SUPERFLUIDITY AND TEMPERATURE EFFECTS

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1. Introduction

This is a joint work with Anne de Bouard (Ecole Polytechnique, France) and Arnaud Debussche (Univ Rennes/IUF, France).

In this talk, we will present our mathematical analysis of the Gross-Pitaevskii equation, which is a model for Bose-Einstein condensates. In particular, we are interested in a model with stochastic effects, e.g. temperature effects arising around the critical temperature of condensation. Interactions of the condensate with the "thermal cloud" formed by non-condensed atoms need to be taken into account in this situation. Those interactions should preserve the principles of the fluctuation-dissipation theorem, which ensures formally the relaxation of the system to the expected physical equilibrium (see [2, 11, 12]), leading to the so-called Projected Stochastic Gross-Pitaevskii Equation:

$$d\psi = \mathcal{P}\Big\{ -\frac{i}{\hbar} L_{GP}\psi dt + \frac{G(x)}{k_B T}(\nu - L_{GP})\psi dt + dW_G(t,x)\Big\} \tag{1.1}$$

where

$$L_{GP} = -\frac{\hbar^2}{2m}\Delta + V(x) + g|\psi|^2, \quad \langle dW_G^*(s,y), dW_G(t,x)\rangle = 2G(x)\delta_{t-s}\delta_{x-y}\, dt.$$

Here, m is the mass of an atom, $V(x)$ is the trapping, generally harmonic, potential, ν is the chemical potential, g characterizes the strength of atomic interactions related to the s-wave scattering length. The second and third terms in the right hand side of (1.1) represent growth processes, i.e., collisions that transfer atoms from the thermal cloud to the classical field and vice versa. The form of $G(x)$ may be determined from kinetic theory, and is often taken as a constant, and dW_G is the complex-valued Gaussian noise associated with the condensate growth. Lastly, $\mathcal{P}$ is a projection which restricts the dynamics to the low-energy region defined by the harmonic oscillator modes, or Fourier modes, depending on the situation. At zero temperature $T = 0$, the statistics of the atoms is well represented by a single condensate wave function, and the standard Gross-Pitaevskii equation (i.e. (1.1) without dissipation and noise) describes the coherent evolution of the wave function in a quite good manner since, for ex. the effect of thermal cloud may be neglected.

In (1.1), neglecting the projection operator on the lowest energy modes, and setting $G(x) = \gamma$, also the chemical potential to zero, the equation for the macroscopic wave function ψ may be written in its simplest dimensionless form:

$$\partial_t \psi = (i+\gamma)\big(\Delta\psi - V(x)\psi - g|\psi|^2\psi\big) + \dot{W}_\gamma(t,x), \quad t>0, \quad x \in \mathbb{R}^d. \tag{1.2}$$

When $\gamma = 0$, one recovers the standard Gross-Pitaevskii equation for the wave function ψ, and in this case the hamiltonian

$$\mathcal{H}(\psi) = \frac{1}{2}\int_{\mathbb{R}^d} |\nabla\psi(x)|^2 dx + \frac{1}{2}\int_{\mathbb{R}^d} V(x)|\psi(x)|^2 dx + \frac{g}{4}\int_{\mathbb{R}^d} |\psi(x)|^4 dx$$

is conserved. Note that for $\gamma > 0$, "the statistical equilibrium", i.e. formal Gibbs measure for (1.2) is given by

$$\rho(d\psi) = \Gamma \exp\left[-\mathcal{H}(\psi)\right] d\psi,$$

for some normalizing constant Γ.

In this talk, we consider the harmonic potential $V(x) = |x|^2$. Equation (1.2) has been studied in space dimension one in [3], and the existence of global solutions for all initial data was proved. The exponential convergence to equilibrium was also obtained in [3], thanks to a Poincaré inequality, and to the properties of the Gibbs measure previously established in [5], where its support was in particular shown to contain $L^p(\mathbb{R})$ for any $p > 2$.

We have recently succeeded to extend part of those results to the two-dimensional case, and we will report the results in this talk. Remark that the Gaussian measure generated by the linear equation is only supported in $\mathcal{W}^{-s,q}$, with $s > 0$, $q \geq 2$, and $sq > 2$. Here, $\mathcal{W}^{-s,q}$ is a Sobolev space based on the operator $-H = -\Delta + V$:

$$\mathcal{W}^{s,p}(\mathbb{R}^2) = \{v \in \mathcal{S}'(\mathbb{R}^2),\ |v|_{\mathcal{W}^{s,p}(\mathbb{R}^2)} := |(-H)^{s/2} v|_{L^p(\mathbb{R}^2)} < +\infty\},$$

for $1 \leq p \leq +\infty$, and $s \in \mathbb{R}$, where $\mathcal{S}$ and $\mathcal{S}'$ denote the Schwartz space and its dual space, respectively.

Hence, as is the case for the stochastic quantization equations, the use of renormalization is necessary in order to give a meaning to the solutions of (1.2) in the support of the Gaussian measure. Renormalization procedures, using Wick products, have been by now widely used in the context of stochastic partial differential equations (see e.g. for the case of dimension 2 considered here [6, 7, 9, 16, 19] and references therein), in particular for parabolic equations based on gradient flows. The complex Ginzburg-Landau equation driven by space-time white noise, i.e. (1.2) without the confining potential, posed on the three-dimensional torus, was studied in [13] and for the two-dimensional torus in [15, 18]. The main difference in our case is the presence of the harmonic potential V. It is thus natural to use functional spaces based on the operator $-H = -\Delta + V$, rather than on standard Sobolev or Besov spaces; we chose to work on Sobolev spaces based on $-H$ since it is enough for our analysis, especially in the two-dimensional setting.

Several difficulties arise when trying to adapt the previous methods to the present two-dimensional case. First, the diverging constant in the definition of the Wick product is no more a constant, but rather a function of the space variable x; This is already the case for SPDEs on manifolds for instance but does not imply many difficulties. Up to our knowledge, Wick products corresponding to the Gaussian measure associated to the operator $-H$ considered here have never been constructed. An essential tool in the definition of the Wick products is the kernel $K(x, y)$ of the operator $(-H)^{-1}$, which is defined by

$$(-H)^{-1}f(x) = \int_{\mathbb{R}^2} K(x, y)f(y)dy,$$

and in particular its integrability properties. It appears ([17]) that K is never in $L^p(\mathbb{R}_x^2 \times \mathbb{R}_y^2)$, for any $p \geq 1$, but we only have $K \in L^r(\mathbb{R}_x^2; L^p(\mathbb{R}_y^2))$ for $r > p \geq 2$ (see Proposition 2 below).

Using these properties of the kernel K, we construct the Wick products with respect to the Gaussian measure with covariance $(-H)^{-1}$ and use the method of [7] to construct local solutions. Then using ideas from [16], we are able to prove that the solutions are global when γ is sufficiently large. Moreover, we prove that (1.2) has an invariant measure which is the limit of Gibbs measures corresponding to finite dimensional approximations of this equation. This can be seen as a construction of the infinite dimensional Gibbs measure ρ. Details of our work presented in this talk can be referred to [4].

2. Main Results.

Writing equation (1.2) in a more mathematical form, we will consider in what follows the infinite dimensional, stochastic complex Ginzburg-Landau equation, with a harmonic potential in the case of $d = 2$:

$$dX = (\gamma_1 + i\gamma_2)(HX - |X|^2X)dt + \sqrt{2\gamma_1}dW, \quad t > 0, \quad x \in \mathbb{R}^2, \tag{2.1}$$

where $H = \Delta - |x|^2$, $x \in \mathbb{R}^2$. We consider a more general equation with parameters $\gamma_1 > 0$, and $\gamma_2 \in \mathbb{R}$, in order to clarify the effects of the dissipation induced by γ_1. Let $\{h_k\}_{k\in\mathbb{N}^2}$ be the orthonormal basis of $L^2(\mathbb{R}^2, \mathbb{R})$, consisting of eigenfunctions of $-H$ with corresponding eigenvalues $\{\lambda_k^2\}_{k\in\mathbb{N}^2}$, i.e. $-Hh_k = \lambda_k^2 h_k$. It is known that $\lambda_k^2 = 2|k| + 2$ with $k = (k_1, k_2) \in \mathbb{N}^2$, and the functions $h_k(x)$ are the Hermite functions. The unknown function X is a complex valued random field on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ endowed with a standard filtration $(\mathcal{F}_t)_{t\geq 0}$.

We take $\{h_k, ih_k\}_{k\in\mathbb{N}^2}$ as a complete orthonormal system in $L^2(\mathbb{R}^2, \mathbb{C})$, and we may write the cylindrical Wiener process as

$$W(t, x) = \sum_{k\in\mathbb{N}^2} (\beta_{k,R}(t) + i\beta_{k,I}(t))h_k(x). \tag{2.2}$$

Here, $(\beta_{k,R}(t))_{t\geq 0}$ and $(\beta_{k,I}(t))_{t\geq 0}$ are sequences of independent real-valued Brownian motions, on the stochastic basis $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\geq 0})$. In all what follows, the notation $\mathbb{E}$ stands for the expectation with respect to $\mathbb{P}$. We denote by $E_N^{\mathbb{C}}$ the complex vector space spanned by the Hermite functions, $E_N^{\mathbb{C}} = \text{span}\{h_k\}_{|k|\le N}$.

For a function series of the form $u = \sum_{k\in\mathbb{N}^2} c_k h_k$, we introduce, for any $N \in \mathbb{N}$ fixed, the spectral projector Π_N by

$$\Pi_N\Big[\sum_{k\in\mathbb{N}^2} c_k h_k\Big] := \sum_{k\in\mathbb{N}^2, |k|\le N} c_k h_k.$$

Also we define, for any $N \in \mathbb{N}$ fixed, a smooth projection operator $S_N : L^2(\mathbb{R}^2, \mathbb{C}) \to E_N^{\mathbb{C}}$ by

$$S_N\Big[\sum_{k\in\mathbb{N}^2} c_k h_k\Big] := \sum_{k\in\mathbb{N}^2} \chi\Big[\frac{\lambda_k^2}{\lambda_N^2}\Big] c_k h_k = \chi\Big[\frac{-H}{\lambda_N^2}\Big]\Big[\sum_{k\in\mathbb{N}^2} c_k h_k\Big],$$

where $\chi \ge 0$ is a cut-off function such that $\chi \in C_0^\infty(-1,1)$, $\chi = 1$ on $[-\frac{1}{2}, \frac{1}{2}]$. Note that here and in what follows, we denote by λ_N the value $\lambda_{(N,0)}$, for simplicity. A modification of Theorem 1.1 of [14] implies that S_N is a bounded operator from L^p to L^p, uniformly in N, for any $p \in [1, \infty]$.

As was pointed out in the introduction, due to the space-time white noise, the solution of (2.1) is expected to have negative space regularity, and thus the nonlinear term $-|X|^2X$ is ill-defined. In order to make sense of this term, we use a renormalization procedure based on Wick products. This amounts to "subtract an infinite constant" from the nonlinear term in (2.1). More precisely, writing the solution $X = u + Z_\infty^{\gamma_1,\gamma_2}$ with

$$Z_\infty^{\gamma_1,\gamma_2}(t) = \sqrt{2\gamma_1}\int_{-\infty}^t e^{(t-\tau)(\gamma_1+i\gamma_2)H} dW(\tau), \tag{2.3}$$

which is the stationary solution for the linear stochastic equation

$$dZ = (\gamma_1 + i\gamma_2)HZdt + \sqrt{2\gamma_1}dW,$$

we find out the following random partial differential equation for u:

$$\partial_t u = (\gamma_1 + i\gamma_2)(Hu - |u + Z_\infty^{\gamma_1,\gamma_2}|^2(u + Z_\infty^{\gamma_1,\gamma_2})), \quad u(0) = u_0 := X(0) - Z_\infty^{\gamma_1,\gamma_2}(0).$$

We are therefore required to solve this random partial differential equation. Here, by standard arguments, it can be seen that the best regularity we may expect for $Z_\infty^{\gamma_1,\gamma_2}$ is almost surely: $Z_\infty^{\gamma_1,\gamma_2} \in \mathcal{W}^{-s,q}(\mathbb{R}^2)$ for $s > 0$, $q \ge 2$, $sq > 2$. If we develop

$$\begin{aligned} |u + Z_\infty^{\gamma_1,\gamma_2}|^2(u + Z_\infty^{\gamma_1,\gamma_2}) &= |u|^2u + 2|u|^2 Z_\infty^{\gamma_1,\gamma_2} + \bar{u}(Z_\infty^{\gamma_1,\gamma_2})^2 + u^2\overline{Z_\infty^{\gamma_1,\gamma_2}} \\ &\quad + 2u|Z_\infty^{\gamma_1,\gamma_2}|^2 + |Z_\infty^{\gamma_1,\gamma_2}|^2 Z_\infty^{\gamma_1,\gamma_2}, \end{aligned}$$

we are led to multiply functions having both negative Sobolev regularity, which cannot be defined in the usual distribution sense. Note that $Z_\infty^{\gamma_1,\gamma_2}$ is complex-valued centered Gaussian (see also (2.6)).

We need some preliminaries to introduce the renormalization procedure. Let us recall a few facts about Hermite polynomials $H_n(x)$, $n \in \mathbb{N}$, These are defined through the generating functions

$$e^{-\frac{t^2}{2}+tx} = \sum_{n=0}^{\infty} \frac{t^n}{\sqrt{n!}} H_n(x), \quad x, t \in \mathbb{R},$$

where

$$H_n(x) = \frac{(-1)^n}{\sqrt{n!}} e^{\frac{x^2}{2}} \frac{d^n}{dx^n}(e^{-\frac{x^2}{2}}), \quad n \geq 1$$

and $H_0(x) = 1$. The Wick products of $Z_\infty^{\gamma_1,\gamma_2}$ are defined as follows. We write $Z_{R,\infty} = \mathrm{Re} Z_\infty^{\gamma_1,\gamma_2}$ and $Z_{I,\infty} = \mathrm{Im} Z_\infty^{\gamma_1,\gamma_2}$. For any $k, l \in \mathbb{N}$, we define, if the limit of the right hand side below exists in a suitable topology,

$$:(Z_{R,\infty})^k (Z_{I,\infty})^l: \quad := \lim_{N\to\infty} :(S_N Z_{R,\infty})^k :: (S_N Z_{I,\infty})^l : .$$

In the right hand side, the notation $:(S_N z)^n:(x)$ for $n \in \mathbb{N}$, $N \in \mathbb{N}$, $x \in \mathbb{R}^2$, and for a real-valued centered Gaussian white noise z, means

$$:(S_N z)^n:(x) = \rho_N(x)^n \sqrt{n!} H_n \left[\frac{1}{\rho_N(x)} S_N z(x) \right], \quad x \in \mathbb{R}^2$$

with

$$\rho_N(x) = \left[\sum_{k \in \mathbb{N}^2} \chi^2 \left(\frac{\lambda_k^2}{\lambda_N^2} \right) \frac{1}{\lambda_k^2} (h_k(x))^2 \right]^{\frac{1}{2}} .$$

Actually this Wick product is indeed well-defined in $\mathcal{W}^{-s,q}$, as soon as $s > 0$, $q \geq 4$ and $qs > 8$:

Proposition 1. *For any $k, l \in \mathbb{N}$, the sequence $\{: (S_N Z_{R,\infty})^k :: (S_N Z_{I,\infty})^l :\}_{N\in\mathbb{N}}$ is a Cauchy sequence in $L^q(\Omega, \mathcal{W}^{-s,q}(\mathbb{R}^2))$, for $q \geq 4$, $s > 0$ with $qs > 8$.*

Moreover, defining then, for any $k, l \in \mathbb{N}$, for any fixed t,

$$:(Z_{R,\infty})^k (Z_{I,\infty})^l: \quad := \lim_{N\to\infty} :(S_N Z_{R,\infty})^k :: (S_N Z_{I,\infty})^l :, \quad \text{in} \quad L^q(\Omega, \mathcal{W}^{-s,q}(\mathbb{R}^2)),$$

where $s > 0, q \geq 4$ and $sq > 8$, there exists a constant $M_{s,q,k,l}$ such that

$$\mathbb{E}\left[|:(Z_{R,\infty})^k (Z_{I,\infty})^l:|_{\mathcal{W}^{-s,q}}^q \right] \leq M_{s,q,k,l}.$$

Remark that higher order moments may also be estimated thanks to the Nelson formula.

For the proof of Proposition 1, the key ingredients are estimates on the kernel K of the operator $(-H)^{-1}$ via Sobolev embeddings:

Proposition 2. *For any $n \in \mathbb{N} \setminus \{0\}$, we have $K^n \in L_x^r \mathcal{W}_y^{\alpha,2}$ for any $r \geq 2$ and $\alpha < 1 - \frac{2}{r}$.*

We then consider the following renormalized equation in the space $\mathcal{W}^{-s,q}(\mathbb{R}^2)$:

$$\begin{aligned} dX &= (\gamma_1 + i\gamma_2)(HX - :|X|^2X:)dt + \sqrt{2\gamma_1}dW, \quad t>0, \quad x \in \mathbb{R}^2, \\ X(0) &= X_0, \end{aligned} \tag{2.4}$$

in the sense that we solve the shifted equation (2) with $|u + Z_\infty^{\gamma_1,\gamma_2}|^2(u + Z_\infty^{\gamma_1,\gamma_2})$ replaced by $:|u + Z_\infty^{\gamma_1,\gamma_2}|^2(u + Z_\infty^{\gamma_1,\gamma_2}):$ defined by replacing in (2.4) all the terms involving $Z_\infty^{\gamma_1,\gamma_2}$ by the corresponding Wick products. Hence we consider

$$\partial_t u = (\gamma_1 + i\gamma_2)\left[Hu - :|u + Z_\infty^{\gamma_1,\gamma_2}|^2(u + Z_\infty^{\gamma_1,\gamma_2}):\right], \tag{2.5}$$

supplemented with the initial condition

$$u(0) = u_0 = X_0 - Z_\infty^{\gamma_1,\gamma_2}(0).$$

We first have the following local well-posedness result of equation (2.5).

Theorem 1. *Fix any $T > 0$. Let $\gamma_1 > 0$, $\gamma_2 \in \mathbb{R}$ and $q > p > 3r$, $r > 6$. Assume $0 < s < \beta < 2/p$, $qs > 8$, $\beta - s > \frac{2}{p} - \beta$ and $s + 2\big(\frac{2}{p} - \beta\big) < 2\big(1 - \frac{1}{q}\big)$. Let $u_0 \in \mathcal{W}^{-s,q}(\mathbb{R}^2)$. Then there exists a random stopping time $T_0^*(\omega) > 0$, which depends on u_0 and $(:(Z_{R,\infty})^k(Z_{I,\infty})^l:)_{0\le k+l\le 3}$, and a unique solution u of (2.5) such that $u \in C([0,T_0^*), W^{-s,q}(\mathbb{R}^2)) \cap L^r(0,T_0^*, W^{\beta,p}(\mathbb{R}^2))$ a.s. We have moreover almost surely $T_0^* = T$ or $\lim_{t\uparrow T_0^*} |u(t)|_{\mathcal{W}^{-s,q}} = +\infty$.*

When the dissipation coefficient γ_1 is sufficiently large, an energy estimate allows us to get a bound on the L^q norm of the solution, and to obtain a global existence result, as is stated in the next Propositions and Theorem. This method of globalization has been widely used for the complex Ginzburg-Landau equation (see [1, 10]) and has been adapted in the renormalized case ([13, 16]).

Proposition 3. *Let $\gamma_1 > 0$, $\gamma_2 \in \mathbb{R}$ and $q > p > 3r$, $r > 6$. Assume $0 < s < \beta < 2/p$ satisfy the assumptions of Theorem 1 and that we have in addition $s \le \frac{2}{p} - \beta < \frac{1}{12}$, with $2(\frac{2}{p} - \beta) < \beta$ and $3(\frac{2}{p} - \beta) < 2(1 - \frac{1}{q})$. Let $u_0 \in L^q(\mathbb{R}^2)$. Then the solution u of (2.5) given by Theorem 1 satisfies : $u \in C([0,T_0^*), L^q(\mathbb{R}^2))$.*

In the next Proposition we give the L^q a priori bound.

Proposition 4. *[L^q a priori estimate]. Let $\gamma_1 > 0$ and $q > p > 3r$, $r > 6$. Assume $0 < s < \beta < 2/p$ satisfy the assumptions of Theorem 1 and we have in addition $s \le \frac{2}{p} - \beta < \frac{1}{12}$, with $2(\frac{2}{p} - \beta) < \beta$ and $3(\frac{2}{p} - \beta) < 2(1 - \frac{1}{q})$. Moreover assume $\gamma_2 = 0$, or $q < 2 + 2(\kappa^2 + \kappa\sqrt{1+\kappa^2})$ with $\kappa = |\gamma_1/\gamma_2|$ if $\gamma_2 \ne 0$. Let $u_0 \in L^q(\mathbb{R}^2)$, and let u be the unique solution constructed in Theorem 1. Then, there exists a constant $C > 0$ depending on γ_1, γ_2, q and $(:(Z_{R,\infty})^k(Z_{I,\infty})^l:)_{0\le k+l\le 3}$, such that for any t with $0 < t < T_0^*$,*

$$|u(t)|_{L^q}^q \le e^{-\frac{\gamma_1 t\delta}{4}}|u_0|_{L^q}^q + C,$$

where T_0^* is the maximal existence time given in Theorem 1. The coefficient δ is given by $\delta = 1$ if $\gamma_2 = 0$, and $\delta = 1 - \frac{q-2}{2(\kappa^2+\kappa\sqrt{1+\kappa^2})}$ if $\gamma_2 \neq 0$.

By the previous results and using the smoothing properties of the heat semi-group, we finally obtain the global existence of solutions in the case of large $\gamma_1 > 0$.

Theorem 2. *Let $\gamma_1 > 0$ and $q > p > 3r$, $r > 6$. Assume $0 < s < \beta < 2/p$ satisfy the assumptions of Theorem 1 and we have in addition $s \leq \frac{2}{p} - \beta < \frac{1}{12}$, with $2(\frac{2}{p} - \beta) < \beta$ and $3(\frac{2}{p} - \beta) < 2(1 - \frac{1}{q})$. Moreover assume $\gamma_2 = 0$, or $q < 2 + 2(\kappa^2 + \kappa\sqrt{1+\kappa^2})$ with $\kappa = |\frac{\gamma_1}{\gamma_2}|$ if $\gamma_2 \neq 0$. Let $u_0 \in \mathcal{W}^{-s,q}(\mathbb{R}^2)$. Then there exists a unique global solution u of (2.5) in $C([0,T], \mathcal{W}^{-s,q}) \cap L^r(0,T; \mathcal{W}^{\beta,p})$ a.s. for any $T > 0$.*

The next step is the construction of a Gibbs measure. The Gibbs measure is formally written as an infinite-dimensional measure of the following form:

$$\rho(du) = \Gamma e^{-\mathcal{H}(u)} du,$$

where

$$\mathcal{H}(u) = \frac{1}{2}\int_{\mathbb{R}^2} |\nabla u(x)|^2 dx + \frac{1}{2}\int_{\mathbb{R}^2} |xu(x)|^2 dx + \frac{1}{4}\int_{\mathbb{R}^2} |u(x)|^4 dx,$$

and Γ is a normalizing constant. We will make sense of this infinite dimensional measure as follows. Using (2.2), it may be easily seen that (2.3) can be written as

$$Z_\infty^{\gamma_1,\gamma_2}(t) = \sum_{k\in\mathbb{N}^2} \frac{\sqrt{2}}{\lambda_k} g_k(\omega,t) h_k(x), \tag{2.6}$$

where $\{g_k(\omega,t)\}_{k\in\mathbb{N}^2}$ is a system of independent, complex-valued random variables with law $\mathcal{N}_{\mathbb{C}}(0,1)$. Thus, the projection onto $E_N^{\mathbb{C}}$ of the stationary solution $Z_\infty^{\gamma_1,\gamma_2}(t)$ has the same law as the Gaussian measure μ_N induced by a random series

$$\varphi_N(\omega,x) := \sum_{k\in\mathbb{N}^2, |k|\leq N} \frac{\sqrt{2}}{\lambda_k} g_k(\omega) h_k(x)$$

defined on $(\Omega, \mathcal{F}, \mathbb{P})$ where $\{g_k(\omega)\}_{k\in\mathbb{N}^2}$ is a system of independent, complex-valued random variables with the law $\mathcal{N}_{\mathbb{C}}(0,1)$. Thanks to the same argument as Lemma 2.1 in [3], we see that the series converges in $L^q(\Omega, \mathcal{W}^{-s,q})$ if $s > 0$, $q \geq 2$ and $sq > 2$, and the limit defines the infinite-dimensional Gaussian measure μ on $\mathcal{W}^{-s,q}$. However, although the above Gibbs expression may be formally written as

$$\rho(du) = \Gamma e^{-\frac{1}{4}\int_{\mathbb{R}^2} |u(x)|^4 dx} \mu(du),$$

we cannot give a sense for it, since $L^4(\mathbb{R}^2)$ is not in the support of μ. For that reason, we should also renormalize the L^4 norm in the Gibbs measure. This fact leads us to define the Gibbs measure for (2.4) as a limit of

$$\tilde{\rho}_N(dy) = \Gamma_N \exp\left\{ -\int_{\mathbb{R}^2} \left(\frac{1}{4}|S_N y(x)|^4 - 2\rho_N^2(x)|S_N y(x)|^2 + 2\rho_N^4(x) \right) dx \right\} \mu_N(dy), \quad y \in E_N^{\mathbb{C}},$$

where Γ_N is the normalizing constant. This $\tilde{\rho}_N$ is a unique invariant measure for the finite dimensional system:

$$dX = (\gamma_1 + i\gamma_2)(HX - S_N(:|S_N X|^2 S_N X:))dt + \sqrt{2\gamma_1}\Pi_N dW, \tag{2.7}$$

whose solution approximates to the solution of (2.4) as $N \to +\infty$. We can indeed prove the tightness of the family of measures $(\tilde{\rho}_N)_{N\in\mathbb{N}}$.

Theorem 3. *Let γ_1, γ_2, q, s be as in Theorem 2. Then, there exists an invariant measure ρ supported in $\mathcal{W}^{-s,q}(\mathbb{R}^2)$, for the transition semi-group associated with equation* (2.4)*, which is well-defined according to Theorem 2. Moreover, ρ is the weak limit of a subsequence of the family $(\tilde{\rho}_N)_N$ defined above.*

The tightness of $(\tilde{\rho}_N)_{N\in\mathbb{N}}$ is not induced by a L^q bound as in Proposition 4, which does not a priori hold for the finite dimensional approximations (2.7). We thus have to prove an alternative bound considering the coupled evolution on $E_N^{\mathbb{C}} \times E_N^{\mathbb{C}}$ given by

$$\begin{cases} \dfrac{du}{dt} &= (\gamma_1 + i\gamma_2)\left[Hu - S_N\big(: |S_N(u+Z)|^2 S_N(u+Z) : \big)\right] \\ dZ &= (\gamma_1 + i\gamma_2) HZdt + \sqrt{2\gamma_1}\Pi_N dW. \end{cases} \tag{2.8}$$

We denote $Z_\infty^{\gamma_1,\gamma_2}$ by Z for the sake of simplicity. One may easily prove, using similar estimates as in the proof of Proposition 5 below, together with the Gaussian properties of Z, and a Krylov-Bogolyubov argument, that (2.8) has an invariant measure ν_N on $E_N^{\mathbb{C}} \times \mathbb{E}_N^{\mathbb{C}}$. Moreover, by uniqueness of the invariant measure of (2.7), we necessarily have for any bounded continuous function φ on $\mathbb{E}_N^{\mathbb{C}}$:

$$\int_{E_N^{\mathbb{C}}} \varphi(x)\tilde{\rho}_N(dx) = \iint_{E_N^{\mathbb{C}}\times E_N^{\mathbb{C}}} \varphi(u+z)\nu_N(du,dz).$$

The next proposition will imply the tightness of the sequence $(\tilde{\rho}_N)_N$ in $\mathcal{W}^{-s,q}$.

Proposition 5. *Let $(u_N, Z_N{}^1) \in C(\mathbb{R}_+; E_N^{\mathbb{C}} \times E_N^{\mathbb{C}})$ be a stationary solution of* (2.8)*. Then, for any $m > 0$, there is a constant $C_m > 0$ independent of t and N, such that*

$$\mathbb{E}(|(-H)^{\frac{1}{2m}} u_N|_{L^2}^{2m}) \le C_m. \tag{2.9}$$

Corollary 2.1. *The family of finite dimensional Gibbs measures $(\tilde{\rho_N})_N$ is tight in $\mathcal{W}^{-s,q}$ for any $q > 8$ and $s > \frac{8}{q}$.*

Unfortunately, the bound in Proposition 5 does not provide higher moment bounds on the measures $\tilde{\rho}_N$, preventing us to obtain global strong solutions in the small dissipation case, as would be expected. Nevertheless, the bound in Proposition 5 allows us to construct a stationary martingale solution for any dissipation coefficients:

[1] $Z_N = \Pi_N Z_\infty^{\gamma_1,\gamma_2}$

Theorem 4. *Let* $\gamma_1 > 0$ *and* $\gamma_2 \in \mathbb{R}$*, and let* $0 < s < 1$*,* $q > 8$*,* $sq > 8$*. Then, there exists a stationary martingale solution* X *of* (2.4) *having trajectories in* $C(\mathbb{R}_+, \mathcal{W}^{-s,q})$ *and such that for any* $t \geq 0$*,* $\mathcal{L}(X(t)) = \rho$*, the measure constructed in Theorem 3.*

3. Final remarks and perspective

The problem we encounter is that we are not able to prove global existence for any γ_1. We cannot use the same argument as in [3]. In [15] the author obtained a global strong solution for any dissipation parameter in the complex Ginzburg-Landau equation. To apply a similar argument to our case, we would need the integrability q of the Wick products in Proposition 1 to be close to 2. Thus this would need an optimization of Proposition 1. It is also expected that (2.4) has a unique invariant measure. Strong Feller property of the associated transition semigroup can be proved as in [19] or [8]. Irreducibility seems to be much more complex. These questions will be the object of a future work.

We consider $V(x) = |x|^2$, but a generalization of the potential $V(x)$ is of course possible; by a technical reason it could impose more restrictions on the parameter γ_1 for the regularization properties of the heat semigroup.

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