Schrödinger Operators with random point interactions

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Abstract

We consider the Schrödinger operators with random point interactions of Poisson–Anderson type, and review some recent results about the self-adjointness and the spectrum of the operators. Moreover, we shall give an asymptotic formula for the corresponding IDS $N(\lambda)$ as $\lambda \to -\infty$ in the three-dimensional case. We also verify the obtained result numerically using R-Language.

1 Definition of point interactions

Let Γ be a *locally finite* set in $\mathbb{R}^d$ $(d = 1, 2, 3)$, that is,

$$\#(\Gamma \cap B_R(0)) < \infty$$

for every $R > 0$, where $B_R(x) = \{y \in \mathbb{R}^d \mid |y - x| < R\}$. Let $\alpha = (\alpha_\gamma)_{\gamma\in\Gamma}$ be a sequence of real numbers. We consider the Schrödinger operator $H_{\Gamma,\alpha}$ formally given as

$$H_{\Gamma,\alpha} = -\Delta + \text{‘point interactions on } \Gamma\text{’},$$

where α_γ is the parameter representing the interaction at the point γ. Basic facts about $H_{\Gamma,\alpha}$ are found in the monograph by Albeverio et al. [2]

There are several ways of defining $H_{\Gamma,\alpha}$ rigorously. Today we adopt the definition in terms of the *boundary condition* at $\gamma \in \Gamma$, as follows.

$$\begin{aligned} H_{\Gamma,\alpha}u &= -\Delta|_{\mathbb{R}^d\setminus\Gamma}u \quad (u \in D(H_{\Gamma,\alpha})), \\ D(H_{\Gamma,\alpha}) &= \{u \in H^2_{\mathrm{loc}}(\mathbb{R}^d \setminus \Gamma) \cap L^2(\mathbb{R}^d) \mid -\Delta|_{\mathbb{R}^d\setminus\Gamma}u \in L^2(\mathbb{R}^d), \\ &\qquad u \text{ satisfies } (BC)_\gamma \text{ for every } \gamma \in \Gamma\}. \end{aligned}$$

Here, $-\Delta|_{\mathbb{R}^d\setminus\Gamma}u$ is defined as a Schwartz distribution on $\mathbb{R}^d \setminus \Gamma$. The boundary condition $(BC)_\gamma$ is defined as follows:

$\boxed{d = 1}$ $u(\gamma+) = u(\gamma-) = u(\gamma)$, $u'(\gamma+) - u'(\gamma-) = \alpha_\gamma u(\gamma)$.

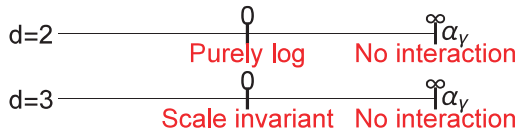
$\boxed{d = 2}$ $u(x) = u_{\gamma,0} \log|x - \gamma| + u_{\gamma,1} + o(1)$ as $x \to \gamma$, and $2\pi\alpha_\gamma u_{\gamma,0} + u_{\gamma,1} = 0$.

$\boxed{d = 3}$ $u(x) = u_{\gamma,0}|x - \gamma|^{-1} + u_{\gamma,1} + o(1)$ as $x \to \gamma$, and $-4\pi\alpha_\gamma u_{\gamma,0} + u_{\gamma,1} = 0$.

We give a remark about the above parametrization. When $d = 1$, ‘no point interaction’ at γ $(u'(\gamma+) = u'(\gamma-))$ corresponds to $\alpha_\gamma = 0$, and ‘Dirichlet condition at γ’ corresponds to $\alpha_\gamma = \infty$.

Moreover, the negative eigenvalues of $H_{\Gamma,\alpha}$ are monotone increasing with respect to α_γ. This means the sign of α_γ coincides with the sign of interaction at γ.

On the other hand, when $d = 2, 3$, 'no point interaction' at γ ($u_{\gamma,0} = 0$) corresponds to $\alpha_\gamma = +\infty$. The negative eigenvalues of $H_{\Gamma,\alpha}$ are still monotone increasing with respect to α_γ in these cases. This means the point interaction is always *negative* when $d = 2, 3$.



Later this fact affects the spectrum of the Schrödinger operators with Poisson–Anderson point interactions (see Theorem 3).

2 Random point interactions

There are many results about the Schrödinger operators with *random point interactions*, that is, the operator $H_{\Gamma,\alpha}$ such that the set Γ or the sequence $\alpha = (\alpha_\gamma)_{\gamma\in\Gamma}$ depends on some random parameter. We list some papers about the Schrödinger operators with random point interactions.

- When $d = 1$, Frisch–Lloyd [8], Luttinger–Sy [10], Kotani [9], Delyon–Simon–Souillard [5], Minami [18], Drabkin–Kirsch–Schulz-Baldes [7], ...
- When $d = 2, 3$, Albeverio–Høegh-Krohn–Kirsch–Martinelli [1], Boutet de Monvel–Grinshpun [4], Dorlas–Macris–Pulé [6], Hislop–Kirsch–Krishna [11, 12], ...

In the above papers, the case Γ_ω is the *Poisson configuration* is well-studied when $d = 1$, but it is not studied when $d = 2, 3$. The definition of the Poisson configuration is as follows (see e.g. Reiss [20]).

Definition 1. *Let μ_ω be a measure on $\mathbb{R}^d$ dependent on a random parameter ω, and ρ be a positive constant. We call μ_ω a Poisson point process measure with intensity measure ρdx, if the following holds.*

(i) For any Lebesgue measurable set E with Lebesgue measure $|E| < \infty$, the random variable $\mu_\omega(E)$ obeys the Poisson distribution with parameter $\rho|E|$, that is,

$$\mathbb{P}(\mu_\omega(E) = k) = \frac{(\rho|E|)^k}{k!} e^{-\rho|E|} \quad (k = 0, 1, 2, \ldots)$$

(ii) For any disjoint Lebesgue measurable sets $E_1, \ldots, E_n$ with $|E_j| < \infty$, the random variables $\{\mu_\omega(E_j)\}_{j=1}^n$ are independent.

We denote the support of μ_ω by Γ_ω, which is called the Poisson configuration.

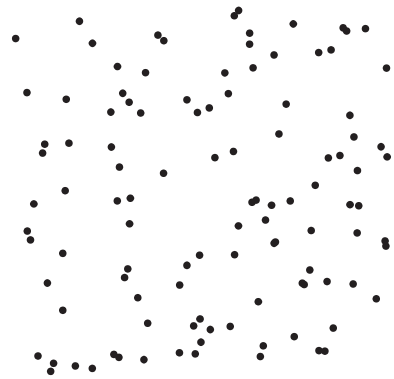


Figure 1: A sample of 2-dim Poisson configuration Γ_ω in $[0,10]^2$ with intensity $1dx$.

Today we assume the following on the pair $(\Gamma_\omega, \alpha_\omega)$.

Assumption 2. *(i) Γ_ω is the Poisson configuration with intensity ρdx, for some positive constant ρ.*

(ii) $\alpha_\omega = (\alpha_{\omega,\gamma})_{\gamma\in\Gamma_\omega}$ is a sequence of independent random variables with common distribution measure ν on $\mathbb{R}$, and α_ω is independent of Γ_ω.

Sometimes Assumption 2 is called 'Poisson–Anderson type'. When $\operatorname{supp}\nu$ is a one-point set (that is, $\alpha_{\omega,\gamma}$ is a constant independent of ω,γ), we say α is a *constant sequence*, and denote $\alpha_{\omega,\gamma} = \alpha$ for simplicity.

Theorem 3 (Kaminaga–M–Nakano [14])**.** *Let $d = 1,2,3$, and suppose $(\Gamma_\omega, \alpha_\omega)$ satisfies Assumption 2.*

(1) The operator $H_{\Gamma_\omega,\alpha}$ is self-adjoint for any sequence $\alpha = (\alpha_\gamma)_{\gamma\in\Gamma_\omega}$, almost surely.

(2) (i) When $d = 1$, we have almost surely

$$\sigma(H_{\Gamma_\omega,\alpha_\omega}) = \begin{cases} [0,\infty) & (\operatorname{supp}\nu \subset [0,\infty)), \\ \mathbb{R} & (\operatorname{supp}\nu \cap (-\infty,0) \neq \emptyset). \end{cases}$$

(ii) When $d = 2,3$, we have $\sigma(H_{\Gamma_\omega,\alpha_\omega}) = \mathbb{R}$ almost surely.

When $d = 1$, the result for $H_{\Gamma_\omega,\alpha_\omega}$ is obtained in Minami [18], and more general result is obtained in Kostenko–Malamud [15]. Concerning the spectrum, Pastur–Figotin's book [19], Ando–Iwatsuka–Kaminaga–Nakano [3], and Kaminaga–M [13] obtain similar results for the Schrödinger operator with random *scalar* potential of Poisson–Anderson type

$$H_\omega = -\Delta + V_\omega, \quad V_\omega(x) = \sum_{\gamma\in\Gamma_\omega} \alpha_{\omega,\gamma} V_0(x-\gamma),$$

where V_0 is a real-valued scalar function having some regularity and decaying property. Their results say *'the spectrum equals* $[0,\infty)$ *if* V_ω *is non-negative, and it equals* $\mathbb{R}$ *if* V_ω *has negative part'*. The result (2) of Theorem 3 can be regarded as a generalization of these results. As stated in the introduction, the sign of the point interaction at γ has the same as the sign of α_γ when $d = 1$, and the sign of point interaction at γ is always negative when $d = 2, 3$.

3 Integrated density of states

In order to define the integrated density of states (IDS), we prepare some notations. Let Γ be a locally finite set in $\mathbb{R}^d$, and $\alpha = (\alpha_\gamma)_{\gamma\in\Gamma}$ a sequence of real numbers. For a bounded open set U in $\mathbb{R}^d$ and $\lambda \in \mathbb{R}$, let

$$N_U^D(\lambda) = N_{U,\Gamma,\alpha}^D(\lambda) = \#\{\mu \le \lambda \mid \mu \text{ is an eigenvalue of } H_{U,\Gamma,\alpha}^D\},$$
$$N_U^N(\lambda) = N_{U,\Gamma,\alpha}^N(\lambda) = \#\{\mu \le \lambda \mid \mu \text{ is an eigenvalue of } H_{U,\Gamma,\alpha}^N\},$$

where $H_{U,\Gamma,\alpha}^D$ (resp. $H_{U,\Gamma,\alpha}^N$) be the restriction of $H_{\Gamma,\alpha}$ on U with Dirichlet boundary conditions (resp. Neumann boundary conditions) on ∂U, and the eigenvalues are counted with multiplicity.

Moreover, for $\lambda < 0$, let

$$N_U(\lambda) = N_{U,\Gamma,\alpha}(\lambda) = \#\{\mu \le \lambda \mid \mu \text{ is an eigenvalue of } H_{\Gamma\cap U,\alpha|_{\Gamma\cap U}}\}.$$

Notice that the operator $H_{\Gamma\cap U,\alpha|_{\Gamma\cap U}}$ is defined on the *whole space* $\mathbb{R}^d$. Then, a standard ergodicity argument and the Dirichlet–Neumann bracketing technique leads the following result.

Proposition 4 (Existence of IDS)**.** *Let* $d = 1, 2, 3$. *Suppose* $(\Gamma_\omega, \alpha_\omega)$ *satisfies Assumption 2. For* $L > 0$, *let* $Q_L = (0, L)^d$. *Then, for any* $\lambda \in \mathbb{R}$, *the equations*

$$N^D(\lambda) := \lim_{L\to\infty} \frac{N_{Q_L}^D(\lambda)}{L^d} = \lim_{L\to\infty} \frac{\mathbb{E}\left[N_{Q_L}^D(\lambda)\right]}{L^d},$$
$$N^N(\lambda) := \lim_{L\to\infty} \frac{N_{Q_L}^N(\lambda)}{L^d} = \lim_{L\to\infty} \frac{\mathbb{E}\left[N_{Q_L}^N(\lambda)\right]}{L^d},$$

hold a.s. (in particular, the limit functions $N^D(\lambda)$ *and* $N^N(\lambda)$ *are independent of* ω *a.s.) If one of* $N^D(\lambda)$ *and* $N^N(\lambda)$ *is continuous at* λ, *then* $N^D(\lambda) = N^N(\lambda)$.

Moreover, if $\lambda < 0$ *and* $N^D(\lambda) = N^N(\lambda)$, *then*

$$N(\lambda) := \lim_{L\to\infty} \frac{N_{Q_L}(\lambda)}{L^d} = \lim_{L\to\infty} \frac{\mathbb{E}\left[N_{Q_L}(\lambda)\right]}{L^d} \tag{1}$$

holds and $N(\lambda) = N^D(\lambda) = N^N(\lambda)$, *almost surely.*

We call the functions $N^D(\lambda)$, $N^N(\lambda)$ and $N(\lambda)$ the *integrated density of states (IDS)* (these functions are equal almost everywhere, by Proposition 4). The definition (1) is useful both for mathematical analysis and numerical analysis.

We list some known results about the asymptotics of IDS as $\lambda \to -\infty$, for the Schrödinger operators with Poisson random potentials.

- When $d = 1$, α is a negative constant, and $H = H_{\Gamma_\omega,\alpha}$ (1-dim Poisson point interactions)

$$N(\lambda) \sim \sqrt{|\lambda|} \exp\left[-2|\alpha|^{-1}|\lambda|^{1/2} \log\left(\frac{|\lambda|}{|\alpha|\rho}\right)\right] \text{ as } \lambda \to -\infty$$

(Magarill–Entin, see [16]).

- When $d \geq 1$ and $H_\omega = -\Delta + \sum_{\gamma\in\Gamma_\omega} V_0(x-\gamma)$, V_0 is a continuous, real-valued function, $V_0 \leq 0$, and V_0 has global non-degenerate minimum at $x = 0$,

$$\log N(\lambda) = -\frac{|\lambda|}{|V_0(0)|} \log|\lambda| \cdot (1 + o(1)) \quad \text{as } \lambda \to -\infty$$

(Pastur, see [19]).

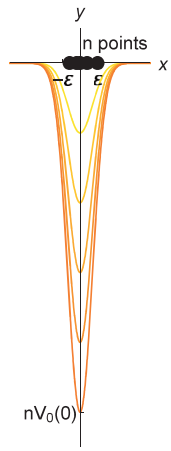
We give a heuristic explanation of the above result by Pastur. When $H_\omega = -\Delta + \sum_{\gamma\in\Gamma_\omega} V_0(x-\gamma)$, $V_0 \leq 0$, and $V_0(0)$ is the global minimum, negative spectrum λ is created by at least

$$n = \frac{|\lambda|}{|V_0(0)|}$$

Poisson points in a small ball B_ϵ. The probability of this event is

$$p \doteqdot e^{-\rho|B_\epsilon|} \frac{(\rho|B_\epsilon|)^n}{n!}.$$

Combining this estimate with the Stirling formula $n! \sim (2\pi n)^{1/2}(n/e)^n$, we get the super exponential decay of IDS.



Surprisingly, the author recently finds that IDS decays *polynomially* as $N(\lambda) \to -\infty$, in the case of the three-dimensional point interactions of Poisson–Anderson type.

Theorem 5. *Let $d = 3$. Suppose $(\Gamma_\omega, \alpha_\omega)$ satisfies Assumption 2, and $\operatorname{supp}\nu$ is a bounded set in $\mathbb{R}$. Let $N(\lambda)$ be the corresponding IDS for $H_{\Gamma_\omega,\alpha_\omega}$. Let t_0 be the unique positive solution of $t = e^{-t}$ ($t_0 = 0.5671...$). Then, we have*

$$\lim_{\lambda\to-\infty} \frac{N(\lambda)}{|\lambda|^{-3/2}} = \frac{2\pi}{3} t_0^3 \rho^2. \tag{2}$$

Theorem 5 gives the principal term of the asymptotics of $N(\lambda)$ as $\lambda \to -\infty$. The second term is unknown at present (it may depend on $\operatorname{supp}\nu$). The limit is independent of $\operatorname{supp}\nu$, provided that $\operatorname{supp}\nu$ is *bounded*. But the speed of convergence depends on $\operatorname{supp}\nu$.

4 Outline of the proof of Theorem 5

In this section, we give an outline of the proof of Theorem 5. The detailed proof will be given elsewhere.

It is known that the negative eigenvalues of $H_{Q_L,\Gamma,\alpha}$ are monotone increasing with respect to each α_γ. If $\alpha_- \le \alpha_\gamma \le \alpha_+$ for every γ, then

$$N_{Q_L,\Gamma,\alpha_+}(\lambda) \le N_{Q_L,\Gamma,\alpha}(\lambda) \le N_{Q_L,\Gamma,\alpha_-}(\lambda).$$

Since the right hand side of (2) is independent of α_γ, we may assume α is a constant sequence.

Of course, the heuristic explanation in the previous section does not work in the case of point interactions. However, there are several tools available only in the case of point interactions. For example, the spectrum $\sigma(H_{\Gamma,\alpha})$ for finite Γ can be calculated explicitly (see [2]).

Theorem 6 (Spectrum of $H_{\Gamma,\alpha}$ for finite Γ)**.** *Let $d = 3$. Let $\Gamma = \{\gamma_j\}_{j=1}^N$ be a finite set and $\alpha = (\alpha_j)_{j=1}^N$ (we write $\alpha_j = \alpha_{\gamma_j}$). Then, for $\lambda = -s^2$ $(s > 0)$, λ is an eigenvalue of $H_{\Gamma,\alpha}$ if and only if* $\det A = 0$*, where $A = (a_{jk})$ is the $N \times N$ matrix given by*

$$a_{jk} = \begin{cases} \alpha_j + \dfrac{s}{4\pi} & (j = k), \\ -\dfrac{e^{-s|\gamma_j - \gamma_k|}}{4\pi|\gamma_j - \gamma_k|} & (j \ne k). \end{cases}$$

In the case $\#\Gamma = 2$, $|\gamma_1 - \gamma_2| = R$ and $\alpha_1 = \alpha_2 = \alpha$, the value $\lambda = -s^2$ $(s > 0)$ is an eigenvalue of $H_{\Gamma,\alpha}$ if and only if 0 is an eigenvalue of

$$A = \begin{pmatrix} \alpha + \dfrac{s}{4\pi} & -\dfrac{e^{-sR}}{4\pi R} \\ -\dfrac{e^{-sR}}{4\pi R} & \alpha + \dfrac{s}{4\pi} \end{pmatrix}.$$

In other words,

$$\alpha + \frac{1}{4\pi}\left(s + \frac{e^{-sR}}{R}\right) = 0 \Leftrightarrow s + \frac{e^{-sR}}{R} = -4\pi\alpha, \text{ or}$$
$$\alpha + \frac{1}{4\pi}\left(s - \frac{e^{-sR}}{R}\right) = 0 \Leftrightarrow s - \frac{e^{-sR}}{R} = -4\pi\alpha.$$

A short analysis of these equations leads the following result.

Lemma 7. *Let $d = 3$, $\Gamma = \{\gamma_1, \gamma_2\}$, $|\gamma_1 - \gamma_2| = R$ and $\alpha_1 = \alpha_2 = \alpha$. Let t_0 be the unique positive solution of $t = e^{-t}$ ($t_0 \doteqdot 0.5671...$).*

(i) When $\alpha = 0$, $H_{\Gamma,\alpha}$ has only one negative eigenvalue

$$\lambda_{1,R} = -\left(\frac{t_0}{R}\right)^2$$

for every $R > 0$.

(ii) When $\alpha \neq 0$, for any $0 < \epsilon < t_0$, there exists $R_0 = R_0(\alpha, \epsilon) > 0$ such that $H_{\Gamma,\alpha}$ has at least one eigenvalue $\lambda_{1,R}$ satisfying

$$-\left(\frac{t_0+\epsilon}{R}\right)^2 \leq \lambda_{1,R} \leq -\left(\frac{t_0-\epsilon}{R}\right)^2 \quad \text{for } 0 < R < R_0.$$

In both cases, the eigenvalue $\lambda_{1,R} \to -\infty$ as $R \to +0$. On the other hand, in the case of the operator $-\Delta + V_0(x-\gamma_1) + V_0(x-\gamma_2)$ with scalar potential V_0, the infimum of the spectrum is always bounded below by $2 \inf V_0$, independent of the positions of γ_1 and γ_2. This fact shows us a qualitative difference between the Schrödinger operators with point interactions and those with scalar potentials.

According to Lemma 7, roughly speaking, we get a very small eigenvalue $\lambda_{1,R} = -(t_0/R)^2$ if we find a very close pair $\{\gamma_1, \gamma_2\}$ with $|\gamma_1 - \gamma_2| = R$. So, in order to count the eigenvalues less than λ, it is sufficient to count the number of *pairs* $\{\gamma_1, \gamma_2\} \subset \Gamma_\omega \cap Q_L$ such that $|\gamma_1 - \gamma_2| = R$ and

$$\lambda_{1,R} < \lambda \Leftrightarrow R < \frac{t_0}{\sqrt{|\lambda|}}.$$

In the sequel, we denote $B_\gamma(R) = \{x \in \mathbb{R}^d \mid |x-\gamma| < R\}$ and

$$n_\gamma(R) = \#(B_\gamma(R) \cap \Gamma_\omega).$$

If there is a pair $\{\gamma_1, \gamma_2\} \subset \Gamma_\omega \cap Q_L$ with $|\gamma_1 - \gamma_2| < R$, then $n_{\gamma_1}(R) \geq 2$ and $n_{\gamma_2}(R) \geq 2$. The following proposition shows that the effect of the points γ with $n_\gamma(R) \geq 3$ is smaller than the main term, in the estimate of IDS.

Proposition 8. *Let $d = 3$ and t_0 as in Lemma 7. Then, for $0 < \epsilon < t_0$, there exists a constant $R_0 > 0$ dependent only on ϵ and α such that*

$$\mathbb{E}\left[N_{Q_L}\left(-\left(\tfrac{t_0-\epsilon}{R}\right)^2\right)\right] \geq \frac{1}{2}\mathbb{E}\left[\#\{\gamma \in \Gamma_\omega \cap Q_L \mid n_\gamma(R) = 2\}\right] - O(R^6|Q_L|),$$
$$\mathbb{E}\left[N_{Q_L}\left(-\left(\tfrac{t_0+\epsilon}{R}\right)^2\right)\right] \leq \frac{1}{2}\mathbb{E}\left[\#\{\gamma \in \Gamma_\omega \cap Q_L \mid n_\gamma(R) = 2\}\right] + O(R^6|Q_L|)$$

for any R with $0 < R < R_0$.

Thus the proof is reduced to the estimate of $\mathbb{E}\left[\#\{\gamma \in \Gamma_\omega \cap Q_L \mid n_\gamma(R) = 2\}\right]$. Estimates of this kind are well studied in percolation theory (see e.g. Meester–Roy [17]), and we have the following result.

Proposition 9. *For $L > 1$ and sufficiently small $\epsilon > 0$, there exists $R_0 > 0$ dependent only on ρ and ϵ (independent of L, R) such that*

$$\left(\frac{4\pi}{3}\rho^2 - \epsilon\right) R^3|Q_L| \leq \mathbb{E}\left[\#\{\gamma \in \Gamma_\omega \cap Q_L \mid n_\gamma(R) = 2\}\right] \leq \left(\frac{4\pi}{3}\rho^2 + \epsilon\right) R^3|Q_L|$$

for any $0 < R < R_0$.

Combining Proposition 8 and Proposition 9, we obtain Theorem 5.

5 Numerical Verification

In order to verify Theorem 5, we shall calculate IDS numerically, in the following way.

(i) Generate $n = \#(\Gamma_\omega \cap Q_L)$, which obeys the Poisson distribution with parameter ρL^3.

(ii) Generate n random points γ in Q_L, obeying the uniform distribution in Q_L independently.

(iii) Divide the interval $[0, 10]$ into 1000 small intervals.

(iv) At each break point s_j, calculate the determinant $\det A(s_j)$.

(v) If the signs of $\det A(s_j)$ and $\det A(s_{j+1})$ are different, then there exists at least one eigenvalue in $[-s_{j+1}^2, -s_j^2]$. In this case, we count *one* eigenvalue in this interval (if more than one eigenvalues are in this interval, we miss some eigenvalues).

(vi) Calculate the number of eigenvalues less than $-s_j^2$, and divide it by L^3.

(vii) Repeat the procedure (i)-(vi) many times, and take the average.

We code the above algorithm by R-language. The above procedure gives us an approximation of

$$\frac{\mathbb{E}\left[N_{Q_L}(-s^2)\right]}{L^3} \quad (0 \leq s \leq 10),$$

which is close to $N(-s^2)$ if L is sufficiently large and the intervals $|s_{j+1} - s_j|$ are sufficiently small. We choose the parameters as follows.

Size of box Q_L	$L = 5$
Intensity of the Poisson configuration	$\rho = 1$
Interaction parameter	$\alpha = 0$ (constant)

After the calculation, we compare the results with the asymptotic formula

$$N(-s^2) \sim \frac{2\pi}{3} t_0^3 \rho^2 s^{-3} \quad (s \to \infty). \tag{3}$$

The result is given in Figure 2. The curve converging to finite positive value as $s \to +0$ is the graph of the numerical IDS $N(-s^2)$, and the curve diverging to ∞ as $s \to +0$ is the graph of the asymptotic formula $N_{\mathrm{asymp}}(-s^2) = \frac{2\pi}{3} t_0^3 \rho^2 s^{-3}$. And the third curve is the ratio $N(-s^2)/N_{\mathrm{asymp}}(-s^2)$, which is close to 1 for large s, as stated in Theorem 5. We do not count the eigenvalues λ with $\lambda < -100$, so we *assume* both sides of (3) coincide at $s = 10$, in Figure 2.

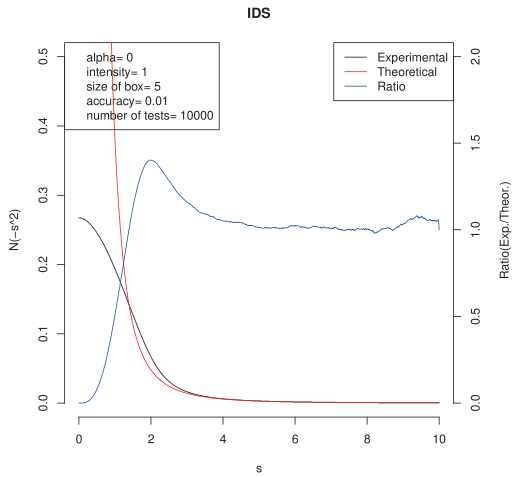


Figure 2: Comparison of the numerical IDS and the asymptotic formula.

6 Future objective

There are still many unsolved problems about the Poisson–Anderson point interactions.

(1) How is the asymptotics of IDS as $\lambda \to -\infty$ in the two-dimensional case?

(2) How is the higher order asymptotics of $N(\lambda)$?

(3) Does the Anderson localization occur near $\lambda = -\infty$?

(4) How is the level statistics of negative eigenvalues?

(5) Does the absolutely continuous spectrum appear near $\lambda = \infty$ in the three-dimensional case?

The problem (1) can be solved in a similar way, but the order of the asymptotics should be different. (2) might be a difficult problem, since we must analyze $\sigma(H_{\Gamma,\alpha})$ for $\#\Gamma \geq 3$. (3) and (4) are solved in the case of Anderson point interaction (see [4], [11, 12]), and the generalization to the Poisson–Anderson point interaction is an interesting problem. (5) is a challenging problem, not solved even in the case of the Schrödinger operators with Poisson random *scalar* potentials. However, there is an explicit resolvent formula for the operator $H_{\Gamma,\alpha}$, so we can analyze the operator $H_{\Gamma,\alpha}$ in more detail than the usual Schrödinger operator $-\Delta + V$. The author hopes that the analysis of $H_{\Gamma,\alpha}$ gives a clue to solve the problem (5).

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