Note on weighted q-Fock spaces and q-orthogonal polynomials

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Abstract

In this short note, we shall discuss weighted q-Fock spaces, field operators and their vacuum distributions, which have strong connections with q-orthogonal polynomials including discrete q-Hermite I polynomials. One can see that our general approach can treat not only known examples scattered in [1][5][8][9][10][13], but also can involve non-trivial and interesting examples, which were not referred in previous works [5][11]. This is a summary paper of our paper [4].

1 Weighted q-Deformation

Let $\mathscr{H}$ be a complex Hilbert space equipped with the inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, where the inner product is linear on the right and conjugate linear on the left. Let $\mathcal{F}_{\text{fin}}(\mathscr{H})$ denote the algebraic full Fock space over $\mathscr{H}$,

$$\mathcal{F}_{\text{fin}}(\mathscr{H}) := \mathbb{C}\Omega \oplus \bigoplus_{n=1}^{\infty} \mathscr{H}^{\otimes n},$$

where Ω denotes the vacuum vector. We note that elements of $\mathcal{F}_{\text{fin}}(\mathscr{H})$ are expressed as finite linear combinations of the elementary vectors $f_1 \otimes \cdots \otimes f_n \in \mathscr{H}^{\otimes n}$. We equip $\mathcal{F}_{\text{fin}}(\mathscr{H})$ with the inner product

$$\langle f_1 \otimes \cdots \otimes f_m, g_1 \otimes \cdots \otimes g_n \rangle_0 := \delta_{m,n} \prod_{k=1}^{n} \langle f_k, g_k \rangle, \quad f_k, g_k \in \mathscr{H}.$$

For $q \in (-1,1)$, define the q-symmetrization operator on $\mathscr{H}^{\otimes n}$ as

$$P_q^{(n)} = \sum_{\sigma \in \mathfrak{S}_n} q^{\ell(\sigma)} \sigma, \quad n \geq 1,$$
$$P_q^{(0)} = I_{\mathscr{H}^{\otimes 0}}, \ P_0^{(n)} = I_{\mathscr{H}^{\otimes n}},$$

where we put $0^0 = 1$ and $\mathscr{H}^{\otimes 0} = \mathbb{C}\Omega$ by convention, $\mathfrak{S}_n$ denotes the n-th symmetric group of permutations and $\ell(\sigma)$ means the number of inversion of a permutation $\sigma \in \mathfrak{S}_n$ defined by

$$\ell(\sigma) = \#\{(i,j) \mid 1 \leq i < j \leq n, \ \sigma(i) > \sigma(j)\}.$$

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Definition 1.1 ([11])**.** Let $\{\tau_n\}_{n=1}^{\infty}$ be a sequence of strictly positive numbers and $[\tau_n]! := \prod_{i=1}^{n} \tau_i$. The τ-weighted q-symmetrization operators on $\mathscr{H}^{\otimes n}$ and $\mathcal{F}(\mathscr{H})$, respectively, are defined by

$$T_q^{(0)} = P_q^{(0)}, \quad T_q^{(n)} = [\tau_n]! P_q^{(n)}, \ n \geq 1,$$
$$T_q = \bigoplus_{n=0}^{\infty} T_q^{(n)}.$$

Since $P_q^{(n)}$ and $\{\tau_n\}_{n=1}^{\infty}$ are a strictly positive operator and sequence, respectively, the τ-weighted q-inner product is defined by

$$\langle f_1 \otimes \cdots \otimes f_m, g_1 \otimes \cdots \otimes g_n \rangle_{q,\{\tau_n\}} := \ \delta_{m,n} \langle f_1 \otimes \cdots \otimes f_m, T_q^{(n)} (g_1 \otimes \cdots \otimes g_n) \rangle_0.$$

Let $\mathcal{F}_{q,\{\tau_n\}}(\mathscr{H})$ denote the τ-weighted (generalized) q-Fock space. In this paper, we do not take completion. The τ-weighted q-creation operator $b^{\dagger}_{q,\{\tau_n\}}(f)$ is defined as the usual left creation operator and $b_{q,\{\tau_n\}}(f)$ is its adjoint with respect to $\langle \cdot, \cdot \rangle_{q,\{\tau_n\}}$, that is, $b_{q,\{\tau_n\}} = (b^{\dagger}_{q,\{\tau_n\}})^*$.

Proposition 1.2. *(1) The τ-weighted q-annihilation operator $b_{q,\{\tau_n\}}$ acting on the elementary vectors is given as follows:*

$$b_{q,\{\tau_n\}}(f)\Omega = 0, \quad b_{q,\{\tau_n\}}(f) f_1 = \tau_1 \langle f, f_1 \rangle \, \Omega, \quad f \in \mathscr{H},$$
$$b_{q,\{\tau_n\}}(f)(f_1 \otimes \cdots \otimes f_n) = \tau_n \sum_{k=1}^{n} q^{k-1} \langle f, f_k \rangle \, f_1 \otimes \cdots \otimes \overset{\vee}{f}_k \otimes \cdots \otimes f_n, \quad n \geq 2,$$

where $\overset{\vee}{f}_k$ means that f_k should be deleted from the tensor product.
(2) The τ-weighted q-creation and annihilation operators satisfy

$$b_{q,\{\tau_n\}}(f) b^{\dagger}_{q,\{\tau_n\}}(g) - q\beta_N b^{\dagger}_{q,\{\tau_n\}}(g) b_{q,\{\tau_n\}}(f) = \langle f, g \rangle \tau_{N+1}, \quad f, g \in \mathscr{H},$$

where $\{\beta_n := \tau_{n+1}/\tau_n\}_{n=1}^{\infty}$ and operators β_N and τ_N are defined as

$$\begin{cases} \varphi_N \Omega = \Omega, \ \varphi_N (f_1 \otimes \cdots \otimes f_n) = \varphi_n (f_1 \otimes \cdots \otimes f_n), \quad n \geq 1, \\ \varphi \in \{\beta, \tau\}. \end{cases}$$

Corollary 1.3. *Suppose $\tau_1 = 1$ and $\beta_n = Q > 0$ for $n \geq 1$. The following commutation relation holds:*

$$b_{q,\{\tau_n\}}(f) b^{\dagger}_{q,\{\tau_n\}}(g) - qQ b^{\dagger}_{q,\{\tau_n\}}(g) b_{q,\{\tau_n\}}(f) = \langle f, g \rangle \tau_{N+1}, \quad f, g \in \mathscr{H}.$$

2 Examples

Let us begin with the following examples to proceed our discussion.

Example 2.1. Suppose $\tau_1 = 1$ and $q \in (-1, 1)$.
(1) $Q = 1$ implies $\tau_n = \tau_2 > 0, \ n \geq 2$. If we set $\tau_2 = t$, then one can get $T_q^{(n)} = t^{n-1} P_q^{(n)}$ and the $(q,t)_W$-Fock space in the sense of Wojakowski [12]. If we take $q = 0$, one can derive the t-free deformation done by Bożejko-Wysoczańsky [9][10]. Moreover, if $\tau_n = 1$ for all $n \geq 1$, one can recover the well-known q-Fock space of Bożejko-Speicher [8] (See also [7]).
(2) If $Q = s^2, s \in (0, 1]$, then we have $\tau_n = s^{2(n-1)}, \ n \geq 1$. One can get $T_q^{(n)} = s^{n(n-1)} P_q^{(n)}$ and the $(q,s)_{BY}$-Fock space by Bożejko-Yoshida [11]. The s-free deformation of Yoshida [13] can be

derived if $q = 0$. Moreover, one can see that a limiting case of $(q, s)_{BY}$ as $q \to 1$ coincides with the Q^N-deformation of the Boson Fock space [1].
(3) The Boolean Fock space can be derived as a limiting case of the $(0, t)_W$-Fock space as $t \to 0$ and also $(0, s)_{BY}$-Fock space as $s \to 0$.

One can derive a further deformation from (2) in Example 2.1. We shall show the relationship between Blitvić [5] construction and ours.

Remark 2.2. In [6], Bożejko-Ejsmont-Hasebe constructed the (α, q)-Fock space, which is different from the (q, t)-Fock space by Blitvić [5]. In this note, the expression "$\{q, t\}$" will be used to refer a symbol "(q, t)" to avoid confusions with the (α, q)-deformation.

In fact, if we replace s^2 by $t > 0$ in (2) of Example 2.1, then we have $\tau_n = t^{n-1}$ and $[\tau_n]! = t^{\binom{n}{2}}$ for $n \geq 1$. In addition, if one considers the (q/t)-symmetrization operator $P^{(n)}_{q/t}$, which is strictly positive for $|q| < t$, then one can consider the weighted (q/t)-symmetrization operator in forms of $T^{(0)}_{q/t} = P^{(0)}_{q/t}$ and $T^{(n)}_{q/t} = t^{\binom{n}{2}} P^{(n)}_{q/t}$, $n \geq 1$. From now on, we set

$$Q^{(0)}_{q,t} = T^{(0)}_{q/t}, \quad Q^{(n)}_{q,t} = T^{(n)}_{q/t}, \; n \geq 1, \; |q| < t,$$
$$Q_{q,t} = \bigoplus_{n=0}^{\infty} Q^{(n)}_{q,t},$$

which are called the $\{q, t\}$-symmetrization operators on $\mathscr{H}^{\otimes n}$ and $\mathcal{F}(\mathscr{H})$, respectively. An inner product defined by

$$\langle f_1 \otimes \cdots \otimes f_m, g_1 \otimes \cdots \otimes g_n \rangle_{q,t} := \; \delta_{m,n} \langle f_1 \otimes \cdots \otimes f_m, Q^{(n)}_{q,t}(g_1 \otimes \cdots \otimes g_n) \rangle_0$$

is called the $\{q, t\}$-inner product, which is the $(q/t, \sqrt{t})_{BY}$-inner product. The free Fock space equipped with this $\{q, t\}$-inner product is called the $\{q, t\}$-Fock space denoted by $\mathcal{F}_{q,t}(\mathscr{H})$. Therefore, we have seen the following propositions:

Proposition 2.3. *Suppose $q \in (-1, 1)$, $t \in (0, 1]$ and $|q| < t$. The $(q/t, \sqrt{t})_{BY}$-Fock space is equivalent to the $\{q, t\}$-Fock space in the sense of [5].*

The $\{q, t\}$-creation operator $a^{\dagger}_{q,t}(f)$ is defined as the usual left creation operator and $\{q, t\}$-annihilation operator $a_{q,t}(f)$ as its adjoint with respect to $\langle \cdot, \cdot \rangle_{q,t}$. By replacing q by q/t and setting $Q = t$, $\tau_n = t^{n-1}$ in Proposition 1.2 and Corollary 1.3, then one can get the following proposition.

Proposition 2.4. *(1) The $\{q, t\}$-annihilation operator $a_{q,t}$ acting on the elementary vectors is given as follows:*

$$a_{q,t}(f)\Omega = 0, \quad a_{q,t}(f)f_1 = \langle f, f_1 \rangle \, \Omega, \quad f \in \mathscr{H},$$
$$a_{q,t}(f)(f_1 \otimes \cdots \otimes f_n) = t^{n-1} \sum_{k=1}^{n} \left(\frac{q}{t}\right)^{k-1} \langle f, f_k \rangle \, f_1 \otimes \cdots \otimes \overset{\vee}{f_k} \otimes \cdots \otimes f_n \quad n \geq 2, \tag{2.1}$$

where $\overset{\vee}{f_k}$ means that f_k should be deleted from the tensor product.
(2) The $\{q, t\}$-creation and annihilation operators satisfy

$$a_{q,t}(f)a^{\dagger}_{q,t}(g) - q a^{\dagger}_{q,t}(g) a_{q,t}(f) = \langle f, g \rangle t^N, \quad f, g \in \mathscr{H},$$

where the operator t^N is defined by

$$t^N \Omega = \Omega, \quad t^N(f_1 \otimes \cdots \otimes f_n) = t^n f_1 \otimes \cdots \otimes f_n, \quad n \geq 1.$$

We would like to consider the spectral measure (vacuum distribution) of the $\{q,t\}$-Gaussian (field) operator $g_{q,t}(f)$ on $\mathcal{F}_{q,t}(\mathscr{H})$ defined by

$$g_{q,t}(f) := a^{\dagger}_{q,t}(f) + a_{q,t}(f), \quad f \in \mathscr{H},$$

with respect to the vacuum state $\langle \Omega, \cdot \Omega \rangle_{q,t}$. Orthogonal polynomials play important roles to compute a distribution of such a field operator with respect to the vacuum state. In [5], the $\{q,t\}$-Hermite polynomials given by the recurrence relation,

$$\begin{aligned} H_0(x;q,t) &= 1,\ H_1(x;q,t) = x, \\ xH_n(x;q,t) &= H_{n+1}(x;q,t) + [n]_{q,t} H_{n-1}(x;q,t),\ n \geq 1, \end{aligned}$$

where $[n]_{q,t} := t^{n-1}[n]_{q/t}$ are mentioned. Note that $[n]_q := [n]_{q,1} = \sum_{k=0}^{n-1} q^k$ and $[n]_{q,q} = q^{n-1}n$. However, concrete densities of orthogonalizing measures are not mentioned except for a very restricted case, $0 = q < t$. We have been seeking examples for $q \neq 0$, which can be treated within the $\{q,t\}$-deformation. In this paper, we shall present not only recognized examples, but also unrecognized ones in [5][11] as follows.

Example 2.5. Let us consider the $\{qs^2, s^2\}$-deformation for $q \in (-1,1), s \in (0,1]$. This deformation is of interest and quite fruitful.

(I) The $\{qs^2, s^2\}$-Gaussian (field) operator is equal to the $(q,s)_{BY}$-Gaussian (field) operator. The $\{q, s^2\}$-deformation is different from the $(q,s)_{BY}$ except for $q = 0$ or $s = 1$.

(II) In addition, the probability density for $(q,s)_{BY}$ case is known for the following three cases: (1) $s = 1$, $q \in (-1,1)$ in [7][8], (2) $s \in (0,1]$, $q = 0$ in [5][13], and (3) $s = \sqrt{|q|}$, $|q| \in (0,1)$.

The case (1) is obvious at this time and provides the (Roger's continuous) q-Hermite polynomials. Therefore, one can obtain the q-Gaussian operator ([7][8]).

In case (2), it is known that the $\{0,t\}$-Hermite polynomials are the t-Chebyshev II polynomials ($q = 0 < t \leq 1$ and set $t = s^2$) . The $\{0,1\}$-Gaussian measure is the semicircular measure. If $t \neq 1$, the $\{0,t\}$-Gaussian measure is known to be a discrete probability measure with atoms at which are represented by the zeros of the t-Airy function (See [5] and references cited therein). The $\{0,t\}$-Gaussian (field) operator is the same as the $(0,\sqrt{t})_{BY}$-Gaussian (field) operator, which is nothing but the s-free Gaussian (field) operator [13]. Moreover, the limiting case $s \to 0$ implies the Boolean Gauss (field) operator, whose distribution is $\frac{1}{2}(\delta_1 + \delta_{-1})$. The case (3) is not referred as a particular example in [5][11]. One can see that the $\{q^2, |q|\}$-Hermite polynomials are identified as a rescaled version of discrete $|q|$-Hermite I polynomials[1]. Let μ_q denote the orthogonalizing measure[2] for the discrete q-Hermite I polynomials. Correspondingly, the rescaled orthogonalizing measure of $\mu_{|q|}$ is given by $D_{1/\sqrt{1-|q|}}\mu_{|q|}, |q| \in (0,1)$, where D_λ denotes the dilation of a probability measure μ by $D_\lambda \mu(\cdot) = \mu(\cdot/\lambda)$, $\lambda \neq 0$. Moreover, the $\{q^2, |q|\}$-Gaussian (field) operator coincides with the $(q, \sqrt{|q|})_{BY}$-Gaussian (field) operator.

(III) Furthermore, since the $(q,s)_{BY}$-Fock space as $q \to 1$ coincides with the Q^N-deformation of the Boson Fock space mentioned in (2) of Example 2.1, a limiting case of the $\{qs^2, s^2\}$-Gaussian (field) operator as $q \to 1$ agrees with the Q^N-deformation of the classical Gaussian (field) operator [1]. It is our paper [4] which first points out this nontrivial relationship of interest.

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[1] It is known that the discrete q-Hermite I polynomials are a symmetric case of Al-Salam-Carlitz I polynomials. In addition, the discrete q-Hermite I polynomials belong to the class IV of Brenke-Chihara polynomials. See [2][4] and references therein.

[2] μ_q is expressed as an infinite sum of atoms on $\{0, \pm q^k : k = 0, 1, \dots\}$.

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