however, the decay condition of the short-range part was $1/2 < \alpha \leqslant \gamma$. Therefore, Theorem 3.1 extends the short-range class introduced in [AM] to the broader $\mathscr{V}^{\mathrm{s}}$. For $V^{\mathrm{l}} \in \mathscr{V}_{\mathrm{D}}^{\mathrm{l}}$, the uniqueness of the short-range interactions was also proved in [AFI] for the time-dependent electric fields (2.5), in which α satisfied $\tilde{\alpha}_{\mu,\mathrm{D}} < \alpha \leqslant \gamma$ with $1/(2-\mu) < \gamma \leqslant 1$ and

$$\tilde{\alpha}_{\mu,\mathrm{D}} = \begin{cases} \dfrac{13 - 5\mu - \sqrt{(1-\mu)(41-25\mu)}}{8(2-\mu)} & \text{if } 0 \leqslant \mu \leqslant 5/7, \\ \dfrac{1+\mu}{2(2-\mu)} & \text{if } 5/7 < \mu < 1, \end{cases} \tag{3.4}$$

and γ_{D} satisfied $\tilde{\gamma}_\mu < \gamma_{\mathrm{D}} \leqslant 1/(2-\mu)$ with

$$\tilde{\gamma}_\mu = \frac{1}{2(2-\mu)} + \frac{1-\mu}{4(2-\mu)}. \tag{3.5}$$

The smallest $\tilde{\alpha}_{\mu,\mathrm{D}}$ and $\tilde{\gamma}_\mu$ are when $\mu = 0$, and this case corresponds to a constant electric field (1.2). In comparison with our result, let us substitute $\mu = 0$ for (3.4) and (3.5). Although $\tilde{\gamma}_0 = 3/8$ says that the condition on the long-range class is the same as our assumption (1.8), for the short-range class, Theorem 3.1 makes true improvement because

$$\tilde{\alpha}_{0,\mathrm{D}} = (13 - \sqrt{41})/16. \tag{3.6}$$

We prove that this $\tilde{\alpha}_{0,\mathrm{D}}$ is allow to be equal to zero.

The following reconstruction theorem yields the proof of Theorem 3.1.

Theorem 3.2. *Let* $\omega \in \mathbb{R}^n$ *be given such that* $|\omega| = 1$ *and* $|\omega \cdot e_1| < 1$. *Put* $v = |v|\omega$. *Suppose* $\Phi_0, \Psi_0 \in L^2(\mathbb{R}^n)$ *such that* $\mathscr{F}\Phi_0, \mathscr{F}\Psi_0 \in C_0^\infty(\mathbb{R}^n)$ *with* $\operatorname{supp}\mathscr{F}\Phi_0, \operatorname{supp}\mathscr{F}\Psi_0 \subset \{\xi \in \mathbb{R}^n \mid |\xi| < \eta\}$ *for the given* $\eta > 0$. *Put* $\Phi_v = e^{\mathrm{i}v\cdot x}\Phi_0$, $\Psi_v = e^{\mathrm{i}v\cdot x}\Psi_0$. *Then*

$$\begin{aligned} &|v|(\mathrm{i}[S_{\mathrm{D}}, p_j]\Phi_v, \Psi_v) \\ &= \int_{-\infty}^{\infty} \Big\{ (V^{\mathrm{vs}}(x+\omega t)p_j\Phi_0, \Psi_0) - (V^{\mathrm{vs}}(x+\omega t)\Phi_0, p_j\Psi_0) \\ &\qquad + (\mathrm{i}(\partial_{x_j} V^{\mathrm{s}})(x+\omega t)\Phi_0, \Psi_0) + (\mathrm{i}(\partial_{x_j} V^{\mathrm{l}})(x+\omega t)\Phi_0, \Psi_0) \Big\} dt + o(1) \end{aligned} \tag{3.7}$$

holds as $|v| \to \infty$ *for* $V^{\mathrm{vs}} \in \mathscr{V}^{\mathrm{vs}}$, $V^{\mathrm{s}} \in \mathscr{V}^{\mathrm{s}}$, *and* $V^{\mathrm{l}} \in \mathscr{V}_{\mathrm{G}}^{\mathrm{l}} \cup \mathscr{V}_{\mathrm{D}}^{\mathrm{l}}$.

We define a class of long-range potentials $\hat{\mathscr{V}}_{\mathrm{D}}^{\mathrm{l}}$ as follows. $V^{\mathrm{l}} \in \hat{\mathscr{V}}_{\mathrm{D}}^{\mathrm{l}}$ belongs to $C^2(\mathbb{R}^n)$ and satisfies that

$$|\partial_x^\beta V^{\mathrm{l}}(x)| \leqslant C_\beta \langle x \rangle^{-\hat{\gamma}_{\mathrm{D}} - |\beta|/2} \tag{3.8}$$

for $|\beta| \leqslant 2$, where $1/4 < \hat{\gamma}_{\mathrm{D}} \leqslant 1/2$. Clearly, $\mathscr{V}_{\mathrm{D}}^{\mathrm{l}} \subsetneq \hat{\mathscr{V}}_{\mathrm{D}}^{\mathrm{l}}$. Moreover, we denote the Dollard-type modifier $M_{\mathrm{D}}(t)$ by

$$M_{\mathrm{D}}(t) = e^{-\mathrm{i}\int_0^t V^{\mathrm{l}}(p\tau + e_1\tau^2/2)d\tau}, \tag{3.9}$$

for $V^{\mathrm{l}} \in \mathscr{V}_{\mathrm{G}}^{\mathrm{l}} \cup \hat{\mathscr{V}}_{\mathrm{D}}^{\mathrm{l}}$.

The next propagation estimate for V^{s} along the modified time evolution by $e^{-\mathrm{i}H_0^{\mathrm{S}}} M_{\mathrm{D}}(t)$ when $V^{\mathrm{l}} \in \hat{\mathscr{V}}_{\mathrm{D}}^{\mathrm{l}}$ is one of the main techniques in this report, and is also one of the improvements on the previous work.

Proposition 3.3. *Let v and Φ_v be as in Theorem 2.2. Then*

$$\int_{-\infty}^{\infty} \|\{V^{\mathrm{s}}(x) - V^{\mathrm{s}}(vt + e_1t^2/2)\}e^{-\mathrm{i}tH_0^{\mathrm{S}}} M_{\mathrm{D}}(t)\Phi_v\| dt = O(|v|^{-1}) \tag{3.10}$$

holds as $|v| \to \infty$ for $V^{\mathrm{s}} \in \mathscr{V}^{\mathrm{s}}$ and $V^{\mathrm{l}} \in \hat{\mathscr{V}}_{\mathrm{D}}^{\mathrm{l}}$.

In [AFI, Lemma 4.4], when $\mu = 0$ of (2.5), the estimate of (3.10) was $O(|v|^{\Theta_{0,\mathrm{D}}(\alpha)+\epsilon})$ with any small $\epsilon > 0$ and

$$\Theta_{0,\mathrm{D}}(\alpha) = -\alpha - \frac{\alpha(1-\alpha)}{4-3\alpha}. \tag{3.11}$$

The number $(13-\sqrt{41})/16$ in (3.6) comes from the inequality $\Theta_{0,\mathrm{D}}(\alpha) < -1/2$. Our key ideas for this improvement are the efficient use of the propagation estimate of the free Schrödinger dynamics (2.12) and the Hölder inequality as with Proposition 2.3.

There are several other studies concerning the uniqueness of the interaction potentials in the external electric fields. Nicoleau [N2] considered the time-periodic electric field and obtained the same result given in [N1]. Valencia and Weder [VW] applied the result obtained in [AM] to the N-body case (see also [We]). Adachi, Kamada, Kazuno, and Toratani [AKKT] also treated the time-dependent electric field, which is the same as in (2.5), however, the case where $\mu = 0$, that is, the constant electric field (1.2) was not included.

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