Stochastic quantization associated with the $\exp(\Phi)_2$-quantum field model and related topics (II) *

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1 Introduction

In this proceeding, we give a summary of our papers [HKK20, HKK21] on stochastic quantization associated with the $\exp(\Phi)_2$-quantum field model as a sequel of the review paper contributed to the proceedings of "Probability Symposium in 2020" (RIMS Kôkyûroku, No.2177). For the background of our problem and related previous results, we refer the interested readers to it.

Let $\Lambda = \mathbb{T}^2 = (\mathbb{R}/2\pi\mathbb{Z})^2$ be the two-dimensional torus and $W = \{W_t = (W_t(x))_{x\in\Lambda}\}_{t\geq 0}$ be a standard $L^2(\Lambda)$-cylindrical Brownian motion defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$. We consider the following stochastic partial differential equation (SPDE in short) driven by an $\mathbb{R}$-valued Gaussian space-time white noise $(\dot{W}_t)_{t\geq 0}$:

$$\partial_t \Phi_t(x) = \frac{1}{2}(\triangle - 1)\Phi_t(x) - \frac{\alpha}{2}\exp^{\diamond}\big(\alpha\Phi_t\big)(x) + \dot{W}_t(x), \qquad t > 0,\ x \in \Lambda, \tag{1.1}$$

where $\triangle = (\frac{\partial}{\partial x_1})^2 + (\frac{\partial}{\partial x_2})^2$, $x = (x_1, x_2) \in \Lambda$ be the Laplacian on Λ with periodic boundry condition, $\alpha \in \mathbb{R}$ is a fixed parameter, called the *charge parameter*, and the rigorous meaning of the drift term $\exp^{\diamond}(\alpha\Phi_t)$ is given in the next section. This SPDE is a called the (parabolic) *stochastic quantization equation* associated with the $\exp(\Phi)_2$-*quantum field model* in finite volume. The $\exp(\Phi)_2$-*quantum field* (or the $\exp(\Phi)_2$-*measure*) $\mu^{(\alpha)}_{\mathsf{exp}}$ is a probability measure on $\mathcal{D}'(\Lambda)$, the space of distributions on Λ, which is given by

$$\mu^{(\alpha)}_{\mathsf{exp}}(d\phi) = \frac{1}{Z^{(\alpha)}_{\mathsf{exp}}}\exp\Big(-\int_\Lambda \exp^{\diamond}(\alpha\phi)(x)dx\Big)\mu_0(d\phi), \tag{1.2}$$

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where *the massive Gaussian free field* μ_0 is the Gaussian measure on $\mathcal{D}'(\Lambda)$ with zero mean and the covariance operator $(1-\triangle)^{-1}$, the *Wick exponential* $\exp^\diamond(\alpha\phi)(x)$ is formally introduced by the expression

$$\exp^\diamond(\alpha\phi)(x) = \exp\Big(\alpha\phi(x) - \frac{\alpha^2}{2}\mathbb{E}^{\mu_0}[\phi(x)^2]\Big), \qquad x \in \Lambda, \tag{1.3}$$

and

$$Z^{(\alpha)}_{\mathsf{exp}} = \int_{\mathcal{D}'(\Lambda)} \exp\Big(-\int_\Lambda \exp^\diamond(\alpha\phi)(x)dx\Big)\mu_0(d\phi) > 0$$

is the normalizing constant. The diverging term $\mathbb{E}^{\mu_0}[\phi(x)^2]$ in (1.3) plays a role of the Wick renormalization. We remark that the SPDE (1.1) is regarded as a random perturbation of the gradient flow of the functional

$$\mathcal{H}(\phi) = \frac{1}{2}\int_\Lambda \big\{\phi(x)^2 + |\nabla\phi(x)|^2_{\mathbb{R}^2} + \exp^\diamond(\alpha\phi(x))\big\}dx, \quad \phi \in \mathcal{D}'(\Lambda)$$

driven by the Gaussian space-time white noise $(\dot{W}_t)_{t\geq 0}$.

Since the measure $\mu^{(\alpha)}_{\mathsf{exp}}$ was first introduced by Høegh-Krohn [Høe71] in the "L^2-regime" $|\alpha| < \sqrt{4\pi}$, it is also called the *Høegh-Krohn model*. On the other hand, Kahane [Kah85] constructed the (law of the) random measure "$\exp^\diamond(\alpha\phi)(x)dx$" on Λ, called the *Gaussian mulptiplicative chaos* (GMC in short), in the "L^1-regime" $|\alpha| < \sqrt{8\pi}$. After that, GMC has been received much attention in connection with topics like the Liouville conformal field theory and the stochastic Ricci flow. See e.g., [DS11, RV14, Ber17, DS19] and references therein. We should also mention that Kusuoka [Kus92] independently studied the measure $\mu^{(\alpha)}_{\mathsf{exp}}$ in L^1-regime.

Due to the singularity of the nonlinear drift term $\exp^\diamond(\alpha\Phi_t)$, the construction of a solution to the SPDE (1.1) has been a challenging problem over the past years. Albeverio and Röckner [AR91] first constructed a weak solution of (1.1) (in $\mathbb{R}^2$ instead of $\mathbb{T}^2$) in L^2-regime by using the Dirichlet form theory. We also mention [AKMR20] as a related paper. On the other hand, based on the idea of Da Prato and Debussche [DPD03], Garban [Gar20] constructed the unique strong solution to (1.1) (for the case where $(\triangle - 1)$ is replaced by $\triangle$) in a more restrictive condition than $|\alpha| < \sqrt{4\pi}$.

In [HKK21], we constructed the unique time-global solution to (1.1) in the full L^2-regime. After that, we obtained a similar result in the full L^1-regime in [HKK20]. This paper consists of the following three parts.

(i) Existence and uniqueness of the time-global strong solution to the SPDE (1.1) (Section 2).

(ii) Invariance of the $\exp(\Phi)_2$-measure $\mu^{(\alpha)}_{\mathsf{exp}}$ under the strong solution (Section 3).

(iii) Relation between the strong solution and a weak solution obtained via the Dirichlet form approach (Section 4).

We mention that, after [HKK21], Oh, Robert and Wang [ORW19] independently obtained the time-global unique solution to (1.1) in the L^2-regime. Later in [ORTW20], together with Tzvetkov, they studied the massless case on two-dimensional compact Riemannian manifolds in the L^2-regime. In [ADG19], Albeverio, De Vecchi and Gubinelli [ADG19] studied the *elliptic* stochastic quantization equation associated with the $\exp(\Phi)_2$-measure $\mu^{(\alpha)}_{\mathsf{exp}}$. Beisides, in a quite recent paper [HKK21+], we studied stochastic quantization for *weighted* $\exp(\Phi)_2$-measures.

2 Time-global strong solution

2.1 Main theorem and the strategy of the proof

We first introduce some notations. Let $\{\mathbf{e}_k\}_{k\in\mathbb{Z}^2}$ be a complete orthonormal system of $L^2(\Lambda;\mathbb{C})$ defined by

$$\mathbf{e}_k(x) = \frac{1}{2\pi} e^{\sqrt{-1}k\cdot x}, \qquad k \in \mathbb{Z}^2,\ x \in \Lambda.$$

For $f \in \mathcal{D}'(\Lambda)$ and $k \in \mathbb{Z}^2$, denote by $\hat{f}(k) = \int_\Lambda f(x)\mathbf{e}_{-k}(x)dx$ the k-th Fourier coefficient. Denote by $H^s = (1-\triangle)^{-s/2}L^2(\Lambda)$ $(s \in \mathbb{R})$ the L^2-Sobolev space on Λ, and by $B^s_{p,q}$ $(s \in \mathbb{R}$, $p, q \in [1,\infty])$ the Besov space on Λ. See [HKK20, Section 1.3] for details. Note that, the Gaussian measure μ_0 and the $\exp(\Phi)_2$-measure $\mu^{(\alpha)}_{\mathsf{exp}}$ has a full support in $H^{-\varepsilon}$ for any $\varepsilon > 0$ and $|\alpha| < \sqrt{8\pi}$ (see [HKK20, Corollary 2.2]).

In this section, we construct the strong solution to the SPDE (1.1). Since Φ_t is expected to take values in $\mathcal{D}'(\Lambda) \setminus C(\Lambda)$, we need to consider a renormalization.

Hypothesis 1. $\psi : \mathbb{R}^2 \to [0,1]$ *is a function satisfying the following properties:*

(i) $\psi(0) = 1$ *and* $\psi(x) = \psi(-x)$ *for any* $x \in \mathbb{R}^2$.

(ii) $\sup_{x\in\mathbb{R}^2} |x|^{2+\kappa}|\psi(x)| < \infty$ *for some* $\kappa > 0$.

(iii) $\sup_{x\in\mathbb{R}^2} |x|^{-\zeta}|\psi(x) - 1| < \infty$ *for some* $\zeta > 0$.

In the above hypothesis, ψ need not be continuous except at the origin. For example, the indicator function of the disc $\{x \in \mathbb{R}^2\,;\, |x| < 1\}$ is allowed. For such a function ψ, we define the Fourier cut-off operator P_N on $\mathcal{D}'(\Lambda)$ by

$$P_N f(x) = \sum_{k\in\mathbb{Z}^2} \psi(2^{-N}k)\hat{f}(k)\mathbf{e}_k(x), \qquad N \in \mathbb{N},\ x \in \Lambda. \tag{2.1}$$

It follows from Hypothesis 1 that P_N maps $H^{-1-\varepsilon}$ to $C(\Lambda)$ for small $\varepsilon > 0$. This implies that the regularized cylindrical Brownian motion $(P_N W_t)_{t\geq 0}$ is a continuous function $\mathbb{P}$-almost surely.

Theorem 2.1 ([HKK20, Theorem 1.1])**.** *Assume that* ψ *satisfies Hypothesis* 1*. Let* $|\alpha| < \sqrt{8\pi}$*,* $p \in (1, \frac{8\pi}{\alpha^2} \wedge 2)$*, and* $\varepsilon > 0$*. For any* $N \in \mathbb{N}$*, consider the initial value problem*

$$\begin{cases} \partial_t \Phi^N_t = \dfrac{1}{2}(\triangle - 1)\Phi^N_t - \dfrac{\alpha}{2}\exp\left(\alpha\Phi^N_t - \dfrac{\alpha^2}{2}C_N\right) + P_N \dot{W}_t, & t > 0, \\ \Phi^N_0 = P_N\phi, \end{cases} \tag{2.2}$$

where $\phi \in \mathcal{D}'(\Lambda)$ *and*

$$C_N := \frac{1}{4\pi^2} \sum_{k \in \mathbb{Z}^2} \frac{\psi(2^{-N}k)^2}{1 + |k|^2}.$$

Then for μ_0*-almost every* $\phi \in \mathcal{D}'(\Lambda)$*, the unique time-global classical solution* Φ^N *converges as* $N \to \infty$ *to a* $B^{-\varepsilon}_{p,p}$*-valued stochastic process* Φ *in the space* $C([0,T]; B^{-\varepsilon}_{p,p})$ *for any* $T > 0$*,* $\mathbb{P}$*-almost surely. Moreover, the limit* Φ *is independent of the choice of* ψ*.*

We call this Φ the *strong solution* to the SPDE (1.1) with the initial value ϕ. To prove Theorem 2.1, we use the *Da Prato–Debussche trick* [DPD03], that is, we decompose the solution of (2.2) by $\Phi^N = X^N + Y^N$, $N \in \mathbb{N}$, where X^N and Y^N solve

$$\begin{cases} \partial_t X^N_t = \dfrac{1}{2}(\triangle - 1)X^N_t + P_N \dot{W}_t, & t > 0, \\ X^N_0 = P_N \phi, \end{cases} \tag{2.3}$$

and the "shifted" equation

$$\begin{cases} \partial_t Y^N_t = \dfrac{1}{2}(\triangle - 1)Y^N_t - \dfrac{\alpha}{2}\exp(\alpha Y^N_t)\exp\left(\alpha X^N_t - \dfrac{\alpha^2}{2}C_N\right), & t > 0, \\ Y^N_0 = 0, \end{cases} \tag{2.4}$$

respectively. Note that X^N is equal to $P_N X$, where X is the infinite-dimensional Ornstein–Uhlenbeck process given by the unique solution to the stochastic heat equation

$$\begin{cases} \partial_t X_t = \dfrac{1}{2}(\triangle - 1)X_t + \dot{W}_t, & t > 0, \\ X_0 = \phi, \end{cases} \tag{2.5}$$

The proof of Theorem 2.1 consists of the following two steps.

(i) Convergence of $\exp\left(\alpha X^N_t - \frac{\alpha^2}{2}C_N\right)$ (Section 2.2).

(ii) Convergence of Y^N (Section 2.3).

It is easy to show step (i) from the convergence results for the Wick exponential of the Ornstein–Uhlenbeck process (see Theorems 2.2 and 2.4). To show step (ii), in Theorem 2.5, we solve the *deterministic* "shifted" equation

$$\begin{cases} \partial_t \Upsilon_t = \dfrac{1}{2}(\triangle - 1)\Upsilon_t - \dfrac{\alpha}{2}e^{\alpha\Upsilon_t}\mathcal{X}_t, & t > 0, \\ \Upsilon_0 = \upsilon, \end{cases} \tag{2.6}$$

for a given family $\{\mathcal{X}_t\}_{t>0}$ of nonnegative distributions. Note that nonnegativity of the distribution $\mathcal{X}_t$ enables us to make a rigorous meaning of the multiplication of $e^{\alpha\Upsilon_t}$ and $\mathcal{X}_t$. In contrast to the standard fixed point argument as in [DPD03, Gar20], we do not construct any contraction map for existence and uniqueness of the solution to (2.6). We mention our new approach for this equation briefly in Section 2.3.

2.2 Gaussian multiplicative chaos

To prove the convergence of $\exp\left(\alpha X_t^N - \frac{\alpha^2}{2}C_N\right)$, we recall the almost sure convergence result obtained in [HKK20].

Theorem 2.2 ([HKK20, Theorem 2.1]). *Let $\mathbb{X}$ be an $\mathcal{D}'(\Lambda)$-valued random variable defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with the law μ_0. Assume that ψ satisfies Hypothesis 1 and consider the approximation*

$$\exp_N^\diamond(\alpha\mathbb{X})(x) := \exp\left(\alpha(P_N\mathbb{X})(x) - \frac{\alpha^2}{2}C_N\right), \qquad N \in \mathbb{N},$$

Let $|\alpha| < \sqrt{8\pi}$ and choose parameters p, β such that

$$p \in \left(1, \frac{8\pi}{\alpha^2} \wedge 2\right), \qquad \beta \in \left(\frac{\alpha^2}{4\pi}(p-1), \frac{2}{p}(p-1)\right). \tag{2.7}$$

Then the sequence $\{\exp_N^\diamond(\alpha\mathbb{X})\}_{N\in\mathbb{N}}$ converges in the space $B_{p,p}^{-\beta}$, $\mathbb{P}$-almost surely and in $L^p(\mathbb{P})$. Moreover, by regarding $\exp_N^\diamond(\alpha\mathbb{X})$ as the random nonnegative Borel measure $\exp_N^\diamond(\alpha\mathbb{X})(x)dx$ on Λ for $N \in \mathbb{N}$, one has the weak convergence of $\{\exp_N^\diamond(\alpha\mathbb{X})\}_{N\in\mathbb{N}}$, $\mathbb{P}$-almost surely. The limits obtained by different ψ's coincide with each other almost surely.

We denote the ($\mathbb{P}$-almost-sure) unique limit by $\exp^\diamond(\alpha\mathbb{X})$. When the probability space $(\Omega, \mathbb{P})$ is $(\mathcal{D}'(\Lambda), \mu_0)$, we may apply this theorem by putting $\mathbb{X}(\phi) = \phi$. We denote by $\exp^\diamond(\alpha\phi)$ the associated Wick exponential. Since $\exp_N^\diamond$ is a nonnegative function for any $N \in \mathbb{N}$, the measure $\mu_{\mathsf{exp}}^{(\alpha)}$ formally given by (1.2) is well-defined for any $|\alpha| < \sqrt{8\pi}$ and equivalent to μ_0 (see [HKK20, Corollary 2.2]).

Our proof is a modification of Berestycki's argument [Ber17]. See [HKK20, Section 2] for the detailed proof of Theorem 2.2. We only mention here that the following estimates has an important role in our proof. We regard a function on $\Lambda \times \Lambda$ as a periodic function on $\mathbb{R}^2 \times \mathbb{R}^2$.

Proposition 2.3. *Let $G_{M,N}$ be the approximated Green function defined by*

$$G_{M,N}(x,y) := \mathbb{E}[(P_M\mathbb{X})(x)(P_N\mathbb{X})(y)] = \frac{1}{2\pi}\sum_{k\in\mathbb{Z}^2}\frac{\psi_M(k)\psi_N(k)}{1+|k|^2}\mathbf{e}_k(x-y).$$

Assume that ψ satisfies Hypothesis 1. Then for any $x, y \in \mathbb{R}^2$ with $|x-y| < 1$ and any $M, N \in \mathbb{N}$,

$$G_{M,N}(x,y) = -\frac{1}{2\pi}\log\left(|x-y| \vee 2^{-M} \vee 2^{-N}\right) + R_{M,N}(x,y),$$

where the remainder term $R_{M,N}(x,y)$ is uniformly bounded over x, y, M, N. Moreover, there exist constants $C > 0$ and $\theta > 0$ such that, for any $M, N \in \mathbb{N}$,

$$\iint_{\Lambda\times\Lambda}|G_{M,N+1}(x,y) - G_{M,N}(x,y)|dxdy \le C2^{-\theta N}.$$

Step (i) of the proof of Theorem 2.1 is an immediate consequence of Theorem 2.2. Denote by $X(\phi)$ the solution to the stochastic heat equation (2.5) with the initial value ϕ. It is known that the Gaussian measure μ_0 is invariant under the process X.

Theorem 2.4 ([HKK20, Theorem 3.2])**.** *Assume that ψ satisfies Hypothesis 1. Let $|\alpha| < \sqrt{8\pi}$ and choose parameters p and β as in (2.7). Then the functions*

$$\mathcal{X}_t^N(\phi)(x) := \exp\left(\alpha\big(P_N X_t(\phi)\big)(x) - \frac{\alpha^2}{2}C_N\right), \qquad N \in \mathbb{N}$$

are uniformly bounded in the space $L^p(\mathbb{P}\otimes\mu_0; L^p([0,T]; B_{p,p}^{-\beta}))$ for any $T>0$. Moreover, the function $\mathcal{X}^N$ converges as $N\to\infty$ in the space $L^p([0,T]; B_{p,p}^{-\beta})$, $\mathbb{P}\otimes\mu_0$-almost surely and in $L^p(\mathbb{P}\otimes\mu_0)$. The limits obtained by different ψ's coincide with each other, $\mathbb{P}\otimes\mu_0$-almost surely.

Denote by $\mathcal{X}^\infty := \lim_{N\to\infty}\mathcal{X}^N$ the $\mathbb{P}\otimes\mu_0$-almost-sure limit. In particular, we have that the random function $\mathcal{X}^N(\phi)$ converges to $\mathcal{X}^\infty(\phi)$ in the space $L^p([0,T]; B_{p,p}^{-\beta})$ almost surely, for μ_0-almost every $\phi\in\mathcal{D}'(\Lambda)$.

2.3 Well-posedness of the shifted equation

In this section, we solve the "shifted" equation (2.6) for a family $\{\mathcal{X}_t\}_{t>0}$ of nonnegative distributions. Since any nonnegative distribution is regarded as a nonnegative Borel measure, the product of a continuous function $f\in C(\Lambda)$ and a nonnegative distribution $\xi\in\mathcal{D}'(\Lambda)$ is well-defined as a signed Borel measure $\mathcal{M}(f,\xi)$. In this sense, the right hand side of the equation (2.6) is well-defined and we can show the following theorem. Denote by $B_{p,q}^{\alpha,+}$ the space of nonnegative distributions in $B_{p,q}^{\alpha}$.

Theorem 2.5 ([HKK20, Theorem 4.4])**.** *Let $p\in(1,\infty)$, $\beta\in(0,2-2/p)$ and $T>0$. Let $\mathcal{X}\in L^p([0,T]; B_{p,p}^{-\beta,+})$ and $\upsilon\in B_{p,p}^{2-\beta}$. Then there exists a unique element Υ in the space*

$$\mathscr{Y}_T = \left\{\Upsilon\in L^p([0,T];C(\Lambda))\cap C([0,T];L^p)\ ; e^{\alpha\Upsilon}\in L^\infty([0,T];C(\Lambda))\right\}$$

such that the mild equation

$$\Upsilon_t = e^{t(\triangle-1)/2}\upsilon - \frac{\alpha}{2}\int_0^t e^{(t-s)(\triangle-1)/2}\mathcal{M}(e^{\alpha\Upsilon_s},\mathcal{X}_s)ds$$

holds for any $t\in(0,T]$. Moreover, this element belongs to the space

$$L^p([0,T]; B_{p,p}^{2/p+\delta})\cap C([0,T]; B_{p,p}^{\delta})$$

for any $\delta\in(0,\frac{2}{p}(p-1)-\beta)$, and the solution map

$$\mathcal{S}: B_{p,p}^{2-\beta}\times L^p([0,T]; B_{p,p}^{-\beta,+})\ni(\upsilon,\mathcal{X})\mapsto\Upsilon\in L^p([0,T]; B_{p,p}^{2/p+\delta})\cap C([0,T]; B_{p,p}^{\delta})$$

is continuous.

As mentioned in Section 2.1, our proof is different from the standard fixed-point argument applied in [DPD03, Gar20]. The key idea of our proof is summarized as follows.

For given $\mathcal{X}$, we take an approximating sequence $\{\mathcal{X}^N\}_{N\in\mathbb{N}}$ of nonnegative continuous functions on $[0,T]\times\Lambda$, and consider

$$\begin{cases} \partial_t \Upsilon_t^N = \frac{1}{2}(\triangle - 1)\Upsilon_t^N - \frac{\alpha}{2} e^{\alpha \Upsilon_t^N} \mathcal{X}_t^N, & 0 < t \le T,\ N \in \mathbb{N}, \\ \Upsilon_0^N = \upsilon. \end{cases}$$

By making use of nonnegativity of $\mathcal{X}^N$, we have

$$\Upsilon_t^N \ge e^{\frac{1}{2}(\Delta-1)t}\upsilon \ge -\|\upsilon\|_{C(\Lambda)}, \qquad 0 \le t \le T$$

in the case $\alpha < 0$. Similarly, we also have $\Upsilon_t^N \le \|\upsilon\|_{C(\Lambda)}$, $0 \le t \le T$ in the case $\alpha \ge 0$. Hence

$$\|e^{\alpha \Upsilon^N}\|_{C([0,T],C(\Lambda))} \le e^{|\alpha|\cdot\|\upsilon\|_{C(\Lambda)}}$$

holds. Once we have that $e^{\alpha\Upsilon^N}$ is bounded, by the Schauder estimate, Υ^N is bounded in a suitable space. Then by using a compact embedding, we have a convergent subsequence of $\{\Upsilon^N\}_{N\in\mathbb{N}}$ and then identify the limit as the unique solution to the desired equation (2.6). See [HKK20, Section 4] for the detail of the proof of Theorem 2.5.

3 Invariance of the $\exp(\Phi)_2$-measure

In this section, we prove the invariance of the $\exp(\Phi)_2$-measure with respect to the strong solution Φ obtained by Theorem 2.1. We follow the argument by Albeverio and Kusuoka [AK20]. Instead of the approximation (2.2), we approximates the $\exp(\Phi)_2$-measure $\mu_{\mathsf{exp}}^{(\alpha)}$ by

$$\mu_N^{(\alpha)}(d\phi) := \frac{1}{Z_N^{(\alpha)}} \exp\left\{ -\int_\Lambda \exp\left(\alpha P_N \phi(x) - \frac{\alpha^2}{2} C_N \right) dx \right\} \mu_0(d\phi), \qquad N \in \mathbb{N},$$

where $Z_N^{(\alpha)} > 0$ is the normalizing constant. It follows from Theorem 2.2 that the sequence $\{\mu_N^{(\alpha)}\}_{N\in\mathbb{N}}$ of probability measures weakly converges to $\mu_{\mathsf{exp}}^{(\alpha)}$ ([HKK20, Corollary 2.2]).

Hypothesis 2. *The operators P_N defined by* (2.1) *satisfy the following properties.*

(i) *P_N is nonnegative, that is, $P_N f \ge 0$ if $f \ge 0$.*

(ii) *For any $p \in (1,2)$, $s \in \mathbb{R}$, there exists a constant $C > 0$ such that*

$$\sup_{N\in\mathbb{N}} \|P_N f\|_{B_{p,p}^s} \le C\|f\|_{B_{p,p}^s}, \qquad \lim_{N\to\infty} \|P_N f - f\|_{B_{p,p}^s} = 0$$

for any $f \in B_{p,p}^s$.

If ψ is a Schwartz function and the inverse Fourier transform of ψ is a nonnegative function, then Hypothesis 2 holds.

Theorem 3.1 ([HKK20, Theorem 1.4, Corollary 1.5])**.** *Assume that ψ satisfies Hypotheses 1 and 2. Let $|\alpha| < \sqrt{8\pi}$ and $\varepsilon > 0$. For any $N \in \mathbb{N}$, consider the solution $\widetilde{\Phi}^N = \widetilde{\Phi}^N(\phi)$ of the SPDE*

$$
(3.1)\qquad \begin{cases} \partial_t \widetilde{\Phi}_t^N = \frac{1}{2}(\triangle - 1)\widetilde{\Phi}_t^N - \frac{\alpha}{2} P_N \exp\left(\alpha P_N \widetilde{\Phi}_t^N - \frac{\alpha^2}{2} C_N\right) + \dot{W}_t, & t > 0, \\ \widetilde{\Phi}_0^N = \phi \in \mathcal{D}'(\Lambda). \end{cases}
$$

Let ξ_N be a random variable with the law $\mu_N^{(\alpha)}$ independent of W. Then $\widetilde{\Phi}^{N,\mathrm{stat}} = \widetilde{\Phi}^N(\xi_N)$ is a stationary process and converges in law as $N \to \infty$ to the strong solution Φ^{stat} of the SPDE (1.1) *with an initial law $\mu_{\mathsf{exp}}^{(\alpha)}$, on the space $C([0,T]; H^{-\varepsilon})$ for any $T > 0$. Beisides, the law of the random variable Φ_t^{stat} is $\mu_{\mathsf{exp}}^{(\alpha)}$ for any $t \geq 0$. Moreover, the strong solution Φ of* (1.1) *belongs to the space $C([0,T]; H^{-\varepsilon})$, $\mathbb{P}$-almost surely, for μ_0-almost every (or $\mu^{(\alpha)}$-almost every) initial value $\phi \in \mathcal{D}'(\Lambda)$.*

See [HKK20, Section 5] for the proof of the above theorem. Our proof consists of the following two steps:

(i) $\{\widetilde{\Phi}^{N,\mathrm{stat}}\}_{N\in\mathbb{N}}$ is tight in the space $C([0,T]; H^{-\varepsilon})$ for any $\varepsilon > 0$.

(ii) $\widetilde{\Phi}^{N,\mathrm{stat}}$ converges in law to Φ^{stat} in the space $C([0,T]; B_{p,p}^{-\varepsilon})$ for any $\varepsilon > 0$.

Step (ii) is a consequence of Theorem 2.1. Once step (i) is proved, then we can show that the convergence (ii) indeed holds in a smaller space $C([0,T]; H^{-\varepsilon})$. In the proof of (i), the following estimates has a crucial role. Recall that Theorem 2.2 asserts only that the random variable $\phi \mapsto \exp^{\diamond}(\alpha\phi)$ belongs to $L^p(\mu_0; B_{p,p}^{-\beta})$.

Proposition 3.2 ([HKK20, Corollary 2.3])**.** *If $|\alpha| < \sqrt{8\pi}$, then there exists an exponent $s \in (0,1)$ such that*

$$
\sup_{N\in\mathbb{N}} \int_{\mathcal{D}'(\Lambda)} \| \exp_N^{\diamond}(\alpha\phi)\|_{H^{-s}}^2 \mu_N^{(\alpha)}(d\phi) < \infty.
$$

Moreover, the random variable $\phi \mapsto \exp^{\diamond}(\alpha\phi)$ belongs to $L^2(\mu_{\mathsf{exp}}^{(\alpha)}; H^{-s})$.

4 Relation with the Dirichlet form theory

As discussed in [AR91, AKMR20], we can construct a weak solution to the equation (1.1) via the Dirichlet form theory. Let $s \in (0,1)$ be an exponent used in Proposition 3.2 and set $H = L^2(\Lambda)$ and $E = H^{-s}(\Lambda)$. Let $\{e_k\}_{k\in\mathbb{Z}^2}$ be a real-valued complete orthonormal system of H and denote by $\mathfrak{F}C_b^\infty$ the space of all smooth cylinder functions $F : E \to \mathbb{R}$ having the form

$$
F(\phi) = f(\langle \phi, l_1\rangle, \dots, \langle \phi, l_n\rangle), \qquad \phi \in E,
$$

with $n \in \mathbb{N}$, $f \in C_b^\infty(\mathbb{R}^n; \mathbb{R})$ and $l_1, \dots, l_n \in \mathrm{Span}\{e_k; k \in \mathbb{Z}^2\}$. For $F \in \mathfrak{F}C_b^\infty$, we define the H-derivative $D_H F : E \to H$ by

$$
D_H F(\phi) := \sum_{j=1}^n \partial_j f\big(\langle \phi, l_1\rangle, \dots, \langle \phi, l_n\rangle\big) l_j, \qquad \phi \in E.
$$

We then consider a pre-Dirichlet form $(\mathcal{E}, \mathfrak{F}C_b^\infty)$ defined by

$$\mathcal{E}(F,G) = \frac{1}{2}\int_E \big(D_H F(\phi), D_H G(\phi)\big)_H \, \mu_{\mathsf{exp}}^{(\alpha)}(d\phi), \qquad F,G \in \mathfrak{F}C_b^\infty,$$

where $(\cdot,\cdot)_H$ is the inner product of H. Applying the integration by parts formula for $\mu_{\mathsf{exp}}^{(\alpha)}$ ([HKK20, Proposition 6.1]), we obtain that $(\mathcal{E}, \mathfrak{F}C_b^\infty)$ is closable on $L^2(\mu_{\mathsf{exp}}^{(\alpha)})$, so we can define $\mathcal{D}(\mathcal{E})$ as the completion of $\mathfrak{F}C_b^\infty$ with respect to $\mathcal{E}_1^{1/2}$-norm. Thus, by directly applying the general methods in the theory of Dirichlet forms (cf. [MR92, CF12]), we can prove quasi-regularity of $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ and the existence of a diffusion process $\mathbb{M} = (\Theta, \mathcal{G}, (\mathcal{G}_t)_{t\ge 0}, (\Psi_t)_{t\ge 0}, (\mathbb{Q}_\phi)_{\phi\in E})$ properly associated with $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$.

In this section, we discuss a relation between the strong solution obtained in Theorem 2.1 and the diffusion process obtained as above.

Theorem 4.1 ([HKK20, Theorem 1.6])**.** *Let $|\alpha| < \sqrt{8\pi}$. Then for $\mu_{\mathsf{exp}}^{(\alpha)}$-almost every ϕ, the diffusion process Ψ coincides $\mathbb{Q}_\phi$-almost surely with the strong solution Φ to the SPDE* (1.1) *with the initial value ϕ, driven by some $L^2(\Lambda)$-cylindrical $(\mathcal{G}_t)_{t\ge0}$-Brownian motion $\mathcal{W} = (\mathcal{W}_t)_{t\ge 0}$.*

We provide a sketch of the proof. For details, see [HKK20, Section 6]. By recalling Proposition 3.2 and applying [AR91, Lemma 4.2], we have

$$\mathbb{E}^{\mathbb{Q}_\phi}\Big[\int_0^T \big\|\exp^\diamond(\alpha\Psi_t)\big\|_E \, dt\Big] < \infty, \qquad T>0,\ \mu_{\mathsf{exp}}^{(\alpha)}\text{-a.e. } \phi.$$

Thus applying [AR91, Lemma 6.1 and Theorem 6.2] and [Ond04, Theorem 13], we have that there exists an H-cylindrical $(\mathcal{G}_t)$-Brownian motion $\mathcal{W} = (\mathcal{W}_t)_{t\ge0}$ defined on $(\Theta, \mathcal{G}, \mathbb{Q}_\phi)$ such that

$$\Psi_t = e^{t(\Delta-1)/2}\phi - \frac{\alpha}{2}\int_0^t e^{(t-s)(\Delta-1)/2}\exp^\diamond(\alpha\Psi_s)ds + \int_0^t e^{(t-s)(\Delta-1)/2} d\mathcal{W}_s, \qquad t\ge 0,$$

$\mathbb{Q}_\phi$-almost surely, for $\mu_{\mathsf{exp}}^{(\alpha)}$-almost every ϕ.

Once we decompose $\Psi = \mathfrak{X}(\phi) + \mathfrak{Y}$, where

$$\mathfrak{X}(\phi)_t := e^{t(\Delta-1)/2}\phi + \int_0^t e^{(t-s)(\Delta-1)/2} d\mathcal{W}_s,$$

then by uniqueness of the solution to the shifted equation (2.6), we can complete the proof of Theorem 4.1 by showing that

$$\mathbb{Q}_\phi\Big(\exp^\diamond(\alpha\Psi_t) = e^{\alpha\mathfrak{Y}_t}\cdot\exp^\diamond(\alpha\mathfrak{X}_t),\ \text{a.e. } t\in[0,T]\Big) = 1, \qquad \mu_{\mathsf{exp}}^{(\alpha)}\text{-a.e. } \phi.$$

This is a consequence of the following result.

Lemma 4.2 ([HKK20, Lemma 6.2])**.** *Assume that the mollifier ψ satisfies Hypothesis* 1*. Let E_0 be the set of all $\phi \in E$ such that the convergence*

$$\exp^\diamond(\alpha\phi) = \lim_{N\to\infty}\exp_N^\diamond(\alpha\phi)$$

holds in $B_{p,p}^{-\beta}$. Then, for any $f \in H^{1+\varepsilon}$ and $\phi\in E_0$ such that $f+\phi\in E_0$, one has

$$\exp^\diamond(\alpha(f+\phi)) = \exp(\alpha f)\exp^\diamond(\alpha\phi).$$

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