An extension of the Bloch-Floquet theory to the Heisenberg group and its applications to asymptotic problems for heat kernels and prime closed geodesics

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Abstract

The Bloch-Floquet theory are popular tools for the investigation of materials with periodic structures. For example, we can show that the spectrum of periodic Schrödinger operators have band structures. Here we shall extend the Bloch-Floquet theory, which is appicable to abelian groups, to the Heisenberg group. Our method is based on a combination of the representations of the discrete Heisenberg groups and of the Heisenberg Lie group. We apply this method to asymptotic problems for heat kernels and for counting prime closed geodesics. In this application, we need additional ingradients, the semi-classical analysis and the Chen's iterated integrals. As a by-product, we give another mathematical explanation of the semi-classical asymptotic expansion formula for the Hofstadter butterfly of Wilkinson, which is originally due to Helffer-Sjöstrand.

1 Introduction

As in the abstract, although our main concern in this paper is an extension of the Bloch Floquet theory to the Heisenberg group, we start from the principal application here.

Let M be a compact Riemannian manifold of negative curvature and let $\pi(x)$ denote the number of prime closed geodesics on M whose length is at most x. Celebrated results of Selberg [38], Huber [16], Margulis [27],[28], Parry and Pollicott [29] asserts that

2020 *Mathematics Subject Classification.* Primary 58J50 Secondary 58J37, 58J35, 30F99

Key words and phrases. Bloch-Floquet theory, Heisenberg group, Asymptotic expansion

*This work was supported by JSPS KAKENHI Grant Number JP18K03282 and the Research Institute for Mathematical Sciences, an International Joint Usage/Research Center located in Kyoto University.

THEOREM 1.1.
$$\pi(x) \sim \frac{e^{hx}}{hx},$$
where $h > 0$ is the topological entropy of the geodesics flow on M.

This is called the prime geodesic theorem which is geometric analogue of the prime number theorem. As a variants, there are several results in geometry, which are analogue of the Dirichlet density theorem for arithmetic progressins or the Chebotarev density theorem for algebraic extensions of number fields.

Let us formulate problems precisely. Taking into account that there are one to one correspondence among the following triplet, closed geodesics of M, free homotopy classes of closed curves in M and conjugacy classes of an element in the fundamental group $\pi_1(M)$ of M, we consider a surjective homomorphism

$$\Phi : \pi_1(M) \to \Gamma$$

for a finitely generated discrete group Γ,

For a conjugacy class α in Γ, let $\pi(x, \alpha)$ denote the number of prime closed geodesics γ on M whose length is at most x and satisfies $\Phi([\gamma]) \subset \alpha$, where $[\gamma]$ denotes the corresponding conjugacy class of γ in $\pi_1(M)$. Then our problems is as follows:

PROBLEM 1.2. What is the asymptotic behavior of $\pi(x, \alpha)$ as $x \to \infty$?

When Γ is the trivial group, $\pi(x, \alpha) = \pi(x)$ and the answer of the above problem is given by the prime geodesic theorem. First answer of the above problem is given when Γ is a finite group by Parry and Pollicott [30] and Adachi and Sunada [1] independently as follows:

THEOREM 1.3.
$$\pi(x, \alpha) \sim \frac{\sharp\alpha}{\sharp\Gamma} \frac{e^{hx}}{hx},$$
where $\sharp\alpha$ and $\sharp\Gamma$ denote the cardinals of α and Γ respectively.

This theorem can be considered as a geomtric analogue of usual Chebotarev density theorem for finite extensions of number fields.

Next answer is given when Γ is an infinite abelian group. In the case when M is a compact Riemann surface of genus g with constant negative curvature -1, the following asymptotic result is given by Phillips and Sarnak [31],

THEOREM 1.4.

$$\pi(x, \alpha) \sim (g-1)^g \frac{e^x}{x^{g+1}} \left(1 + \frac{c_1}{x} + \frac{c_2}{x^2} + \ldots\right)$$

The leading term in the above theorem also obtained by Katsuda and Sunada[19] independently. The essential invariant to the numerator $(g-1)^g$ of the leading term is the volume of the Jacobi torus which is equalto 1 in this case. The genus g in the exponent of the denominator is the half of the first Betti number i.e. the rank of the first integral homology group $H_1(M,\mathbb{Z})$ of M.

This result is generalized to the case of prime closed geodesics in compact Riemannian manifolds with variable negative curvatuire or more generally, prime closed orbits of weakly mixing Anosov flows on compact Riemannian manifolds by Lalley [26], Pollicott Pollicott1, Katsuda and Sunada [20] independently for the leading terms and Anantharaman [3], Pollicott and Sharp [33] and Kotani [21] for asymptotic expansions. Furthermore, there are genelarizations to the central limit theorems and the large deviations [26], [6], [4].

The case we wish to consider here is that Γ is a discrete nilpotent group. This generalization seems to be natural by the following reason: when Γ is an abelian group, it is a quotient of the maximal abelian group which is isomorphic to $H_1(M,\mathbb{Z})$. By the Hurewicz theorem, we have $H_1(M,\mathbb{Z}) \simeq \pi_1(M)/[\pi_1(M),\pi_1(M)]$ where $[\pi_1(M),\pi_1(M)]$ denotes the commutator subgroup of $\pi_1(M)$ and $\pi_1(M)/[\pi_1(M),\pi_1(M)]$ is the abelianeization of $\pi_1(M)$. An example of nilpotent group naturally arises as $\Gamma \simeq \pi_1(M)/[\pi_1(M),[\pi_1(M),\pi_1(M)]]$ where $[\pi_1(M),[\pi_1(M),\pi_1(M)]]$ denotes the double commutator subgroup of $\pi_1(M)$.

For general nilpotent groups, we have the following conjecture:

Conjecture 1.5. Let Γ be a finitely generated nilpotent group and α be a conjugacy class of a central element of Γ,

$$\pi(x,\alpha) \sim C\frac{e^{hx}}{x^{1+d/2}}\left(1+\frac{c_1}{x}+\frac{c_2}{x^2}+\dots\right)$$

, where C is a constant depending on the geometry of M and the special values $\zeta_H(d/2)$ of the spectral zeta function of a hypo-elliptic operator H related to some irreducible representaions the simply connected nilpotent Lie group G which contains Γ as a lattice subgroup, i.e. G is the Malcev completion of Γ, and d is the polynomial growth order of Γ. Moreover constants $c_1, c_2, \dots$ are also written in the quantities of M related to the Chen's iterated integrals.

Note that for a finitely generated discrete nilpotent group Γ, the polynomial growth order d of Γ can be defined as follows: Take a finite generating set S of Γ and let $\omega(k)$ be the number of elements of Γ whose word length with respect to S are less than or equal to k. Then it is known that there are constants $C_1, C_2 > 0$ and positive integer d such that

$$C_1k^d \le \omega(k) \le C_2k^d$$

and d is independent to the choice of S.

As an application of our extension of Bloch-Floquet theory, we can show that Conjecture 1.5 holds in the case when Γ is the three dimensional discrete Heisenberg group $\mathrm{Heis}_3(\mathbb{Z})$ and M is a compact Riemann surface with constant negative curvature -1 as follows:

Theorem 1.6. *Let M be a compact Rieman surface with the constant negative curvature -1 and Γ be $Heis_3(\mathbb{Z})$. For a conjugacy class α of a central element of Γ, we have the following asymptotic expansion:*

$$\pi(x,\alpha) \sim \frac{Ce^x}{x^3}(1+\frac{c_1}{x}+\frac{c_2}{x^2}+\ldots),$$

where constants $C, c_1, c_2\ldots$ are described in Conjecture 1.5.

We notice that there is a surjective homomorphism from $\pi_1(M)$ of Riemannian surface M of genus $g \geq 2$ to $\Gamma = \mathrm{Heis}_3(\mathbb{Z})$ through the surjective homomorphism from $\pi_1(M)$ to the free group F_g of rank g. Since the polynomial growth order d is 4 in this case and thus, the exponent of denominator of the leading term is $3 = 1+4/2$. Moreover, the hypoelliptic operator H in Conjecture 1.5 is essentially the harmonic oscillator $\mathcal{H} := -\frac{d^2}{du^2}+u^2$ and the special value $\zeta_H(2)$ can be expressed by the value $\zeta(2) = \pi^2/6$ of the Riemann zeta function $\zeta(s)$.

If a conjugacy class α is not coming from central elements, then the asymptotic behavior of $\pi(x,\alpha)$ can be reduced to the analysis of the case for abelian groups, which is the following.

Proposition 1.7. *Let M be a compact Rieman surface with the constant negative curvature -1 and Γ be $Heis_3(\mathbb{Z})$. For a conjugacy class α of a non-central element of Γ, we have the following asymptotic expansion:*

$$\pi(x,\alpha) \sim \frac{Ce^x}{x^2}(1+\frac{c_1}{x}+\frac{c_2}{x^2}+\ldots). \tag{1.1}$$

In this case, the exponent of denominator of the leading term is express as $2 = 1+2/2$ whose numerator of the second term 2 is the rank of an abelian group $\Gamma/[\Gamma,\Gamma]$.

Next we consider the long time asymptotics of the heat kernels on Γ-covering manifolds. We can also apply our extension of the Bloch-Floquet analysis. In the case when Γ is an abelian group, the similarity of the analysis of the closed geodesics and the heat kernels was pointed by Sunada. In the case when Γ is nilpotent, Boulanger [9] reduces the asymptotics of the prime closed geodesics for manifolds with costant negative curvature to that of the heat kernels.

Let M be a compact Riemannian manifolds or a finite unorieted graph. For a discrete group Γ, take a normal covering $\Pi : X \to M$ with the covering transformation group

Γ. We denote by $k_X(t,p,q)$ the heat kernel (resp. the transition probability of simple random walks) of X when M is a compact Riemannian manifold (resp. a finite graph). For the brevity of the description of results, we assume that M is not a bipartite graph.

Our problem here is as follows:

PROBLEM 1.8. What is the asymptotic behavior of $k_X(t,p,q)$ as $t \to \infty$?

In the case when Γ is an abelian group, the following results are known.

THEOREM 1.9.

$$k_X(t,p,q) \sim \frac{C}{t^{d/2}}(1 + \frac{c_1}{t} + \frac{c_2}{t^2} + \ldots), \tag{1.2}$$

where d is the rank of abelian group Γ and $C, c_1, c_2, \ldots$ are constants depending on the geometry of M.

This result is due to Kotani and Sunada [23]. The leading term of the above theorem is obtained by several authors [14], [18], [25] and [39] independently. In particular, Kotani, Shirai and Sunada [22] focus on the geometric nature of C which is written in terms of the volume of "Jacobi torus". This was inspired by the corresponding case for closed geodesics in [31], [19] as in Theorem 1.4.

In the case of nilpotent groups, we have the following conjecture for heat kernels, which corresponds to Conjecture 1.5,

CONJECTURE 1.10. Let Γ be a finitely generated nilpotent group.

$$k_X(t,p,q) \sim \frac{C}{t^{d/2}}\left(1 + \frac{c_1}{t} + \frac{c_2}{t^2} + \ldots\right),$$

where constants $C, d, c_1, c_2, \ldots$ are similar to Conjecture 1.5.

In the case when $\Gamma = \mathrm{Heis}_3(\mathbb{Z})$, our result for heat kernels is the following, which corresponds Theorem 1.6.

THEOREM 1.11. *Let Γ be $Heis_3(\mathbb{Z})$.*

$$k_X(t,p,q) \sim \frac{C}{t^2}\left(1 + \frac{c_1}{t} + \frac{c_2}{t^2} + \ldots\right),$$

where constants $C, c_1, c_2 \ldots$ are similar to Theorem 1.6.

It should be noted that in the case when M is a finite graph, the asymtotics for the leading term is already obtained by a combination of Alexopoulos [2] and Ishiwata [17] for general nilpotent groups Γ. However, their method seems not to give geometric nature of the leading coefficient C. In fact, their results is a comparison between $k_X(t,p,q)$ and the heat kernel of a stratified nilpotent Lie groups but gives no information for the latter.

Finally, we point out a relation to the our methods for the Heisenberg group and the analysis of the discrete magnetic Laplacian or the Harper operator on the square lattice $\mathbb{Z}^2$. Spectrum of the latter operators are expressed by the celebrated Hofstadter butterfly as the following figure:

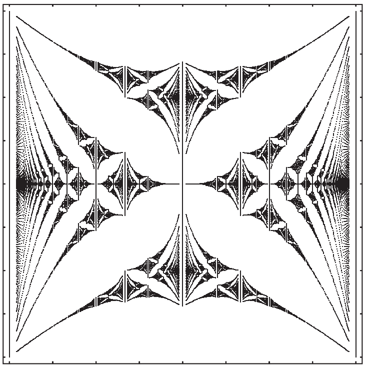

Figure 1: the Hofstadter's butterfly (created by Hisashi Naito). The horizontal axis express the value of θ corresponding to the strength of a magnetic field or the magnetic flux density in the interval $[0, 2\pi]$ and the vertical axis express the spectrum of H_θ which is a subset of the interval $[-4, 4]$.

The Harper operator $H_\theta : \ell^2(\mathbb{Z}^2) \to \ell^2(\mathbb{Z}^2)$ is defined as follows:

$$(H_\theta u)(m, n) = u(m + 1, n) + u(m - 1, n) + e^{\sqrt{-1}n\theta}u(m, n + 1) + e^{-\sqrt{-1}n\theta}u(m, n - 1)$$

This operator is a discrete analogue of the Laplacian on the plane under constant magnetic field.

The structure of the spectrum of H_θ are already extensively studied (cf.[40]). For example, if the parameter θ is a rational number, then the spectrum of H_θ is a union of finite intervals, namely it has band structure and θ is a irrational number, it was conjectured that the spectrum is a Cantor set, which was known as M. Kac's "Ten Martini Problem" and was finially settled by Avila and Jitomirskaya [5]. Our interest here is the following semi-classical asymtoptotic expansion formula of the spectrum of H_θ as $\theta \to 0$ where we regard θ as semi classical parameters (3.9 [35], similar formulas (6.9 in [43] and 6.3.1 in [15]),

$$E_n = -4 + (2n + 1)\theta + O(\theta^2) \qquad n = 0, 1, 2, \ldots \tag{1.3}$$

This formula first obtained by Wilkinson [43] by WKB arguments. Rammal and Bellissard [35] also derived this formula by a similar method. Formal derivation of the above expansion is not difficult as follows: it is known that H_θ is unitally equivalent to the following operator acting on $L^2(\mathbb{R})$,

$$h_\theta = -2\cos(\sqrt{\theta}\frac{d}{\sqrt{-1}du}) - 2\cos(\sqrt{\theta}u).$$

By the (formal) Taylor expansion formula, this operator can be expressed as

$$h_\theta = -4 + \left(-\frac{d^2}{du^2} + u^2\right)\theta + O(\theta^2). \tag{1.4}$$

The coefficient of the linear part in θ is the harmonic oscillator $\mathcal{H} := -\frac{d^2}{du^2} + u^2$ whose eigenvalues are $n + \frac{1}{2}, n = 0, 1, 2, \ldots$, which implies the above expansion (1.3). However, (1.4) is a form that a bounded operator h_θ is approximated by a unbounded operator $\mathcal{H}$. We need to justify the above arguments mathematically. Note that there are other reasons of necessity for mathematical justifications.

This is first given by Helffer and Sjöstrand [15] using semi-classical analysis. Thier method is described very roughly as follows: If the domain $L^2(\mathbb{R})$ of both operators h_θ and $\mathcal{H}$ are exhauseted by a sequence of common invariant finite dimensional spaces $V_k, k = 1, 2, \ldots$, then the expansion 1.4 can be considered as a limit of Taylor expansion of the restriction $h_\theta|_{V_k}$ of h_θ. However there is no such sequence. In stead of V_k, Helffer and Sjöstrand used the space $\tilde{V}_k$ consisting of eigenfunctions of $\mathcal{H}$ associated with eigenvalues less than or equal to k. This space is not invariant by h_θ but the error is $O(h^\infty)$ and thus does not effect the asymptotic expansion 1.3.

We shall give another proof based on a comparison between unitary representations of $\mathrm{Heis}_3(\mathbb{Z})$ and the Heisenberg Lie group $\mathrm{Heis}_3(\mathbb{R})$. We believe our method has some advantage to the original proof, which we shall explain in later.

2 Outline of the proof for the asymptotics of the heat kernels and other results

Since we take the similar strategy in a broad sense, as in the case when Γ is an abelian group, let us recall the proof of Theorem 1.9 for asymptotics of the leading term.

Our concern is a long time asymptotics of the heat kernel $k_X(t, p, q)$ on X for a Γ-covering $\pi : X \to M$. We start from easier case that Γ is a finite group. In this case, X is compact and thus, $k_X(t, p, q)$ can be expressed as follows:

$$k_X(t, p, q) = \sum_{i=0}^{\infty} e^{-\lambda_i t}\varphi_i(p)\varphi_i(q), \tag{2.1}$$

where

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots$$

are eigenvalues of the Laplacian Δ_X on X and $\{\varphi_i\}_{i=0}^{\infty}$ is a complete orthonormal system of eigenfuctions φ_i associated to eigenvalues λ_i. Note that the 0-th (normalized) eigenfuction φ_0 is a costant function $1/\sqrt{\mathrm{vol}(X)}$ with the volume $\mathrm{vol}(X)$ of X. If $i \geq 1$, then $\lambda_i > 0$ and thus $e^{-\lambda_i t}$ decay to 0 if $t \to \infty$. Then, we have

$$\begin{aligned} \lim_{t\to\infty} k_X(t,p,q) &= \lim_{t\to\infty} e^{-\lambda_0 t}\varphi_0(p)\varphi_0(q) \\ &= \varphi_0(p)\varphi_0(q) = \frac{1}{\mathrm{vol}X}. \end{aligned}$$

Next we consider the case that Γ is an infinite abelian group. For simplicity, we assume $\Gamma = \mathbb{Z}^d = H_1(M,\mathbb{Z})$. Essential points in the arguments are already appeared in this case and it is not so difficult to extend to general finitely generated abelian groups. Here X is noncompact. Although there exists a similar formula to (2.1) in this case using the spectral decomposition of Δ_X, it seems useless for our problem. In stead of this formula, we decompose $L^2(X)$ into in some sense "$L^2(M) \otimes L^2(\Gamma)$". In the case when Γ is abelian, the latter space can be analyzed by the usual Bloch-Floquet theory in conjunction with some perturbation arguments, which we explain in the following two steps:

The first step is a decomposition of $L^2(X)$ as above. For this purpose, we formulate the Bloch-Floquet theory geometrically as follows: First we identify $L^2(X)$ with the space $L^2(E_R)$ of sections of the flat vector bundle E_R over M associated with the right regular representation R of Γ. Here E_R is described as follows;

$$E_R \simeq X \times L^2(\Gamma)/\sim, \quad (p,\varphi) \sim (\gamma p, R(\gamma^{-1})\varphi)$$

for $\gamma \in \Gamma$. Since $\Gamma \simeq \mathbb{Z}^d$, R can be written as a direct integral of one dimensional irreducible unitary representations, i.e. characters over the unitary dual $\hat{\Gamma}$ of Γ which is the space of equivalence class of characters $\chi : \Gamma \to U(1)$. Note that $\hat{\Gamma}$ is isomorphic to a d-dimetional torus $U(1)^d$, which can be idetified with the Brillouin zone in usual terminology of the Bloch-Floquet theory in condensed matter physics. Associated with this formula, we have the following decomposition

$$L^2(E_R) \simeq \int_{\hat{\Gamma}}^{\oplus} L^2(E_\chi) d\chi,$$

where $L^2(E_\chi)$ is the space of sections of the flat line bundle associated with χ, which is similarly defined as $L^2(E_R)$. The space $L^2(E_\chi)$ can be identified with the space

$$H_\chi = \{f : X \to \mathbb{C} \mid f(\gamma p) = \chi(\gamma) f(p) \text{ for all } \gamma \in \Gamma\}.$$

Using this identification, the heat kernel $k_X(t,p,q)$ can be decomposed as

$$k_X(t,p,q) = \int_{\hat{\Gamma}} k_\chi(t,p,q)d\chi \tag{2.2}$$

where

$$\begin{aligned} k_\chi(t,p,q) &= \sum_{\gamma\in\Gamma} \chi(\gamma) k_X(t,p,\gamma q) \\ &= \sum_{i=0}^{\infty} e^{-\lambda_i(\chi)t}\varphi_{i,\chi}(p)\varphi_{i,\chi}(q) \end{aligned} \tag{2.3}$$

where $\lambda_i(\chi)$ and $\varphi_{i,\chi}$ are i-th eigenvalue and eigenfunction of the twisted Laplacian $\Delta_\chi := \Delta_X\,\big|_{H_\chi}$. Note that $\lambda_i(\chi)$ are discrete as eigenvalues λ_i of the usual Laplacian Δ_M on M and $\lambda_i = \lambda_i(\mathbf{1})$ for the trivial character $\mathbf{1}$.

The second step is an analysis of eigenvalues $\lambda_i(\chi)$ of the twisted Laplacian Δ_χ with respect to χ. It is not difficult to show that there is a positive constant c such that $\lambda_i(\chi) \geq c$ for $i \geq 1$. We also have $\lambda_0(\chi) \geq 0$ and the equality holds if and only if $\chi = \mathbf{1}$, $\lambda_0(\chi)$ depend smoothly on χ and the first derivative with respect to χ at $\mathbf{1}$ is zero. The essential point is to show the hessian of $\lambda_0(\chi)$ at $\chi = \mathbf{1}$ is positive definite.

To prove this, we use perturbation arguments as follows: Although the usual setting of the perturbation theory is that operators defined on the fixed domain are varied, our situation here are converse that defining domains H_χ are varied but the operator Δ_χ are fixed since it is the restriction of the Laplacian Δ_X on $L^2(X)$ to H_χ.

To reduce usual setting of the perturbation theory, we construct a canonical section s_χ of the line bundle E_χ and identify $L^2(M)$ with $L^2(E_\chi)$ by the correspondence

$$L^2(M) \ni f \longleftrightarrow f s_\chi \in L^2(E_\chi) \simeq H_\chi.$$

Associated with this correspondence, we have a unitary equivalence between the twisted operator $L_\chi := s_\chi^{-1} \circ \Delta_\chi \circ s_\chi$ acting on $L^2(M)$ and the twisted Laplacian Δ_χ acting on $L^2(E_\chi) \simeq H_\chi$.

To construct s_χ, by de Rham-Hodge Theorem, we can take a harmonic 1-form ω satisfying

$$\chi([\gamma]) := \chi_\omega([\gamma]) := \exp(2\pi\sqrt{-1}\int_\gamma \omega)$$

for a closed curve γ and its homology class $[\gamma] \in H_1(M,\mathbb{Z})$.

Since χ is a homomorphism from $\Gamma \simeq H_1(M,\mathbb{Z}) \simeq \pi_1/[\pi_1,\pi_1]$ to an abelian group $U(1)$ where $\pi_1/[\pi_1,\pi_1]$ is the abelianeization of the fundamental group $\pi_1 = \pi_1(M)$ of M, we can lift χ to a character $\tilde{\chi} : \pi_1(M) \to U(1)$ canonically. Then, taking a lift $\tilde{\omega}$ of ω to the universal covering $\tilde{M}$ of M, we define a function $\tilde{s}_{\tilde{\omega}}$ on $\tilde{M}$ by

$$\tilde{s}_{\tilde{\omega}}(p) = \exp(2\pi\sqrt{-1}\int_{p_0}^{p} \tilde{\omega}) \tag{2.4}$$

for some reference point $p_0 \in \tilde{M}$. Note that this is well defined since the line integral $\int_{p_0}^{p} \tilde{\omega}$ does not depend on the choice of a curve from p_0 to p by the homotopy invariance property due to the fact that $\tilde{\omega}$ is a closed form. Since $\tilde{s}_{\tilde{\omega}}(\gamma p) = \chi_\omega(\gamma)\tilde{s}_{\tilde{\omega}}(p)$, this function can be identified with a section $s_\omega = s_{\chi_\omega}$ of $L^2(E_\chi)$.

Then $L_\chi = s_{\chi_\omega}^{-1} \circ \Delta_{\chi_\omega} \circ s_{\chi_\omega}$ can be written as

$$L_\chi f = \Delta_M f - 4\pi\langle \omega, df\rangle + 4\pi^2|\omega|^2 f$$

where $\langle\cdot,\cdot\rangle$ and $|\cdot|$ are the inner product and the norm of cotangent space induced from the Riemannian metric of M respectively.

From this, we can compute the hessian $\mathrm{Hess}_0\lambda_0$ of $\lambda_0(\chi)$ at $\chi = \chi_\omega = \mathbf{1}$ (i.e. $\omega = 0$) which is written as

$$\mathrm{Hess}_0\lambda_0(\omega,\omega) = \frac{8\pi^2}{\mathrm{vol}(M)}\int_M |\omega|^2 dv_g$$

where $\mathrm{vol}(M)$ is the Riemannian volume and dv_g is the Riemannian measure of M.

By the Morse lemma, we can take a local coordinates $(U, (x^1, x^2, \ldots, x^d))$ near the trivial character $\mathbf{1}$ on $\hat{\Gamma} \simeq (U(1))^d$ such that

$$\lambda_0(\chi) = \lambda_0(x^1, x^2, \ldots, x^d) = \sum_{k=1}^{d}(x^i)^2.$$

Then we get the conclusion by the following computation

$$\begin{aligned} k_X(t,p,q) &= \int_{\hat{\Gamma}} k_\chi(t,p,q)d\chi \\ &= \int_{\hat{\Gamma}} \sum_{n=0}^{\infty} e^{-\lambda_n(\chi)t}\varphi_{n,\chi}(p)\varphi_{n,\chi}(q)d\chi \\ &= \int_U e^{-\sum_{i=1}^{d}(x^i)^2 t}dx^1dx^2\ldots dx^d + \text{error term} \\ &\sim \prod_{i=1}^{d}\left\{\frac{1}{\sqrt{t}}\int_{-\infty}^{\infty} e^{-{y^i}^2}dy^i\right\} \\ &= \frac{C}{t^{d/2}}. \end{aligned}$$

where U is a neighborhood of $\mathbf{1}$ in $\hat{\Gamma}$.

For generalization of the above arguments to nilpotent groups, we point out arising difficulties and give a strategy to overcome them in the case when Γ is the three dimensional discrete Heisenberg group $\mathrm{Heis}_3(\mathbb{Z})$.

As for the first step in previous arguments, we have used the irreducible decomposition formula of the right regular representation R of abeilan groups. However, if Γ is non abelian nilpotent, its representation theory is not so called of type I. In this case, although there exists abstract decompositon formula of the regular representation, it is not unique and the unitary dual is a wild space. As a conclusion, it seems that there is no computable formula in practice for full unitary dual $\hat{\Gamma}$.

In the case that $\Gamma = \mathrm{Heis}_3(\mathbb{Z})$, if we restrict to finite dimensional unitary representations, we can use the Plancherel formula due to Pytlik [34]. In this formula, the Plancherel measure μ is finitely additive and supported on finite dimensional irreducible representations. This formula is indeed useful. However dimensions of representations appearing in the formula are varied, which makes some difficulties to apply perturbation arguments similar to the second step in the above. To overcome this point, we relate these finite dimensional representations to infinite dimensional irreducible unitary representaions whcih are called the Schrödinger representations of the Heisenberg Lie group $\mathrm{Heis}_3(\mathbb{R})$. These representations ρ_h are parametrized by $h \in \mathbb{R} \setminus \{0\}$ and their representaion spaces are common $L^2(\mathbb{R})$ for all $h \in \mathbb{R} \setminus \{0\}$

Let us explain briefly the above relation. Finite dimensional irreducible unitary represenntations $\rho_{\mathrm{fin},x}$ of Γ are parametrized by $x = (x_1, x_2, x_3) \in \hat{X} := [0,1] \times [0,1] \times (\mathbb{Q} \cap [0,1])$ and the above Plancherel measure μ is the product measure of the Lebesgues measure m on the interval $[0,1]$ for first and second factors and the finitely additive measure $\tilde{m}$ on the third factor $\mathbb{Q} \cap [0,1]$ determined by

$$\tilde{m}(\mathbb{Q} \cap [a,b]) = b - a.$$

The representations $\rho_{\mathrm{fin},x}$ are essentially determined from x_3 which is, in some sense, a discrete version of the Schrödinger representation. If we express x_3 as irreducible fraction p/q, then the dimension of $\rho_{\mathrm{fin},x}$ is q and the role of x_1, x_2 for $\rho_{\mathrm{fin},x}, x = (x_1, x_2, x_3)$ are abelian perturbations of $\rho_{\mathrm{fin},(0,0,x_3)}$. The point of our arguments is the fuctuation by x_1, x_2 is like $O(1/q)$ which is independent to the numerator p of $x_3 = p/q$. Moreover, the subset of X consisting $x = (x_1, x_2, x_3)$ whose third component $x_3 = p/q$ has the denominator smaller than a fixed constant is a null set with rerspect to the above Plancherel measure μ. To relate the Schrödinger representation ρ_h, we note that if we restrict ρ_h to the discrete subgroup Γ when the parameter h is a rational number p/q, the restriction $\rho_h|_\Gamma$ to Γ of ρ_h can be decomposed as a direct integral

$$\rho_h|_\Gamma = \int_0^1 \int_0^1 \rho_{\mathrm{fin},(x_1,\{qx_3x_2\},x_3)} dm(x_1) dm(x_2),$$

where $\{a\}$ denotes the fractional part $a - [a]$. We view this decomposition formula as the approximation formula of the left hand side by $\rho_{\mathrm{fin},x}$. It means that the left hand side is the "mean" of the integrand in the right hand side and the fluctuation of the integrand are $O(1/q)$. Moreover, for any h near 0, it can be approximated by rational numbers with arbitarily large denominators, which implies an infinite dimensional representation ρ_h can be approximated by finite dimensional representations $\rho_{\mathrm{fin},x}$ by arbitarily orders. This means that for a mathematical justification of the arguments (e.g. problem of convergence), we use the above Plancherel formula which include only finite dimensional representations but for some formal compuation of the eigenvalues, we can use the Schrödinger representation since it varies smoothly in h. Namely we can exchange freely finite dimensional representations and infinite dimensional representations. This is what we call the extension of the Bloch-Floquet theory to the Herisenberg group.

As in the second step of previous arguments for abelian groups, we need to investigate the behavior of the eigenvalues of Laplacian $\Delta_{\rho_{\mathrm{fin},x}}$ twisted by $\rho_{\mathrm{fin},x}$ near the trivial representation $\mathbf{1}$. By the first step, It can reduced to formal computaion replacing $\Delta_{\rho_{\mathrm{fin},x}}$ to the twisted Laplacian Δ_{ρ_h} associated to ρ_h, which are, in spirits, same as abelian case. However, in technically, it is necessary to modify the arguments. One of them is to replace the line integral $\int_{p_0}^{p} \tilde{\omega}$ in (2.4) by the Lie integral of the Lie algebra valued 1-form (cf. [7]), which is equivalent to Chen's iterated integrals [10]. Other part of the actual computation is somewhat complicated since we need to carry out in infinite rank vector bundles. For example, in this procedure, the hessian $\mathrm{Hess}_0\lambda_0$ should be replaced by the quadratic forms associated with the harmonic oscillator acting on the fiber $L^2(\mathbb{R})$ of $E_{\rho_h}(L^2(\mathbb{R}))$. However, the arguments are essentially a hybrid of the so called Schrödinger method in physics literatures (cf. [35],[36]) and a computation carried in [23] in abelian case.

For the asymptotic formula for counting prime geodesics in Theorem 1.6, we use the Selberg trace formula as in the proof of Theorem 1.4 for abelian groups. The spectral side of the trace formula can be treated similarly as the above heat kernel asymptotics. In the geometric side, the assumption that α is a conjugacy class of a central element of Γ in Theorem 1.4 are used in the fact that the following formula holds for such α

$$\mathrm{tr}(\rho_{\mathrm{fin},x}(f)\rho_{\mathrm{fin},x}(\alpha)) = \mathrm{tr}(\rho_{\mathrm{fin},x}(f))\mathrm{tr}(\rho_{\mathrm{fin},x}(\alpha))$$

where $\mathrm{tr}(A) = \frac{1}{\dim A}\mathrm{Tr}(A)$ is th e normalized trace of the matrix A which is normalization of the usual matrix trace $\mathrm{Tr}(A)$ satisfying $\mathrm{tr}(I) = 1$ for the identity matrix I and $\rho_{\mathrm{fin},x}(f)$ is the Fourier transform

$$\rho_{\mathrm{fin},x}(f) = \int_\Gamma f(\sigma)\rho_{\mathrm{fin},x}(\sigma)d\sigma = \sum_{\sigma\in\Gamma} f(\sigma)\rho_{\mathrm{fin},x}(\sigma).$$

Then, similar arguments to abelian case in [31], which is essentially due to Dirichlet, implies Theorem 1.6.

In the case when α is coming from non-central elements σ, then we can identify the conjugacy class $\alpha = [\sigma]$ with $\Pi(\alpha)$ for the canonical projection $\Pi : \Gamma \to \Gamma/[\Gamma,\Gamma] \simeq \mathbb{Z}^2$. Therefore we can apply same arguments as in abelian case.

As for our mathematical justification of the semiclassical expansion (1.4) and (1.3), we first point out that analysis of the Harper operator corresponds the case when M is the graph $\mathcal{G} = (V, E)$ such that V has only one element p and E has two loops u, v at p, namely if we realize G as a one dimensional complex, G is the bouquet of two $S^1{}'s$, i.e. one point sum of two circles. If $\Gamma = \mathrm{Heis}_3(\mathbb{Z})$, then the Γ covering space X is the Cayley graph of Γ with generators $\{u, v\}$,

$$u = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad v = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

which is described by the following figuire:

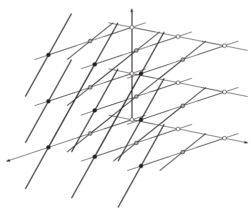

Figure 2: created by Satoshi Ishiwata

Moreover, the square lattice $\mathbb{Z}^2$ appears as an intermediate covering $\varpi_1 : X \to \mathbb{Z}^2$ and $\varpi_2 : \mathbb{Z}^2 \to M$ of $\pi = \varpi_2 \circ \varpi_1 : X \to M$. Moreover for $\theta \in \mathbb{Q}$, the Harper operator H_θ is a lift (with a shift by $4 \times$ identity) of the twist discrete Laplacian $\Delta_{\rho_{\mathrm{fin},x}}$ on M with $x = (0, 0, \theta/2\pi) \in \hat{X}$. Then, the semiclassical expansion (1.4) and 1.3) is essentially same as the above investigation of the behavior of the eigenvalue of the twisted Laplacian $\Delta_{\rho_{\mathrm{fin},x}}$ and Δ_{ρ_h}. Namely our mathematical justification comes from a simple application of the above freeness of the interchange between ρ_h and $\rho_{\mathrm{fin},x}$.

We think our method has some advantage to the original proof in [15] since we can justufy approximations of the differential operator $fracd\sqrt{-1}du$ and the multiplication operator u separately while the original case is an approximation conjointly, i.e. by the harmonic oscillator $\mathcal{H} = -\frac{d^2}{du^2} + u^2$.

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An extension of the Bloch-Floquet theory to the Heisenberg group

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