GRADED QUIVER VARIETIES AND SINGULARITIES OF NORMALIZED R-MATRICES FOR FUNDAMENTAL MODULES

RYO FUJITA

ABSTRACT. We present a simple unified formula expressing the denominators of the normalized R-matrices between the fundamental modules over the quantum loop algebras of type ADE. It has an interpretation in terms of representations of Dynkin quivers and can be proved in a unified way using geometry of the graded quiver varieties. As a by-product, we obtain a geometric interpretation of Kang-Kashiwara-Kim's generalized quantum affine Schur-Weyl duality functor when it arises from a family of the fundamental modules. We also study several cases when the graded quiver varieties are isomorphic to unions of the graded nilpotent orbits of type A.

CONTENTS

1. INTRODUCTION

1.1. For a complex finite-dimensional simple Lie algebra $\mathfrak{g}$, we can consider its (untwisted) quantum loop algebra $U_q(L\mathfrak{g})$ as a certain quantum affinization of the universal enveloping algebra $U(\mathfrak{g})$. It is a Hopf algebra defined over the field $\Bbbk = \overline{\mathbb{Q}(q)}$, where q is the generic quantum parameter. The structure of the monoidal abelian category $\mathcal{C}$ of finite-dimensional $U_q(L\mathfrak{g})$-modules is much more complicated than that of $U(\mathfrak{g})$. Indeed, the category $\mathcal{C}$ is neither semisimple as an abelian category, nor braided as a monoidal category. It has been studied by many researchers in connection with various research topics such as quantum integrable systems, combinatorics and cluster algebras.

The normalized R-matrices are constructed as intertwining operators between tensor products of (relatively generic) simple objects of the category $\mathcal{C}$, satisfying the quantum Yang-Baxter equation. They can be seen as matrix-valued rational

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functions in the spectral parameters, whose singularities strongly reflect the structure of tensor product modules (cf. [1, 24]). Thus, the singularities of normalized R-matrices carry some important information on the monoidal structure of $\mathcal{C}$.

1.2. **A unified denominator formula.** In this paper, we focus on the normalized R-matrices between the fundamental modules over $U_q(L\mathfrak{g})$ associated with $\mathfrak{g}$ of type ADE. Note that every simple object of $\mathcal{C}$ is obtained as a head of a suitably ordered tensor product of the fundamental modules. Thus, studying tensor products of the fundamental modules can be thought of a first step toward a better understanding of the monoidal structure of the whole category $\mathcal{C}$.

From now on, we assume that $\mathfrak{g}$ is of type ADE. Let I be the set of Dynkin indices and $(c_{ij})_{i,j\in I}$ the Cartan matrix of $\mathfrak{g}$. For each $i \in I$, the i-th fundamental module $V_i(a)$ is a simple object of $\mathcal{C}$, which has a canonical highest weight vector v_i and depends on a non-zero scalar $a \in \Bbbk^\times$ called the spectral parameter. Making the spectral parameters formal, for each $(i,j) \in I^2$, the normalized R-matrix $R_{ij}(z_2/z_1)$ is defined to be the unique $U_q(L\mathfrak{g}) \otimes \Bbbk(z_1, z_2)$-linear isomorphism

$$R_{ij}(z_2/z_1)\colon V_i(z_1) \otimes V_j(z_2) \to V_j(z_2) \otimes V_i(z_1)$$

satisfying the condition $R_{ij}(z_2/z_1)(v_i \otimes v_j) = v_j \otimes v_i$. Since the normalized R-matrix $R_{ij}(z_2/z_1)$ only rationally depends on the ratio $u = z_2/z_1$ of the spectral parameters, one can consider its denominator $d_{ij}(u) \in \Bbbk[u]$. Explicit computations of these denominators $d_{ij}(u)$ have been accomplished in the separate works by Date-Okado [8] for type A, by Kang-Kashiwara-Kim [19] for type D, and by Oh-Scrimshaw [36, 37] for type E. Note that these computations relied on case-by-case arguments, which also required a use of computer particularly for type E.

The main theorem of this paper asserts that these denominators $d_{ij}(u)$ can be expressed in a simple unified formula.

Theorem 1.1 (= Theorem 2.10)**.** *For each $(i,j) \in I^2$, we have*

$$d_{ij}(u) = \prod_{\ell=1}^{h-1} (u - q^{\ell+1})^{\widetilde{c}_{ij}(\ell)},$$

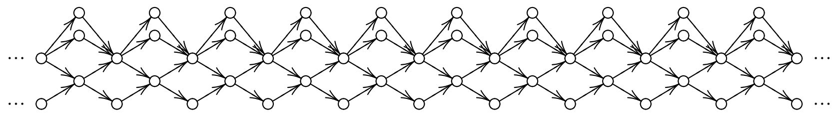
where h is the Coxeter number of $\mathfrak{g}$ and $\widetilde{c}_{ij}(\ell)$ is the coefficient of z^ℓ in the formal expansion at $z = 0$ of the (i,j)-entry of the inverse of the quantum Cartan matrix $\left(\frac{z^{c_{ij}} - z^{-c_{ij}}}{z - z^{-1}}\right)_{i,j\in I}$.

Note that the quantum Cartan matrix has appeared several times as a key combinatorial ingredient in the study of the category $\mathcal{C}$. For example, it already appeared in the work of Frenkel-Reshetikhin [10], which introduced the notion of q-characters for finite-dimensional $U_q(L\mathfrak{g})$-modules.

1.3. **An interpretation by quiver representations.** An advantage of our denominator formula is that it admits an interpretation in terms of representations of a Dynkin quiver Q of type $\mathfrak{g}$. To describe it, we need additional notation. Let us choose an I-tuple $(\epsilon_i)_{i\in I} \in \{0,1\}^I$ such that $\epsilon_i \neq \epsilon_j$ whenever $c_{ij} = -1$. Then we define an infinite quiver $\Delta = (\Delta_0, \Delta_1)$ by

$$\Delta_0 := \{(i,p) \in I \times \mathbb{Z} \mid p - \epsilon_i \in 2\mathbb{Z}\},$$
$$\Delta_1 := \{(i,p) \to (j, p+1) \mid (i,p), (j,p+1) \in \Delta_0,\ c_{ij} = -1\}.$$

For instance, when $\mathfrak{g}$ is of type D_5, the quiver Δ looks like:



It was shown by Happel [16, 17] that the quiver Δ is isomorphic to the Auslander-Reiten quiver of the bounded derived category $\mathcal{D}_Q := D^b(\mathbb{C}Q\text{-}\mathsf{mod})$ of representations of the Dynkin quiver Q. In particular, there is a nice bijection $\mathtt{H}_Q$ from the vertex set Δ_0 to the set of isomorphism classes of indecomposable objects of $\mathcal{D}_Q$.

An intimate connection between the Auslander-Reiten quiver of $\mathcal{D}_Q$ and the category $\mathcal{C}$ was originally observed by Hernandez-Leclerc [18]. In that paper, it was shown that the integers $\widetilde{c}_{ij}(\ell)$ can be expressed as Euler-Poincaré characteristics of suitable pairs of indecomposable objects of $\mathcal{D}_Q$. Using this interpretation, one can see that the following assertion is equivalent to Theorem 1.1.

Theorem 1.2 (= Theorem 3.9)**.** *For any $(i,p),(j,r) \in \Delta_0$, the pole order of the normalized R-matrix $R_{ij}(u)$ at $u = q^r/q^p$ is equal to $\dim \mathrm{Ext}^1_{\mathcal{D}_Q}(\mathtt{H}_Q(j,r), \mathtt{H}_Q(i,p))$.*

This yields the following interesting corollary.

Corollary 1.3 (= Corollary 3.10)**.** *For any $(i,p),(j,r) \in \Delta_0$, the following conditions are mutually equivalent:*

- *The tensor product $V_i(q^p) \otimes V_j(q^r)$ is irreducible;*
- *$V_i(q^p) \otimes V_j(q^r) \cong V_j(q^r) \otimes V_i(q^p)$ as $U_q(L\mathfrak{g})$-modules;*
- *$\mathrm{Ext}^1_{\mathcal{D}_Q}(\mathtt{H}_Q(i,p), \mathtt{H}_Q(j,r)) = 0$ and $\mathrm{Ext}^1_{\mathcal{D}_Q}(\mathtt{H}_Q(j,r), \mathtt{H}_Q(i,p)) = 0$.*

1.4. **Graded quiver varieties.** In this paper, we give a unified proof of Theorem 1.2 (and hence Theorem 1.1) without using a computer. Instead, we use geometry of the graded quiver varieties.

The graded quiver varieties were originally defined by Nakajima [32] as suitable torus fixed loci of the usual Nakajima quiver varieties, which provide a useful geometric setting to study finite-dimensional $U_q(L\mathfrak{g})$-modules when $\mathfrak{g}$ is of type ADE. Given a finite-dimensional Δ_0-graded $\mathbb{C}$-vector space $W = \bigoplus_{x \in \Delta_0} W_x$, one can associate the graded quiver variety $\mathfrak{M}_0^\bullet(W)$, which is an affine complex algebraic variety equipped with an action of the group $G_W = \prod_{x \in \Delta_0} GL(W_x)$.

Our proof of Theorem 1.2 is based on the following beautiful result obtained by Keller-Scherotzke [29] in their categorical study of the graded quiver varieties. It also generalizes an important result by Hernandez-Leclerc [18, Section 9]. Let $\Gamma = (\Gamma_0, \Gamma_1)$ be another infinite quiver with the vertex set $\Gamma_0 := \Delta_0$, whose arrow set Γ_1 is given by the following condition: for each $x, y \in \Delta_0$ the number of arrows from x to y is equal to $\dim \mathrm{Ext}^1_{\mathcal{D}_Q}(\mathtt{H}_Q(x), \mathtt{H}_Q(y))$. With this notation, Keller-Scherotzke's theorem tells us that, for each Δ_0-graded vector space W, there exists a G_W-equivariant closed embedding of varieties

$$\mathfrak{M}_0^\bullet(W) \hookrightarrow \mathsf{rep}_W(\Gamma), \tag{1.1}$$

where $\mathsf{rep}_W(\Gamma)$ denotes the affine space parametrizing representations of the quiver Γ realized on W.

In the special case when $W = W_{(i,p)} \oplus W_{(j,r)}$ for some $(i,p),(j,r) \in \Delta_0$ with $p \le r$ and $\dim W_{(i,p)} = \dim W_{(j,r)} = 1$, the above embedding (1.1) becomes an isomorphism. Namely, the graded quiver variety $\mathfrak{M}_0^\bullet(W)$ in this case is just an affine space whose dimension is equal to $\dim \mathrm{Ext}^1_{\mathcal{D}_Q}(\mathtt{H}_Q(j,r), \mathtt{H}_Q(i,p))$. This simple situation enables us to prove Theorem 1.2 by using Nakajima's theory [32] and a standard technique in geometric representation theory.

1.5. **Generalized quantum affine Schur-Weyl duality.** In the paper [20], Kang-Kashiwara-Kim gave a general construction of a monoidal functor $\mathscr{F}_J$, called the generalized quantum affine Schur-Weyl duality functor, associated with a given family $\{\mathsf{V}_j\}_{j\in J}$ of real simple objects of $\mathcal{C}$. It connects the category $\mathcal{C}$ with a category of modules over the symmetric quiver Hecke algebra H_J associated with a quiver Γ_J determined by the singularities of normalized R-matrices between the simple objects in $\{\mathsf{V}_j\}_{j\in J}$. The quiver Hecke algebras are $\mathbb{Z}$-graded algebras introduced by Khovanov-Lauda [30] and by Rouquier [38] independently to establish a categorification of the half of the quantized enveloping algebra associated with a general symmetrizable Kac-Moody algebra. In this sense, the quiver Hecke algebra H_J is a generalization of the affine Hecke algebra of type A, and Kang-Kashiwara-Kim's construction can be thought of a generalization of the usual quantum affine Schur-Weyl duality between $U_q(L\mathfrak{sl}_n)$ and the affine Hecke algebra of type A.

In the subsequent works by Kang, Kashiwara, Kim, Oh, Park and Scrimshaw [19, 21, 22, 25, 27, 36, 26], many interesting examples of the functor $\mathscr{F}_J$ are constructed. In these nice examples, the functor $\mathscr{F}_J$ induces an isomorphism of Grothendieck rings between a category of finite-dimensional H_J-modules (or its suitable modification), and a certain monoidal subcategory $\mathcal{C}_J$ of $\mathcal{C}$.

In this paper, we give a geometric interpretation of the functor $\mathscr{F}_J$ whenever it arises from a family $\{\mathsf{V}_j\}_{j\in J}$ of fundamental modules of type ADE. More precisely, we realize the bimodule corresponding to the functor $\mathscr{F}_J$ via the equivariant K-theory of the graded quiver varieties, mimicking Ginzburg-Reshetikhin-Vasserot's geometric realization of the usual quantum affine Schur-Weyl duality [15]. This is a generalization of the author's previous result [12]. A key fact in our construction is that the quiver Γ_J defining the quiver Hecke algebra H_J is identical to a full subquiver of the quiver Γ that appeared in Keller-Scherotzke's theorem above. This is a direct consequence of Theorem 1.2 and explains the appearance of the quiver Hecke algebra H_J from a geometric point of view.

1.6. **Type A subquivers and graded nilpotent orbits.** As an example of the above construction, with a given subquiver Q' of a Dynkin quiver Q which is isomorphic to a quiver of type A with monotone orientation, we associate a specific family $\{\mathsf{V}_j\}_{j\in\mathbb{Z}}$ of fundamental modules labeled by the set of integers $\mathbb{Z}$. We prove that the associated quiver Γ_J is of type A_∞ with monotone orientation, and the corresponding graded quiver varieties are isomorphic to unions of graded nilpotent orbits of type A. Moreover, we show that the associated functor $\mathscr{F}_J$ induces an isomorphism of Grothendieck rings between a certain localization $\mathcal{T}_N$ of the module category of H_J constructed in [20] and the monoidal full subcategory $\mathcal{C}_{\mathcal{D}_{Q'}}$ of $\mathcal{C}$ generated by the fundamental modules $V_i(q^p)$ such that $\mathtt{H}_Q(i,p) \in \mathcal{D}_{Q'} \subset \mathcal{D}_Q$. In some special cases of type AD, the associated functors $\mathscr{F}_J$ coincide with the ones studied in [20, 25, 26]. Recently, Kashiwara-Kim-Oh-Park [26] proved that the localized category $\mathcal{T}_N$ gives a monoidal categorification of a certain cluster algebra of infinite rank. Therefore, we conclude that our monoidal category $\mathcal{C}_{\mathcal{D}_{Q'}}$ always inherits the same cluster structure from the category $\mathcal{T}_N$ via the monoidal functor $\mathscr{F}_J$.

1.7. **Remark.** Note that explicit computations of the denominators $d_{ij}(u)$ for the other non-symmetric affine types have been also accomplished in the separate works by Akasaka-Kashiwara [1] for type C, by Oh [35] for type B and for doubly-twisted type AD, and by Oh-Scrimshaw [36, 37] for all the remaining cases. Unfortunately, our geometric approach using the graded quiver varieties is applicable only to the cases of untwisted type ADE (i.e. symmetric affine types). At this moment, it is unclear whether there is an analogous geometric approach to compute the denominators $d_{ij}(u)$ for the non-symmetric types.

1.8. **Organization.** This paper is organized as follows. In Section 2, we recall some known facts about the representation theory of the quantum loop algebras $U_q(L\mathfrak{g})$ of type ADE and state our main theorem. In Section 3, we present an interpretation of our denominator formula in terms of representations of Dynkin quivers. After reviewing the graded quiver varieties in Section 4, we give a geometric proof of our denominator formula in Section 5.1. In Section 5.2, we add a remark on the case when the normalized R-matrix has a simple pole. Section 6 is devoted to a study of the generalized quantum affine Schur-Weyl duality. In Section 6.2, we give a geometric interpretation of the functor $\mathscr{F}_J$ when it arises from a family of fundamental modules. Finally, we study some examples where the graded quiver varieties are isomorphic to unions of graded nilpotent orbits of type A in Section 6.3.

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1.10. **Overall convention.** Working over a base field $\mathbb{F}$, we often write $\otimes$ (resp. Hom, dim) instead of $\otimes_{\mathbb{F}}$ (resp. $\mathrm{Hom}_{\mathbb{F}}$, $\dim_{\mathbb{F}}$) suppressing the symbol $\mathbb{F}$ for simplicity. For an algebra A over a field $\mathbb{F}$, we denote by A-mod the category of left A-modules which are finite-dimensional over $\mathbb{F}$. We denote by A^{op} (resp. $A^{\times}$) the opposite algebra (resp. the multiplicative group of invertible elements) of A.

2. A unified denominator formula

In this section, we recall some known facts on representation theory of the quantum loop algebras of type ADE and state our main theorem.

2.1. **Notation.** Throughout this paper, we fix a finite-dimensional complex simple Lie algebra $\mathfrak{g}$ of type A_n ($n \in \mathbb{Z}_{\ge 1}$), D_n ($n \in \mathbb{Z}_{\ge 4}$), or E_n ($n = 6, 7, 8$). Let $I := \{1, 2, \dots, n\}$ be the set of Dynkin indices. The Cartan matrix of $\mathfrak{g}$ is denoted by $(c_{ij})_{i,j \in I}$. We write $i \sim j$ if $c_{ij} = -1$.

Let $\mathsf{P}^\vee = \bigoplus_{i \in I} \mathbb{Z} h_i$ be the coroot lattice of $\mathfrak{g}$. The fundamental weights $\{\varpi_i\}_{i \in I}$ form a basis of the weight lattice $\mathsf{P} = \mathrm{Hom}_{\mathbb{Z}}(\mathsf{P}^\vee, \mathbb{Z})$ which is dual to $\{h_i\}_{i \in I}$. Let $\alpha_i = \sum_{j \in I} c_{ij} \varpi_j$ be the i-th simple root and $\mathsf{Q} = \bigoplus_{i \in I} \mathbb{Z}\alpha_i \subset \mathsf{P}$ be the root lattice. We put $\mathsf{P}^+ = \sum_{i \in I} \mathbb{Z}_{\ge 0} \varpi_i$ and $\mathsf{Q}^+ = \sum_{i \in I} \mathbb{Z}_{\ge 0} \alpha_i$. Denote by $(-,-)$ the symmetric bilinear form on $\mathsf{P} \otimes_{\mathbb{Z}} \mathbb{Q}$ given by $(\alpha_i, \varpi_j) = \delta_{ij}$, or equivalently $(\alpha_i, \alpha_j) = c_{ij}$. Let W be the Weyl group of $\mathfrak{g}$. It is a finite group of linear transformations on P generated by the simple reflections $\{r_i\}_{i \in I}$ defined by $r_i(\lambda) := \lambda - \lambda(h_i)\alpha_i$ for $\lambda \in \mathsf{P}$. The set R^+ of positive roots is defined by $\mathsf{R}^+ = (\mathsf{W}\{\alpha_i\}_{i \in I}) \cap \mathsf{Q}^+$. Let $h := 2|\mathsf{R}^+|/n$ be the Coxeter number of $\mathfrak{g}$.

We fix an I-tuple $\epsilon = (\epsilon_i)_{i\in I} \in \{0,1\}^I$ such that $\epsilon_i \neq \epsilon_j$ whenever $i \sim j$. We refer to such an ϵ as a *parity function*. Note that we have only two possible choices of ϵ and the difference does not affect the main results of this paper.

2.2. **Quantum loop algebra.** Let q be an indeterminate and $\Bbbk := \overline{\mathbb{Q}(q)}$ be the algebraic closure of the field $\mathbb{Q}(q)$ of rational functions in q with rational coefficients inside the ambient field $\bigcup_{m\in\mathbb{Z}_{>0}} \overline{\mathbb{Q}}(q^{1/m})$.

Definition 2.1. The quantum loop algebra $U_q(L\mathfrak{g})$ associated with $\mathfrak{g}$ is defined as a $\Bbbk$-algebra with the generators:

$$\left\{x^+_{i,r}, x^-_{i,r} \mid i \in I, r \in \mathbb{Z}\right\} \cup \{q^y \mid y \in \mathsf{P}^\vee\} \cup \{h_{i,m} \mid i \in I, m \in \mathbb{Z}\setminus\{0\}\}$$

satisfying the following relations:

$$q^0 = 1, \quad q^y q^{y'} = q^{y+y'}, \quad [q^y, h_{i,m}] = [h_{i,m}, h_{j,l}] = 0, \quad q^y x^\pm_{i,r} q^{-y} = q^{\pm\alpha_i(y)} x^\pm_{i,r},$$
$$(z - q^{\pm c_{ij}} w)\phi^\varepsilon_i(z) x^\pm_j(w) = (q^{\pm c_{ij}} z - w) x^\pm_j(z)\phi^\varepsilon_i(w),$$
$$(z - q^{\pm c_{ij}} w) x^\pm_i(z) x^\pm_j(w) = (q^{\pm c_{ij}} z - w) x^\pm_j(w) x^\pm_i(z),$$
$$[x^+_i(z), x^-_j(w)] = \frac{\delta_{ij}}{q-q^{-1}}\left(\delta\left(\frac{w}{z}\right)\phi^+_i(w) - \delta\left(\frac{z}{w}\right)\phi^-_i(z)\right),$$

$$\left\{x^\pm_i(z_1)x^\pm_i(z_2)x^\pm_j(w) - (q+q^{-1})x^\pm_i(z_1)x^\pm_j(w)x^\pm_i(z_2) + x^\pm_j(w)x^\pm_i(z_1)x^\pm_i(z_2)\right\}$$
$$+ \{z_1 \leftrightarrow z_2\} = 0 \qquad \text{if } i \sim j,$$

where $\varepsilon \in \{+,-\}$ and $\delta(z), x^\pm_i(z), \phi^\pm_i(z)$ are the formal series defined as follows:

$$\delta(z) := \sum_{r=-\infty}^{\infty} z^r, \quad x^\pm_i(z) := \sum_{r=-\infty}^{\infty} x^\pm_{i,r} z^{-r},$$
$$\phi^\pm_i(z) := q^{\pm h_i}\exp\left(\pm(q-q^{-1})\sum_{m=1}^{\infty} h_{i,\pm m} z^{\mp m}\right).$$

In the last relation, the second term $\{z_1 \leftrightarrow z_2\}$ means the exchange of z_1 with z_2 in the first term.

Let $\widehat{\mathfrak{g}}$ be the (untwisted) affine Lie algebra associated with $\mathfrak{g}$. It is realized as

$$\widehat{\mathfrak{g}} = L\mathfrak{g} \oplus \mathbb{C}c \oplus \mathbb{C}d$$

with a suitable Lie algebra structure, where $L\mathfrak{g} := \mathfrak{g}\otimes\mathbb{C}[z^{\pm1}]$ is the loop algebra of $\mathfrak{g}$, c is a central element and $d := z\frac{\mathrm{d}}{\mathrm{d}z}$ is the degree operator. The derived subalgebra $\widehat{\mathfrak{g}}' = [\widehat{\mathfrak{g}}, \widehat{\mathfrak{g}}] = L\mathfrak{g} \oplus \mathbb{C}c$ is a central extension of the loop algebra $L\mathfrak{g}$. Let $U_q(\widehat{\mathfrak{g}})$ be the quantized enveloping algebra of $\widehat{\mathfrak{g}}$. This is a Hopf algebra over $\Bbbk$ presented by the Chevalley type generators $\{e_i, f_i \mid i \in I\cup\{0\}\} \cup \{q^y \mid y \in \mathsf{P}^\vee \oplus \mathbb{Z}c \oplus \mathbb{Z}d\}$ and the well-known relations. The coproduct $\triangle\colon U_q(\widehat{\mathfrak{g}}) \to U_q(\widehat{\mathfrak{g}}) \otimes U_q(\widehat{\mathfrak{g}})$ is given by:

$$\triangle(e_i) = e_i \otimes q^{-h_i} + 1 \otimes e_i, \quad \triangle(f_i) = f_i \otimes 1 + q^{h_i} \otimes f_i, \quad \triangle(q^y) = q^y \otimes q^y$$

for $i \in I\cup\{0\}, y \in \mathsf{P}^\vee \oplus \mathbb{Z}c \oplus \mathbb{Z}d$. The subalgebra $U'_q(\widehat{\mathfrak{g}})$ generated by the generators $\{e_i, f_i, q^{\pm h_i} \mid i \in I\cup\{0\}\}$ is a Hopf subalgebra of $U_q(\widehat{\mathfrak{g}})$, which is regarded as a q-deformation of the universal enveloping algebra of $\widehat{\mathfrak{g}}'$. By Beck [3], we have a $\Bbbk$-algebra isomorphism $U_q(L\mathfrak{g}) \cong U'_q(\widehat{\mathfrak{g}})/\langle q^c - 1\rangle$, via which the quantum loop algebra $U_q(L\mathfrak{g})$ inherits a structure of Hopf algebra. Actually this isomorphism depends on the choice of a function $o\colon I \to \{\pm1\}$ satisfying $o(i) = -o(j)$ if $i \sim j$. In this paper, we set $o(i) := (-1)^{\epsilon_i}$ by using the parity function ϵ we fixed in the last subsection.

2.3. **Simple and fundamental modules.** A $U_q(L\mathfrak{g})$-module is said to be of type **1** if, for each $i \in I$, the element q^{h_i} acts on it as a semisimple linear operator whose eigenvalues belong to $q^{\mathbb{Z}}$. Let $\mathcal{C}$ denote the category of finite-dimensional $U_q(L\mathfrak{g})$-modules of type **1**. The category $\mathcal{C}$ is a $\Bbbk$-linear abelian monoidal category.

It is well-known that the simple modules of the category $\mathcal{C}$ are parametrized by so-called Drinfeld polynomials [5], or equivalently by the dominant monomials [10], which we recall here. Let $\mathcal{M}$ be the abelian (multiplicative) group freely generated by the symbols $\{Y_{i,a}\}_{(i,a)\in I\times\Bbbk^\times}$ and $\mathcal{M}^+$ be the submonoid of $\mathcal{M}$ generated by $\{Y_{i,a}\}_{(i,a)\in I\times\Bbbk^\times}$. We refer to an element of $\mathcal{M}^+$ as a *dominant monomial*.

Theorem 2.2 (Chari-Pressley [5, Theorem 3.3]). *For each dominant monomial $m = \prod_{(i,a)} Y_{i,a}^{m_{i,a}}$, there exists a simple module $L(m) \in \mathcal{C}$ with a non-zero vector $v_m \in L(m)$ satisfying*

$$x_i^+(z)v_m = 0, \quad \phi_i^\pm(z)v_m = \left(\prod_{a\in\Bbbk^\times}\left(\frac{q-q^{-1}az^{-1}}{1-az^{-1}}\right)^{m_{i,a}}\right)^\pm v_m$$

for each $i \in I$, where $(-)^\pm$ denotes the formal expansion at $z^{\mp1}=0$. Such a vector $v_m \in L(m)$ is unique up to $\Bbbk^\times$. Moreover, the correspondence $m \mapsto L(m)$ gives a bijection between the set $\mathcal{M}^+$ of dominant monomials and the set of isomorphism classes of simple modules of $\mathcal{C}$.

For each $(i,a) \in I\times\Bbbk^\times$, we define an element $A_{i,a} \in \mathcal{M}$ by

$$A_{i,a} := Y_{i,qa}Y_{i,q^{-1}a}\cdot\prod_{j\sim i}Y_{j,a}^{-1}.$$

For $m, m' \in \mathcal{M}$, we write $m \le m'$ if $m'm^{-1}$ is a monomial in $\{A_{i,a}\}_{(i,a)\in I\times\Bbbk^\times}$. This defines a partial ordering on the set $\mathcal{M}^+$ of dominant monomials.

The simple modules $L(Y_{i,a})$ corresponding to the degree 1 dominant monomials $Y_{i,a}\in\mathcal{M}^+$, $(i,a)\in I\times\Bbbk^\times$, are called the *fundamental modules*. The next theorem shows their importance in the monoidal category $\mathcal{C}$.

Theorem 2.3 (Frenkel-Reshetikhin [10], Frenkel-Mukhin [9], Nakajima [32]). *Let $K(\mathcal{C})$ denote the Grothendieck ring of the monoidal abelian category $\mathcal{C}$.*

1. *The ring $K(\mathcal{C})$ is isomorphic to the polynomial ring $\mathbb{Z}[t_{i,a} \mid (i,a)\in I\times\Bbbk^\times]$ in infinitely many variables, where the variable $t_{i,a}$ corresponds to the class of the fundamental module $L(Y_{i,a})$. In particular, $K(\mathcal{C})$ is commutative;*
2. *For each dominant monomial $m = \prod_{i,a} Y_{i,a}^{m_{i,a}} \in \mathcal{M}^+$, we have*

$$\prod_{i,a}[L(Y_{i,a})]^{m_{i,a}} = [L(m)] + \sum_{m'\in\mathcal{M}^+, m'\lneq m} c(m,m')[L(m')]$$

in the Grothendieck ring $K(\mathcal{C})$, where $c(m,m')\in\mathbb{Z}_{\ge0}$.

Proof. (1) is [10, Corollary 2]. (2) was originally conjectured by [10] and proved by [9, Theorem 4.1] and [32, Proposition 5.2] independently. □

2.4. **Normalized R-matrices and their denominators.**

Definition 2.4. Let M be a $U_q(L\mathfrak{g})$-module and z be a formal parameter. We equip the free $\Bbbk[z^{\pm1}]$-module $M[z^{\pm1}] := M \otimes_\Bbbk \Bbbk[z^{\pm1}]$ with a left $U_q(L\mathfrak{g})$-module structure by

$$x_{i,r}^{\pm} \cdot (v \otimes f(z)) := x_{i,r}^{\pm} v \otimes z^r f(z),$$
$$q^y \cdot (v \otimes f(z)) := q^y v \otimes f(z),$$
$$h_{i,m} \cdot (v \otimes f(z)) := h_{i,m} v \otimes z^m f(z),$$

where $v \in M, f(z) \in \Bbbk[z^{\pm 1}]$. We refer to the resulting $U(L\mathfrak{g}) \otimes_{\Bbbk} \Bbbk[z^{\pm 1}]$-module $M[z^{\pm 1}]$ as the *affinization* of M.[1]

To simplify the notation, we denote the *affinized fundamental module* and its generating vector by

$$V_i[z^{\pm 1}] := L(Y_{i,1})[z^{\pm 1}], \quad v_i := (v_{Y_{i,1}}) \otimes 1$$

for each $i \in I$. In addition, for any non-zero scalar $a \in \Bbbk^{\times}$, we set

$$V_i(a) := V_i[z^{\pm 1}]/(z-a)V_i[z^{\pm 1}]$$

and denote by $v_i(a)$ the image of the vector v_i under the canonical quotient map $V_i[z^{\pm 1}] \to V_i(a)$. With this notation, we have an isomorphism $V_i(a) \cong L(Y_{i,a})$ of $U_q(L\mathfrak{g})$-modules via which the vector $v_i(a)$ corresponds to the vector $v_{Y_{i,a}}$.

Remark 2.5. The affinized fundamental module $V_i[z^{\pm 1}]$ is known to be isomorphic to the following modules:

- the level zero extremal weight module of extremal weight ϖ_i introduced by Kashiwara [23, 24];
- the global Weyl module of highest weight ϖ_i introduced by Chari-Pressley [6];
- the standard module associated with ϖ_i, realized via the equivariant K-theory of quiver varieties by Nakajima [32] (see Section 4.5 below).

For a proof, see [24, Section 5] and [34, Remark 2.15].

For each pair $(i, j) \in I^2$, there is a unique $U_q(L\mathfrak{g}) \otimes_{\Bbbk} \Bbbk[z_1^{\pm 1}, z_2^{\pm 1}]$-homomorphism called the *normalized R-matrix*

$$R_{ij} \colon V_i[z_1^{\pm 1}] \otimes V_j[z_2^{\pm 1}] \to \Bbbk(z_2/z_1) \otimes_{\Bbbk[(z_2/z_1)^{\pm 1}]} \left(V_j[z_2^{\pm 1}] \otimes V_i[z_1^{\pm 1}]\right),$$

satisfying the condition $R_{ij}(v_i \otimes v_j) = v_j \otimes v_i$ (see [1, Appendix A] or [24, Section 8]). The *denominator* of the normalized R-matrix R_{ij} is a unique monic polynomial $d_{ij}(u) \in \Bbbk[u]$ of the smallest degree among polynomials satisfying

$$d_{ij}(z_2/z_1) R_{ij} \left(V_i[z_1^{\pm 1}] \otimes V_j[z_2^{\pm 1}]\right) \subset 1 \otimes \left(V_j[z_2^{\pm 1}] \otimes V_i[z_1^{\pm 1}]\right).$$

Remark 2.6. In the same way, we can define the normalized R-matrix

$$R_{M,M'} \colon M[z_1^{\pm 1}] \otimes M'[z_2^{\pm 1}] \to \Bbbk(z_2/z_1) \otimes_{\Bbbk[(z_2/z_1)^{\pm 1}]} \left(M'[z_2^{\pm 1}] \otimes M[z_1^{\pm 1}]\right)$$

and its denominator $d_{M,M'}(u) \in \Bbbk[u]$ for any simple modules $M, M' \in \mathcal{C}$.

In the rest of this subsection, we recall some properties of the normalized R-matrices R_{ij} and their denominators $d_{ij}(u)$ for future use. Let $a, b \in \Bbbk^{\times}$ be non-zero scalars such that $d_{ij}(b/a) \neq 0$. Then the normalized R-matrix R_{ij} can be specialized to yield a non-zero $U_q(L\mathfrak{g})$-homomorphism $R_{ij}(b/a) \colon V_i(a) \otimes V_j(b) \to V_j(b) \otimes V_i(a)$ which sends the vector $v_i(a) \otimes v_j(b)$ to the vector $v_j(b) \otimes v_i(a)$.

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[1] In [24], the affinization is defined in terms of the Chevalley type generators of the algebra $U_q'(\widehat{\mathfrak{g}})$. One can easily see that it coincides with our affinization in Definition 2.4 under the isomorphism $U_q(L\mathfrak{g}) \cong U_q'(\widehat{\mathfrak{g}})/\langle q^c - 1\rangle$ in [3].

Theorem 2.7 ([1, 4, 9, 24, 39])**.** *Let $i, j \in I$ and $a, b \in \Bbbk^\times$.*

(1) *As a $U_q(L\mathfrak{g})$-module, $V_i(a) \otimes V_j(b)$ is generated by the vector $v_i(a) \otimes v_j(b)$ if and only if $d_{ij}(b/a) \neq 0$. If this is the case, the module $V_i(a) \otimes V_j(b)$ has a simple head $\mathrm{Im}(R_{ij}(b/a))$.*
(2) *Any non-zero $U_q(L\mathfrak{g})$-submodule of $V_i(a) \otimes V_j(b)$ contains the vector $v_i(a) \otimes v_j(b)$ if and only if $d_{ji}(a/b) \neq 0$. If this is the case, the module $V_i(b) \otimes V_i(a)$ has a simple socle $\mathrm{Im}(R_{ji}(a/b))$.*

In particular, the following conditions are mutually equivalent:

- *The tensor product $V_i(a) \otimes V_j(b)$ is irreducible;*
- *$V_i(a) \otimes V_j(b) \cong V_j(b) \otimes V_i(a)$ as $U_q(L\mathfrak{g})$-modules;*
- *$d_{ij}(b/a) \neq 0$ and $d_{ji}(a/b) \neq 0$.*

Proof. This is a special case of Akasaka-Kashiwara's conjecture [1], which was proved by Chari [4], Kashiwara [24] and Varagnolo-Vasserot [39] independently. The irreducibility of $\mathrm{Im}(R_{ij}(b/a))$ and $\mathrm{Im}(R_{ji}(a/b))$ was proved in [1, Corollary 2.3]. Note that Frenkel-Mukhin [9] also proved the last assertion. □

Theorem 2.8 (Chari [4], Kashiwara [24])**.** *Let $i, j \in I$ and $a \in \Bbbk$. If $d_{ij}(a) = 0$, we have $a \in \{q^k \in \Bbbk^\times \mid k + \epsilon_i + \epsilon_j \in 2\mathbb{Z}, k > 0\}$.*

Proof. Assume that $d_{ij}(a) = 0$. By [4, Theorem 4.4] and Theorem 2.7 (1) above, we see that $a = q^k$ for some integer k satisfying $k + \epsilon_i + \epsilon_j \in 2\mathbb{Z}$. On the other hand, [24, Proposition 9.3] implies that $a \in \bigcup_{m \in \mathbb{Z}_{>0}} q^{1/m}\overline{\mathbb{Q}}[\![q^{1/m}]\!]$. Therefore k should be positive. □

Remark 2.9. In [4, Section 6], Chari further computed all the zeros of $d_{ij}(u)$ by a type-by-type argument. However, we do not use this fact in this paper.

2.5. **Main theorem.** Let z be a formal parameter. The *quantum Cartan matrix* $C(z) = (C_{ij}(z))_{i,j \in I}$ of $\mathfrak{g}$ is defined by

$$C_{ij}(z) := \begin{cases} z + z^{-1} & (i = j); \\ c_{ij} & (i \neq j). \end{cases}$$

We regard $C(z)$ as an element of the group $GL_n(\mathbb{Z}(\!(z)\!))$ and denote its inverse by $\widetilde{C}(z) = (\widetilde{C}_{ij}(z))_{i,j \in I}$. The (i, j)-entry $\widetilde{C}_{ij}(z) \in \mathbb{Z}(\!(z)\!)$ can be written as

$$\widetilde{C}_{ij}(z) = \sum_{\ell=1}^{\infty} \widetilde{c}_{ij}(\ell) z^\ell.$$

In this way, we get a collection of integers $\{\widetilde{c}_{ij}(\ell)\}_{i,j \in I, \ell \geq 1}$.

Now we can state the main theorem of this paper.

Theorem 2.10. *For each pair $(i, j) \in I^2$, the denominator $d_{ij}(u) \in \Bbbk[u]$ of the normalized R-matrix R_{ij} is given by the following formula:*

$$d_{ij}(u) = \prod_{\ell=1}^{h-1} (u - q^{\ell+1})^{\widetilde{c}_{ij}(\ell)}, \tag{2.1}$$

where h is the Coxeter number of $\mathfrak{g}$.

Theorem 2.10 is equivalent to Theorem 3.9 below, whose proof is given later in Section 5.1 using geometry of the graded quiver varieties.

Remark 2.11. The RHS of the formula (2.1) is actually a polynomial because we have $\widetilde{c}_{ij}(\ell) \in \mathbb{Z}_{\geq 0}$ for $1 \leq \ell \leq h-1$ by Lemma 3.7 (7) below.

Remark 2.12. Note that our denominator $d_{ij}(u)$ is different from the denominator $d_{V(\varpi_i),V(\varpi_j)}(u)$, which has been written by the same symbol $d_{ij}(u)$ in the works of Kashiwara and his collaborators (e.g. [1, 24, 20]). Here $V(\varpi_i)$ denotes the i-th fundamental module in the sense of Kashiwara [23], which has a global crystal basis. It was shown by Nakajima [34] that Kashiwara's fundamental module $V(\varpi_i)$ is isomorphic to our fundamental module $V_i(a_i)$ with $a_i := (-1)^{\epsilon_i}(-q)^{1-h}$. Moreover, we see in Proposition 3.5 below that $\widetilde{c}_{ij}(\ell) \neq 0$ only if $\ell + \epsilon_i + \epsilon_j + 1 \in 2\mathbb{Z}$. Therefore, our denominator formula (2.1) is equivalent to the formula

$$d_{V(\varpi_i),V(\varpi_j)}(u) = \prod_{\ell=1}^{h-1} (u - (-q)^{\ell+1})^{\widetilde{c}_{ij}(\ell)}. \tag{2.2}$$

By making the values $\widetilde{c}_{ij}(\ell)$ explicit, we can check that the formula (2.2) certainly recovers the known type-by-type denominator formulas obtained in [8, 20, 36, 37]. However we do not use this fact in this paper.

3. An interpretation by quiver representations

In this section, we give an interpretation of our denominator formula (2.1) in terms of homological properties of representations of a Dynkin quiver of type $\mathfrak{g}$. We keep the notation from the previous section.

3.1. **Convention.** First, we fix our convention on quivers and their representations. A quiver $Q = (Q_0, Q_1)$ is an oriented graph, consisting of the set Q_0 of vertices and the set Q_1 of arrows. Here the sets Q_0 and Q_1 can be infinite. For an arrow $a \in Q_1$, let $a', a'' \in Q_0$ denote its origin and goal respectively. We always assume that the set $\{a \in Q_1 \mid a' = x, a'' = y\}$ is finite for each $x, y \in Q_0$.

We equip the vector space $\mathbb{C}Q_0 := \bigoplus_{x \in Q_0} \mathbb{C}e_x$ with a structure of $\mathbb{C}$-algebra by $e_x \cdot e_y = \delta_{xy} e_x$. This is non-unital if Q_0 is infinite. We equip the vector space $\mathbb{C}Q_1 := \bigoplus_{a \in Q_1} \mathbb{C}a$ with a structure of $\mathbb{C}Q_0$-bimodule by setting $a \cdot e_x = \delta_{a',x} a$ and $e_x \cdot a = \delta_{x,a''} a$ for $x \in Q_0, a \in Q_1$. The path algebra $\mathbb{C}Q$ of Q is defined to be the tensor algebra $T_{\mathbb{C}Q_0}(\mathbb{C}Q_1) := \bigoplus_{d \geq 0} (\mathbb{C}Q_1)^{\otimes d}$, where tensor products are taken over $\mathbb{C}Q_0$. Given a quotient algebra $A = \mathbb{C}Q/\mathcal{I}$ by an ideal $\mathcal{I} \subset \bigoplus_{d \geq 1} (\mathbb{C}Q_1)^{\otimes d}$, we denote by $A\text{-}\mathsf{mod}$ the $\mathbb{C}$-linear abelian category of finite-dimensional left A-modules M satisfying $M = \bigoplus_{x \in Q_0} e_x M$. For each vertex $x \in Q_0$, we denote by S_x the simple object of $A\text{-}\mathsf{mod}$ associated with x, i.e. satisfying $\dim(e_y S_x) = \delta_{xy}$. For a finite-dimensional Q_0-graded $\mathbb{C}$-vector space $V = \bigoplus_{x \in Q_0} V_x$, we denote by $\mathsf{rep}_V(A)$ the variety of representations of the algebra A realized on V. By definition, this is the closed subvariety of the affine space $\mathsf{rep}_V(Q) := \prod_{a \in Q_1} \mathrm{Hom}_{\mathbb{C}}(V_{a'}, V_{a''})$ consisting of points $(f_a)_{a \in Q_1}$ such that all the polynomials in the linear maps f_a corresponding to elements in $\mathcal{I}$ vanish.

3.2. **Dynkin quiver.** In this subsection, we fix a Dynkin quiver $Q = (Q_0, Q_1)$ of type $\mathfrak{g}$, i.e. $Q_0 := I = \{1, \dots n\}$ and the arrow set Q_1 satisfies the condition $c_{ij} = 2\delta_{ij} - \#\{a \in Q_1 \mid \{a', a''\} = \{i, j\}\}$ for each $i, j \in I$. We write $i \to j$ if there is an arrow $a \in Q_1$ such that $a' = i, a'' = j$. For $M \in \mathbb{C}Q\text{-}\mathsf{mod}$, we define its dimension vector by $\underline{\dim}\,(M) := \sum_{i \in I} \dim(e_i M) \alpha_i \in \mathsf{Q}^+$.

By Gabriel's theorem [13], for each $\alpha \in \mathsf{R}^+$, there exists an indecomposable object $M_\alpha \in \mathbb{C}Q\text{-}\mathsf{mod}$ such that $\underline{\dim}\,(M_\alpha) = \alpha$ uniquely up to isomorphism. The correspondence $\alpha \mapsto M_\alpha$ gives a bijection between the set R^+ of positive roots and the set of isomorphism classes of indecomposable objects of $\mathbb{C}Q\text{-}\mathsf{mod}$. In particular, we have $S_i = M_{\alpha_i}$ for each $i \in I$.

Let $\mathcal{D}_Q$ denote the bounded derived category $D^b(\mathbb{C}Q\text{-}\mathsf{mod})$ of the abelian category $\mathbb{C}Q\text{-}\mathsf{mod}$. The category $\mathcal{D}_Q$ is a $\mathbb{C}$-linear triangulated category with Krull-Schmidt property. The category $\mathbb{C}Q\text{-}\mathsf{mod}$ is naturally identified with a full subcategory of $\mathcal{D}_Q$ consisting of complexes concentrated on the cohomological degree 0. We denote by $X[k]$ the cohomological degree shift of $X \in \mathcal{D}_Q$ by $k \in \mathbb{Z}$. Then the set $\{M_\alpha[k] \mid \alpha \in \mathsf{R}^+, k \in \mathbb{Z}\}$ forms a complete collection of indecomposable objects of $\mathcal{D}_Q$ (see [16, Lemma 4.1]). Extending the definition of $\underline{\dim}$, for each $X \in \mathcal{D}_Q$, we define its dimension vector $\underline{\dim}\,(X) \in \mathsf{Q}$ by

$$\underline{\dim}\,(X) := \sum_{k \in \mathbb{Z}} (-1)^k \underline{\dim}\, H^k(X),$$

where $H^k(X) \in \mathbb{C}Q\text{-}\mathsf{mod}$ denotes the k-th cohomology of X. For $X, Y \in \mathcal{D}_Q$, we define the Euler-Poincaré characteristic $\langle X, Y \rangle \in \mathbb{Z}$ by

$$\langle X, Y \rangle := \sum_{k \in \mathbb{Z}} (-1)^k \dim \operatorname{Ext}^k_{\mathcal{D}_Q}(X, Y),$$

where $\operatorname{Ext}^k_{\mathcal{D}_Q}(X, Y) := \operatorname{Hom}_{\mathcal{D}_Q}(X, Y[k])$.

3.3. **Happel's equivalence.** Let Q be a Dynkin quiver of type $\mathfrak{g}$. In this subsection, we recall the description of the full subcategory $\mathsf{ind}(\mathcal{D}_Q) \subset \mathcal{D}_Q$ consisting of indecomposable objects in $\mathcal{D}_Q$ due to Happel [16, 17] .

Let $\xi = (\xi_i)_{i \in I} \in \mathbb{Z}^I$ be an I-tuple of integers such that $\xi_i - \epsilon_i \in 2\mathbb{Z}$ and $\xi_i = \xi_j + 1$ if $i \to j$. Such an I-tuple ξ is called a *height function* of Q and determined up to a simultaneous shift by an even integer. Choose a total ordering $I = \{i_1, i_2, \dots, i_n\}$ satisfying $\xi_{i_1} \ge \xi_{i_2} \ge \cdots \ge \xi_{i_n}$ and consider the Coxeter element $\tau := r_{i_1} r_{i_2} \cdots r_{i_n} \in \mathsf{W}$. The element τ depends only on Q (independent from the choice of the above total ordering of I). By an abuse of notation, we use the same symbol τ for the corresponding Coxeter functor, which is an auto-equivalence of $\mathcal{D}_Q$. Under this convention, we have $\underline{\dim}\,(\tau X) = \tau \underline{\dim}\,(X)$ for any $X \in \mathcal{D}_Q$. For an indecomposable object $X \in \mathsf{ind}(\mathcal{D}_Q)$, its Coxeter transformation τX coincides with the Auslander-Reiten translation of X (see [2, Lemma VII.5.8] for example).

For each $i \in I$, we define a positive root γ_i to be the sum of simple roots α_j labeled by the vertices j such that there exists an oriented path in Q from j to i. Then the corresponding indecomposable representation $I_i := M_{\gamma_i}$ is an injective hull of the simple representation S_i in $\mathbb{C}Q\text{-}\mathsf{mod}$. Note that we have $\langle X, I_i \rangle = (\underline{\dim}\,(X), \varpi_i)$ for any $X \in \mathcal{D}_Q$ and $i \in I$.

Definition 3.1. We define an infinite quiver $\Delta = (\Delta_0, \Delta_1)$ by

$$\begin{aligned} \Delta_0 &:= \{(i, p) \in I \times \mathbb{Z} \mid p - \epsilon_i \in 2\mathbb{Z}\}, \\ \Delta_1 &:= \{(i, p) \to (j, p+1) \mid (i, p), (j, p+1) \in \Delta_0,\ i \sim j\}. \end{aligned}$$

Let $\mathbb{C}(\Delta)$ denote the $\mathbb{C}$-linear category whose set of objects is Δ_0 and whose morphisms are generated by Δ_1 satisfying the so-called *mesh relations*, i.e. the sum of all paths from (i, p) to $(i, p+2)$ vanishes for each $(i, p) \in \Delta_0$. Note that the quiver Δ and the category $\mathbb{C}(\Delta)$ are independent from the choice of the Dynkin quiver Q (depends only on $\mathfrak{g}$).

Theorem 3.2 (Happel [16, 17]). *For a Dynkin quiver Q of type $\mathfrak{g}$ with a height function ξ, there is an equivalence of $\mathbb{C}$-linear categories*

$$\mathtt{H}_Q \colon \mathbb{C}(\Delta) \simeq \mathsf{ind}(\mathcal{D}_Q)$$

satisfying $\mathtt{H}_Q(i,p) = \tau^{(\xi_i - p)/2}(I_i)$ for each $(i,p) \in \Delta_0$.

Proof. See [16, Proposition 4.6] or [17, Theorem 5.6]. □

Remark 3.3. Although the equivalence $\mathtt{H}_Q$ depends on the choice of the height function ξ, this choice does not affect on the results in the present paper essentially and hence we suppress it from the notation. In addition, the Euler-Poincaré characteristic $\langle \mathtt{H}_Q(i,p), \mathtt{H}_Q(j,r)\rangle$ does not depend on the choice of the Dynkin quiver Q because, for any two Dynkin quivers Q and Q' of type $\mathfrak{g}$, we have a natural isomorphism

$$\operatorname{Ext}^k_{\mathcal{D}_Q}(\mathtt{H}_Q(i,p), \mathtt{H}_Q(j,r)) \cong \operatorname{Ext}^k_{\mathcal{D}_{Q'}}(\mathtt{H}_{Q'}(i,p), \mathtt{H}_{Q'}(j,r)) \tag{3.1}$$

for any $(i,p), (j,r) \in \Delta_0$ and $k \in \mathbb{Z}$.

Remark 3.4. As explained in [14, Section 6.5], we have

$$\mathtt{H}_Q(i,p)[1] = \mathtt{H}_Q(i^*, p+h) \tag{3.2}$$

for any $(i,p) \in \Delta_0$. Here $i \mapsto i^*$ is the involution on I given by $\mathsf{w}_0(\alpha_i) = -\alpha_{i^*}$, where w_0 denotes the longest element of the Weyl group W.

3.4. **Quiver interpretation of quantum Cartan matrix.** In this subsection, we give an interpretation of the integers $\{\widetilde{c}_{ij}(\ell)\}_{i,j\in I, \ell \geq 1}$ defined in Section 2.5 in terms of representations of a Dynkin quiver. Our discussion is based on the following observation due to Hernandez-Leclerc.

Proposition 3.5 (Hernandez-Leclerc [18]). *Take a Dynkin quiver Q of type $\mathfrak{g}$ together with a height function ξ as in the previous subsection. Then, for any $i, j \in I$ and $\ell \in \mathbb{Z}_{\geq 1}$, we have*

$$\widetilde{c}_{ij}(\ell) = \begin{cases} \left(\tau^{(\ell + \xi_i - \xi_j - 1)/2}(\gamma_i), \varpi_j\right) & \text{if } \ell + \epsilon_i + \epsilon_j + 1 \in 2\mathbb{Z}; \\ 0 & \text{otherwise.} \end{cases}$$

Proof. This is [18, Proposition 2.1]. Note that the condition $\ell + \epsilon_i + \epsilon_j + 1 \in 2\mathbb{Z}$ here is equivalent to the condition $\ell + \xi_i - \xi_j - 1 \in 2\mathbb{Z}$ therein. □

Thanks to Proposition 3.5, once we depict the Auslander-Reiten quiver of $\mathcal{D}_Q$, we can easily compute the explicit values of the integers $\{\widetilde{c}_{ij}(\ell)\}_{i,j\in I,\ell\geq 1}$. See [18, Example 2.2] for an example of such a computation.

Corollary 3.6. *For $(i,p), (j,r) \in \Delta_0$ with $r \geq p$, we have*

$$\langle \mathtt{H}_Q(i,p), \mathtt{H}_Q(j,r)\rangle = \widetilde{c}_{ij}(r-p+1)$$

for any Dynkin quiver Q of type $\mathfrak{g}$.

Proof. We compute as:

$$\begin{aligned}\langle \mathtt{H}_Q(i,p), \mathtt{H}_Q(j,r)\rangle &= \left\langle \tau^{(\xi_i-p)/2}(I_i), \tau^{(\xi_j-r)/2}(I_j)\right\rangle \\ &= \left\langle \tau^{((r-p+1)+\xi_i-\xi_j-1)/2}(I_i), I_j\right\rangle \\ &= \left(\tau^{((r-p+1)+\xi_i-\xi_j-1)/2}(\gamma_i), \varpi_j\right).\end{aligned}$$

Since $r-p+1 \geq 1$ by assumption, the RHS is equal to $\widetilde{c}_{ij}(r-p+1)$ by Proposition 3.5. □

Here we record some basic properties of the integers $\{\widetilde{c}_{ij}(\ell)\}_{i,j\in I,\ell\geq 1}$.

Lemma 3.7. *The integers $\{\widetilde{c}_{ij}(\ell)\}_{i,j\in I,\ell\geq 1}$ satisfy the following properties:*

(1) $\widetilde{c}_{ij}(\ell) = \widetilde{c}_{ji}(\ell)$;
(2) $\widetilde{c}_{ij}(\ell) = \widetilde{c}_{\sigma(i),\sigma(j)}(\ell)$ *for any automorphism σ of the Dynkin diagram;*
(3) $\widetilde{c}_{ij}(\ell) = \widetilde{c}_{ij}(\ell+2h)$;
(4) $\widetilde{c}_{ij}(\ell) = -\widetilde{c}_{ij}(2h-\ell)$ *for* $1 \leq \ell \leq 2h-1$;
(5) $\widetilde{c}_{ij}(\ell) = \widetilde{c}_{ji^*}(h-\ell)$ *for* $1 \leq \ell \leq h-1$;
(6) $\widetilde{c}_{ij}(kh) = 0$ *for any* $k \in \mathbb{Z}_{\geq 1}$;
(7) $\widetilde{c}_{ij}(\ell) \geq 0$ *if* $1 \leq \ell \leq h-1$;
(8) $\widetilde{c}_{ij}(\ell) \leq 0$ *if* $h+1 \leq \ell \leq 2h-1$.

Proof. (1) and (2) are immediate from the definition.

Let us take Q and ξ as in Proposition 3.5. (3) is a direct consequence of Proposition 3.5 and the well-known fact $\tau^h = 1$.

To prove (4), we may assume $\ell+\epsilon_i+\epsilon_j+1 \in 2\mathbb{Z}$. Then we can pick $(i,p),(j,r) \in \Delta_0$ such that $\ell = r-p+1$. By Corollary 3.6 and (3.2), we have

$$\widetilde{c}_{ij}(\ell) = \langle \mathtt{H}_Q(i,p), \mathtt{H}_Q(j,r)\rangle = \langle \mathtt{H}_Q(i,p), \mathtt{H}_Q(j,r-2h)\rangle$$

for any Dynkin quiver Q. Using the Auslander-Reiten duality $\langle X,Y\rangle = -\langle Y,\tau X\rangle$, $X,Y \in \mathcal{D}_Q$, the RHS is further computed as:

$$\begin{aligned}\langle \mathtt{H}_Q(i,p), \mathtt{H}_Q(j,r-2h)\rangle &= -\langle \mathtt{H}_Q(j,r-2h), \tau\mathtt{H}_Q(i,p)\rangle \\ &= -\langle \mathtt{H}_Q(j,r-2h), \mathtt{H}_Q(i,p-2)\rangle.\end{aligned}$$

Since $(p-2)-(r-2h) = 2h-1-\ell \geq 0$ by assumption, the RHS of the last equation is equal to $-\widetilde{c}_{ij}(2h-\ell)$ again by Corollary 3.6. This proves (4).

Let us prove (5). As before, we may assume $\ell+\epsilon_i+\epsilon_j+1 \in 2\mathbb{Z}$. For a Dynkin quiver Q and $(i,p),(j,r) \in \Delta_0$ with $r-p+1 = \ell$, we have $\widetilde{c}_{ij}(\ell) = \langle \mathtt{H}_Q(i,p), \mathtt{H}_Q(j,r)\rangle$ by Corollary 3.6. Using $\langle X,Y\rangle = -\langle Y,\tau X\rangle = \langle Y,\tau X[1]\rangle$ and (3.2), we further compute as:

$$\langle \mathtt{H}_Q(i,p), \mathtt{H}_Q(j,r)\rangle = \langle \mathtt{H}_Q(j,r), \tau\mathtt{H}_Q(i,p)[1]\rangle = \langle \mathtt{H}_Q(j,r), \mathtt{H}_Q(i^*,p+h-2)\rangle.$$

Since $(p+h-2)-r = (h-1)-\ell \geq 0$ by assumption, we get $\langle \mathtt{H}_Q(j,r), \mathtt{H}_Q(i^*,p+h-2)\rangle = \widetilde{c}_{ji^*}(h-\ell)$ again by Corollary 3.6. This proves (5).

To prove (6), it suffices to check that $\widetilde{c}_{ij}(h) = \widetilde{c}_{ij}(2h) = 0$ thanks to (3). Specializing $\ell = h$ in (4), we get $\widetilde{c}_{ij}(h) = -\widetilde{c}_{ij}(h)$ and hence $\widetilde{c}_{ij}(h) = 0$. Let us verify $\widetilde{c}_{ij}(2h) = 0$. When $\epsilon_i = \epsilon_j$, the number $2h+\epsilon_i+\epsilon_j+1$ is always odd. Therefore we have $\widetilde{c}_{ij}(2h) = 0$ by Proposition 3.5. When $\epsilon_i \neq \epsilon_j$, let us choose Q with the *sink-source* orientation such that i is a source and j is a sink. Namely, we choose Q and its height function ξ so that we have $\xi_i = \xi_j + 1$, and $\xi_k = \xi_i$ (resp. $\xi_k = \xi_j$) if

$\epsilon_k = \epsilon_i$ (resp. $\epsilon_k = \epsilon_j$). With such a choice, we have $\gamma_i = \alpha_i$ and $2h+\xi_i-\xi_j-1 = 2h$. Therefore, by Proposition 3.5, we get

$$\widetilde{c}_{ij}(2h) = (\tau^{2h/2}(\gamma_i), \varpi_j) = (\alpha_i, \varpi_j) = 0.$$

Let us prove (7). Assume $\widetilde{c}_{ij}(\ell) \neq 0$. First we consider the case $\epsilon_i = \epsilon_j$. In this case, Proposition 3.5 implies that ℓ is odd. Let us take a Dynkin quiver Q with a sink-source orientation. In particular, we have $\xi_i = \xi_j$. By the description of the Auslander-Reiten quiver of $\mathcal{D}_Q$ in [14, Proposition 6.5], we see that the set

$$\{\tau^{(\ell-1)/2}(\gamma_i) \mid 1 \le \ell \le h-1,\ \ell \text{ is odd}\} = \{\gamma_i, \tau(\gamma_i), \dots, \tau^{\lfloor (h-2)/2 \rfloor}(\gamma_i)\}$$

consists of positive roots. Therefore we have $\widetilde{c}_{ij}(\ell) = \left(\tau^{(\ell-1)/2}(\gamma_i), \varpi_j\right) \ge 0$ for any $1 \le \ell \le h-1$. For the other case $\epsilon_i \neq \epsilon_j$, Proposition 3.5 implies that ℓ is even. Let us take Q with a sink-source orientation with i being a sink. Then j is a source and hence $\ell + \xi_i - \xi_j - 1 = \ell - 2$. Again we can see that the set

$$\{\tau^{(\ell-2)/2}(\gamma_i) \mid 1 \le \ell \le h-1,\ \ell \text{ is even}\} = \{\gamma_i, \tau(\gamma_i), \dots, \tau^{\lfloor (h-3)/2 \rfloor}(\gamma_i)\}$$

consists of positive roots. Therefore we get $\widetilde{c}_{ij}(\ell) = \left(\tau^{(\ell-2)/2}(\gamma_i), \varpi_j\right) \ge 0$.

The last item (8) follows from (4) and (7). □

Thanks to the above lemma, we can recover all the integers $\{\widetilde{c}_{ij}(\ell) \mid \ell \ge 1\}$ for each $(i,j) \in I^2$ from the first $h-1$ integers $\{\widetilde{c}_{ij}(\ell) \mid 1 \le \ell \le h-1\}$, for which we have the following simple representation-theoretic interpretation.

Proposition 3.8. *Let Q be a Dynkin quiver of type $\mathfrak{g}$. For any $(j,r),(i,p) \in \Delta_0$, we have*

$$\dim \operatorname{Ext}^1_{\mathcal{D}_Q}(\mathtt{H}_Q(j,r), \mathtt{H}_Q(i,p)) = \begin{cases} \widetilde{c}_{ij}(r-p-1) & \text{if } 1 \le r-p-1 \le h-1; \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Using (3.2), we have

$$\begin{aligned} \dim \operatorname{Ext}^1_{\mathcal{D}_Q}(\mathtt{H}_Q(j,r), \mathtt{H}_Q(i,p)) &= \dim \operatorname{Hom}_{\mathcal{D}_Q}(\mathtt{H}_Q(j,r), \mathtt{H}_Q(i,p)[1]) \\ &= \dim \operatorname{Hom}_{\mathcal{D}_Q}(\mathtt{H}_Q(j,r), \mathtt{H}_Q(i^*,p+h)). \end{aligned} \tag{3.3}$$

On the other hand, using the Auslander-Reiten duality, we have

$$\begin{aligned} \dim \operatorname{Ext}^1_{\mathcal{D}_Q}(\mathtt{H}_Q(j,r), \mathtt{H}_Q(i,p)) &= \dim \operatorname{Hom}_{\mathcal{D}_Q}(\mathtt{H}_Q(i,p), \tau\mathtt{H}_Q(j,r)) \\ &= \dim \operatorname{Hom}_{\mathcal{D}_Q}(\mathtt{H}_Q(i,p), \mathtt{H}_Q(j,r-2)). \end{aligned} \tag{3.4}$$

Now we assume that $\operatorname{Ext}^1_{\mathcal{D}_Q}(\mathtt{H}_Q(j,r), \mathtt{H}_Q(i,p)) \neq 0$. Then we have

$$\operatorname{Hom}_{\mathcal{D}_Q}(\mathtt{H}_Q(j,r), \mathtt{H}_Q(i^*,p+h)) \neq 0, \quad \operatorname{Hom}_{\mathcal{D}_Q}(\mathtt{H}_Q(i,p), \mathtt{H}_Q(j,r-2)) \neq 0$$

by the above equations (3.3) and (3.4) respectively. In view of Theorem 3.2, they imply that $r \le p+h$ and $p \le r-2$, or equivalently $1 \le r-p-1 \le h-1$.

Conversely, let us assume $1 \le r-p-1 \le h-1$. We continue (3.4) as:

$$\begin{aligned} \dim \operatorname{Hom}_{\mathcal{D}_Q}(\mathtt{H}_Q(i,p), \mathtt{H}_Q(j,r-2)) &= \dim \operatorname{Hom}_{\mathcal{D}_Q}(\tau^{(\xi_i-p)/2}(I_i), \tau^{(\xi_j-r+2)/2}(I_j)) \\ &= \dim \operatorname{Hom}_{\mathcal{D}_Q}(\tau^{((r-p-1)+\xi_i-\xi_j-1)/2}(I_i), I_j). \end{aligned}$$

Because of (3.1), we may assume that our Dynkin quiver Q has the sink-source orientation with the vertex i being a source. Then the object $\tau^{((r-p-1)+\xi_i-\xi_j-1)/2}(I_i)$ remains inside $\mathbb{C}Q\text{-}\mathsf{mod} \subset \mathcal{D}_Q$. Therefore we have

$$\dim \operatorname{Hom}_{\mathcal{D}_Q}(\tau^{((r-p-1)+\xi_i-\xi_j-1)/2}(I_i), I_j) = \left\langle \tau^{((r-p-1)+\xi_i-\xi_j-1)/2}(I_i), I_j \right\rangle.$$

The RHS is equal to $\widetilde{c}_{ij}(r-p-1)$ by Corollary 3.6. □

3.5. **Quiver interpretation of the denominator formula.** Thanks to Theorem 2.8 and Proposition 3.8, we see that the following assertion is equivalent to our main theorem (=Theorem 2.10).

Theorem 3.9. *Let Q be a Dynkin quiver of type $\mathfrak{g}$. For any $(i,p),(j,r) \in \Delta_0$, the pole order of the normalized R-matrix R_{ij} at $z_2/z_1 = q^r/q^p$ (i.e. the zero order of $d_{ij}(u)$ at $u = q^r/q^p$) is equal to $\dim \mathrm{Ext}^1_{\mathcal{D}_Q}(\mathtt{H}_Q(j,r), \mathtt{H}_Q(i,p))$.*

A proof of Theorem 3.9 is given in Section 5.1 below.

In what follows, for each $(i,p) \in I \times \mathbb{Z}$, we simplify the notation by setting

$$Y_{(i,p)} := Y_{i,q^p}.$$

Recall that we have an identification $L(Y_{(i,p)}) = V_i(q^p)$ for each $(i,p) \in I \times \mathbb{Z}$.

Corollary 3.10. *Let Q be a Dynkin quiver of type $\mathfrak{g}$. For any $x, y \in \Delta_0$, the following conditions are mutually equivalent:*

- *The tensor product $L(Y_x) \otimes L(Y_y)$ is irreducible;*
- *$L(Y_x) \otimes L(Y_y) \cong L(Y_y) \otimes L(Y_x)$ as $U_q(L\mathfrak{g})$-modules;*
- *$\mathrm{Ext}^1_{\mathcal{D}_Q}(\mathtt{H}_Q(x), \mathtt{H}_Q(y)) = 0$ and $\mathrm{Ext}^1_{\mathcal{D}_Q}(\mathtt{H}_Q(y), \mathtt{H}_Q(x)) = 0$.*

Proof. This follows from Theorem 2.7 and Theorem 3.9. □

4. Graded quiver varieties

In this section, we collect some known facts about the graded quiver varieties which we need in this paper. We keep the notation from the previous sections.

4.1. **Notation on the equivariant K-theory.** Let G be a complex linear algebraic group. In the present paper, a G-variety X always means a quasi-projective complex algebraic variety equipped with an algebraic action of the group G. We set $\mathrm{pt} := \operatorname{Spec} \mathbb{C}$ with the trivial G-action. The equivariant K-group $K^G(X)$ is defined to be the Grothendieck group of the abelian category of G-equivariant coherent sheaves on X, which is a module over the representation ring $R(G) = K^G(\mathrm{pt})$.

Let $\mathbb{F}$ be a field of characteristic zero. We put

$$K^G(X)_{\mathbb{F}} := K^G(X) \otimes_{\mathbb{Z}} \mathbb{F}, \quad R(G)_{\mathbb{F}} := R(G) \otimes_{\mathbb{Z}} \mathbb{F}.$$

Let $\mathfrak{a} \subset R(G)_{\mathbb{F}}$ be the augmentation ideal, i.e. the ideal consisting of virtual representations of dimension 0. We define the $\mathfrak{a}$-adic completions by

$$\widehat{K}^G(X)_{\mathbb{F}} := \varprojlim_k K^G(X)_{\mathbb{F}}/\mathfrak{a}^k K^G(X)_{\mathbb{F}}, \quad \widehat{R}(G)_{\mathbb{F}} := \varprojlim_k R(G)_{\mathbb{F}}/\mathfrak{a}^k.$$

The completed K-group $\widehat{K}^G(X)_{\mathbb{F}}$ is a module over the algebra $\widehat{R}(G)_{\mathbb{F}}$.

4.2. **Convolution product.** We recall the definition of the *convolution product* for the equivariant K-groups (see [7, Chapter 5] and [32, Section 6, 8] for details). Let M_i be a non-singular G-variety for $i = 1,2,3$. We denote by $p_{ij} \colon M_1 \times M_2 \times M_3 \to M_i \times M_j$ the natural projection for $(i,j) = (1,2),(2,3),(1,3)$. Let $Z_{12} \subset M_1 \times M_2$ and $Z_{23} \subset M_2 \times M_3$ be G-stable closed subvarieties such that the morphism $p_{13} \colon p_{12}^{-1}(Z_{12}) \cap p_{23}^{-1}(Z_{23}) \to Z_{13} := p_{13}(p_{12}^{-1}(Z_{12}) \cap p_{23}^{-1}(Z_{23}))$ is proper. Then we define the convolution product $*\colon K^G(Z_{12}) \otimes_{R(G)} K^G(Z_{23}) \to K^G(Z_{13})$ relative to M_1, M_2, M_3 by

$$\zeta * \eta := p_{13*}(p_{12}^*\zeta \otimes^{\mathbb{L}}_{M_1 \times M_2 \times M_3} p_{23}^*\eta), \tag{4.1}$$

where $\zeta \in K^G(Z_{12}), \eta \in K^G(Z_{23})$. This naturally induces the convolution product on the completed G-equivariant K-groups $\widehat{K}^G(Z_{12})_{\mathbb{F}} \otimes_{\widehat{R}(G)_{\mathbb{F}}} \widehat{K}^G(Z_{23})_{\mathbb{F}} \to \widehat{K}^G(Z_{13})_{\mathbb{F}}$. Note that the convolution product $*$ depends on the ambient smooth spaces M_1, M_2, M_3.

4.3. **Quiver varieties.** In this subsection, we recall the definition of the (usual) Nakajima quiver varieties. A basic reference is [32].

We fix I-graded finite-dimensional complex vector spaces $\bar{V} = \bigoplus_{i \in I} \bar{V}_i, \bar{W} = \bigoplus_{i \in I} \bar{W}_i$ and consider the following space of linear maps:

$$\mathbf{M}(\bar{V}, \bar{W}) := \left(\bigoplus_{i \sim j} \mathrm{Hom}(\bar{V}_i, \bar{V}_j) \right) \oplus \left(\bigoplus_{i \in I} \mathrm{Hom}(\bar{W}_i, \bar{V}_i) \right) \oplus \left(\bigoplus_{i \in I} \mathrm{Hom}(\bar{V}_i, \bar{W}_i) \right)$$

On the $\mathbb{C}$-vector space $\mathbf{M}(\bar{V}, \bar{W})$, the groups $G_{\bar{V}} := \prod_{i \in I} GL(\bar{V}_i)$, $G_{\bar{W}} := \prod_{i \in I} GL(\bar{W}_i)$ act by conjugation and the 1-dimensional torus $\mathbb{C}^\times$ acts by the scalar multiplication. We write an element of $\mathbf{M}(\bar{V}, \bar{W})$ as a triple (B, a, b) of linear maps $B = \bigoplus B_{ij}$, $a = \bigoplus a_i$ and $b = \bigoplus b_i$. Let $\mu = \bigoplus_{i \in I} \mu_i \colon \mathbf{M}(\bar{V}, \bar{W}) \to \bigoplus_{i \in I} \mathfrak{gl}(\bar{V}_i)$ be the map given by

$$\mu_i(B, a, b) = a_i b_i + \sum_{j \sim i} B_{ij} B_{ji}.$$

A point $(B, a, b) \in \mu^{-1}(0)$ is said to be stable if there exists no non-zero I-graded subspace $\bar{V}' \subset \bar{V}$ such that $B(\bar{V}') \subset \bar{V}'$ and $\bar{V}' \subset \operatorname{Ker} b$. Let $\mu^{-1}(0)^{\mathrm{st}}$ be the set of stable points, on which $G_{\bar{V}}$ acts freely. Then we consider a set-theoretic quotient

$$\mathfrak{M}(\bar{V}, \bar{W}) := \mu^{-1}(0)^{\mathrm{st}} / G_{\bar{V}}.$$

It is known that this quotient has a structure of a non-singular quasi-projective variety which is isomorphic to a quotient in the geometric invariant theory. We also consider the categorical quotient

$$\mathfrak{M}_0(\bar{V}, \bar{W}) := \mu^{-1}(0) /\!/ G_{\bar{V}} = \operatorname{Spec} \mathbb{C}[\mu^{-1}(0)]^{G_{\bar{V}}},$$

together with a canonical projective morphism $\mathfrak{M}(\bar{V}, \bar{W}) \to \mathfrak{M}_0(\bar{V}, \bar{W})$. These quotients $\mathfrak{M}(\bar{V}, \bar{W})$, $\mathfrak{M}_0(\bar{V}, \bar{W})$ naturally inherit the actions of the group $\mathbb{G}_{\bar{W}} := G_{\bar{W}} \times \mathbb{C}^\times$, which makes the canonical projective morphism $\mathbb{G}_{\bar{W}}$-equivariant.

For any two I-graded vector spaces $\bar{V}, \bar{V}'$ such that $\dim \bar{V}_i \le \dim \bar{V}'_i$ for each $i \in I$, there is a natural closed embedding $\mathfrak{M}_0(\bar{V}, \bar{W}) \hookrightarrow \mathfrak{M}_0(\bar{V}', \bar{W})$. With respect to these embeddings, the family $\{\mathfrak{M}_0(\bar{V}, \bar{W})\}_{\bar{V}}$ forms an inductive system, which stabilizes at some large $\bar{V}$. We consider the union (inductive limit) and obtain the following combined $\mathbb{G}_{\bar{W}}$-equivariant morphism:

$$\pi \colon \mathfrak{M}(\bar{W}) := \bigsqcup_{\bar{V}} \mathfrak{M}(\bar{V}, \bar{W}) \to \mathfrak{M}_0(\bar{W}) := \bigcup_{\bar{V}} \mathfrak{M}_0(\bar{V}, \bar{W}).$$

We refer to these varieties $\mathfrak{M}(\bar{W}), \mathfrak{M}_0(\bar{W})$ as the *quiver varieties*. Note that the component $\mathfrak{M}(0, \bar{W})$ consists of a single point, which we denote by $\hat{0}$. We call its image $\pi(\hat{0}) = 0$ the origin of $\mathfrak{M}_0(\bar{W})$.

4.4. **Graded quiver varieties.** Next we define the graded quiver varieties. Recall the infinite set $\Delta_0 = \{(i,p) \in I \times \mathbb{Z} \mid p - \epsilon_i \in 2\mathbb{Z}\}$ in Definition 3.1. We fix a Δ_0-graded finite-dimensional complex vector space $W = \bigoplus_{(i,p)\in\Delta_0} W_{(i,p)}$. Let $\bar{W} = \bigoplus_{i\in I} \bar{W}_i$ be the underlying I-graded vector space of W, i.e. $\bar{W}_i := \bigoplus_p W_{(i,p)}$ for each $i \in I$. We define a 1-dimensional algebraic subtorus $T_W \subset \mathbb{G}_{\bar{W}}$ by

$$T_W := \left\{ \left(\bigoplus_{(i,p)\in\Delta_0} t^p \mathrm{id}_{W_{(i,p)}}, t \right) \in \mathbb{G}_{\bar{W}} \;\middle|\; t \in \mathbb{C}^\times \right\}. \tag{4.2}$$

Note that the centralizer of T_W inside $\mathbb{G}_{\bar{W}}$ is $\mathbb{G}_W := G_W \times \mathbb{C}^\times$, where $G_W := \prod_{(i,p)\in\Delta_0} GL(W_{(i,p)}) \subset G_{\bar{W}}$. We consider the T_W-fixed loci:

$$\pi^\bullet := \pi^{T_W} : \mathfrak{M}^\bullet(W) := \mathfrak{M}(\bar{W})^{T_W} \to \mathfrak{M}_0^\bullet(W) := \mathfrak{M}_0(\bar{W})^{T_W},$$

and refer to these varieties $\mathfrak{M}^\bullet(W), \mathfrak{M}_0^\bullet(W)$ as the *graded quiver varieties*. The centralizer $\mathbb{G}_W$ naturally acts on the varieties $\mathfrak{M}^\bullet(W)$, $\mathfrak{M}_0^\bullet(W)$ and the proper morphism $\pi^\bullet$ is $\mathbb{G}_W$-equivariant.

4.5. **Nakajima's homomorphism.** Take a finite-dimensional I-graded $\mathbb{C}$-vector space $\bar{W}$ and consider the quiver varieties $\pi \colon \mathfrak{M}(\bar{W}) \to \mathfrak{M}_0(\bar{W})$. We define

$$Z(\bar{W}) := \mathfrak{M}(\bar{W}) \times_{\mathfrak{M}_0(\bar{W})} \mathfrak{M}(\bar{W}), \quad \mathfrak{L}(\bar{W}) := \pi^{-1}(0) = \mathfrak{M}(\bar{W}) \times_{\mathfrak{M}_0(\bar{W})} \{0\}.$$

Applying the convolution construction in Section 4.2, we obtain an $R(\mathbb{G}_{\bar{W}})$-algebra $K^{\mathbb{G}_{\bar{W}}}(Z(\bar{W}))$ and a left module $K^{\mathbb{G}_{\bar{W}}}(\mathfrak{L}(\bar{W}))$ over it.

We set $A := R(\mathbb{C}^\times)$ and regard $K^{\mathbb{G}_{\bar{W}}}(-)$ as an A-module via the inclusion $A = R(\mathbb{C}^\times) \hookrightarrow R(\mathbb{G}_{\bar{W}})$ arising from the second projection $\mathbb{G}_{\bar{W}} = G_{\bar{W}} \times \mathbb{C}^\times \to \mathbb{C}^\times$. Also, we regard the field $\Bbbk = \overline{\mathbb{Q}(q)}$ as an A-algebra via the homomorphism $A = R(\mathbb{C}^\times) \to \Bbbk$ sending the class of the 1-dimensional natural $\mathbb{C}^\times$-module to the parameter q. After the base change, we obtain the $\Bbbk$-algebra $K^{\mathbb{G}_{\bar{W}}}(Z(\bar{W})) \otimes_A \Bbbk$ and the left module $K^{\mathbb{G}_{\bar{W}}}(\mathfrak{L}(\bar{W})) \otimes_A \Bbbk$ over it.

Theorem 4.1 (Nakajima [32])**.** *There exists a $\Bbbk$-algebra homomorphism*

$$\Psi_{\bar{W}} \colon U_q(L\mathfrak{g}) \to K^{\mathbb{G}_{\bar{W}}}(Z(\bar{W})) \otimes_A \Bbbk$$

via which the $K^{\mathbb{G}_{\bar{W}}}(Z(\bar{W}))\otimes_A\Bbbk$-module $K^{\mathbb{G}(\bar{W})}(\mathfrak{L}(\bar{W}))\otimes_A\Bbbk$ is regarded as a $U_q(L\mathfrak{g})$-module and isomorphic to the level-zero extremal weight module of extremal weight $\sum_{i\in I}(\dim \bar{W}_i)\varpi_i$ in the sense of Kashiwara [23].

Proof. See [32, Section 9] and [34, Theorem 2]. □

Let us describe some more details in the special case when $\bar{W} = \bar{W}_i$ and $\dim \bar{W}_i = 1$ for some $i \in I$. In this case, the extremal weight module of extremal weight ϖ_i is isomorphic to the affinized fundamental module $V_i[z^{\pm 1}]$. By Theorem 4.1, we have a $U_q(L\mathfrak{g})$-isomorphism

$$K^{\mathbb{G}_{\bar{W}}}(\mathfrak{L}(\bar{W})) \otimes_A \Bbbk \cong V_i[z^{\pm 1}]. \tag{4.3}$$

Under this isomorphism, the vector $v_i \in V_i[z^{\pm 1}]$ corresponds to the class $[\mathcal{O}_{\{\hat{0}\}}]$ of the structure sheaf of $\mathfrak{M}(0, \bar{W}) = \{\hat{0}\}$. The action of $R(\mathbb{G}_{\bar{W}})_\Bbbk$ on the LHS is identified with the action of $\Bbbk[z^{\pm 1}]$ on the RHS via the isomorphism $\psi_i \colon R(\mathbb{G}_{\bar{W}}) \otimes_A \Bbbk \to \Bbbk[z^{\pm 1}]$ which sends the class $[\bar{W}_i]$ of the natural representation given by the first projection $\mathbb{G}_{\bar{W}} = GL(\bar{W}_i) \times \mathbb{C}^\times \to GL(\bar{W}_i)$ to the formal parameter z.

4.6. **Completion.** As in Section 4.4, we fix a finite-dimensional Δ_0-graded vector space $W = \bigoplus_{x\in\Delta_0} W_x$ and denote by $\bar{W} = \bigoplus_{i\in I} \bar{W}_i$ its underlying I-graded vector space. Recall the 1-dimensional subtorus $T_W \subset \mathbb{G}_{\bar{W}}$ and its centralizer $\mathbb{G}_W = G_W \times \mathbb{C}^\times \subset \mathbb{G}_{\bar{W}}$. Note that the multiplication induces a group isomorphism

$$G_W \times T_W \cong \mathbb{G}_W, \quad (g,t) \mapsto gt. \tag{4.4}$$

Let $\mathfrak{r}_W$ be the kernel of the restriction $R(\mathbb{G}_W)\otimes_A \Bbbk \to R(T_W)\otimes_A \Bbbk = \Bbbk$. Note that the decomposition (4.4) yields an isomorphism

$$K^{\mathbb{G}_W}(X)\otimes_A \Bbbk \cong K^{G_W}(X)_\Bbbk$$

for any $\mathbb{G}_W$-variety X with a trivial T_W-action. In particular, we have an isomorphism $R(\mathbb{G}_W)\otimes_A \Bbbk \cong R(G_W)_\Bbbk$ of $\Bbbk$-algebras, via which the maximal ideal $\mathfrak{r}_W \subset R(\mathbb{G}_W)\otimes_A \Bbbk$ corresponds to the augmentation ideal $\mathfrak{a} \subset R(G_W)_\Bbbk$. Therefore we have an isomorphism

$$\left[K^{\mathbb{G}_W}(X)\otimes_A \Bbbk\right]^\wedge_{\mathfrak{r}_W} \cong \widehat{K}^{G_W}(X)_\Bbbk, \tag{4.5}$$

where $[-]^\wedge_{\mathfrak{r}_W}$ denotes the $\mathfrak{r}_W$-adic completion.

We consider the fiber product

$$Z^\bullet(W) := \mathfrak{M}^\bullet(W) \times_{\mathfrak{M}_0^\bullet(W)} \mathfrak{M}^\bullet(W) = Z(\bar{W})^{T_W}.$$

The completed equivariant K-group $\widehat{K}^{G_W}(Z^\bullet(W))_\Bbbk$ becomes a $\Bbbk$-algebra via the convolution product. We define the $\Bbbk$-algebra homomorphism $\widehat{\Psi}_W\colon U_q(L\mathfrak{g}) \to \widehat{K}^{G_W}(Z^\bullet(W))_\Bbbk$ to be the following composition:

$$\begin{aligned}
U_q(L\mathfrak{g}) &\to K^{\mathbb{G}_{\bar{W}}}(Z(\bar{W}))\otimes_A \Bbbk && \text{(Nakajima's homomorphism } \Psi_{\bar{W}}\text{)}\\
&\to K^{\mathbb{G}_W}(Z(\bar{W}))\otimes_A \Bbbk && \text{(restriction to } \mathbb{G}_W \subset \mathbb{G}_{\bar{W}}\text{)}\\
&\to \left[K^{\mathbb{G}_W}(Z(\bar{W}))\otimes_A \Bbbk\right]^\wedge_{\mathfrak{r}_W} && (\mathfrak{r}_W\text{-adic completion})\\
&\cong \left[K^{\mathbb{G}_W}(Z^\bullet(W))\otimes_A \Bbbk\right]^\wedge_{\mathfrak{r}_W} && \text{(localization theorem)}\\
&\cong \widehat{K}^{G_W}(Z^\bullet(W))_\Bbbk. && \text{(isomorphism (4.5))}
\end{aligned}$$

We refer to the homomorphism $\widehat{\Psi}_W$ as the *completed Nakajima homomorphism*.

Let us describe the special case when $W = W_x$ and $\dim W_x = 1$ for some $x = (i,p) \in \Delta_0$. In this case, we have $\mathfrak{M}_0^\bullet(W) = \{0\}$ and hence $\mathfrak{M}^\bullet(W) = \mathfrak{L}(\bar{W})^{T_W}$. Note that the composition of $R(G_W)_\Bbbk \cong R(\mathbb{G}_W)\otimes_A \Bbbk = R(\mathbb{G}_{\bar{W}})\otimes_A \Bbbk$ arising from (4.4) and ψ_i in the previous subsection yields an isomorphism $\psi_x\colon R(G_W)_\Bbbk \cong \Bbbk[z^{\pm 1}]$. Since the group homomorphism $\mathbb{G}_W \cong G_W \times T_W \to G_W$ obtained by composing the inverse of (4.4) and the natural projection is given by $(g,t) \mapsto gt^{-p}$ for $(g,t) \in \mathbb{G}_W = (\mathbb{C}^\times)^2$, the isomorphism ψ_x sends the class $[W_x]$ of the natural representation of G_W to the element $q^{-p}z$. After the completion, we get an isomorphism $\widehat{\psi}_x\colon \widehat{R}(G_W)_\Bbbk \cong \Bbbk[\![z-q^p]\!]$. In the sequel, we identify them via $\widehat{\psi}_x$. By completing the isomorphism (4.3) $\mathfrak{r}_W$-adically, we obtain the following.

Lemma 4.2. *We have an isomorphism of* $U_q(L\mathfrak{g})\otimes_\Bbbk \Bbbk[\![z-q^p]\!]$*-modules*

$$\widehat{K}^{G_W}(\mathfrak{M}^\bullet(W))_\Bbbk \cong V_i[z^{\pm1}]\otimes_{\Bbbk[z^{\pm1}]} \Bbbk[\![z-q^p]\!],$$

under which the vector $v_i \otimes 1$ *in the RHS corresponds to the class* $[\mathcal{O}_{\{\hat{0}\}}]$ *of the structure sheaf of* $\{\hat{0}\}$ *in the LHS.*

4.7. **Keller-Scherotzke's theorem.** In this subsection, we recall a description of the affine graded quiver variety $\mathfrak{M}_0^\bullet(W)$ due to Keller-Scherotzke [29], which plays a crucial role in this paper. Recall the notation on quivers in Section 3.1.

Definition 4.3. We define an infinite quiver $\widetilde{\Delta} = (\widetilde{\Delta}_0, \widetilde{\Delta}_1)$ whose set of vertex is $\widetilde{\Delta}_0 := I \times \mathbb{Z}$. Let Δ_0^+ denote the complement of the subset Δ_0 of $\widetilde{\Delta}_0$:

$$\widetilde{\Delta}_0 = \Delta_0 \sqcup \Delta_0^+, \quad \Delta_0^+ = \{(i, p+1) \in I \times \mathbb{Z} \mid (i,p) \in \Delta_0\}.$$

The arrow set $\widetilde{\Delta}_1$ consists of the following three kinds of arrows:

- $a_i(p)\colon (i,p) \to (i, p-1)$ for each $(i,p) \in \Delta_0$;
- $b_i(p)\colon (i,p) \to (i, p-1)$ for each $(i,p) \in \Delta_0^+$;
- $B_{ji}(p)\colon (i,p) \to (j, p-1)$ for each $(i,p) \in \Delta_0^+$ and $j \in I$ with $j \sim i$.

Let $\mathfrak{I}$ be a two sided ideal of the path algebra $\mathbb{C}\widetilde{\Delta}$ generated by the elements

$$a_i(p-1)b_i(p) + \sum_{j \sim i} B_{ij}(p-1)B_{ji}(p)$$

for $(i,p) \in \Delta_0^+$. Then, we define the (non-unital) $\mathbb{C}$-algebras $\widetilde{\Lambda}$ and Λ by

$$\widetilde{\Lambda} := \mathbb{C}\widetilde{\Delta}/\mathfrak{I}, \quad \Lambda := \bigoplus_{x,y \in \Delta_0} e_x \widetilde{\Lambda} e_y.$$

For a Δ_0-graded finite-dimensional $\mathbb{C}$-vector space $W = \bigoplus_{x \in \Delta_0} W_x$, we can consider the variety $\mathsf{rep}_W(\Lambda)$ of representations of the algebra Λ realized on W. We have the natural conjugation action of the group G_W on the variety $\mathsf{rep}_W(\Lambda)$.

Proposition 4.4 (Leclerc-Plamondon [31])**.** *There is an isomorphism of G_W-varieties:*

$$\mathfrak{M}_0^\bullet(W) \cong \mathsf{rep}_W(\Lambda).$$

Proof. This is [31, Theorem 2.4]. Note that the graded quiver variety $\mathfrak{M}_0^\bullet(W)$ defined therein is naturally isomorphic to our graded quiver variety defined as the T_W-fixed locus of $\mathfrak{M}_0(\bar{W})$ (see [32, Section 4.1] for details). □

Let C be a subset of the vertex set Δ_0. We denote by Λ_C the quotient of the algebra Λ by the ideal generated by all the idempotents e_x with $x \notin C$. We consider the following condition (R) on the subset C:

(R) For each vertex $x \in \Delta_0$, there is a vertex $c \in C$ such that the space $\mathrm{Hom}_{\mathbb{C}(\Delta)}(x,c)$ of morphisms in the category $\mathbb{C}(\Delta)$ does not vanish.

Theorem 4.5 (Keller-Scherotzke [29, Corollary 3.10])**.** *Assume that our subset $C \subset \Delta_0$ satisfies the above condition* (R)*. Then we have a canonical isomorphism*

$$\mathrm{Ext}^k_{\Lambda_C}(S_x, S_y) \cong \mathrm{Ext}^k_{\mathcal{D}_Q}(\mathtt{H}_Q(x), \mathtt{H}_Q(y))$$

for any Dynkin quiver Q of type $\mathfrak{g}$, vertices $x, y \in C$ and $k \in \mathbb{Z}_{\geq 1}$.

Remark 4.6. The condition (R) is obviously satisfied when $C = \Delta_0$.

Definition 4.7. We define an infinite quiver Γ with $\Gamma_0 = \Delta_0$ whose arrow set Γ_1 is determined by the following condition:

$$\#\{a \in \Gamma_1 \mid a' = x, a'' = y\} = \dim \mathrm{Ext}^1_{\mathcal{D}_Q}(\mathtt{H}_Q(x), \mathtt{H}_Q(y)) \quad \text{for each } x, y \in \Delta_0,$$

where Q is a Dynkin quiver of type $\mathfrak{g}$.

Note that, by Proposition 3.8, there is no arrow from (i,p) to (j,r) in the quiver Γ unless $p > r$. In particular, the quiver Γ has neither loops nor oriented cycles.

Corollary 4.8. *For any Δ_0-graded finite-dimensional complex vector space W, there is a G_W-equivariant closed embedding*

$$\mathfrak{M}_0^\bullet(W) \hookrightarrow \mathsf{rep}_W(\Gamma).$$

Moreover, if W is supported on just two vertices $(i,p),(j,r) \in \Delta_0$, i.e. $W = W_{(i,p)} \oplus W_{(j,r)}$ with $r \geq p$, the graded quiver variety $\mathfrak{M}_0^\bullet(W)$ is G_W-equivariantly isomorphic to the affine space

$$\mathrm{Hom}_{\mathbb{C}}(W_{(j,r)}, W_{(i,p)}) \otimes \mathrm{Ext}^1_{\mathcal{D}_Q}(\mathtt{H}_Q(j,r), \mathtt{H}_Q(i,p)),$$

where G_W acts trivially on the second tensor factor.

Proof. By a general theory (see [2, Theorem 3.7] for example), the algebra Λ can be written as a quotient of the path algebra of a quiver Q_Λ by an admissible ideal $\mathfrak{J} \subset \mathbb{C}Q_\Lambda$. By Theorem 4.5 (in the case $C = \Delta_0$), we have $Q_\Lambda = \Gamma$ and hence the variety $\mathsf{rep}_W(\Lambda)$ is a closed subvariety of the affine space

$$\begin{aligned} \mathsf{rep}_W(\Gamma) &= \prod_{a \in \Gamma_1} \mathrm{Hom}_{\mathbb{C}}(W_{a'}, W_{a''}) \\ &= \prod_{(i,p),(j,r) \in \Delta_0, r>p} \mathrm{Hom}_{\mathbb{C}}(W_{(j,r)}, W_{(i,p)}) \otimes_{\mathbb{C}} \mathrm{Ext}^1_{\mathcal{D}_Q}(\mathtt{H}_Q(j,r), \mathtt{H}_Q(i,p)) \end{aligned}$$

Combining with Proposition 4.4, we obtain a G_W-equivariant closed embedding $\mathfrak{M}_0^\bullet(W) = \mathsf{rep}_W(\Lambda) \hookrightarrow \mathsf{rep}_W(\Gamma)$.

If $W = W_{(i,p)} \oplus W_{(j,r)}$ with $r \geq p$, we have

$$\mathsf{rep}_W(\Gamma) = \mathrm{Hom}_{\mathbb{C}}(W_{(j,r)}, W_{(i,p)}) \otimes \mathrm{Ext}^1_{\mathcal{D}_Q}(\mathtt{H}_Q(j,r), \mathtt{H}_Q(i,p)).$$

In addition, all the polynomials corresponding to elements of $\mathfrak{J}$ vanish because $\mathfrak{J} \subset \bigoplus_{d \geq 2} (\mathbb{C}\Gamma_1)^{\otimes d} \subset \mathbb{C}\Gamma$. Therefore the closed embedding is an isomorphism in this case. □

Remark 4.9. By the same argument, we can show the following more general assertion. If a Δ_0-graded vector space W is supported on a subset $C \subset \Delta_0$ satisfying the condition (R), the graded quiver variety $\mathfrak{M}_0^\bullet(W)$ is identical to the space of representations of the full subquiver $\Gamma|_C \subset \Gamma$ satisfying some relations corresponding to elements of an admissible ideal $\mathfrak{J}_C$ of the path algebra $\mathbb{C}\Gamma|_C$.

4.8. **Stratification.** Let W be a Δ_0-graded vector space as above and V be a Δ_0^+-graded vector space. We set

$$Y^W := \prod_{x \in \Delta_0} Y_x^{\dim W_x}, \quad A^{-V} := \prod_{y \in \Delta_0^+} A_y^{-\dim V_y},$$

where $Y_{(i,p)} := Y_{i,q^p}, A_{(i,p)} := A_{i,q^p} \in \mathcal{M}$ for $(i,p) \in I \times \mathbb{Z}$.

Let $\mathsf{rep}_{W \oplus V}(\widetilde{\Lambda})$ be the space of representations of the algebra $\widetilde{\Lambda}$ on the $\widetilde{\Delta}_0$-graded vector space $W \oplus V$. We have a natural G_W-equivariant forgetful morphism $\mathsf{rep}_{W \oplus V}(\widetilde{\Lambda}) \to \mathsf{rep}_W(\Lambda) = \mathfrak{M}_0^\bullet(W)$. We consider the subvariety of $\mathsf{rep}_{W \oplus V}(\widetilde{\Lambda})$ consisting of modules M whose stabilizer in the group G_V is trivial, and denote its image under the forgetful morphism by $\mathfrak{M}_0^{\bullet\mathrm{reg}}(V,W) \subset \mathfrak{M}_0^\bullet(W)$. Note that $\mathfrak{M}_0^{\bullet\mathrm{reg}}(V,W)$ can be empty.

Theorem 4.10 (Nakajima [32])**.** *The collection $\{\mathfrak{M}_0^{\bullet\mathrm{reg}}(V,W)\}_V$ of locally closed smooth G_W-subvarieties gives a stratification of the variety $\mathfrak{M}_0^{\bullet}(W)$ with finitely many (non-empty) strata, satisfying the following properties:*

(1) *$\mathfrak{M}_0^{\bullet\mathrm{reg}}(V,W) \neq \varnothing$ if and only if $Y^W A^{-V} \in \mathcal{M}^+$ and $c(Y^W, Y^W A^{-V}) \neq 0$ (see Theorem 2.3 (2) for the notation). If this is the case, we have*
$$c(Y^W, Y^W A^{-V}) = \dim \iota^! IC(\mathfrak{M}_0^{\bullet\mathrm{reg}}(V,W), \Bbbk),$$
where $IC(\mathfrak{M}_0^{\bullet\mathrm{reg}}(V,W), \Bbbk)$ denotes the intersection cohomology complex associated with the constant $\Bbbk$-sheaf on $\mathfrak{M}_0^{\bullet\mathrm{reg}}(V,W)$ and $\iota\colon \{0\} \hookrightarrow \mathfrak{M}_0^{\bullet}(W)$ denotes the inclusion of the origin;

(2) *Let $\mathscr{L}_W$ be the (derived) push-forward of the constant sheaf $\underline{\Bbbk}_{\mathfrak{M}^\bullet(W)}$ along the proper morphism $\pi^\bullet\colon \mathfrak{M}^\bullet(W) \to \mathfrak{M}_0^\bullet(W)$. Then it has a decomposition:*
$$\mathscr{L}_W \cong \bigoplus_V IC(\mathfrak{M}_0^{\bullet\mathrm{reg}}(V,W), \Bbbk) \otimes_{\Bbbk} L_V^\bullet,$$
where $L_V^\bullet \in D^b(\Bbbk\text{-}\mathsf{mod})$ is a finite-dimensional $\mathbb{Z}$-graded $\Bbbk$-vector space. Moreover, $L_V^\bullet \neq 0$ if and only if $\mathfrak{M}_0^{\bullet\mathrm{reg}}(V,W) \neq \varnothing$;

(3) *The closure inclusion $\mathfrak{M}_0^{\bullet\mathrm{reg}}(V,W) \subset \overline{\mathfrak{M}_0^{\bullet\mathrm{reg}}(V',W)}$ between non-empty strata implies $Y^W A^{-V} \geq Y^W A^{-V'}$.*

Proof. See [32, Section 14.3] for (1), (2), and [32, Section 3.3] for (3). □

The next theorem describes the stratification $\{\mathfrak{M}_0^{\bullet\mathrm{reg}}(V,W)\}_V$ in terms of the algebra Λ.

Theorem 4.11 (Keller-Scherotzke [29])**.** *Let Q be a Dynkin quiver of type $\mathfrak{g}$. There exists a canonical δ-functor $\Phi_Q\colon \Lambda\text{-}\mathsf{mod} \to \mathcal{D}_Q$ such that $\Phi_Q(S_x) \cong \mathtt{H}_Q(x) \in \mathcal{D}_Q$ for each $x \in \Delta_0$ and satisfies the following properties:*

(1) *Two representations $M_1, M_2 \in \mathsf{rep}_W(\Lambda) = \mathfrak{M}_0^\bullet(W)$ belong to a common stratum $\mathfrak{M}_0^{\bullet\mathrm{reg}}(V,W)$ if and only if we have $\Phi_Q(M_1) \cong \Phi_Q(M_2)$;*

(2) *If we write $\Phi_Q(M) \cong \bigoplus_{x\in\Delta_0} \mathtt{H}_Q(x)^{\oplus m_x}$ for $M \in \mathfrak{M}_0^{\bullet\mathrm{reg}}(V,W)$, we have $Y^W A^{-V} = \prod_{x\in\Delta_0} Y_x^{m_x} \in \mathcal{M}^+$.*

Proof. See [29, Theorem 2.7] for (1), and [29, Lemma 4.14] for (2). □

We refer to the above δ-functor $\Phi_Q\colon \Lambda\text{-}\mathsf{mod} \to \mathcal{D}_Q$ as the *stratification functor*. For a concrete construction of Φ_Q, see [29, Sections 4 and 5].

5. A geometric proof of the denominator formula

In this section, we give a proof of Theorem 3.9, which is equivalent to our main theorem (= Theorem 2.10). We also describe a structure of the tensor product module $V_i(a) \otimes V_j(b)$ when R_{ij} has a simple pole at $z_2/z_1 = b/a$ in Section 5.2 .

5.1. **Proof of Theorem 3.9.** For $x = (i,p), y = (j,r) \in \Delta_0$, we set
$$V(x,y) := V_i(q^p) \otimes V_j(q^r) = L(Y_x) \otimes L(Y_y),$$
$$\widehat{V}(x,y) := \mathbb{O} \otimes_{\Bbbk[z_1^{\pm1}, z_2^{\pm1}]} \left(V_i[z_1^{\pm1}] \otimes V_j[z_2^{\pm1}]\right),$$
where $\mathbb{O} := \Bbbk[\![z_1 - q^p, z_2 - q^r]\!]$. We have $V(x,y) \cong \widehat{V}(x,y)/\mathfrak{m}\widehat{V}(x,y)$ as $U_q(L\mathfrak{g})$-modules, where $\mathfrak{m} \subset \mathbb{O}$ is the maximal ideal. Since the module $V_i[z_1^{\pm1}] \otimes V_j[z_2^{\pm1}]$ is free over $\Bbbk[z_1^{\pm1}, z_2^{\pm1}]$, the module $\widehat{V}(x,y)$ is free over $\mathbb{O}$. We denote the image of

the vector $v_i \otimes v_j$ under the natural homomorphism $V_i[z_1^{\pm 1}] \otimes V_j[z_2^{\pm 1}] \to V(x,y)$ (resp. $V_i[z_1^{\pm 1}] \otimes V_j[z_2^{\pm 1}] \to \widehat{V}(x,y)$) by $v_{x,y}$ (resp. $\hat{v}_{x,y}$).

Let $\mathbb{K}$ be the fraction field of $\mathbb{O}$. We set

$$\widehat{V}(x,y)_{\mathbb{K}} := \mathbb{K} \otimes_{\mathbb{O}} \widehat{V}(x,y) = \mathbb{K} \otimes_{\Bbbk[z_1^{\pm 1}, z_2^{\pm 1}]} (V_i[z_1^{\pm 1}] \otimes V_j[z_2^{\pm 1}]).$$

We can naturally regard $\widehat{V}(x,y)$ as a submodule of $\widehat{V}(x,y)_{\mathbb{K}}$. The normalized R-matrix $R_{ij} \colon V_i[z_1^{\pm 1}] \otimes V_j[z_2^{\pm 1}] \to \Bbbk(z_2/z_1) \otimes_{\Bbbk[(z_2/z_1)^{\pm 1}]} \left(V_j[z_2^{\pm 1}] \otimes V_i[z_1^{\pm 1}]\right)$ induces a unique $U_q(L\mathfrak{g}) \otimes \mathbb{K}$-homomorphism

$$\widehat{R}_{x,y} \colon \widehat{V}(x,y)_{\mathbb{K}} \to \widehat{V}(y,x)_{\mathbb{K}}$$

characterized by the property $\widehat{R}_{x,y}(\hat{v}_{x,y}) = \hat{v}_{y,x}$. Since the $U_q(L\mathfrak{g}) \otimes \mathbb{K}$-modules $\widehat{V}(x,y)_{\mathbb{K}}$ and $\widehat{V}(y,x)_{\mathbb{K}}$ are irreducible (see [24, Lemma 8.1] for example), the homomorphism $\widehat{R}_{x,y}$ is an isomorphism, and we have

$$\operatorname{Hom}_{U_q(L\mathfrak{g}) \otimes \mathbb{K}} \left(\widehat{V}(x,y)_{\mathbb{K}}, \widehat{V}(y,x)_{\mathbb{K}}\right) = \mathbb{K}\widehat{R}_{x,y}. \tag{5.1}$$

Let $d_{x,y} := \dim \operatorname{Ext}^1_{\mathcal{D}_Q}(\mathtt{H}_Q(y), \mathtt{H}_Q(x))$, where Q is a Dynkin quiver of type $\mathfrak{g}$. If $r \le p$, we have $d_{x,y} = 0$ by Proposition 3.8. On the other hand, we know that $d_{ij}(q^r/q^p) \neq 0$ for $r \le p$ by Theorem 2.8. Therefore, to prove Theorem 3.9, we may assume that $r > p$. Then, it suffices to verify the following two properties:

$$(z_2/z_1 - q^r/q^p)^{d_{x,y}} \widehat{R}_{x,y} \left(\widehat{V}(x,y)\right) \subset \widehat{V}(y,x), \tag{5.2}$$

$$(z_2/z_1 - q^r/q^p)^{d_{x,y}} \widehat{R}_{x,y} \left(\widehat{V}(x,y)\right) \not\subset \mathfrak{m}\widehat{V}(y,x). \tag{5.3}$$

We prove these properties by using geometry of the graded quiver varieties.

Let $W = W_x \oplus W_y$ be the Δ_0-graded vector space supported on $\{x,y\} \subset \Delta_0$ satisfying $\dim W_x = \dim W_y = 1$. In this case, we have $G_W = GL(W_x) \times GL(W_y) = (\mathbb{C}^\times)^2$. In what follows, we identify the completed representation ring $\widehat{R}(G_W)_\Bbbk$ with the ring $\mathbb{O}$ by the isomorphism

$$\widehat{R}(G_W)_\Bbbk \cong \mathbb{O}, \quad [W_x] \leftrightarrow q^{-p} z_1, \ [W_y] \leftrightarrow q^{-r} z_2, \tag{5.4}$$

where $[W_x]$ and $[W_y]$ denote the classes of the natural 1-dimensional representations of $G_W = GL(W_x) \times GL(W_y)$. By Corollary 4.8, the graded quiver variety $\mathfrak{M}_0^\bullet(W)$ is identified with the affine space $E = \mathbb{C}^{d_{x,y}}$ of dimension $d_{x,y}$ as a G_W-variety. Here the action of the group $G_W = (\mathbb{C}^\times)^2$ on E is given by $(s_1, s_2) \cdot e = s_1 s_2^{-1} e$, where $(s_1, s_2) \in (\mathbb{C}^\times)^2$ and $e \in E$. Let $\iota \colon \{0\} \hookrightarrow E$ denote the inclusion of the origin. From the morphisms $\pi^\bullet \colon \mathfrak{M}^\bullet(W) \to \mathfrak{M}_0^\bullet(W) = E$, $\mathrm{id} \colon E \to E$ and $\iota \colon \{0\} \to E$, we make the fiber products $\mathfrak{M}^\bullet(W) \times_E E \subset \mathfrak{M}^\bullet(W) \times E$ and $\mathfrak{M}^\bullet(W) \times_E \{0\} \subset \mathfrak{M}^\bullet(W) \times \{0\}$. The convolution product makes the completed equivariant K-groups $\widehat{K}^{G_W}(\mathfrak{M}^\bullet(W) \times_E E)_\Bbbk$ and $\widehat{K}^{G_W}(\mathfrak{M}^\bullet(W) \times_E \{0\})_\Bbbk$ into left $\widehat{K}^{G_W}(Z^\bullet(W))_\Bbbk$-modules. Via the completed Nakajima homomorphism $\widehat{\Psi}_W \colon U_q(L\mathfrak{g}) \to \widehat{K}^{G_W}(Z^\bullet(W))_\Bbbk$, they are regarded as left $U_q(L\mathfrak{g})$-modules.

Lemma 5.1. *There are isomorphisms of $U_q(L\mathfrak{g}) \otimes \mathbb{O}$-modules*

$$\widehat{K}^{G_W}(\mathfrak{M}^\bullet(W) \times_E E)_\Bbbk \cong \widehat{V}(x,y), \tag{5.5}$$

$$\widehat{K}^{G_W}(\mathfrak{M}^\bullet(W) \times_E \{0\})_\Bbbk \cong \widehat{V}(y,x), \tag{5.6}$$

under which the class $[\mathcal{O}_{\{\hat{0}\}}]$ of the structure sheaf of $\{\hat{0}\} \subset \mathfrak{M}^\bullet(W)$ corresponds to the vectors $\hat{v}_{x,y}$ and $\hat{v}_{y,x}$ respectively.

Proof. Define the 1-parameter subgroups $\lambda_{x,y}\colon \mathbb{C}^\times \to G_W$ and $\lambda_{y,x}\colon \mathbb{C}^\times \to G_W$ by $\lambda_{x,y}(t) := (t,1)$ and $\lambda_{y,x}(t) := (1,t)$ respectively. Since $\lambda_{x,y}(t)\cdot e = te$ and $\lambda_{y,x}(t)\cdot e = t^{-1}e$ for any point $e \in E = \mathfrak{M}_0^\bullet(W)$, we have

$$\mathfrak{M}^\bullet(W) \times_E E = \left\{ m \in \mathfrak{M}^\bullet(W) \;\middle|\; \lim_{t\to 0} \lambda_{x,y}(t)\pi^\bullet(m) = 0 \right\},$$
$$\mathfrak{M}^\bullet(W) \times_E \{0\} = \left\{ m \in \mathfrak{M}^\bullet(W) \;\middle|\; \lim_{t\to 0} \lambda_{y,x}(t)\pi^\bullet(m) = 0 \right\}.$$

From these descriptions, we see that they coincide with the T_W-fixed loci of the tensor product varieties introduced by Nakajima [33]. More precisely, they are $\widetilde{\mathfrak{Z}}(W_x; W_y)^{T_W}$ and $\widetilde{\mathfrak{Z}}(W_y; W_x)^{T_W}$ respectively in the notation loc. cit. Therefore, we can proceed the similar argument as the proof of [33, Theorem 6.12] in the $\mathfrak{r}_W$-adically completed setting to obtain the following isomorphisms of $U_q(L\mathfrak{g}) \otimes \mathbb{O}$-modules:

$$\widehat{K}^{G_W}(\mathfrak{M}^\bullet(W) \times_E E)_{\Bbbk} \cong \widehat{K}^{G_{W_x}}(\mathfrak{M}^\bullet(W_x))_{\Bbbk} \hat{\otimes} \widehat{K}^{G_{W_y}}(\mathfrak{M}^\bullet(W_y))_{\Bbbk},$$
$$\widehat{K}^{G_W}(\mathfrak{M}^\bullet(W) \times_E \{0\})_{\Bbbk} \cong \widehat{K}^{G_{W_y}}(\mathfrak{M}^\bullet(W_y))_{\Bbbk} \hat{\otimes} \widehat{K}^{G_{W_x}}(\mathfrak{M}^\bullet(W_x))_{\Bbbk},$$

where $K \hat{\otimes} K'$ denotes the completion of $K \otimes_{\Bbbk} K'$. On the other hand, there are isomorphisms $\widehat{K}^{G_{W_x}}(\mathfrak{M}^\bullet(W_x))_{\Bbbk} \cong V_i[z_1^{\pm 1}] \otimes_{\Bbbk[z_1^{\pm 1}]} \Bbbk[\![z_1 - q^p]\!]$ and $\widehat{K}^{G_{W_y}}(\mathfrak{M}^\bullet(W_y))_{\Bbbk} \cong V_j[z_2^{\pm 1}] \otimes_{\Bbbk[z_2^{\pm 1}]} \Bbbk[\![z_2 - q^r]\!]$ by Lemma 4.2. Thus we obtain the desired isomorphisms (5.5) and (5.6). □

Now we consider the completed equivariant K-group $\widehat{K}^{G_W}(E \times_E \{0\})_{\Bbbk}$. Since $E \times_E \{0\} = \{0\}$, this is a free $\mathbb{O}$-module of rank 1 generated by the class $[\mathcal{O}_{\{0\}}]$ of the structure sheaf. The convolution product with the class $[\mathcal{O}_{\{0\}}]$ from the right

$$(-) * [\mathcal{O}_{\{0\}}]\colon \widehat{K}^{G_W}(\mathfrak{M}^\bullet(W) \times_E E)_{\Bbbk} \to \widehat{K}^{G_W}(\mathfrak{M}^\bullet(W) \times_E \{0\})_{\Bbbk}$$

is identified via the isomorphisms (5.5) and (5.6) with a $U_q(L\mathfrak{g}) \otimes \mathbb{O}$-homomorphism

$$\mathbf{r}\colon \widehat{V}(x,y) \to \widehat{V}(y,x).$$

By the base change $\mathbb{O} \to \mathbb{K}$ (resp. $\mathbb{O} \to \Bbbk$), the homomorphism $\mathbf{r}$ gives rise to $\mathbf{r}_{\mathbb{K}} \in \mathrm{Hom}_{U_q(L\mathfrak{g}) \otimes \mathbb{K}}(\widehat{V}(x,y)_{\mathbb{K}}, \widehat{V}(y,x)_{\mathbb{K}})$ (resp. $\bar{\mathbf{r}} \in \mathrm{Hom}_{U_q(L\mathfrak{g})}(V(x,y), V(y,x))$).

The following two lemmas prove the properties (5.2) and (5.3) respectively, and hence complete the proof of Theorem 3.9.

Lemma 5.2. *Up to $\Bbbk^\times$-multiplication, the homomorphism $\mathbf{r}_{\mathbb{K}}$ is equal to the homomorphism $(z_2/z_1 - q^r/q^p)^{d_{x,y}} \widehat{R}_{x,y}$. In particular, the property (5.2) holds.*

Proof. In this proof, we identify $\widehat{V}(x,y)$ and $\widehat{V}(y,x)$ with the completed equivariant K-groups via the isomorphisms (5.5) and (5.6) respectively. Let us compute the operator $\mathbf{r} = (-) * [\mathcal{O}_{\{0\}}]$ following the definition of the convolution product (4.1). For any $\zeta \in \widehat{K}^{G_W}(\mathfrak{M}^\bullet(W) \times_E E)$, we have

$$\begin{aligned}\mathbf{r}(\zeta) &= p_{13*}(p_{12}^*\zeta \otimes^{\mathbb{L}}_{\mathfrak{M}^\bullet(W) \times E \times \{0\}} p_{23}^*[\mathcal{O}_{\{0\}}]) \\ &= p'_{1*}(\zeta \otimes^{\mathbb{L}}_{\mathfrak{M}^\bullet(W) \times E} p_2'^*[\mathcal{O}_{\{0\}}]),\end{aligned}$$

where $p'_1\colon \mathfrak{M}^\bullet(W) \times E \to \mathfrak{M}^\bullet(W)$ and $p'_2\colon \mathfrak{M}^\bullet(W) \times E \to E$ are the natural projections. By the Koszul resolution, we have

$$[\mathcal{O}_{\{0\}}] = \left(\sum_{k=1}^{d_{x,y}} (-1)^k \left[\wedge^k T_0^* E\right] \right) [\mathcal{O}_E] = \left(1 - q^{-r} z_2/q^{-p} z_1\right)^{d_{x,y}} [\mathcal{O}_E],$$

where we regard $[\wedge^k T_0^* E] = [\wedge^k E^*] \in R(G_W)$ and used the identification (5.4). Thus we obtain $\mathbf{r}(\zeta) = (1 - q^{-r}z_2/q^{-p}z_1)^{d_{x,y}} p'_{1*}(\zeta)$. In the special case $\zeta = [\mathcal{O}_{\{\hat{0}\}}] = \hat{v}_{x,y}$, we have $p'_{1*}[\mathcal{O}_{\{\hat{0}\}}] = [\mathcal{O}_{\{\hat{0}\}}]$ and hence $\mathbf{r}(\hat{v}_{x,y}) = (1 - q^{-r}z_2/q^{-p}z_1)^{d_{x,y}} \hat{v}_{y,x}$. Thanks to (5.1), we conclude that $\mathbf{r}_{\mathbb{K}} = (1 - q^{-r}z_2/q^{-p}z_1)^{d_{x,y}} \widehat{R}_{x,y}$. □

Lemma 5.3. *The homomorphism $\bar{\mathbf{r}}$ is non-zero. In particular, the property (5.3) holds.*

Proof. Applying the base change $\mathbb{O} \to \Bbbk$ to the isomorphisms (5.5) and (5.6) in Lemma 5.1, we obtain

$$K(\mathfrak{M}^\bullet(W) \times_E E)_\Bbbk \cong V(x,y), \quad K(\mathfrak{M}^\bullet(W) \times_E \{0\})_\Bbbk \cong V(y,x).$$

Here we used the freeness of the equivariant K-groups of the quiver varieties (see [32, Theorem 7.3.5]). Under these isomorphisms, the homomorphism $\bar{\mathbf{r}}\colon V(x,y) \to V(y,x)$ is identified with the convolution operation

$$(-) * [\mathcal{O}_{\{0\}}]\colon K(\mathfrak{M}^\bullet(W) \times_E E)_\Bbbk \to K(\mathfrak{M}^\bullet(W) \times_E \{0\})_\Bbbk.$$

Here the class $[\mathcal{O}_{\{0\}}] \neq 0$ is regarded as an element of $K(E \times_E \{0\})_\Bbbk = K(\mathrm{pt})_\Bbbk$.

Let $D^b_c(E)$ denote the bounded derived category of constructible complexes of $\Bbbk$-vector spaces on E. For $\mathscr{F}, \mathscr{G} \in D^b_c(E)$, we denote by $\mathrm{Ext}^*(\mathscr{F}, \mathscr{G})$ the direct sum of the spaces $\mathrm{Hom}_{D^b_c(E)}(\mathscr{F}, \mathscr{G}[k])$ over $k \in \mathbb{Z}$. By the Chern character map (with a certain modification, see [7, Section 5.11]) and a standard isomorphism explained in [7, Section 8.6], we obtain the following commutative diagram:

$$\begin{array}{ccc} K(\mathfrak{M}^\bullet(W) \times_E E)_\Bbbk \otimes K(E \times_E \{0\})_\Bbbk & \xrightarrow{\ *\ } & K(\mathfrak{M}^\bullet(W) \times_E \{0\})_\Bbbk \\ \downarrow \cong & & \downarrow \cong \\ \mathrm{Ext}^*(\underline{\Bbbk}_E, \mathscr{L}^\bullet_W) \otimes \mathrm{Ext}^*(\underline{\Bbbk}_{\{0\}}, \underline{\Bbbk}_E) & \xrightarrow{\ \circ\ } & \mathrm{Ext}^*(\underline{\Bbbk}_{\{0\}}, \mathscr{L}^\bullet_W), \end{array}$$

where $\mathscr{L}^\bullet_W$ denotes the (derived) proper push-forward along $\pi^\bullet$ of the constant sheaf on $\mathfrak{M}^\bullet(W)$. The lower horizontal arrow denotes the Yoneda product $(\varphi_1, \varphi_2) \mapsto \varphi_1 \circ \varphi_2$. The vertical arrows are isomorphisms thanks to [32, Theorem 7.4.1]. Note that the $\Bbbk$-vector space $\mathrm{Ext}^*(\underline{\Bbbk}_{\{0\}}, \underline{\Bbbk}_E) = \mathrm{Hom}_{D^b_c(E)}\left(\iota_! \iota^! \underline{\Bbbk}_E, \underline{\Bbbk}_E\right)$ is 1-dimensional and spanned by the adjoint morphism $\eta\colon \iota_! \iota^! \underline{\Bbbk}_E \to \underline{\Bbbk}_E$. Under the vertical isomorphism $K(E \times_E \{0\})_\Bbbk \cong \mathrm{Ext}^*(\underline{\Bbbk}_{\{0\}}, \underline{\Bbbk}_E)$, the class $[\mathcal{O}_{\{0\}}]$ corresponds to the adjoint morphism η up to $\Bbbk^\times$. Since $E = \mathfrak{M}^\bullet_0(W)$ is an affine space, there exists a unique open dense stratum $\mathfrak{M}^{\bullet\mathrm{reg}}_0(V,W) \subset E$ and the corresponding intersection cohomology complex $IC(\mathfrak{M}^{\bullet\mathrm{reg}}_0(V,W), \Bbbk)$ is just a shift of the constant sheaf $\underline{\Bbbk}_E$. By Theorem 4.10 (2), we see that $\mathscr{L}^\bullet_W$ contains a certain shift of the constant sheaf $\underline{\Bbbk}_E$ as a direct summand. This implies that the Yoneda product

$$(-) \circ \eta\colon \mathrm{Ext}^*\left(\underline{\Bbbk}_E, \mathscr{L}^\bullet_W\right) \to \mathrm{Ext}^*(\underline{\Bbbk}_{\{0\}}, \mathscr{L}^\bullet_W)$$

is non-zero because $\mathrm{id}_{\underline{\Bbbk}_E} \circ \eta = \eta \neq 0$. Thus the homomorphism $\bar{\mathbf{r}} = (-) * [\mathcal{O}_{\{0\}}]$ is also non-zero. □

5.2. **A remark on the case of simple pole.** Let $x = (i,p), y = (j,r) \in \Delta_0$ and assume that the normalized R-matrix R_{ij} has a simple pole at $z_2/z_1 = q^r/q^p$. By Theorem 3.9, this assumption is equivalent to the condition $\dim \mathrm{Ext}^1_{\mathcal{D}_Q}(\mathtt{H}_Q(y), \mathtt{H}_Q(x)) = 1$ for a Dynkin quiver Q of type $\mathfrak{g}$. Therefore there exists the following non-split exact triangle in $\mathcal{D}_Q$:

$$\mathtt{H}_Q(x) \to \bigoplus_{w \in \Delta_0} \mathtt{H}_Q(w)^{\oplus \mu_w} \to \mathtt{H}_Q(y) \xrightarrow{+1},$$

where the multiplicities $(\mu_w)_{w \in \Delta_0}$ are uniquely determined by the pair (x,y) and independent from the choice of Q. Then we define a dominant monomial $m_{[y,x]} \in \mathcal{M}^+$ by

$$m_{[y,x]} := \prod_{w \in \Delta_0} Y_w^{\mu_w}. \tag{5.7}$$

Proposition 5.4. *Let $x = (i,p), y = (j,r) \in \Delta_0$ and assume that the normalized R-matrix R_{ij} has a simple pole at $z_2/z_1 = q^r/q^p$. With the above notation, we have the following non-split short exact sequences in $\mathcal{C}$:*

$$0 \to L(Y_xY_y) \to L(Y_x) \otimes L(Y_y) \to L(m_{[y,x]}) \to 0,$$
$$0 \to L(m_{[y,x]}) \to L(Y_y) \otimes L(Y_x) \to L(Y_xY_y) \to 0.$$

Proof. As in Section 5.1 above, we consider the graded quiver variety $E := \mathfrak{M}_0^\bullet(W)$ associated with a Δ_0-graded vector space $W = W_x \oplus W_y$ such that $\dim W_x = \dim W_y = 1$. By Corollary 4.8, our assumption implies that E is just a 1-dimensional affine space. Since the action of $(s_1, s_2) \in G_W = (\mathbb{C}^\times)^2$ on E is given by the multiplication of $s_1 s_2^{-1} \in \mathbb{C}^\times$, there are only two G_W-orbits $\{0\}$ and $E \setminus \{0\}$. On the other hand, $E = \mathfrak{M}_0^\bullet(W)$ is stratified by the G_W-stable subvarieties $\mathfrak{M}_0^{\bullet \mathrm{reg}}(V,W)$ and we know that $\{0\}$ coincides with the stratum $\mathfrak{M}_0^{\bullet \mathrm{reg}}(0,W)$. Therefore there exists a unique V such that $E \setminus \{0\} = \mathfrak{M}_0^{\bullet \mathrm{reg}}(V,W)$. Since $IC(\mathfrak{M}_0^{\bullet \mathrm{reg}}(V,W), \Bbbk) = \underline{\Bbbk}_E[1]$, we have $c(Y_xY_y, Y_xY_yA^{-V}) = \dim \iota^!(\underline{\Bbbk}_E[1]) = 1$ by Theorem 4.10 (1) and hence

$$[L(Y_x) \otimes L(Y_y)] = [L(Y_xY_y)] + [L(Y_xY_yA^{-V})]$$

in the Grothendieck ring $K(\mathcal{C})$. Then, in view of Theorem 2.7, we have

$$0 \to L(Y_xY_y) \to L(Y_x) \otimes L(Y_y) \to L(Y_xY_yA^{-V}) \to 0,$$
$$0 \to L(Y_xY_yA^{-V}) \to L(Y_y) \otimes L(Y_x) \to L(Y_xY_y) \to 0,$$

which are exact and non-split. It remains to show $Y_xY_yA^{-V} = m_{[y,x]}$. Recall that $E = \mathsf{rep}_W(\Lambda)$ and pick a Λ-module M corresponding to a point of $E \setminus \{0\}$. Then there exists a non-split short exact sequence in Λ-mod:

$$0 \to S_x \to M \to S_y \to 0.$$

By applying the stratifying functor $\Phi_Q \colon \Lambda\text{-}\mathsf{mod} \to \mathcal{D}_Q$ in Theorem 4.11, we obtain an exact triangle in $\mathcal{D}_Q$:

$$\mathtt{H}_Q(x) \to \Phi_Q(M) \to \mathtt{H}_Q(y) \xrightarrow{+1}. \tag{5.8}$$

By Theorem 4.11 (1), we see that the isomorphism class of $\Phi_Q(M)$ does not depend on the choice of $M \in E \setminus \{0\}$ and the exact triangle (5.8) does not split. Thus we get $Y_xY_yA^{-V} = m_{[y,x]}$ by Theorem 4.11 (2) and the definition (5.7). □

6. Generalized quantum affine Schur-Weyl duality

In this section, as an application of our discussion so far, we give a geometric interpretation of the generalized quantum affine Schur-Weyl duality functor when it arises from a family of fundamental modules.

6.1. KKK-functors.

First, we shall outline the original construction by Kang-Kashiwara-Kim. Let $\{(\mathsf{V}_j, \mathsf{a}_j)\}_{j\in J}$ be a family indexed by an arbitrary set J, consisting of pairs of a real simple module $\mathsf{V}_j \in \mathcal{C}$ (i.e. a simple object in $\mathcal{C}$ whose tensor square remains simple) and a non-zero scalar $\mathsf{a}_j \in \Bbbk^\times$. Recall that we have the normalized R-matrix $R_{\mathsf{V}_i,\mathsf{V}_j}$ and its denominator $d_{\mathsf{V}_i,\mathsf{V}_j}(u) \in \Bbbk[u]$ for each $(i,j) \in J^2$ (cf. Remark 2.6).

Definition 6.1. Given a family $\{(\mathsf{V}_j, \mathsf{a}_j)\}_{j\in J}$ as above, we define a quiver Γ_J with $(\Gamma_J)_0 := J$ whose arrow set $(\Gamma_J)_1$ is determined by the following condition:

$$\#\{a \in (\Gamma_J)_1 \mid a' = j, a'' = i\} = (\text{zero order of } d_{\mathsf{V}_i,\mathsf{V}_j}(u) \text{ at } u = \mathsf{a}_j/\mathsf{a}_i),$$

for each $(i,j) \in J^2$.

The quiver Γ_J has no loops since each V_j is real. Let $\mathfrak{g}_J$ be the Kac-Moody algebra associated with the underlying graph of Γ_J and $\{\alpha_j^J\}_{j\in J}$ its set of simple roots. We put $\mathsf{Q}_J^+ := \sum_{j\in J} \mathbb{Z}_{\geq 0}\alpha_j^J$. For each $\beta \in \mathsf{Q}_J^+$, we denote by $H_J(\beta)$ the corresponding quiver Hecke algebra. This is a $\mathbb{Z}$-graded $\Bbbk$-algebra defined by generators and relations (see [20, Section 1.2] for instance). Let $H_J(\beta)\text{-}\mathsf{gmod}$ denote the category of finite-dimensional graded $H_J(\beta)$-modules. The direct sum $H_J\text{-}\mathsf{gmod} := \bigoplus_{\beta\in\mathsf{Q}_J^+} H_J(\beta)\text{-}\mathsf{gmod}$ carries a structure of a $\Bbbk$-linear monoidal category with respect to the so-called *convolution product*, which is an analogue of the parabolic induction for the affine Hecke algebras.

In the above setting, Kang-Kashiwara-Kim [20] constructed a bimodule

$$U_q(L\mathfrak{g}) \quad \curvearrowright \quad \widehat{\mathsf{V}}^{\otimes\beta} \quad \curvearrowleft \quad \widehat{H}_J(\beta) \tag{6.1}$$

with some good properties. Here $\widehat{H}_J(\beta)$ denotes the completion of $H_J(\beta)$ along the $\mathbb{Z}$-grading. As a left $U_q(L\mathfrak{g})$-module, $\widehat{\mathsf{V}}^{\otimes\beta}$ is a direct sum of suitable tensor products of the completed modules $\mathsf{V}_j[z^{\pm1}] \otimes_{\Bbbk[z^{\pm1}]} \Bbbk[\![z - \mathsf{a}_j]\!]$ for various $j \in J$. The right action of $\widehat{H}_J(\beta)$ is given by an explicit formula involving the normalized R-matrices $R_{\mathsf{V}_i,\mathsf{V}_j}$. See [20, Section 3] for details. The assignment $M \mapsto \widehat{\mathsf{V}}^{\otimes\beta} \otimes_{\widehat{H}_J(\beta)} M$ combined with the forgetful functor $H_J(\beta)\text{-}\mathsf{gmod} \to \widehat{H}_J(\beta)\text{-}\mathsf{mod}$ yields a $\Bbbk$-linear functor $H_J(\beta)\text{-}\mathsf{gmod} \to \mathcal{C}$. Summing up over $\beta \in \mathsf{Q}_J^+$, we obtain a $\Bbbk$-linear monoidal functor

$$\mathscr{F}_J \colon H_J\text{-}\mathsf{gmod} \to \mathcal{C},$$

which we refer to as the *generalized quantum affine Schur-Weyl duality functor*, or simply the *KKK-functor* associated with the family $\{(\mathsf{V}_j, \mathsf{a}_j)\}_{j\in J}$.

6.2. A geometric interpretation.

In this subsection, we give a geometric interpretation of the KKK-functor when it arises from a family of fundamental modules. Namely, we restrict ourselves to the case when $\mathsf{V}_j \in \{V_i(1) \mid i \in I\}$ for every $j \in J$. Furthermore, we focus on the case when the associated quiver Γ_J is connected. Then, in view of our denominator formula (2.2), we may assume that there exists an injective map $\mathsf{x} \colon J \hookrightarrow \Delta_0$ which determines the family $\{(\mathsf{V}_j, \mathsf{a}_j)\}_{j\in J}$ by

$$(\mathsf{V}_j, \mathsf{a}_j) = (V_{\mathsf{x}_1(j)}(1), q^{\mathsf{x}_2(j)})$$

for every $j \in J$, where we write $\mathsf{x}(j) = (\mathsf{x}_1(j), \mathsf{x}_2(j)) \in I \times \mathbb{Z}$.

The next lemma is a key for our construction. Recall the quiver Γ from Definition 4.7.

Lemma 6.2. *Under the above assumption, the quiver Γ_J in Definition 6.1 is identical to the full subquiver $\Gamma|_{\mathsf{x}(J)}$ of the quiver Γ whose vertex set is the image $\mathsf{x}(J)$ of the injective map $\mathsf{x}\colon J \hookrightarrow \Delta_0 = \Gamma_0$.*

Proof. This is a consequence of Theorem 3.9. □

In what follows, we often identify a J-graded vector space $D = \bigoplus_{j\in J} D_j$ with the Δ_0-graded vector space $\bigoplus_{x\in\Delta_0} D_x$ defined by

$$D_x := \begin{cases} D_j & \text{if } x = \mathsf{x}(j) \text{ with } j \in J; \\ 0 & \text{if } x \notin \mathsf{x}(J). \end{cases}$$

Under this convention, we have $\mathsf{rep}_D(\Gamma) = \mathsf{rep}_D(\Gamma_J)$ by Lemma 6.2 above.

Now we fix $\beta = \sum_{j\in J} d_j \alpha_j^J \in \mathsf{Q}_J^+$ and a J-graded complex vector space $D_\beta = \bigoplus_{j\in J}(D_\beta)_j$ such that $\dim(D_\beta)_j = d_j$ for each $j \in J$. To simplify the notation, we set $G_\beta := G_{D_\beta}$ and $E_\beta := \mathsf{rep}_{D_\beta}(\Gamma_J)$. Let us consider the following two nonsingular G_β-varieties:

$$\mathcal{B}_\beta = \{F^\bullet = (D_\beta = F^0 \supsetneq F^1 \supsetneq \cdots \supsetneq F^d = 0) \mid F^k \text{ is a } J\text{-graded subspace of } D_\beta\},$$
$$\mathcal{F}_\beta = \{(F^\bullet, X) \in \mathcal{B}_\beta \times E_\beta \mid X(F^k) \subset F^k \text{ for any } 1 \le k \le d\},$$

where $d := \sum_{j\in J} d_j = \dim D_\beta$. We denote by $\mu\colon \mathcal{F}_\beta \to E_\beta$ the second projection $\mu(F^\bullet, X) = X$. This is a G_β-equivariant proper morphism since $\mathcal{B}_\beta$ is a projective variety. Combined with Corollary 4.8, we have obtained the following diagram

$$\mathfrak{M}^\bullet(D_\beta) \xrightarrow{\pi^\bullet} \mathfrak{M}_0^\bullet(D_\beta) \hookrightarrow E_\beta \xleftarrow{\mu} \mathcal{F}_\beta \tag{6.2}$$

consisting of G_β-equivariant proper morphisms. Applying the convolution construction, we get a bimodule

$$K^{G_\beta}(Z^\bullet(D_\beta))_\Bbbk \quad \curvearrowright \quad K^{G_\beta}(\mathfrak{M}^\bullet(D_\beta) \times_{E_\beta} \mathcal{F}_\beta)_\Bbbk \quad \curvearrowleft \quad K^{G_\beta}(\mathcal{Z}_\beta)_\Bbbk, \tag{6.3}$$

where we set $Z^\bullet(D_\beta) := \mathfrak{M}^\bullet(D_\beta) \times_{E_\beta} \mathfrak{M}^\bullet(D_\beta)$ and $\mathcal{Z}_\beta := \mathcal{F}_\beta \times_{E_\beta} \mathcal{F}_\beta$.

For each $\beta \in \mathsf{Q}_J^+$, we denote by $\mathrm{Ext}^k_{G_\beta}(\mathscr{F}, \mathscr{G})$ the k-th Ext-space in the G_β-equivariant bounded derived category of complexes of $\Bbbk$-sheaves on E_β and define

$$\mathrm{Ext}^*_{G_\beta}(-,-) := \bigoplus_{k\in\mathbb{Z}} \mathrm{Ext}^k_{G_\beta}(-,-), \quad \mathrm{Ext}^*_{G_\beta}(-,-)^\wedge := \prod_{k\in\mathbb{Z}} \mathrm{Ext}^k_{G_\beta}(-,-).$$

Let $\mathscr{L}_\beta$ denote the push-forward of the constant perverse $\Bbbk$-sheaf on the smooth variety $\mathcal{F}_\beta$ along the G_β-equivariant proper morphism $\mu_\beta\colon \mathcal{F}_\beta \to E_\beta$. The following theorem establishes a geometric realization of the quiver Hecke algebra $H_J(\beta)$.

Theorem 6.3 (Varagnolo-Vasserot [40, Theorem 3.6])**.** *There is an isomorphism*

$$H_J(\beta) \cong \mathrm{Ext}^*_{G_\beta}(\mathscr{L}_\beta, \mathscr{L}_\beta) \tag{6.4}$$

of $\mathbb{Z}$-graded $\Bbbk$-algebras.

After completing the above isomorphism (6.4), we obtain

$$\widehat{H}_J(\beta) \cong \mathrm{Ext}^*_{G_\beta}(\mathscr{L}_\beta, \mathscr{L}_\beta)^\wedge \cong \widehat{K}^{G_\beta}(\mathcal{Z}_\beta)_\Bbbk, \tag{6.5}$$

where the second isomorphism is given by the equivariant Chern character map (see [12, Corollary 3.9] for details).

Now we give a geometric interpretation of the bimodule $\widehat{\mathsf{V}}^{\otimes\beta}$.

Theorem 6.4. *With the above notation, there exists an isomorphism*

$$\widehat{\mathsf{V}}^{\otimes\beta} \cong \widehat{K}^{G_\beta}(\mathfrak{M}^\bullet(D_\beta) \times_{E_\beta} \mathcal{F}_\beta)_{\Bbbk}$$

which makes the following diagram commute

$$\begin{array}{ccccc}
U_q(L\mathfrak{g}) & \longrightarrow & \mathrm{End}\left(\widehat{\mathsf{V}}^{\otimes\beta}\right) & \longleftarrow & \widehat{H}_J(\beta)^{\mathrm{op}} \\
\downarrow{\scriptstyle \widehat{\Psi}_{D_\beta}} & & \downarrow{\scriptstyle \cong} & & {\scriptstyle (6.5)}\downarrow{\scriptstyle \cong} \\
\widehat{K}^{G_\beta}(Z^\bullet(D_\beta))_{\Bbbk} & \longrightarrow & \mathrm{End}\left(\widehat{K}^{G_\beta}(\mathfrak{M}^\bullet(D_\beta) \times_{E_\beta} \mathcal{F}_\beta)_{\Bbbk}\right) & \longleftarrow & \widehat{K}^{G_\beta}(\mathcal{Z}_\beta)_{\Bbbk}^{\mathrm{op}},
\end{array}$$

where the first and second rows denote the structure homomorphisms of the bimodules (6.1) *and* (6.3) *respectively.*

We omit a proof because one can prove the assertion along the same lines as the proof of [12, Theorem 1.1], which is a spacial case (see Example 6.5 below).

Example 6.5. Let Q be a Dynkin quiver of type $\mathfrak{g}$. We take $J = I$ and define an injective map $\mathsf{x}\colon J = I \hookrightarrow \Delta_0$ by $\mathsf{x}(i) := \mathtt{H}_Q^{-1}(S_i)$ for each $i \in I$. This is the case Kang-Kashiwara-Kim considered in [19]. Then, we have $\Gamma_J = Q$ and hence $\mathfrak{g}_J = \mathfrak{g}$. Moreover the embedding $\mathfrak{M}_0^\bullet(D) \hookrightarrow \mathsf{rep}_D(Q)$ in Corollary 4.8 becomes an isomorphism for any I-graded vector space D (see [18, Theorem 9.11]). Theorem 6.4 for this special case was established in [12]. In this case, we can further prove (see [11, 12]) that the corresponding KKK-functor $\mathscr{F}_J$ induces an equivalence of monoidal categories $\bigoplus_{\beta\in\mathsf{Q}^+} \widehat{H}_J(\beta)\text{-}\mathsf{mod} \simeq \mathcal{C}_Q$, where $\mathcal{C}_Q$ is the monoidal full subcategory of $\mathcal{C}$ introduced by Hernandez-Leclerc [18], consisting of modules whose composition factors are isomorphic to $L(m)$ for some dominant monomial m in variables Y_x labeled by $x \in \Delta_0$ such that $\mathtt{H}_Q(x) \in \mathbb{C}Q\text{-}\mathsf{mod} \subset \mathcal{D}_Q$.

6.3. **Type A subquivers and graded nilpotent orbits.** In this subsection, we study some examples of the KKK-functors when the corresponding graded quiver varieties $\mathfrak{M}_0^\bullet(D_\beta)$ are isomorphic to graded nilpotent orbits of type A.

Let Q be a Dynkin quiver of type $\mathfrak{g}$ and fix an integer N such that $1 \le N-1 \le n$. We assume that the full subquiver Q' of Q supported on the subset $I' := \{1, 2, \dots, N-1\} \subset I = \{1, 2, \dots, n\}$ is of type A_{N-1} with a monotone orientation, i.e.

$$Q' = \left(\overset{1}{\circ} \longrightarrow \overset{2}{\circ} \longrightarrow \overset{3}{\circ} \longrightarrow \cdots \longrightarrow \overset{N-1}{\circ} \right) \subset Q.$$

Note that we have a natural fully faithful embedding $\varepsilon\colon \mathcal{D}_{Q'} \hookrightarrow \mathcal{D}_Q$ of triangulated categories. We fix a height function ξ as in Section 3.2. The restriction of ξ to the subset I' gives a height function for Q'. With these choices, we have the equivalences $\mathtt{H}_Q\colon \mathbb{C}(\Delta) \simeq \mathsf{ind}(\mathcal{D}_Q)$ and $\mathtt{H}_{Q'}\colon \mathbb{C}(\Delta') \simeq \mathsf{ind}(\mathcal{D}_{Q'})$ in Theorem 3.2, where Δ' denotes the counterpart of Δ for the subquiver Q'.

Let $J := \mathbb{Z}$. With the above notation, we define an injective map $\mathsf{x}\colon J \hookrightarrow \Delta_0$ by $\mathsf{x}(j) := (\mathtt{H}_Q^{-1} \circ \varepsilon \circ \mathtt{H}_{Q'})(1, \xi_1 - 2j + 2)$ for each $j \in J$, or equivalently, we define

$$\mathsf{x}(j) := \begin{cases} \mathtt{H}_Q^{-1}(S_i[-2k]) & \text{if } j = i + kN, 1 \le i < N, k \in \mathbb{Z}; \\ \mathtt{H}_Q^{-1}(M_\theta[-2k+1]) & \text{if } j = kN, k \in \mathbb{Z}, \end{cases} \tag{6.6}$$

where $\theta := \sum_{i=1}^{N-1} \alpha_i \in \mathsf{R}^+$.

Lemma 6.6. *The quiver Γ_J associated with the injective map $\mathsf{x}\colon J \hookrightarrow \Delta_0$ given by* (6.6) *is equal to the quiver of type A_∞ with a monotone orientation, namely*

$$\Gamma_J = \left(\cdots \longrightarrow \overset{-2}{\circ} \longrightarrow \overset{-1}{\circ} \longrightarrow \overset{0}{\circ} \longrightarrow \overset{1}{\circ} \longrightarrow \overset{2}{\circ} \longrightarrow \cdots \right).$$

Proof. By Lemma 6.2, it suffices to prove that

$$\dim \mathrm{Ext}^1_{\mathcal{D}_Q}\left(\mathtt{H}_Q(\mathsf{x}(j)), \mathtt{H}_Q(\mathsf{x}(j'))\right) = \begin{cases} 1 & \text{if } j' = j+1; \\ 0 & \text{otherwise.} \end{cases} \tag{6.7}$$

Using the fully faithful embedding $\varepsilon\colon \mathcal{D}_{Q'} \hookrightarrow \mathcal{D}_Q$, we can reduce the situation to the special case $Q' = Q$. In this case, we can easily check (6.7) since the $\mathbb{C}Q$-module M_θ is a projective cover of the simple module S_1 and at the same time it is an injective hull of the simple module S_{N-1}. □

By Lemma 6.6, the Kac-Moody algebra $\mathfrak{g}_J$ is of type A_∞. Let R_J^+ denote the set of positive roots of $\mathfrak{g}_J$, which is given by

$$\mathsf{R}_J^+ = \{\alpha(j;\ell) \in \mathsf{Q}_J^+ \mid j \in J, \ell \in \mathbb{Z}_{\geq 1}\}, \quad \text{where } \alpha(j;\ell) := \sum_{k=0}^{\ell-1} \alpha^J_{j+k}.$$

We also consider the subsets

$$\mathsf{R}^+_{J,N} := \{\alpha(j;N) \mid j \in J\}, \quad \mathsf{R}^+_{J,\leq N} := \{\alpha(j;\ell) \mid j \in J, 1 \leq \ell \leq N\}.$$

For a fixed element $\beta = \sum_{j \in J} d_j \alpha_j^J \in \mathsf{Q}_J^+$, we define a finite set

$$\mathsf{KP}(\beta) := \left\{ \nu = (\nu_\alpha) \in (\mathbb{Z}_{\geq 0})^{\mathsf{R}_J^+} \;\middle|\; \textstyle\sum_{\alpha \in \mathsf{R}_J^+} \nu_\alpha \alpha = \beta \right\}.$$

An element ν of $\mathsf{KP}(\beta)$ is called a *Kostant partition* of β. We also consider the subset

$$\mathsf{KP}_{\leq N}(\beta) := \{\nu \in \mathsf{KP}(\beta) \mid \nu_\alpha = 0 \text{ unless } \alpha \in \mathsf{R}^+_{J,\leq N}\}.$$

Let $D_\beta = \bigoplus_{j \in J} (D_\beta)_j$ be a J-graded vector space such that $\dim(D_\beta)_j = d_j$ for each $j \in J$ as in the previous subsection. We set

$$E_\beta := \mathsf{rep}_{D_\beta}(\Gamma_J) = \prod_{j \in J} \mathrm{Hom}_{\mathbb{C}}((D_\beta)_j, (D_\beta)_{j+1})$$

and regard it as a G_β-stable closed subvariety of $\mathfrak{gl}(D_\beta)$. A G_β-orbit in E_β can be realized as a component of a certain $\mathbb{C}^\times$-fixed locus of a nilpotent orbit of $\mathfrak{gl}(D_\beta)$ and hence called a *graded nilpotent orbit*. By Gabriel's theorem [13], the set of G_β-orbits in E_β is in bijection with the set $\mathsf{KP}(\beta)$. For an element $\nu \in \mathsf{KP}(\beta)$, the corresponding G_β-orbit $\mathfrak{O}_\nu$ contains the $\mathbb{C}\Gamma_J$-module $\bigoplus_{\alpha \in \mathsf{R}_J^+} (M_\alpha^J)^{\oplus \nu_\alpha}$, where M_α^J denotes the unique indecomposable $\mathbb{C}\Gamma_J$-module of dimension vector $\alpha \in \mathsf{R}_J^+$.

Let A' be the quotient of the path algebra $\mathbb{C}\Gamma_J$ by the ideal generated by all the paths of length $\geq N$. We consider the G_β-stable closed subvariety of E_β

$$E'_\beta := \mathsf{rep}_{D_\beta}(A') = \{X \in E_\beta \mid X^N = 0\}.$$

We can naturally regard the category A'-mod as the full subcategory of $\mathbb{C}\Gamma_J$-mod consisting of modules isomorphic to direct sums of M_α^J for various $\alpha \in \mathsf{R}^+_{J,\leq N}$. Thus the variety E'_β is a union of G_β-orbits $\mathfrak{O}_\nu$ with $\nu \in \mathsf{KP}_{\leq N}(\beta)$.

As explained in [17, Chapter II.2.6(a)], the algebra A' coincides with the repetitive algebra of $\mathbb{C}Q'$. Since it is self-injective, the category A'-mod is a Frobenius category and the set $\{M^J_\alpha \mid \alpha \in \mathsf{R}^+_N\}$ forms a complete collection of indecomposable projective A'-modules. We denote by A'-$\underline{\mathsf{mod}}$ the stable category of A'-mod.

Theorem 6.7 (Happel [17])**.** *There exists a δ-functor $\Phi' \colon A'\text{-}\mathsf{mod} \to \mathcal{D}_{Q'}$ which satisfies*

$$\Phi'(M^J_{\alpha(j;\ell)}) \cong \begin{cases} \mathtt{H}_{Q'}(\ell, \xi_\ell - 2j + 2) & \text{if } 1 \le \ell < N; \\ 0 & \text{if } \ell = N \end{cases}$$

for each $j \in J$ and $1 \le \ell \le N$, and induces a triangle equivalence $A'\text{-}\underline{\mathsf{mod}} \simeq \mathcal{D}_{Q'}$.

Proof. Apply the general theory [17, Theorem II.4.9] of the repetitive algebras. □

To each $\alpha \in \mathsf{R}^+_{J,\le N}$, we assign an element $\mathsf{x}(\alpha) \in \Delta_0 \sqcup \{0\}$ by

$$\mathsf{x}(\alpha) := (\mathtt{H}_Q^{-1} \circ \varepsilon \circ \Phi')(M^J_\alpha).$$

Note that we have $\mathsf{x}(\alpha^J_j) = \mathsf{x}(j)$ for each $j \in J$ by definition.

Now we state the main theorem of this subsection.

Theorem 6.8. *For any $\beta \in \mathsf{Q}^+_J$, the following assertions hold:*

(1) *The closed embedding $\mathfrak{M}^\bullet_0(D_\beta) \hookrightarrow E_\beta$ of Corollary 4.8 induces an isomorphism of G_β-varieties*

$$\mathfrak{M}^\bullet_0(D_\beta) \cong E'_\beta. \tag{6.8}$$

(2) *Under the isomorphism (6.8), each non-empty stratum $\mathfrak{M}^{\bullet\mathrm{reg}}_0(V, D_\beta)$ coincides with a single G_β-orbit $\mathfrak{O}_\nu$ associated with the Kostant partition $\nu \in \mathsf{KP}_{\le N}(\beta)$ determined by the relation*

$$Y^{D_\beta} A^{-V} = m_\nu := \prod_{\alpha \in \mathsf{R}^+_{J,\le N}} Y^{\nu_\alpha}_{\mathsf{x}(\alpha)}, \tag{6.9}$$

where we set $Y_{\mathsf{x}(\alpha)} = Y_0 := 1$ for $\alpha \in \mathsf{R}^+_{J,N}$.

For a proof of Theorem 6.8 (1), we need the following lemma.

Lemma 6.9. *We define a subset $C \subset \Delta_0$ by*

$$C := \mathsf{x}(J) \sqcup \{\mathtt{H}_Q^{-1}(S_i[k]) \mid i \in I \setminus I', k \in \mathbb{Z}\}. \tag{6.10}$$

Then the following assertions hold.

(1) *The subset C satisfies the condition* (R) *in Theorem 4.5.*

(2) *For any $i, j \in J$, we have*

$$\dim\left(e_{\mathsf{x}(i)} \cdot \mathbb{C}\Gamma|_C \cdot e_{\mathsf{x}(j)}\right) = \begin{cases} 1 & i \ge j; \\ 0 & i < j. \end{cases} \tag{6.11}$$

Proof. Let $I \setminus I' = I_1 \sqcup \cdots \sqcup I_b$ ($b \in \mathbb{Z}_{\ge 0}$) be a decomposition such that the full subquiver $Q|_{I_k}$ is a connected component of $Q|_{I \setminus I'}$ for each $1 \le k \le b$. Since the Dynkin graph is a tree, there exist unique $i_k \in I'$ and $j_k \in I_k$ satisfying $i_k \sim j_k$ for each $1 \le k \le b$. After reordering if necessary, we may assume that there exists $0 \le b_1 \le b$ such that we have $i_k \leftarrow j_k$ for $1 \le k \le b_1$ and $i_k \to j_k$ for $b_1 < k \le b$. We put $I^\circ := \bigsqcup_{1 \le k \le b_1} I_k$ and $I^\bullet := \bigsqcup_{b_1 < k \le b} I_k$.

To verify the assertion (1), it suffices to prove that for any $K \in \operatorname{ind}\mathcal{D}_Q$ there exists $L \in \mathtt{H}_Q(C)$ such that $\operatorname{Hom}_{\mathcal{D}_Q}(K, L) \neq 0$. Since the set $\mathtt{H}_Q(C)$ is stable under even degree shifts, we may assume that $K \cong M_\alpha$ or $K \cong M_\alpha[1]$ for some $\alpha \in \mathsf{R}^+$. When $K \cong M_\alpha$, we just take a simple quotient $M_\alpha \twoheadrightarrow S_i$ and find $\operatorname{Hom}_{\mathcal{D}_Q}(K, L) \neq 0$ with $L = S_i \in \mathtt{H}_Q(C)$. When $K \cong M_\alpha[1]$, we encounter the following two cases.

Case 1: $(\alpha, \varpi_j) \neq 0$ for some $j \in I^\circ$. In this case, the subspace $M' := \bigoplus_{i \in I' \sqcup I^\bullet} e_i M_\alpha$ is a submodule of M_α and the quotient M_α/M' is non-zero. Therefore there is $j \in I^\circ$ such that $\operatorname{Hom}_{\mathbb{C}Q}(M_\alpha/M', S_j) \neq 0$. This implies that $\operatorname{Hom}_{\mathcal{D}_Q}(K, L) \neq 0$ with $L = S_j[1] \in \mathtt{H}_Q(C)$.

Case 2: $(\alpha, \varpi_j) = 0$ for all $j \in I^\circ$. In this case, M_α can be regarded as a representation of the full subquiver $Q|_{I' \sqcup I^\bullet}$. First, we assume $(\alpha, \varpi_{N-1}) \neq 0$. Then we have $\operatorname{Hom}_{\mathbb{C}Q}(M_\alpha, M_\theta) \neq 0$ because M_θ is an injective hull of the simple module S_{N-1} in the category $(\mathbb{C}Q|_{I' \sqcup I^\bullet})$-$\mathsf{mod}$. Thus we obtain $\operatorname{Hom}_{\mathcal{D}_Q}(K, L) \neq 0$ with $L = M_\theta[1] \in \mathtt{H}_Q(C)$. Next, we assume $(\alpha, \varpi_{N-1}) = 0$ and $(\alpha, \varpi_i) \neq 0$ for some $1 \leq i < N-1$. Let us take such an i as large as possible. Then we see that $\alpha' := \alpha + \alpha_{i+1}$ is a positive root and there is a non-trivial extension $0 \to S_{i+1} \to M_{\alpha'} \to M_\alpha \to 0$. Therefore we obtain $\operatorname{Ext}^1_{\mathbb{C}Q}(M_\alpha, S_{i+1}) \neq 0$. Noting that $i+1 \in I'$, we get $\operatorname{Hom}_{\mathcal{D}_Q}(K, L) \neq 0$ with $L = S_{i+1}[2] \in \mathtt{H}_Q(C)$. Finally, we assume $(\alpha, \varpi_i) = 0$ for all $i \in I'$. Then M_α is supported on $I^\bullet$ and hence $\operatorname{Hom}_{\mathcal{D}_Q}(K, L) \neq 0$ with $L = S_j[1]$ for some $j \in I^\bullet$.

Next we shall prove the assertion (2). We note that the LHS of (6.11) is equal to the number of (oriented) paths from $\mathsf{x}(j)$ to $\mathsf{x}(i)$ in the quiver $\Gamma|_C$. Since $\Gamma_J = \Gamma|_{\mathsf{x}(J)} \subset \Gamma|_C$, there is at least one path from $\mathsf{x}(j)$ to $\mathsf{x}(i)$ when $i \geq j$. On the other hand, we know that the quiver Γ has neither loops nor oriented cycles by Proposition 3.8. In particular, there are no paths from $\mathsf{x}(j)$ to $\mathsf{x}(i)$ in $\Gamma|_C$ when $i < j$. Thus, we only have to prove that there are no two different paths from $\mathsf{x}(j)$ to $\mathsf{x}(i)$ when $j < i$. To see this, we divide the arrows of the quiver $\Gamma|_C$ into the following seven types:

(i) the arrows of the subquiver $\Gamma_J = \Gamma|_{\mathsf{x}(J)} \subset \Gamma|_C$;
(ii) $\mathtt{H}_Q^{-1}(S_{j_k}[2\ell]) \to \mathtt{H}_Q^{-1}(S_{i_k}[2\ell])$ for any $1 \leq k \leq b_1, \ell \in \mathbb{Z}$;
(iii) $\mathtt{H}_Q^{-1}(S_{i_k}[2\ell]) \to \mathtt{H}_Q^{-1}(S_{j_k}[2\ell])$ for any $b_1 < k \leq b, \ell \in \mathbb{Z}$;
(iv) $\mathtt{H}_Q^{-1}(S_j[\ell]) \to \mathtt{H}_Q^{-1}(S_j[\ell-1])$ for any $j \in I \setminus I', \ell \in \mathbb{Z}$;
(v) $\mathtt{H}_Q^{-1}(S_j[\ell]) \to \mathtt{H}_Q^{-1}(S_{j'}[\ell])$ for some $j, j' \in I_k, 1 \leq k \leq b, \ell \in \mathbb{Z}$;
(vi) $\mathtt{H}_Q^{-1}(S_{j_k}[2\ell+1]) \to \mathtt{H}_Q^{-1}(M_\theta[2\ell+1])$ for any $1 \leq k \leq b_1, \ell \in \mathbb{Z}$;
(vii) $\mathtt{H}_Q^{-1}(M_\theta[2\ell+1]) \to \mathtt{H}_Q^{-1}(S_{j_k}[2\ell+1])$ for any $b_1 < k \leq b, \ell \in \mathbb{Z}$.

A path in $\Gamma|_C$ going out from $\mathsf{x}(J)$ should contain an arrow of type (iii) or (vii), and hence go through a vertex belonging to the set $S^\bullet := \{\mathtt{H}_Q^{-1}(S_j[\ell]) \mid j \in I^\bullet, \ell \in \mathbb{Z}\}$. On the other hand, a path in $\Gamma|_C$ coming into $\mathsf{x}(J)$ should contain an arrow of type (ii) or (vi), and hence go through a vertex belonging to the set $S^\circ := \{\mathtt{H}_Q^{-1}(S_j[\ell]) \mid j \in I^\circ, \ell \in \mathbb{Z}\}$. However, there are no paths in $\Gamma|_C$ from a vertex of $S^\bullet$ to a vertex of S°. Therefore there are no paths in $\Gamma|_C$ connecting two different vertices of $\mathsf{x}(J)$ other than the paths in Γ_J. □

Proof of Theorem 6.8 (1). Thanks to Lemma 6.9 (1), we can apply Theorem 4.5 and Remark 4.9 to the algebra Λ_C associated with the subset $C \subset \Delta_0$ given by (6.10). Thus, there exists an admissible ideal $\mathfrak{J}_C$ of the path algebra $\mathbb{C}\Gamma|_C$ such that $\Lambda_C \cong (\mathbb{C}\Gamma|_C)/\mathfrak{J}_C$ and hence $\mathfrak{M}_0^\bullet(D_\beta) = \mathsf{rep}_{D_\beta}(\Lambda_C) \cong \mathsf{rep}_{D_\beta}((\mathbb{C}\Gamma|_C)/\mathfrak{J}_C)$.

Now, we need to prove $\mathsf{rep}_{D_\beta}((\mathbb{C}\Gamma|_C)/\mathfrak{J}_C) = E'_\beta$. Thanks to Lemma 6.9 (2), it suffices to show that $e_{\mathsf{x}(j+N)}(\mathfrak{J}_C)e_{\mathsf{x}(j)} \neq 0$ and $e_{\mathsf{x}(j+i)}(\mathfrak{J}_C)e_{\mathsf{x}(j)} = 0$ for any $j \in J$ and $1 \le i < N$. By Theorem 4.5, these conditions can be verified by checking the following two homological properties:

$$\mathrm{Ext}^2_{\mathcal{D}_Q}(\mathtt{H}_Q(\mathsf{x}(j)), \mathtt{H}_Q(\mathsf{x}(j+N))) \neq 0, \tag{6.12}$$

$$\mathrm{Ext}^2_{\mathcal{D}_Q}(\mathtt{H}_Q(\mathsf{x}(j)), \mathtt{H}_Q(\mathsf{x}(j+i))) = 0 \tag{6.13}$$

for all $j \in J$ and $1 \le i < N$. The property (6.12) follows because we have $\mathtt{H}_Q(\mathsf{x}(j+N)) = \mathtt{H}_Q(\mathsf{x}(j))[-2]$ by definition. We can prove the property (6.13) easily by using the fact that M_θ is both projective and injective in the subcategory $\mathbb{C}Q'$-$\mathsf{mod} \subset \mathbb{C}Q$-$\mathsf{mod}$. □

For a proof of Theorem 6.8 (2), we need the following lemma.

Lemma 6.10. *Let ℓ', ℓ'' be two positive integers such that $\ell := \ell' + \ell'' \le N$. Fix $j \in J$ and set $\alpha' := \alpha(j; \ell'), \alpha'' := \alpha(j + \ell'; \ell''), \alpha := \alpha(j; \ell' + \ell'') = \alpha' + \alpha''$. Then, there is an injective $U_q(L\mathfrak{g})$-homomorphism*

$$L(Y_{\mathsf{x}(\alpha)}) \hookrightarrow L(Y_{\mathsf{x}(\alpha')}) \otimes L(Y_{\mathsf{x}(\alpha'')}).$$

Proof. Under the assumption, the functor $\varepsilon \circ \Phi' \colon A'$-$\mathsf{mod} \to \mathcal{D}_Q$ sends a non-split short exact sequence $0 \to M^J_{\alpha''} \to M^J_\alpha \to M^J_{\alpha'} \to 0$ in A'-mod to a non-split exact triangle $\mathtt{H}_Q(\mathsf{x}(\alpha'')) \to \mathtt{H}_Q(\mathsf{x}(\alpha)) \to \mathtt{H}_Q(\mathsf{x}(\alpha')) \xrightarrow{+1}$ in $\mathcal{D}_Q$, where we understand $\mathtt{H}_Q(\mathsf{x}(\alpha)) = 0$ when $\alpha \in \mathsf{R}^+_{J,N}$, or equivalently $\ell = N$. Moreover, it induces an isomorphism of 1-dimensional vector spaces:

$$\mathrm{Ext}^1_{A'}(M^J_{\alpha'}, M^J_{\alpha''}) \cong \mathrm{Ext}^1_{\mathcal{D}_Q}(\mathtt{H}_Q(\mathsf{x}(\alpha')), \mathtt{H}_Q(\mathsf{x}(\alpha''))).$$

Applying Proposition 5.4, we obtain a short exact sequence in $\mathcal{C}$:

$$0 \to L(m_{[\mathsf{x}(\alpha'),\mathsf{x}(\alpha'')]}) \to L(Y_{\mathsf{x}(\alpha')}) \otimes L(Y_{\mathsf{x}(\alpha'')}) \to L(Y_{\mathsf{x}(\alpha')}Y_{\mathsf{x}(\alpha'')}) \to 0$$

with $m_{[\mathsf{x}(\alpha'),\mathsf{x}(\alpha'')]} = Y_{\mathsf{x}(\alpha)}$. □

Proof of Theorem 6.8 (2). First, we note that the assignment $\mathsf{KP}_{\le N}(\beta) \ni \nu \mapsto m_\nu \in \mathcal{M}^+$ is injective for each fixed element $\beta \in \mathsf{Q}^+_J$.

Since a non-empty stratum $\mathfrak{M}^{\bullet\mathrm{reg}}_0(V, D_\beta)$ is G_β-stable, it is a union of G_β-orbits. In particular, the number of non-empty strata $\mathfrak{M}^{\bullet\mathrm{reg}}_0(V, D_\beta)$ is less than or equal to the number of G_β-orbits in E'_β, which is $\#\mathsf{KP}_{\le N}(\beta)$. On the other hand, we apply Lemma 6.10 repeatedly to find that $c(Y^{D_\beta}, m_\nu) \neq 0$ for all $\nu \in \mathsf{KP}_{\le N}(\beta)$. By Theorem 4.10 (1), this implies that the number of non-empty strata is not less than $\#\mathsf{KP}_{\le N}(\beta)$. Therefore each non-empty stratum consists of a single G_β-orbit.

We shall prove the relation (6.9). Recall the stratifying functor $\Phi_Q \colon \Lambda$-$\mathsf{mod} \to \mathcal{D}_Q$ in Theorem 4.11. Thanks to Theorem 6.8 (1), we can identify the category A'-mod with the full subcategory of Λ-mod consisting of modules supported on the subset $\mathsf{x}(J) \subset \Delta_0$. By Theorem 4.11 (2), the relation (6.9) holds if and only if there is an isomorphism

$$\bigoplus_{\alpha \in \mathsf{R}^+_{J,\le N}} \Phi_Q(M^J_\alpha)^{\oplus \nu_\alpha} \cong \bigoplus_{\alpha \in \mathsf{R}^+_{J,\le N}} (\varepsilon \circ \Phi')(M^J_\alpha)^{\oplus \nu_\alpha}.$$

Thus, it suffices to prove the relation (6.9) for the special case when β is a positive root $\alpha \in \mathsf{R}^+_{J,\le N}$ and ν is the Kostant partition $\delta(\alpha) \in \mathsf{KP}_{\le N}(\alpha)$ given by $\delta(\alpha)_{\alpha'} = \delta_{\alpha,\alpha'}$ for each $\alpha' \in \mathsf{R}^+_{J,\le N}$. Now we concentrate on this special case. As in the previous paragraph, we have $c(Y^{D_\alpha}, Y_{\mathsf{x}(\alpha)}) \neq 0$ and hence there is a non-empty stratum $\mathfrak{M}_0^{\bullet\mathrm{reg}}(V, D_\alpha)$ such that $Y^{D_\alpha}A^{-V} = Y_{\mathsf{x}(\alpha)}$. Note that $Y_{\mathsf{x}(\alpha)}$ is a minimal element of $\mathcal{M}^+$ with respect to the partial ordering $\le$, which implies that $\mathfrak{M}_0^{\bullet\mathrm{reg}}(V, D_\alpha)$ is a maximal stratum with respect to the closure ordering by Theorem 4.10 (3). On the other hand, E'_α is an affine space and $\mathfrak{O}_{\delta(\alpha)}$ is the unique open dense G_α-orbit of E'_α. Therefore we have $\mathfrak{M}_0^{\bullet\mathrm{reg}}(V, D_\alpha) = \mathfrak{O}_{\delta(\alpha)}$. □

Let $\mathscr{L}_\beta$ be the push-forward of the constant perverse $\Bbbk$-sheaf along the proper morphism $\mu_\beta \colon \mathcal{F}_\beta \to E_\beta$ as in the previous subsection. By the decomposition theorem, we have

$$\mathscr{L}_\beta \cong \bigoplus_{\nu \in \mathsf{KP}(\beta)} IC(\mathfrak{O}_\nu, \Bbbk) \otimes_\Bbbk L_\nu, \tag{6.14}$$

where each $L_\nu \in D^b(\Bbbk\text{-}\mathsf{mod})$ is a finite-dimensional $\mathbb{Z}$-graded $\Bbbk$-vector space which is self-dual. Via the isomorphism (6.4), each vector space L_ν is equipped with a structure of graded $H_J(\beta)$-module. It is known that we have $L_\nu \neq 0$ for all $\nu \in \mathsf{KP}(\beta)$ and the set $\{L_\nu \mid \nu \in \mathsf{KP}(\beta)\}$ forms a complete collection of self-dual simple objects in the category $H_J(\beta)\text{-}\mathsf{gmod}$ (see [28, Corollary 2.8] for instance).

On the $U_q(L\mathfrak{g})$-side, we have the following homomorphisms of $\Bbbk$-algebras

$$U_q(L\mathfrak{g}) \xrightarrow{\widehat{\Psi}_{D_\beta}} \widehat{K}^{G_\beta}(Z^\bullet(D_\beta))_\Bbbk \cong \mathrm{Ext}^*_{G_\beta}(\mathscr{L}^\bullet_\beta, \mathscr{L}^\bullet_\beta)^\wedge,$$

where $\mathscr{L}^\bullet_\beta$ is the push-forward of the constant $\Bbbk$-sheaf on $\mathfrak{M}^\bullet(D_\beta)$ along the G_β-equivariant proper morphism $\pi^\bullet \colon \mathfrak{M}^\bullet(D_\beta) \to \mathfrak{M}_0^\bullet(D_\beta) = E'_\beta$ (see [12, Corollary 3.16]). By Theorem 4.10 (2) and Theorem 6.8 (2), the complex $\mathscr{L}^\bullet_\beta$ decomposes as:

$$\mathscr{L}^\bullet_\beta \cong \bigoplus_{\nu \in \mathsf{KP}_{\le N}(\beta)} IC(\mathfrak{O}_\nu, \Bbbk) \otimes_\Bbbk L^\bullet_\nu, \tag{6.15}$$

where each $L^\bullet_\nu$ is a non-zero finite-dimensional $\mathbb{Z}$-graded $\Bbbk$-vector space, which has a natural structure of a simple module over the algebra $\mathrm{Ext}^*_{G_\beta}(\mathscr{L}^\bullet_\beta, \mathscr{L}^\bullet_\beta)^\wedge$. Moreover, by [32, Theorem 14.3.2(3)] and the relation (6.9), we have $(\widehat{\Psi}_{D_\beta})^*(L^\bullet_\nu) \cong L(m_\nu)$ as $U_q(L\mathfrak{g})$-modules.

Let us consider a natural bimodule given by the Yoneda products:

$$\mathrm{Ext}^*_{G_\beta}(\mathscr{L}^\bullet_\beta, \mathscr{L}^\bullet_\beta) \curvearrowright \mathrm{Ext}^*_{G_\beta}(\mathscr{L}_\beta, \mathscr{L}^\bullet_\beta) \curvearrowleft \mathrm{Ext}^*_{G_\beta}(\mathscr{L}_\beta, \mathscr{L}_\beta). \tag{6.16}$$

Comparing the decompositions (6.14) and (6.15), we see that the functor

$$\mathrm{Ext}^*_{G_\beta}(\mathscr{L}_\beta, \mathscr{L}_\beta)\text{-}\mathsf{gmod} \to \mathrm{Ext}^*_{G_\beta}(\mathscr{L}^\bullet_\beta, \mathscr{L}^\bullet_\beta)\text{-}\mathsf{gmod}$$

induced from the bimodule (6.16) sends the simple module L_ν to the simple module $L^\bullet_\nu$ if $\nu \in \mathsf{KP}_{\le N}(\beta)$, or zero otherwise (see [15, Theorem 6.8] for a detailed explanation). Moreover, after the completion, the bimodule (6.16) gets identified with the bimodule (6.3), i.e. we have the following commutative diagram

$$\begin{array}{ccccc}
\widehat{K}^{G_\beta}(Z^\bullet(D_\beta))_\Bbbk & \longrightarrow & \mathrm{End}\left(\widehat{K}^{G_\beta}(\mathfrak{M}^\bullet(D_\beta) \times_{E_\beta} \mathcal{F}_\beta)_\Bbbk\right) & \longleftarrow & \widehat{K}^{G_\beta}(\mathcal{Z}_\beta)^{\mathrm{op}}_\Bbbk \\
\downarrow \cong & & \downarrow \cong & & \downarrow \cong \\
\mathrm{Ext}^*_{G_\beta}(\mathscr{L}^\bullet_\beta, \mathscr{L}^\bullet_\beta)^\wedge & \longrightarrow & \mathrm{End}\left(\mathrm{Ext}^*_{G_\beta}(\mathscr{L}_\beta, \mathscr{L}^\bullet_\beta)^\wedge\right) & \longleftarrow & \mathrm{Ext}^*_{G_\beta}(\mathscr{L}_\beta, \mathscr{L}_\beta)^{\wedge\mathrm{op}}.
\end{array}$$

Combining the above discussion with Theorem 6.4, we obtain the following.

Corollary 6.11. *The KKK-functor $\mathscr{F}_J \colon H_J\text{-}\mathsf{gmod} \to \mathcal{C}$ associated with the injective map $\mathsf{x} \colon J \hookrightarrow \Delta_0$ given by (6.6)satisfies*

$$\mathscr{F}_J(L_\nu) \cong \begin{cases} L(m_\nu) & \text{if } \nu \in \mathsf{KP}_{\le N}(\beta); \\ 0 & \text{otherwise,} \end{cases}$$

for each $\beta \in \mathsf{Q}_J^+$ and $\nu \in \mathsf{KP}(\beta)$.

We define a category $\mathcal{C}_{\mathcal{D}_{Q'}}$ as the full subcategory of $\mathcal{C}$ consisting of modules whose composition factors are isomorphic to $L(m)$ for some dominant monomial m in variables Y_x labeled by $x \in \Delta_0$ such that $\mathtt{H}_Q(x) \in \mathcal{D}_{Q'} \subset \mathcal{D}_Q$. Note that the above KKK-functor $\mathscr{F}_J$ is exact by [20, Theorem 3.8]. By Corollary 6.11, the image of the functor $\mathscr{F}_J$ is contained in the category $\mathcal{C}_{\mathcal{D}_{Q'}}$.

In [20, Section 4], Kang-Kashiwara-Kim introduced a certain localization $\mathcal{T}_N$ of the category H_J-gmod for each $N \in \mathbb{Z}_{\ge 1}$. This is a $\mathbb{Z}$-graded $\Bbbk$-linear abelian monoidal category equipped with a canonical quotient functor $\Omega \colon H_J\text{-}\mathsf{gmod} \to \mathcal{T}_N$ characterized by the following universal property. Suppose that $\mathcal{A}$ is a $\Bbbk$-linear abelian monoidal category and $F \colon H_J\text{-}\mathsf{gmod} \to \mathcal{A}$ is a $\Bbbk$-linear exact monoidal functor such that

(i) $F(L_\nu) = 0$ for any $\nu \in \mathsf{KP}(\beta) \setminus \mathsf{KP}_{\le N}(\beta)$;
(ii) $F(L_{\delta(\alpha)}) \cong \mathbf{1}_{\mathcal{A}}$ for any $\alpha \in \mathsf{R}_{J,N}^+$. Here $\mathbf{1}_{\mathcal{A}}$ denotes the unit object of $\mathcal{A}$;
(iii) F satisfies a certain commutativity condition on the tensoring operations with the modules $\{L_{\delta(\alpha)} \mid \alpha \in \mathsf{R}_{J,N}^+\}$ as in [20, Proposition A.12].

Then there exists a unique $\Bbbk$-linear exact monoidal functor $\widetilde{F} \colon \mathcal{T}_N \to \mathcal{A}$ such that we have $F \simeq \widetilde{F} \circ \Omega$.

Since the category $\mathcal{T}_N$ is $\mathbb{Z}$-graded, its Grothendieck ring $K(\mathcal{T}_N)$ has a structure of $\mathbb{Z}[v^{\pm 1}]$-algebra, where the multiplication of v is given by the grading shift functor. We denote by $K(\mathcal{T}_N)|_{v=1}$ the specialization $K(\mathcal{T}_N)/(v-1)K(\mathcal{T}_N)$.

Corollary 6.12. *After a suitable modification of the isomorphism (6.5) for each $\beta \in \mathsf{Q}_J^+$, the above KKK-functor $\mathscr{F}_J \colon H_J\text{-}\mathsf{gmod} \to \mathcal{C}_{\mathcal{D}_{Q'}}$ factors through the localized category $\mathcal{T}_N$ and yields a ring isomorphism*

$$K(\mathcal{T}_N)|_{v=1} \cong K(\mathcal{C}_{\mathcal{D}_{Q'}}).$$

Sketch of proof. The functor $\mathscr{F}_J \colon H_J\text{-}\mathsf{gmod} \to \mathcal{C}_{\mathcal{D}_{Q'}}$ satisfies the above conditions (i) and (ii) by Corollary 6.11. By the similar argument as in [25, Theorem 2.6.8], we can modify the isomorphism $\widehat{H}_J(\beta) \cong \widehat{K}^{G_\beta}(\mathcal{Z}_\beta)_\Bbbk$ for each $\beta \in \mathsf{Q}_J^+$ to make the functor $\mathscr{F}_J$ satisfy the condition (iii) as well. Thus, by the universal property, the functor $\mathscr{F}_J$ factors through the localization $\mathcal{T}_N$. By [20, Proposition 4.31], the set $\{\Omega(L_\nu) \mid \nu \in \mathsf{KP}_{\le N-1}(\beta), \beta \in \mathsf{Q}_J^+\}$ forms a complete collection of self-dual simple objects of $\mathcal{T}_N$ and hence their classes give a $\mathbb{Z}$-basis of $K(\mathcal{T}_N)|_{v=1}$. On the other hand, the set $\{L(m_\nu) \mid \nu \in \mathsf{KP}_{\le N-1}(\beta), \beta \in \mathsf{Q}_J^+\}$ forms a complete collection of simple modules of $\mathcal{C}_{\mathcal{D}_{Q'}}$ and hence their classes give a $\mathbb{Z}$-basis of $K(\mathcal{C}_{\mathcal{D}_{Q'}})$. Now the desired ring isomorphism follows because the functor $\mathscr{F}_J$ induces a bijection between these two bases again by Corollary 6.11. □

Remark 6.13. In the recent paper [26] by Kashiwara-Kim-Oh-Park, it was shown that the category $\mathcal{T}_N$ gives a monoidal categorification of a cluster algebra associated with a certain infinite quiver. Combined with Corollary 6.12, we conclude that the category $\mathcal{C}_{\mathcal{D}_{Q'}}$ also gives a monoidal categorification of the same cluster algebra.

We finish this subsection with exhibiting a couple of examples of subquivers $Q' \subset Q$ and the corresponding injective maps $\mathsf{x}\colon J \hookrightarrow \Delta_0$.

Example 6.14 (Type A_n)**.** Consider the following case with $N = n+1$:

$$Q' = \left(\overset{1}{\circ} \to \overset{2}{\circ} \to \overset{3}{\circ} \to \cdots \to \overset{n}{\circ} \right) = Q.$$

For simplicity, we choose the height function ξ with $\xi_1 = -2$. Then the corresponding injective map $\mathsf{x}\colon J = \mathbb{Z} \hookrightarrow \Delta_0$ is explicitly given by

$$\mathsf{x}(j) = (1, -2j) \quad \text{for each } j \in \mathbb{Z}.$$

In this case, the associated functor $\mathscr{F}_J$ has been studied in detail by [20, 25, 26]. Moreover, it can be seen as a suitable completion of the usual quantum affine Schur-Weyl duality between the quantum loop algebra $U_q(L\mathfrak{sl}_{n+1})$ and the affine Hecke algebras of GL's. Moreover, our geometric interpretation can be obtained from Ginzburg-Reshetikhin-Vasserot's geometric interpretation [15].

Example 6.15 (Type D_n)**.** Consider the following case with $N = n$:

$$Q' = \left(\overset{1}{\circ} \to \overset{2}{\circ} \to \cdots \to \overset{n-2}{\circ} \to \overset{n-1}{\circ} \right) \subset Q = \left(\overset{1}{\circ} \to \overset{2}{\circ} \to \cdots \to \overset{n-2}{\circ} \begin{smallmatrix} \nearrow \circ\, n-1 \\ \searrow \circ\, n \end{smallmatrix} \right).$$

For simplicity, we choose the height function ξ with $\xi_1 = -2$. Then the corresponding injective map $\mathsf{x}\colon J = \mathbb{Z} \hookrightarrow \Delta_0$ is explicitly given by

$$\begin{aligned} \mathsf{x}(i+kn) &= (1, -2i-2kh) \quad \text{if } 1 \le i \le n-2, \\ \mathsf{x}(n-1+kn) &= ((n-1)^*, -3n+4-2kh), \\ \mathsf{x}(kn) &= ((n-1)^*, n-2-2kh) \end{aligned}$$

where $k \in \mathbb{Z}$, and $h = 2n-2$ is the Coxeter number. Note that $(n-1)^* = n-1$ when n is even, and $(n-1)^* = n$ when n is odd. The associated functor $\mathscr{F}_J$ in this case coincides with the one studied in [26, Section 6.2.4].

Example 6.16 (Type E_n)**.** For $n = 6, 7, 8$, consider the following case with $N = n$:

$$Q' = \left(\overset{1}{\circ} \to \overset{2}{\circ} \to \overset{3}{\circ} \to \cdots \to \overset{n-1}{\circ} \right) \subset Q = \left(\overset{1}{\circ} \to \overset{2}{\circ} \to \underset{\underset{\circ\, n}{\downarrow}}{\overset{3}{\circ}} \to \cdots \to \overset{n-1}{\circ} \right).$$

For simplicity, we choose the height function ξ with $\xi_1 = -2$. Then the corresponding injective map $\mathsf{x}\colon J = \mathbb{Z} \hookrightarrow \Delta_0$ is explicitly given by

$$\mathsf{x}(i+kn) = \begin{cases} (1, -2i-2kh) & \text{if } 0 \le i \le 3; \\ ((n-1)^*, n-h-2i-2kh) & \text{if } 4 \le i \le n-1, \end{cases}$$

where $k \in \mathbb{Z}$, and $h = 12, 18, 30$ is the Coxeter number of type $\mathsf{E}_{6,7,8}$ respectively. Note that $(n-1)^* = 1$ when $n = 6$, and $(n-1)^* = n-1$ when $n = 7, 8$.

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(R. Fujita) Research Institute for Mathematical Sciences, Kyoto University, Oiwake-Kitashirakawa, Sakyo, Kyoto, 606-8502, Japan & Institut de Mathématiques de Jussieu-Paris Rive Gauche, Université de Paris, F-75013, Paris, France
Email address: rfujita@kurims.kyoto-u.ac.jp