Quenched and averaged tails of the heat kernel of the two-dimensional uniform spanning tree*

M. T. Barlow[†], D. A. Croydon[‡] and T. Kumagai[§]

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This paper is dedicated to the memory of Harry Kesten, a pioneer in the study of anomalous random walks in random media.

Abstract

This article investigates the heat kernel of the two-dimensional uniform spanning tree. We improve previous work by demonstrating the occurrence of log-logarithmic fluctuations around the leading order polynomial behaviour for the on-diagonal part of the quenched heat kernel. In addition we give two-sided estimates for the averaged heat kernel, and we show that the exponents that appear in the off-diagonal parts of the quenched and averaged versions of the heat kernel differ. Finally, we derive various scaling limits for the heat kernel, the implications of which include enabling us to sharpen the known asymptotics regarding the on-diagonal part of the averaged heat kernel and the expected distance travelled by the associated simple random walk.

1 Introduction

The focus of this article is the two-dimensional uniform spanning tree (UST), which is a random subgraph of $\mathbb{Z}^2$ that will henceforth be denoted by $\mathcal{U}$. Since the introduction of this object in [34], considerable progress has been made in our understanding of the geometry of USTs (and, more generally, uniform spanning forests), see [11] for background. In this direction, a particularly useful viewpoint was provided by Wilson, who gave a construction of USTs via loop erased random walks (LERWs) [36]. Indeed, the latter description was at the heart of Schramm's seminal work describing the subsequential scaling limits of two-dimensional LERW and $\mathcal{U}$ in terms of what is now called the Schramm-Loewner evolution (SLE) [35], see also [29]. In recent years, building on Lawler and Viklund's convergence result for the LERW in its natural parametrisation [30], a more detailed picture of the scaling limit of $\mathcal{U}$ has been established [4, 18]. And, closely related to this, properties of the simple random walk (SRW) on $\mathcal{U}$ have also been explored [4, 8, 9]. The goal here is to provide further insight into the behaviour of the heat kernel (transition density) of the latter process.

Let us proceed to present some of the basic notation that will be used throughout the article. We will assume that the two-dimensional UST $\mathcal{U}$ is built on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$; we

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write $\mathbf{E}$ for the associated expectation. Note that, $\mathbf{P}$-a.s., $\mathcal{U}$ is a one-ended tree with vertex set $\mathbb{Z}^2$ [34]. We write $\gamma(x,y)$ for the unique self-avoiding path between $x,y \in \mathbb{Z}^2$, and $\gamma(x,\infty)$ for the unique infinite self-avoiding path started at x. By Wilson's algorithm (see [36], and the recollection of this at the start of Section 2), $\gamma(x,y)$ is equal in law to the loop erasure of a SRW started at x and run until it hits y. We will denote by $d_{\mathcal{U}}$ the intrinsic metric on the graph $\mathcal{U}$, so that $d_{\mathcal{U}}(x,y)$ is the length of the geodesic $\gamma(x,y)$. We write $\mu_{\mathcal{U}}$ for the measure on $\mathbb{Z}^2$ such that $\mu_{\mathcal{U}}(\{x\})$ is given by the number of edges of $\mathcal{U}$ that contain x; this is the invariant measure of the simple random walk. We denote balls in the intrinsic metric $d_{\mathcal{U}}$ by $B_{\mathcal{U}}(x,r) = \{y \in \mathbb{Z}^2 : d_{\mathcal{U}}(x,y) \leq r\}$. We use d_∞ to denote the ℓ_∞ metric on $\mathbb{Z}^2$, and $B_\infty(x,r)$ to denote balls in the d_∞-metric; these balls are of course boxes.

Many of the exponents that describe the behaviour of $\mathcal{U}$ and the associated random walk can be expressed in terms of the *growth exponent of the two-dimensional LERW*, which is given by $\kappa = 5/4$. More precisely, let L_n be the loop erasure of a SRW in $\mathbb{Z}^2$ run until its first exit from $[-n,n]^2$, M_n be the number of steps in L_n, and $G(n) = \mathbf{E}(M_n)$. By [26], we have

$$G(n) = \mathbf{E}M_n \asymp n^\kappa, \tag{1.1}$$

where $\asymp$ means 'bounded above and below by constant multiples of'. (This improves earlier estimates in [19, 33], which establish that $\lim_{n\to\infty} \log G(n)/\log n = \kappa$.) The papers [8, 9] gave estimates for the heat kernel of $\mathcal{U}$ in terms of the function G; these can now be written more simply using (1.1). When we cite results from [8, 9] we will give the simplified versions without further comment.

Next, we introduce the simple random walk on $\mathcal{U}$, which is the discrete-time Markov process $X^{\mathcal{U}} = ((X^{\mathcal{U}}_n)_{n\geq 0}, (P^{\mathcal{U}}_x)_{x\in\mathbb{Z}^2})$ that at each time step jumps from its current location to a uniformly chosen neighbour in the graph $\mathcal{U}$. For $x \in \mathbb{Z}^2$, the law $P^{\mathcal{U}}_x$ is called the *quenched* law of the simple random walk on $\mathcal{U}$ started at x, and we write

$$p^{\mathcal{U}}_n(x,y) = \frac{P^{\mathcal{U}}_x\left(X^{\mathcal{U}}_n = y\right)}{\mu_{\mathcal{U}}\left(\{y\}\right)}, \qquad \forall x,y \in \mathbb{Z}^2,$$

for the corresponding quenched heat kernel.

To understand the properties of random walk on a space such as $\mathcal{U}$, a by now well-established approach is to first study volume growth and resistance growth (see, for example, [6, 21, 22]). Regarding the volume growth, one would expect from (1.1) and Wilson's algorithm that $B_{\mathcal{U}}(x,r^\kappa)$ should be approximately the same as $B_\infty(x,r)$, and hence that $|B_{\mathcal{U}}(x,R)|$ should be of order $R^{2/\kappa}$. This expectation was confirmed by [9, Theorem 1.2], which gives stretched exponential estimates for the upper and lower tails of $R^{-2/\kappa}|B_{\mathcal{U}}(0,R)|$. We define the 'fractal dimension' of $\mathcal{U}$ by

$$d_f = \frac{2}{\kappa} = \frac{8}{5}. \tag{1.2}$$

Using the estimates in [9, Theorem 1.2] an easy Borel-Cantelli argument gives that there exist deterministic constants $c_1, c_2 \in (0,\infty)$ such that, $\mathbf{P}$-a.s.,

$$c_1 r^{d_f}(\log\log r)^{-9} \leq \mu_{\mathcal{U}}\left(B_{\mathcal{U}}(0,r)\right) \leq c_2 r^{d_f}(\log\log r)^3$$

for large r. The first main result of this paper is that volume fluctuations of log-logarithmic magnitude really do occur.

Theorem 1.1. $\mathbf{P}$-*a.s.,*

$$\limsup_{r\to\infty}\frac{\mu_{\mathcal{U}}\left(B_{\mathcal{U}}(0,r)\right)}{r^{d_f}(\log\log r)^{1/5}}=\infty, \tag{1.3}$$

and also

$$\liminf_{r\to\infty}\frac{(\log\log r)^{3/5}\mu_{\mathcal{U}}\left(B_{\mathcal{U}}(0,r)\right)}{r^{d_f}}=0. \tag{1.4}$$

Similar fluctuations have also been observed for Galton-Watson trees [7, Proposition 2.8] (see also [16, Lemma 5.1]). The proof here is more complicated as the correlations between different parts of the space are harder to control. The key ingredient is the argument of Section 3 below, in which we provide a general technique for estimating from below the probability of seeing a particular path configuration in the initial stages of the construction of the UST via Wilson's algorithm. This enables us to control the probability of seeing especially short or long paths in some region of $\mathcal{U}$.

The volume fluctuations of Theorem 1.1 are associated with corresponding fluctuations in the on-diagonal part of the quenched heat kernel. From [9, Theorem 4.5], we know there exist deterministic constants $c_1,c_2\in(0,\infty)$ and $\alpha_1,\alpha_2\in(0,\infty)$ such that, $\mathbf{P}$-a.s.,

$$c_1n^{-d_f/d_w}(\log\log n)^{-\alpha_1}\leq p^{\mathcal{U}}_{2n}(0,0)\leq c_2n^{-d_f/d_w}(\log\log n)^{\alpha_2}$$

for large n. Here

$$d_w=1+d_f=\frac{2+\kappa}{\kappa}=\frac{13}{5} \tag{1.5}$$

is the so-called walk dimension; this represents the space-time scaling exponent with respect to the intrinsic metric. Applying Theorem 1.1, we are able to deduce that log-log fluctuations in the quenched heat kernel actually occur.

Corollary 1.2. *There exists $\beta>0$ such that,* $\mathbf{P}$-*a.s.,*

$$\liminf_{n\to\infty}(\log\log n)^{1/13}n^{d_f/d_w}p^{\mathcal{U}}_{2n}(0,0)=0,$$
$$\limsup_{n\to\infty}(\log\log n)^{-\beta}n^{d_f/d_w}p^{\mathcal{U}}_{2n}(0,0)=\infty.$$

These volume and heat kernel fluctuations arise from unlikely configurations of $\mathcal{U}$ inside $B_\infty(0,r_k)$ at a (random) sequence of scales $r_k\to\infty$. Another consequence of the occurrence of such exceptional configurations is the failure of the elliptic Harnack inequality in this setting. For a precise description of the particular form of the elliptic Harnack inequality that we consider, see Definition 7.1 below.

Corollary 1.3. *The large-scale elliptic Harnack inequality does not hold for the random walk on $\mathcal{U}$.*

We now consider the off-diagonal heat kernel. To avoid the issues of parity that arise from the fact $\mathcal{U}$ is a bipartite graph, we introduce the following smoothed version of the heat kernel

$$\tilde{p}^{\mathcal{U}}_n(x,y):=\frac{p^{\mathcal{U}}_n(x,y)+p^{\mathcal{U}}_{n+1}(x,y)}{2}.$$

In [9, Theorem 4.7] it was shown that there exist deterministic constants $\alpha, C \in (0,\infty)$ such that, $\mathbf{P}$-a.s.:

$$\frac{n^{-\frac{d_f}{d_w}}}{A} \exp\left\{-A\left(\frac{d_{\mathcal{U}}(0,x)^{d_w}}{n}\right)^{\frac{1}{d_w-1}}\right\} \leq \tilde{p}_n^{\mathcal{U}}(0,x) \leq An^{-\frac{d_f}{d_w}} \exp\left\{-\frac{1}{A}\left(\frac{d_{\mathcal{U}}(0,x)^{d_w}}{n}\right)^{\frac{1}{d_w-1}}\right\}$$

holds whenever $n \geq d_{\mathcal{U}}(0,x)$ and $\max\{n^{d_w}, |x|\}$ is suitably large, where

$$A = A(n,x) := C\left(\log\left(\max\{n^{d_w}, |x|\}\right)\right)^{\alpha}. \tag{1.6}$$

The logarithmic correction factor A represents the possible influence of exceptional environments on the heat kernel.

We are unlikely to see an exceptional configuration at any particular scale, so it is not surprising that for the averaged heat kernel the fluctuations of Corollary 1.2 disappear: by [9, Theorem 4.4(c)], we have that

$$c_1 n^{-d_f/d_w} \leq \mathbf{E}\, p_{2n}^{\mathcal{U}}(0,0) \leq c_2 n^{-d_f/d_w}, \qquad \forall n \geq 1. \tag{1.7}$$

As for the off-diagonal part of the averaged heat kernel, one might hope that one could replace the random distance $d_{\mathcal{U}}(0,x)$ with its typical order with respect to the Euclidean metric, that is, $|x|^{\kappa}$, and that, as with (1.7) one would be able to remove the errors associated with the term A in (1.6). We show that this is almost the case, however, in the annealed off-diagonal bounds the exponent $\frac{1}{d_w-1}$ needs to be replaced by a strictly smaller number.

Theorem 1.4. *There exist constants $c_1, c_2, c_3, c_4 \in (0,\infty)$ and $0 < \theta_2 \leq \theta_1 < 1$ such that: for every $x = (x_1, x_2) \in \mathbb{Z}^2$ and $n \geq |x_1| + |x_2|$,*

$$n^{-\frac{d_f}{d_w}} \exp\left\{-c_2\left(\frac{|x|^{\kappa d_w}}{n}\right)^{\frac{\theta_1}{d_w-1}}\right\} \leq \mathbf{E}\tilde{p}_n^{\mathcal{U}}(0,x) \leq c_3 n^{-\frac{d_f}{d_w}} \exp\left\{-c_4\left(\frac{|x|^{\kappa d_w}}{n}\right)^{\frac{\theta_2}{d_w-1}}\right\}.$$

Our argument indicates that we can take $\theta_1 < 1$ due to contribution to the averaged heat kernel from realisations of $\mathcal{U}$ where the intrinsic distance from 0 to x is unusually short, and thus where the heat kernel $\tilde{p}_n^{\mathcal{U}}(0,x)$ is unusually large. This phenomenon was not observed in the earlier study of random walk on a Galton-Watson tree of [7] (see Theorem 1.5 in particular), since the intrinsic metric of the trees was the only one involved there.

Remark 1.5. We have $\theta_1 = \frac{d_w-1}{\kappa d_w - 1} = \frac{32}{45}$, and we conjecture that this is also the correct value for θ_2. This would mean that the averaged heat kernel estimates of Theorem 1.4 are of the usual sub-Gaussian form, but with respect to the extrinsic walk dimension κd_w, rather than the intrinsic walk dimension that appears in the quenched bounds.

In the course of our proofs we obtain some new tail estimates on the length of the path $\gamma(x,y)$ between points x and y; by Wilson's algorithm this is also the length of a LERW run from x to y.

Theorem 1.6. *(a) There exist constants c_i such that for $\lambda \geq 1$, $x, y \in \mathbb{Z}^2$,*

$$c_1 e^{-c_2\lambda^4} \leq \mathbf{P}\big(d_{\mathcal{U}}(x,y) < \lambda^{-1} d_{\infty}(x,y)^{\kappa}\big) \leq c_3 e^{-c_4\lambda^4}.$$

(b) There exist constants c, q such that for $\lambda \geq 1$, $x, y \in \mathbb{Z}^2$,

$$\mathbf{P}\left(d_{\mathcal{U}}(x,y) \geq \lambda d_{\infty}(x,y)^{\kappa}\right) \leq c(\log\lambda)^q \lambda^{-(2-\kappa)/\kappa}.$$

The upper bound in (a) is proved in Theorem 2.7, (b) is proved at the end of Section 2, and the lower bound in (a) is proved at the end of Section 3.

We now consider the scaling limit of the UST and its heat kernel. Schramm's original work encoded $\mathcal{U}$ in terms of a path ensemble (consisting of the shortest paths in $\mathcal{U}$ between pairs of vertices), which enabled basic topological properties of any possible scaling limit to be deduced. In [4], building on the work of [8, 9], this scaling picture was extended to incorporate the intrinsic (i.e. shortest path) metric on $\mathcal{U}$, as well as the uniform measure, with the result of [4] being expressed in terms of the tightness under rescaling of $\mathcal{U}$ in a certain Gromov-Hausdorff-type topology for metric-measure spaces with an embedding into Euclidean space. The main obstacle to extending the work of [4] to a full (i.e. non-subsequential) scaling limit was the need to prove the existence of the scaling limit of the two-dimensional LERW as a stochastic process, rather than simply as a compact subset of the plane. This was subsequently established in [30], and Holden and Sun [18] then proved that $\mathcal{U}$ has a full scaling limit as a metric-measure space.

Let us now describe the setting of [4] more precisely. To retain information about $\mathcal{U}$ in the Euclidean topology, $(\mathcal{U}, d_{\mathcal{U}})$ can be considered as a spatial tree (cf. [17]) – that is, as a real tree (see [31, Definition 1.1], for example) obtained from the graph by including unit line segments along edges, embedded into $\mathbb{R}^2$ via a continuous map $\phi_{\mathcal{U}} : \mathcal{U} \to \mathbb{R}^2$, which is given by the identity on vertices, with linear interpolation along edges. In addition, suppose the space is rooted at the origin of $\mathbb{Z}^2$, giving a random 'measured, rooted spatial tree' $(\mathcal{U}, d_{\mathcal{U}}, \mu_{\mathcal{U}}, \phi_{\mathcal{U}}, 0)$. For this quintuplet, it follows from [18, Theorem 1.1] (see also [4, Theorem 1.1]) that

$$\left(\mathcal{U}, \delta^{\kappa} d_{\mathcal{U}}, \delta^2 \mu_{\mathcal{U}}, \delta\phi_{\mathcal{U}}, 0\right) \xrightarrow{d} (\mathcal{T}, d_{\mathcal{T}}, \mu_{\mathcal{T}}, \phi_{\mathcal{T}}, \rho_{\mathcal{T}}) \tag{1.8}$$

as $\delta \to 0$ with respect to the Gromov-Hausdorff-type topology introduced in [4, Section 3]. The random limit space is such that, $\mathbf{P}$-a.s.: $(\mathcal{T}, d_{\mathcal{T}})$ is a complete and locally compact real tree; $\mu_{\mathcal{T}}$ is a locally finite Borel measure on $(\mathcal{T}, d_{\mathcal{T}})$; $\phi_{\mathcal{T}}$ is a continuous map from $(\mathcal{T}, d_{\mathcal{T}})$ into $\mathbb{R}^2$; and $\rho_{\mathcal{T}}$ is a distinguished vertex in $\mathcal{T}$. In the original result of [4], the measure $\mu_{\mathcal{U}}$ considered was the uniform measure on the vertices, but it is no problem to replace this with the measure we consider here since, after scaling the uniform measure by a factor of two, the Prohorov distance between the two measures is bounded above by two, and so the discrepancy disappears in the scaling limit. Moreover, it readily follows from (1.8) that the space $(\mathcal{T}, d_{\mathcal{T}}, \mu_{\mathcal{T}}, \phi_{\mathcal{T}}, \rho_{\mathcal{T}})$ satisfies the following scale invariance property: for any $\lambda > 0$,

$$\left(\mathcal{T}, \lambda^{\kappa} d_{\mathcal{T}}, \lambda^2 \mu_{\mathcal{T}}, \lambda\phi_{\mathcal{T}}, \rho_{\mathcal{T}}\right) \stackrel{d}{=} (\mathcal{T}, d_{\mathcal{T}}, \mu_{\mathcal{T}}, \phi_{\mathcal{T}}, \rho_{\mathcal{T}}) . \tag{1.9}$$

Further from the SLE description of the limit in [18], one also has rotational invariance, i.e. for any $\theta \in [0, 2\pi)$,

$$(\mathcal{T}, d_{\mathcal{T}}, \mu_{\mathcal{T}}, R_\theta\phi_{\mathcal{T}}, \rho_{\mathcal{T}}) \stackrel{d}{=} (\mathcal{T}, d_{\mathcal{T}}, \mu_{\mathcal{T}}, \phi_{\mathcal{T}}, \rho_{\mathcal{T}}) , \tag{1.10}$$

where R_θ is a rotation of Euclidean space about the origin by the angle θ. In Proposition 8.2 below, we further establish an invariance under a rerooting property for the limit space.

One of the motivations for proving (1.8) was to show that the random walks on $\mathcal{U}$ converge to a limiting process. It was shown in [4] that the random walks on $\mathcal{U}$ started from 0 satisfy

$$\left(\delta X^{\mathcal{U}}_{\delta^{-\kappa d_w} t}\right)_{t \geq 0} \to \left(\phi_{\mathcal{T}}\left(X^{\mathcal{T}}_t\right)\right)_{t \geq 0} \tag{1.11}$$

in distribution under the averaged or annealed law. (Cf. the more general statements concerning the convergence of random walks on trees of [2, 13].) Here, for $\mathbf{P}$-a.e. realisation of $(\mathcal{T}, d_{\mathcal{T}}, \mu_{\mathcal{T}}, \phi_{\mathcal{T}}, \rho_{\mathcal{T}})$, $X^{\mathcal{T}} = (X_t^{\mathcal{T}})_{t\geq 0}$ is the canonical diffusion, or Brownian motion, on $(\mathcal{T}, d_{\mathcal{T}}, \mu_{\mathcal{T}})$ started from $\rho_{\mathcal{T}}$, and $\phi_{\mathcal{T}}(X^{\mathcal{T}})$ is the corresponding random element of $C(\mathbb{R}_+, \mathbb{R}^2)$. In this article, we connect the heat kernel of the discrete process $X^{\mathcal{U}}$ to that of $X^{\mathcal{T}}$, for which off-diagonal estimates were given in [4]. As our first result in this direction, we show the convergence of the quenched and averaged on-diagonal part of the heat kernel. From (1.8) and (1.11), the first claim of the following result, which concerns $(p_t^{\mathcal{T}}(x,y))_{x,y\in\mathcal{T},t>0}$, the quenched heat kernel on the tree $\mathcal{T}$ (as defined in [4]), is essentially an application of the local limit theorem of [14]. To adapt this to yield the corresponding statement for the averaged heat kernels, we check the uniform integrability of the on-diagonal part of the discrete heat kernel by applying an argument similar to that applied to deduce averaged heat kernel estimates for Galton-Watson trees in [7, Theorem 1.5]. We note that the exact form of the on-diagonal part of the limiting averaged heat kernel is a simple consequence of the scale invariance property (1.8). Moreover, the result at (1.12) improves part of [9, Theorem 4.4], where it was shown that $n^{d_f/d_w}\mathbf{E}\tilde{p}_n^{\mathcal{U}}(0,0)$ is bounded above and below by constants.

Theorem 1.7. *It holds that*

$$\left(n^{d_f/d_w}\tilde{p}^{\mathcal{U}}_{\lfloor tn\rfloor}(0,0)\right)_{t>0} \xrightarrow{d} \left(p_t^{\mathcal{T}}(\rho_{\mathcal{T}}, \rho_{\mathcal{T}})\right)_{t>0}$$

in distribution with respect to the topology of uniform convergence on compact subsets of $(0,\infty)$*, and moreover,*

$$\left(n^{d_f/d_w}\mathbf{E}\tilde{p}^{\mathcal{U}}_{\lfloor tn\rfloor}(0,0)\right)_{t>0} \to \left(\mathbf{E}p_t^{\mathcal{T}}(\rho_{\mathcal{T}}, \rho_{\mathcal{T}})\right)_{t>0} = \left(Ct^{-d_f/d_w}\right)_{t>0} \tag{1.12}$$

in the same topology, where $C\in(0,\infty)$ *is a constant.*

We next turn our attention to the off-diagonal part of the heat kernel. Whilst it is natural to ask whether the scaling limit of (1.12) can be extended to include the off-diagonal part, we recall that $\phi_{\mathcal{T}}$ is not a bijection (see [4, Theorem 1.3]), and so one cannot *a priori* assume that the limit of $n^{d_f/d_w}\mathbf{E}\tilde{p}^{\mathcal{U}}_{\lfloor tn\rfloor}(0,[xn^{1/\kappa d_w}])$ (where we write $[xn^{1/\kappa d_w}]$ for the closest lattice point to $xn^{1/\kappa d_w}$) can be written as $\mathbf{E}p_t^{\mathcal{T}}(\rho_{\mathcal{T}}, \phi_{\mathcal{T}}^{-1}(x))$, or indeed that this latter expectation is well-defined. This being the case, the following result is presented in terms of the density of the embedded process $\phi_{\mathcal{T}}(X^{\mathcal{T}})$, where $X^{\mathcal{T}}$ is the canonical Brownian motion on the limiting space; we note $\phi_{\mathcal{T}}(X^{\mathcal{T}})$ is not Markov under the annealed law (or strong Markov under the quenched law, see Remark 8.3 below). Nevertheless, as we will show, $\phi_{\mathcal{T}}^{-1}$ is well defined except on a set of Lebesgue measure zero, and so the averaged density of $\phi_{\mathcal{T}}(X^{\mathcal{T}})$ is in fact given by the expression $\mathbf{E}p_t^{\mathcal{T}}(\rho_{\mathcal{T}}, \phi_{\mathcal{T}}^{-1}(x))$. The key additional input to the proof of this result is an equicontinuity property for the discrete heat kernel under scaling (see Proposition 8.1), which in turn depends on our estimate for the probability of seeing long paths in the uniform spanning tree (see Theorem 1.6).

Theorem 1.8. *For each* $t\in(0,\infty)$, $\phi_{\mathcal{T}}(X_t^{\mathcal{T}})$ *admits a continuous probability density* $q_t = (q_t(x))_{x\in\mathbb{R}^2}$ *under the annealed probability law* $\mathbf{P}\cdot P_0^{\mathcal{U}}$*, so that*

$$\mathbf{E}\left(P_0^{\mathcal{U}}(\phi_{\mathcal{T}}(X_t^{\mathcal{T}})\in B)\right) = \int_B q_t(x)dx$$

for all Borel $B\subseteq\mathbb{R}^2$*. The functions* $(q_t)_{t>0}$ *satisfy the following.*
(a) There exists a constant $C\in(0,\infty)$ *such that*

$$|q_t(x)-q_t(y)|\leq Ct^{-d_f/2d_w}|x-y|^{\kappa/2}, \qquad \forall x,y\in\mathbb{R}^2, t>0.$$

(b) For any $\lambda > 0$ and $\theta \in [0, 2\pi)$, it holds that

$$\left(\lambda^{d_f/d_w} q_{t\lambda}\left(\lambda^{\frac{1}{\kappa d_w}} R_\theta x\right)\right)_{x\in\mathbb{R}^2} = (q_t(x))_{x\in\mathbb{R}^2}.$$

(c) For each $t \in (0, \infty)$, it holds that

$$\left(n^{d_f/d_w} \mathbf{E}\tilde{p}^{\mathcal{U}}_{\lfloor tn\rfloor}(0, [xn^{\frac{1}{\kappa d_w}}])\right)_{x\in\mathbb{R}^2} \to (q_t(x))_{x\in\mathbb{R}^2}$$

uniformly on compact subsets of $\mathbb{R}^2$.
(d) For each $t \in (0, \infty)$ and $x \in \mathbb{R}^2$, it holds that

$$q_t(x) = \mathbf{E}\left(p^{\mathcal{T}}_t(\rho_{\mathcal{T}}, \phi^{-1}_{\mathcal{T}}(x))\right).$$

Remark 1.9. By (b) there exists a continuous function $f : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$q_t(x) = t^{-d_f/d_w} f(|x|t^{-1/\kappa d_w}).$$

As an almost immediate corollary of Theorems 1.4 and 1.8, we obtain the following.

Corollary 1.10. *There exist constants $c_1, c_2, c_3, c_4 \in (0, \infty)$ and $\theta_1, \theta_2 \in (0, 1)$ such that the averaged density of $\phi^{\mathcal{T}}(X^{\mathcal{T}}_t)$, as given by Theorem 1.8, satisfies: for every $x \in \mathbb{R}^2$ and $t > 0$,*

$$c_1 t^{-d_f/d_w} \exp\left\{-c_2\left(\frac{|x|^{\kappa d_w}}{t}\right)^{\frac{\theta_1}{d_w-1}}\right\} \le q_t(x) \le c_3 t^{-d_f/d_w} \exp\left\{-c_4\left(\frac{|x|^{\kappa d_w}}{t}\right)^{\frac{\theta_2}{d_w-1}}\right\}.$$

As a further consequence of Theorem 1.4, together with the the estimate on the probability of seeing long paths of Theorem 1.6, we obtain the following upper bounds on the averaged behaviour of the distance travelled by $X^{\mathcal{U}}$ up to a given time, both in terms of the Euclidean and the intrinsic distances. Bounds for $\mathbf{E}\big(E^{\mathcal{U}}_0\big(d_{\mathcal{U}}(0, X^{\mathcal{U}}_n)^p\big)$ were considered in [9, Theorem 4.6], but the upper bound there has an additional term $(\log n)^{cp}$.

Corollary 1.11. *For every $p > 0$, it holds that for $n \ge 1$,*

$$c'_p n^{p/\kappa d_w} \le \mathbf{E}\Big(E^{\mathcal{U}}_0 |X^{\mathcal{U}}_n|^p\Big) \le c_p n^{p/\kappa d_w},$$
$$c'_p n^{p/d_w} \le \mathbf{E}\Big(E^{\mathcal{U}}_0 (d_{\mathcal{U}}(0, X^{\mathcal{U}}_n)^p\Big) \le c_p n^{p/d_w}.$$

It follows from the argument used to establish the random walk convergence result of [4, Theorem 1.4] that, under the averaged distribution, not only do we have (1.11), but also $(n^{-1/\kappa d_w}|X^{\mathcal{U}}_{\lfloor tn\rfloor}|)_{t\ge0} \xrightarrow{d} (|\phi_{\mathcal{T}}(X^{\mathcal{T}}_t)|)_{t\ge0}$ and $(n^{-1/d_w} d_{\mathcal{U}}(0, X^{\mathcal{U}}_{\lfloor tn\rfloor}))_{t\ge0} \xrightarrow{d} (d_{\mathcal{T}}(\rho_{\mathcal{T}}, X^{\mathcal{T}}_t))_{t\ge0}$. Combining this with the integrability given by Corollary 1.11 we obtain the following convergence result.

Corollary 1.12. *(a) For every $p > 0$, it holds that*

$$\left(n^{-p/\kappa d_w}\mathbf{E}\left(E^{\mathcal{U}}_0\left|X^{\mathcal{U}}_{\lfloor tn\rfloor}\right|^p\right)\right)_{t\ge0} \to \left(\mathbf{E}\left(E^{\mathcal{T}}_{\rho_{\mathcal{T}}}\left(\left|\phi_{\mathcal{T}}(X^{\mathcal{T}}_t)\right|^p\right)\right)\right)_{t\ge0} = \left(C_p t^{p/\kappa d_w}\right)_{t\ge0}$$

with respect to the topology of uniform convergence on compact subsets of $[0,\infty)$*, where* $C_p \in (0,\infty)$ *is a constant depending only upon* p*.*
(b) For every $p>0$*, it holds that*

$$\left(n^{-p/d_w}\mathbf{E}\left(E_0^{\mathcal{U}}\left(d_{\mathcal{U}}\left(0,X^{\mathcal{U}}_{\lfloor tn\rfloor}\right)^p\right)\right)\right)_{t\geq 0}\to\left(\mathbf{E}\left(E_{\rho_{\mathcal{T}}}^{\mathcal{T}}\left(d_{\mathcal{T}}\left(\rho_{\mathcal{T}},X_t^{\mathcal{T}}\right)^p\right)\right)\right)_{t\geq 0}=\left(C_p t^{p/d_w}\right)_{t\geq 0}$$

with respect to the topology of uniform convergence on compact subsets of $[0,\infty)$*, where again* $C_p \in (0,\infty)$ *is a constant depending only upon* p*.*

The remainder of the article is organised as follows. In Section 2 we review and refine some previous estimates for LERWs and the two-dimensional UST, proving the upper bound of Theorem 1.6(a) and Theorem 1.6(b) in particular. Section 3 provides an approach to showing that particular anomalous paths occur within the UST. This allows us to check the remaining part of Theorem 1.6, as well as the volume and heat kernel fluctuation results of Theorem 1.1 and Corollary 1.2 respectively, which will be done in Section 4. Section 5 adapts results of [4] concerning the structure of the UST to the case where we condition on a particular path being present, and these preliminary statements are then applied in Section 6 to deduce the heat kernel bounds of Theorem 1.4. Then, in Section 7, we confirm the failure of the elliptic Harnack inequality, as stated in Corollary 1.3. And, in Section 8, we apply the random walk scaling limit result of [4] in conjunction with the estimates of this article to deduce Theorems 1.7 and 1.8, as well as Corollaries 1.10 – 1.12. Finally, we postpone to the appendix the proofs of some estimates from Section 2 that are relatively close variations on the proofs of the corresponding results in [8]. NB. We will often use a continuous variable in places where a discrete one is required; in this case we implicitly mean that the floor of the relevant variable should be considered.

2 Loop erased random walk and the UST

This section contains some refinements of previous estimates on the geometry of the UST and the behaviour of the LERW. The key input we need for the averaged heat kernel upper bound (Proposition 2.9) is a relatively straightforward adaptation of [4, Proposition 2.10], adding resistance estimates to the volume estimates of the latter result. We also set out some new results, which include the upper bounds of Theorem 1.6.

We begin by introducing some notation for paths and operations on paths. A *path* γ is a (finite or infinite) sequence of adjacent vertices in $\mathbb{Z}^2$, i.e. $\gamma=(\gamma_0,\gamma_1,\dots)$ with $\gamma_{i-1}\sim\gamma_i$, where for $x,y\in\mathbb{Z}^2$ we write $x\sim y$ if $|x-y|=1$. Given a set $A\subseteq\mathbb{Z}^2$, we define $\tau_A=\min\{i\geq 0:\gamma_i\notin A\}$, and set $\mathcal{E}_A(\gamma)=(\gamma_0,\dots,\gamma_{\tau_A})$. Given a finite path γ, we write $\mathcal{L}(\gamma)$ for the chronological loop erasure of γ, see [23, 25].

We now recall Wilson's algorithm, see [36]. For $x\in\mathbb{Z}^2$ let S^x be a simple random walk (SRW) on $\mathbb{Z}^2$ started at x; we take $(S^x)_{x\in\mathbb{Z}^2}$ to be independent. Write $\mathbb{Z}^2$ as a sequence $\{z_0,z_1,z_2,\dots\}$, and define a sequence of trees as follows:

$$\begin{aligned}\mathcal{U}_0&=\{z_0\},\\ \mathcal{U}_i&=\mathcal{U}_{i-1}\cup\mathcal{L}(\mathcal{E}_{\mathcal{U}_{i-1}^c}(S^{z_i})),\quad i\geq 1,\\ \mathcal{U}&=\cup_i\mathcal{U}_i.\end{aligned}\tag{2.1}$$

By [36], the random tree $\mathcal{U}$ has the law of the UST. It follows that the law of $\mathcal{U}$ does not depend on the particular sequence (z_i). In fact, more is true: the z_i can be chosen adaptively as a function of $\mathcal{U}_{i-1}$. However, if we use independent (S^x) as above, then the final tree $\mathcal{U}$ depends as a random variable on both the random walks and the sequence (z_i). To circumvent this, for a finite graph Wilson [36] defined a family of random variables (called 'stacks') that enable one to define (non-independent) random walks $(\widetilde{S}^x)_{x\in\mathbb{Z}^2}$, and in this setup the final tree $\mathcal{U}$ does not depend on (z_i) for $i \geq 1$. (Note though that the random walk $\widetilde{S}^{(z_n)}$ does depend on the sequence $(z_1,\dots z_{n-1})$.) It is straightforward to check that this also holds with probability 1 for a recurrent graph, and it will sometimes be useful for us to apply this construction.

We write $L(x,\infty)$ for the loop-erased random walk from x to infinity; this is the weak limit as $m \to \infty$ of $\mathcal{L}(\mathcal{E}_{B_m(x)}(S^x))$. By Wilson's algorithm $L(x,\infty)$ has the same law as the $\gamma(x,\infty)$, the unique injective path from x to infinity in $\mathcal{U}$. (See [11, Proposition 14.1], for example.) We moreover write $\gamma_x = \gamma(x,\infty)$, $\gamma_x[i]$ for the ith point on γ_x, define the segment of the path γ_x between its ith and jth points by $\gamma_x[i,j] = (\gamma_x[i], \gamma_x[i+1], \dots, \gamma_x[j])$, and define $\gamma_x[i,\infty)$ in a similar fashion. Furthermore, we let $\tau_{y,r}(\gamma_x) = \min\{i \geq 0 : \gamma_x[i] \notin B_\infty(y,r)\}$; whenever we use notation such as $\gamma_x[\tau_{y,r}]$, the exit time $\tau_{y,r}$ will always be for the path γ_x. For $x, y \in \mathbb{Z}^2$, we introduce the 'Schramm distance' on $\mathcal{U}$ (after [35]) by setting

$$d^{\mathcal{S}}_{\mathcal{U}}(x,y) := \operatorname{diam}(\gamma(x,y)),$$

where the right-hand side is the diameter of $\gamma(x,y)$ (considered as a subset of $\mathbb{Z}^2$) with respect to d_∞.

In the next sequence of results of this section we collect and refine some properties of loop-erased random walks from [4, 8, 9, 33]. In the following result, we write $\partial B_\infty(0,r)$ for the outer boundary of $B_\infty(0,r)$, i.e. those vertices of $\mathbb{Z}^2 \backslash B_\infty(0,r)$ that have as a neighbour a vertex in $B_\infty(0,r)$.

Lemma 2.1. *Let $\theta > 1$, $n \geq 1$, and suppose that D_1, D_2 are subsets of $\mathbb{Z}^2$ with $B_\infty(0,\theta n) \subseteq D_1 \cap D_2$. There exists a constant $c_1 = c_1(\theta)$ such that if γ is a self-avoiding path from 0 to $\partial B_\infty(0,n)$ then*

$$\mathbf{P}\left(\mathcal{E}_{B_\infty(0,n)}(\mathcal{L}(\mathcal{E}_{D_1}(S^0))) = \gamma\right) \leq c_1 \mathbf{P}\left(\mathcal{E}_{B_\infty(0,n)}(\mathcal{L}(\mathcal{E}_{D_2}(S^0))) = \gamma\right).$$

If for $i=1$ or $i=2$ one has $D_i = \mathbb{Z}^d$, then $\mathcal{L}(\mathcal{E}_{D_i}(S^0))$ should be taken to be $L(0,\infty)$.

Proof. See [33, Proposition 4.4] for the result when $\theta \geq 4$. Checking the proof of the latter result, one finds that the result as stated above holds for any $\theta > 1$. (In [33] the emphasis was on the fact that one can take $C(\theta) = 1 + c(\log\theta)^{-1}$ for large θ.) □

Definition 2.2. Let $D \subseteq \mathbb{Z}^2$. Let $\lambda > 1$, $1 \leq r_1 \leq r_2$. We say that D is (λ, r_1, r_2)-*regular* if we have for each $x, y \in D$,

$$\begin{aligned}
\lambda^{-1} d^{\mathcal{S}}_{\mathcal{U}}(x,y)^\kappa \leq d_{\mathcal{U}}(x,y) &\leq \lambda d^{\mathcal{S}}_{\mathcal{U}}(x,y)^\kappa, && \text{when } r_1 \leq d^{\mathcal{S}}_{\mathcal{U}}(x,y) \leq r_2,\\
d_{\mathcal{U}}(x,y) &\leq \lambda r_1^\kappa, && \text{when } d^{\mathcal{S}}_{\mathcal{U}}(x,y) \leq r_1,\\
d_{\mathcal{U}}(x,y) &\geq \lambda^{-1} r_2^\kappa, && \text{when } d^{\mathcal{S}}_{\mathcal{U}}(x,y) \geq r_2.
\end{aligned}$$

It is straightforward to check the following.

Lemma 2.3. *(a) If D is (λ, r_1, r_2)-regular and $r_2 \geq \lambda^{2/\kappa} r_1$, then*

$$\lambda^{-1} d_{\mathcal{U}}(x,y) \leq d^{\mathcal{S}}_{\mathcal{U}}(x,y)^\kappa \leq \lambda d_{\mathcal{U}}(x,y)^\kappa, \qquad \textit{when } \lambda r_1^\kappa \leq d_{\mathcal{U}}(x,y) \leq \lambda^{-1} r_2^\kappa,$$

$$d_{\mathcal{U}}^{\mathcal{S}}(x,y)^{\kappa} \le \lambda r_1^{\kappa}, \qquad \textit{when } d_{\mathcal{U}}(x,y) \le \lambda r_1^{\kappa},$$
$$d_{\mathcal{U}}^{\mathcal{S}}(x,y) \ge \lambda^{-1} r_2^{\kappa}, \qquad \textit{when } d_{\mathcal{U}}(x,y) \ge \lambda^{-1} r_2^{\kappa}.$$

(b) Let $\mathcal{T} \subset \mathbb{Z}^2$ be a tree, $w \notin \mathcal{T}$, $\gamma(w,\mathcal{T})$ be a self-avoiding path from w to $\mathcal{T}$, and let $\mathcal{T}' = \mathcal{T} \cup \gamma(w,\mathcal{T})$. If $\mathcal{T}$ and $\gamma(w,\mathcal{T})$ are (λ, r_1, r_2)-regular, then $\mathcal{T}'$ is $(2^{\kappa}\lambda, 2r_1, r_2)$-regular.

The next result is a consequence of [4, Proposition 2.8].

Lemma 2.4. *Let $n \ge 1$ and $\lambda \ge \lambda_0$. Then*

$$\mathbf{P}\left(B_\infty(0,n) \textit{ is } (\lambda, e^{-c_1\lambda^{1/2}}n, n)\textit{-regular}\right) \ge 1 - c_2 e^{-c_3\lambda^{1/2}}.$$

Lemma 2.5. *Let $\theta > 1$, $n \ge 1$, and suppose $D \subseteq \mathbb{Z}^2$ is such that $B_\infty(0,\theta n) \subseteq D$. It then holds that there exist constants $c_i = c_i(\theta)$ such that for $\lambda > \lambda_0$, where λ_0 is some large, finite constant,*

$$\mathbf{P}\big(\mathcal{E}_{B_\infty(0,n)}(\mathcal{L}(\mathcal{E}_D(S^0))) \textit{ is } (\lambda, ne^{-c_1\lambda^{1/2}}, n)\textit{-regular}\big) \ge 1 - c_2 e^{-c_3\lambda^{1/2}}.$$

Proof. By Lemma 2.1 it is enough to prove this when $\mathcal{L}(\mathcal{E}_D(S^0))$ is replaced by $L(0,\infty)$. The bound then follows from Lemma 2.4 and Wilson's algorithm. □

Lemma 2.6. *There exists $\lambda_0 \ge 1$ and constant $c_1 > 0$ with the following properties. Let $n \ge 1$, $\lambda \ge \lambda_0$ and $x \in B_\infty(0, 3n/4)$, let π be a shortest path in $\mathbb{Z}^2$ between 0 and x, and set $A = \{y \in \mathbb{Z}^2 : d_\infty(y,\pi) \le n/8\}$. Then*

$$\mathbf{P}\big(\gamma(0,x) \subseteq A \textit{ and is } (\lambda, ne^{-c_1\lambda^{1/2}}, n)\textit{-regular}\big) \ge c_1.$$

Proof. Let G_1 be the event that $B_\infty(0,n)$ is $(\lambda, ne^{-c_1\lambda^{1/2}}, n)$-regular. Choose $k \ge 16$ and let y satisfy $d_\infty(0,y) = n/k$. Let $G_2 = \{\gamma(0,y) \subseteq B_\infty(0,n/8)\}$; by [9, Lemma 2.6] we have $\mathbf{P}(G_2^c) \le c_2 k^{-1/3}$; choose k so that $c_2 k^{-1/3} \le \frac{1}{2}$. Let S^x be a SRW started at x, and G_4 be the event that S^x makes a closed loop around 0 which separates 0 and y before it leaves A; we have $\mathbf{P}(G_4) \ge c_3 > 0$. Then $\mathbf{P}(G_2 \cap G_4) \ge \frac{1}{2}c_3$. We now choose λ_0 large enough so that $\mathbf{P}(G_1^c) \le \frac{1}{4}c_3$ and hence writing $G = G_1 \cap G_2 \cap G_4$ we have $\mathbf{P}(G) \ge \frac{1}{4}c_3$. On the event G the SRW S^x hits $\gamma(0,y)$ before it exits A, so $\gamma(0,x) \subseteq A$. Since $B_\infty(0,n)$ is regular, so is the path $\gamma(0,x)$. □

Theorem 2.7. *Let $n \ge 1$, and suppose $D \subseteq \mathbb{Z}^2$ is such that $B_\infty(0,n) \subseteq D$. If $D \ne \mathbb{Z}^2$, set $L_{n,D} = \mathcal{E}_{B_\infty(0,n)}(\mathcal{L}(\mathcal{E}_D(S^0)))$, and set $L_{n,\mathbb{Z}^2} = \mathcal{E}_{B_\infty(0,n)}(L(0,\infty))$. It then holds that there exist constants c_i such that, for $\lambda \ge 1$,*

$$\mathbf{P}\left(|L_{n,D}| < \lambda^{-1} n^{\kappa}\right) \le c_1 e^{-c_2\lambda^{1/(\kappa-1)}}. \tag{2.2}$$

In particular, for any $x, y \in \mathbb{Z}^2$,

$$\mathbf{P}\left(d_{\mathcal{U}}(x,y) < \lambda^{-1} d_\infty(x,y)^{\kappa}\right) \le c_1 e^{-c_2\lambda^{1/(\kappa-1)}}. \tag{2.3}$$

Proof. A bound with exponent $\lambda^{4/5-\varepsilon}$ is given in [8, Theorem 6.7], and with some more care one can obtain (2.2) by essentially the same arguments – see the Appendix for details. Taking $x = 0$, $D = \mathbb{Z}^2 \backslash \{y\}$ and $n = \lfloor d_\infty(0,y) \rfloor$, we have $d_{\mathcal{U}}(x,y) = |\mathcal{L}(\mathcal{E}_D(S^0))| \ge |L_{n,D}|$, which (with translation invariance) gives (2.3). □

To state our next result, Proposition 2.9, we need to introduce some more notation and basic definitions. Specifically, we write R_{eff} for the effective resistance on $\mathcal{U}$ considered as an electrical network with unit conductances along each edge. (See [3, 32] for background.) We recall from (1.2) and (1.5) the definition of d_f and d_w.

Definition 2.8. We say a ball $B_{\mathcal{U}}(x,r)$ is λ-good if we have the following:
(1) $\lambda^{-1}r^{d_f} \leq |B_{\mathcal{U}}(x,r)| \leq \lambda r^{d_f}$,
(2) $R_{\text{eff}}(x, B_{\mathcal{U}}(x,r)^c) \geq r/\lambda$,
(3) $B_{\mathcal{U}}(x,r) \subseteq B_\infty(x, \lambda r^{1/\kappa})$.

We moreover define

$$F_1(\lambda,n) = \{B_{\mathcal{U}}(x,r) \text{ is } \lambda\text{-good for all } x \in B_\infty(0,n),\ e^{-\lambda^{1/40}}n^\kappa \leq r \leq n^\kappa\}, \tag{2.4}$$

and note that on the event $F_1(\lambda,n)$ we have $B_{\mathcal{U}}(x,n^\kappa) \subseteq B_\infty(x,\lambda n)$ for all $x \in B_\infty(0,n)$.

Proposition 2.9. *There exist constants c_1, c_2, λ_0 such that*

$$\mathbf{P}\left(F_1(\lambda,n)^c\right) \leq c_1 \exp(-c_2\lambda^{1/16}), \qquad \forall n \geq 1,\ \lambda \geq \lambda_0.$$

Proof. The proof below is a modification of that of [4, Proposition 2.10]. Let $r = ne^{-\lambda^{1/32}}$, and assume first that n is large enough so that $r \geq \lambda$. Let $J(x,\lambda)$ be the set of those $r \in [1,\infty)$ such that the three conditions in Definition 2.8 hold.

Set $R_1 = n$, $R_2 = re^{\lambda^{1/16}}$, and let D_2 be as in [4, Proposition 2.9], with $|D_2| \leq c\lambda^4 e^{2\lambda^{1/16}}$. Set $m_0 := \inf\{m : m \geq e^{\kappa\lambda^{1/32}}\}$, and let $E(r,\lambda) := \cap_{x\in D_2} \cap_{m=1}^{m_0+1} \{mr^\kappa \in J(x,\lambda)\}$. A simple union bound allows us to deduce from [9, Theorem 1.1(a) and Proposition 4.2(a)] that

$$\mathbf{P}\left(E(r,\lambda)^c\right) \leq |D_2| e^{\kappa\lambda^{1/32}} ce^{-c'\lambda^{1/9}} \leq Ce^{-c''\lambda^{1/9}}.$$

Let $A_5(r,\lambda)$ be the event given in the statement of [4, Proposition 2.9]; we have $\mathbf{P}(E(r,\lambda)^c \cup A_5(r,\lambda)^c) \leq C\exp(-c\lambda^{1/16})$. Moreover, if $E(r,\lambda) \cap A_5(r,\lambda)$ holds, then, by [4, (2.14) and the last display in the proof of Proposition 2.9], for each $x \in B_\infty(0,n)$, there exists $y = y_x \in D_2$ with $d_{\mathcal{U}}(x,y) \leq 4r^\kappa/\lambda^{1/4}$ and $d_\infty(x,y) \leq 2r/\lambda$. Choosing λ_0 large enough, we have $d_{\mathcal{U}}(x,y_x) \leq r^\kappa$ and $d_\infty(x,y_x) \leq r$.

Now, suppose $E(r,\lambda) \cap A_5(r,\lambda)$ holds, and let $x \in B_\infty(0,n)$, and $s \in [4\lambda r^\kappa, n^\kappa]$. We will prove that $s \in J(x,2\lambda)$ by verifying the conditions (1)–(3) in Definition 2.8. Choose $m \in [4\lambda, m_0+1]$ so that $(m-1)r^\kappa \leq s \leq mr^\kappa$. It then holds that

$$|B_{\mathcal{U}}(x,s)| \leq |B_{\mathcal{U}}(y_x,(m+1)r^\kappa)| \leq \lambda((m+1)/(m-1))^{d_f} s^{d_f} \leq 2\lambda s^{d_f}.$$

Similarly, $|B_{\mathcal{U}}(x,s^\kappa)| \geq (2\lambda)^{-1}s^{d_f}$, so that the volume bound (1) holds. Next, applying the triangle inequality for resistances (and the fact that $R_{\text{eff}}(x,y_x) = d_{\mathcal{U}}(x,y_x)$),

$$\begin{aligned} R_{\text{eff}}\left(x, B_{\mathcal{U}}(x,s)^c\right) &\geq R_{\text{eff}}\left(y_x, B_{\mathcal{U}}(x,s)^c\right) - d_{\mathcal{U}}(x,y_x) \\ &\geq R_{\text{eff}}(y_x, B_{\mathcal{U}}(y_x, s-r^\kappa)^c) - r^\kappa \\ &\geq \lambda^{-1}(m-2)r^\kappa - r^\kappa \geq (2\lambda)^{-1}s, \end{aligned}$$

which gives (2). Finally for (3) we have

$$B_{\mathcal{U}}(x,s) \subseteq B_{\mathcal{U}}\left(y_x,(m+1)r^\kappa\right) \subseteq B_\infty(y_x, \lambda(m+1)^{1/\kappa}r) \subseteq B_\infty(x, 2\lambda s),$$

where the last inclusion holds since $r(1+\lambda(m+1)^{1/\kappa}) \leq 2\lambda(m-1)^{1/\kappa}r \leq 2\lambda s$. Thus, for $\lambda \geq \lambda_0$ with λ_0 suitably large, $E(r,\lambda) \cap A_5(r,\lambda) \subseteq F_1(2\lambda, n)$, and this completes the proof of the proposition in the case when $r \geq \lambda$.

Finally, suppose $r = ne^{-\lambda^{1/32}} < \lambda$, i.e. $n \leq \lambda e^{\lambda^{1/32}}$. A union bound then gives

$$\mathbf{P}\left(F_1(\lambda,n)^c\right) \leq \sum_{x \in B_n(0)} \sum_{s=1}^{\lceil n^\kappa \rceil} \mathbf{P}\left(s \notin J(x,\lambda)\right) \leq cn^{2+\kappa}e^{-c'\lambda^{1/9}} \leq Ce^{-c''\lambda^{1/9}},$$

where the last inequality is again an application of [9, Theorem 1.1(a) and Proposition 4.2(a)]. This is enough to complete the proof. □

A further observation that will be useful in the proof of the averaged heat kernel upper bound is the following.

Lemma 2.10. *Suppose that $F_1(\lambda,n)$ occurs, and let $x,y \in B_\infty(0,n)$. It then holds that $d_\infty(x,y) \in [\lambda e^{-\lambda^{1/40}/\kappa} n, n]$ implies $d_{\mathcal{U}}(x,y) \geq \lambda^{-\kappa} d_\infty(x,y)^\kappa$.*

Proof. Let $r < \lambda^{-\kappa} d_\infty(x,y)^\kappa$, so that $y \notin B_\infty(x, \lambda r^{1/\kappa})$. The condition on $d_\infty(x,y)$ implies that we can choose r so that $r \in [e^{-\lambda^{1/40}} n^\kappa, n^\kappa]$, and thus property (3) in the definition of a good ball implies that $y \notin B_{\mathcal{U}}(x,r)$, and so $d_{\mathcal{U}}(x,y) > r$. □

The next few results will lead to the proof of Theorem 1.6(b), beginning with the case when x and y are neighbours in $\mathbb{Z}^2$.

Proposition 2.11. *There exist constants c_i, q such that*

$$c_1 s^{-(2-\kappa)/\kappa} \leq \mathbf{P}\left(d_{\mathcal{U}}(0,e_1) \geq s\right) \leq c_2 (\log s)^q s^{-(2-\kappa)/\kappa},$$

for all $s \geq 2$, where $e_1 = (1,0)$.

We start with a proof of the lower bound. For this, it will be convenient to introduce $\mathcal{U}'$, the dual of $\mathcal{U}$. This is the graph with vertex set $(\mathbb{Z}+\frac{1}{2})^2$ whose edges are precisely those nearest neighbour edges that do not cross an edge of $\mathcal{U}$. It is known that $\mathcal{U}'$ has the same law as $\mathcal{U}$ (see [11]). We set $0' = (1/2, 1/2)$ for the root of the dual graph.

Proof of the lower bound of Proposition 2.11. Applying [4, Proposition 2.8] and Proposition 2.9, for $r \geq 1$, $\lambda \geq \lambda_0$, we can find an event $G_0(\lambda,r)$ with $\mathbf{P}(G_0(\lambda,r)^c) \leq c_1 e^{-c_2\lambda^{1/40}}$ such that if this event holds then we have $B_\infty(0,R)$ is $(c_3\lambda^{1/20}, r, R)$-regular and also $F_1(\lambda, R)$ holds, where $R := re^{\lambda^{1/40}/\kappa}$. Let $G_0'(\lambda,r)$ be the corresponding event for the dual graph, and define $G(\lambda,r) = G_0(\lambda,r) \cap G_0'(\lambda,r)$.

On $G(\lambda,r)$, if γ' is the unique injective path from $0'$ to infinity in $\mathcal{U}'$, then we have that the section of γ' from its last exit from $B_\infty(0',r)$ to $B_\infty(0',2r)^c$ has length greater than $c\lambda^{-1/20}r^\kappa$. (We assume that λ is chosen large enough so that $e^{\lambda^{1/40}} \geq 2$.) Denote by γ_r' this section of γ', and let $\{x_1', x_2'\}$ be an edge crossed by γ_r'. If $\{x_1, x_2\}$ is the dual edge to $\{x_1', x_2'\}$, then it must be the case that $d_{\mathcal{U}}^S(x_1,x_2) \geq r$, and thus $d_{\mathcal{U}}(x_1,x_2) \geq c\lambda^{-1/20}r^\kappa$.

Finally, for $x_1 \sim x_2$ (in $\mathbb{Z}^2$), set $F(x_1,x_2) = \{d_{\mathcal{U}}(x_1,x_2) \geq c\lambda^{-1/20}r^\kappa\}$. The argument above gives that

$$\sum_{x_1 \in B_\infty(0,2r)} \sum_{x_2 \sim x_1} \mathbf{1}_{F(x_1,x_2)} \geq \mathbf{1}_{G(\lambda,r)} \sum_{x_1 \in B_\infty(0,2r)} \sum_{x_2 \sim x_1} \mathbf{1}_{F(x_1,x_2)} \geq c\lambda^{-1/20} r^\kappa \mathbf{1}_{G(\lambda,r)}.$$

Hence taking expectations

$$\left(1 - c_1 e^{-c_2\lambda^{1/40}}\right) c\lambda^{-1/20} r^\kappa \le \sum_{x_1 \in B_\infty(0,2r)} \sum_{x_2 \sim x_1} \mathbf{P}(F(x_1,x_2)) \le c' r^2 \mathbf{P}(F(0,e_1)),$$

and the result follows by a simple reparameterisation. □

A similar idea gives an upper bound. We begin by looking at the size of the finite component rooted at a vertex. In particular, for $x \in \mathbb{Z}^2$, this is defined to be the set $A_x = \{y : x \in \gamma(y,\infty)\}$. We also define the depth of A_x by $\mathrm{dep}(A_x) = \max\{d_{\mathcal{U}}(x,y) : y \in A_x\}$.

Lemma 2.12. *For $\lambda \in \mathbb{N}$, we have that*

$$\mathbf{P}(\mathrm{dep}(A_0) = \lambda) \le c_1 \lambda^{-2/\kappa}, \tag{2.5}$$

$$\mathbf{P}(\mathrm{dep}(A_0) \ge \lambda) \le c_2 \lambda^{-(2-\kappa)/\kappa}. \tag{2.6}$$

Proof. Suppose that the event $G(\lambda,r)$ defined in the proof of the lower bound holds, and again set $R := re^{\lambda^{1/40}/\kappa}$. NB. We suppose that λ is large enough so that $(32\lambda)^{1/\kappa} r \le R$. Let $x \in B_\infty(0,r)$, and suppose that $\mathrm{dep}(A_x) = s$, where $r^\kappa \le s \le (16\lambda)^{-1} R^\kappa$. We then claim that $A_x \subseteq B_\infty(0,R/2)$. Indeed, suppose $y \in A_x \cap B_\infty(0,R/2)^c$, and let y' be the first point on $\gamma(y,x)$ which is in $B_\infty(0,R)$. Since $r \le \frac{1}{4}R \le d_\infty(x,y') \le d_{\mathcal{U}}^S(x,y')$, we must then have $s = \mathrm{dep}(A_x) \ge d_{\mathcal{U}}(x,y') \ge \min\{\lambda^{-1}R^\kappa, \lambda^{-1} d_{\mathcal{U}}^S(x,y')^\kappa\} \ge \lambda^{-1} 4^{-\kappa} R^\kappa$, which is a contradiction. Now, there exists $y_x \in A_x$ such that $d_{\mathcal{U}}(x,y_x) = s$, and it must be the case that $y_x \in B_\infty(0,R/2)$. Thus the ball $B_{\mathcal{U}}(y_x,s)$ is λ-good, and we obtain

$$|A_x| \ge |B_{\mathcal{U}}(y_x,s)| \ge \lambda^{-1} \mathrm{dep}(A_x)^{2/\kappa}. \tag{2.7}$$

Next, let $s \in \mathbb{N} \cap [r^\kappa, 2r^\kappa]$, and set $\tilde{H}_s = \{x \in B_\infty(0,r) : \mathrm{dep}(A_x) = s\}$. By (2.7), we then have that

$$\mathbf{1}_{G(\lambda,r)} \sum_{x \in \tilde{H}_s} |A_x| \ge \lambda^{-1} s^{2/\kappa} |\tilde{H}_s| \mathbf{1}_{G(\lambda,r)}. \tag{2.8}$$

If $x \in \tilde{H}_s$ and $y \in A_x$, then $y \in B_\infty(0,R/2)$ and $d_{\mathcal{U}}(x,y) \le s$. Hence $d_\infty(x,y)^\kappa \le d_{\mathcal{U}}^S(x,y)^\kappa \le \max\{\lambda r^\kappa, \lambda d_{\mathcal{U}}(x,y)\} \le 2\lambda r^\kappa$, and so $y \in B_\infty(0,(1+(2\lambda)^{1/\kappa})r)$. Since the sets $(A_x)_{x \in \tilde{H}_s}$ are disjoint, it follows that $\mathbf{1}_{G(\lambda,r)} \sum_{x \in \tilde{H}_s} |A_x| \le c\lambda^{2/\kappa} r^2 \mathbf{1}_{G(\lambda,r)}$, and combining this with the estimate at (2.8) yields

$$|\tilde{H}_s| \mathbf{1}_{G(\lambda,r)} \le c\lambda^{1+2/\kappa} \mathbf{1}_{G(\lambda,r)}. \tag{2.9}$$

Finally, let $\Lambda_* = \inf\{\lambda \ge \lambda_0 : G(\lambda,r) \text{ holds}\}$, and note that $\mathbf{P}(\Lambda_* > \lambda) \le \mathbf{P}(G(\lambda,r)^c) \le c_1 e^{-c_2 \lambda^{1/40}}$. Thus Λ_* is almost-surely finite, and there exist finite constants c_p such that $\mathbf{E}(\Lambda_*^p) \le c_p$; note that these constants can be chosen not to depend on r. Hence, from (2.9), we deduce that, for $s \in \mathbb{N} \cap [r^\kappa, 2r^\kappa]$,

$$\mathbf{P}(\mathrm{dep}(A_0) = s) \le r^{-2} |B_\infty(0,r)| \mathbf{P}(\mathrm{dep}(A_0) = s) = r^{-2} \mathbf{E}(|\tilde{H}_s|) \le cr^{-2} \mathbf{E}(\Lambda_*^{1+2/\kappa}) \le cs^{-2/\kappa}.$$

This gives the bound (2.5), and the bound (2.6) readily follows. □

Proof of the upper bound of Proposition 2.11. Again, suppose that the event $G(\lambda,r)$ defined in the proof of the lower bound holds, and λ is chosen large enough so that $2 \le e^{\lambda^{1/40}/\kappa}$. Suppose that $d_{\mathcal{U}}(0,e_1) > 2^\kappa c_3\lambda^{1/20}r^\kappa$ (where c_3 is as in the definition of the aforementioned event). Note that the dual vertices enclosed by the path $\gamma(0,e_1)$ (combined with the edge from 0 to e_1) are all elements of the finite component of $\mathcal{U}'$ rooted at $0'$, which we denote $A'_{0'}$. On $G(\lambda,r)$, we have $d^S_{\mathcal{U}}(0,e_1) \geq 2r$, and so there exists a point $x_1 \in \gamma(0,e_1)$ such that $d_\infty(0,x_1) \geq 2r$. Let x_2 be a point on $\gamma(0,e_1)$ adjacent to x_1, and let $\{x'_1,x'_2\}$ be the edge dual to $\{x_1,x_2\}$. One of the points x'_1, x'_2 is in $A'_{0'}$; we call this point x'. Since $d^S_{\mathcal{U}'}(0',x') \geq r$ and $G(\lambda,r)$ holds, we thus obtain that $\mathrm{dep}(A'_{0'}) \geq c_3^{-1}\lambda^{-1/20}r^\kappa$. Hence $G(\lambda,r)\cap\{d_{\mathcal{U}}(0,e_1) \geq 2^\kappa c_3\lambda^{1/20}r^\kappa\} \subseteq G(\lambda,r)\cap\{\mathrm{dep}(A'_{0'}) \geq c_3^{-1}\lambda^{-1/20}r^\kappa\}$, and so, setting $\tilde{r} := c_3^{-1}\lambda^{-1/20}r^\kappa$ and $\lambda = c'(\log\tilde{r})^{40}$,

$$\begin{aligned}\mathbf{P}(d_{\mathcal{U}}(0,e_1) \geq \tilde{r}) &\leq \mathbf{P}(G(\log\tilde{r},r)^c) + \mathbf{P}(\mathrm{dep}(A'_{0'}) \geq 2^{-\kappa}c_3^{-2}(c')^{-1/10}(\log\tilde{r})^{-4}\tilde{r}^\kappa)\\ &\leq c_1e^{-c_2c'\log\tilde{r}} + c\tilde{r}^{-(2-\kappa)/\kappa}(\log\tilde{r})^{4(2-\kappa)/\kappa},\end{aligned}$$

which, taking c' suitably large, yields the desired result. □

Proof of Theorem 1.6(b). Suppose that the event $G(\lambda,r)$ defined in the proof of Proposition 2.11 holds, and write $R := re^{\lambda^{1/40}/\kappa}$. We will assume that we also have a parameter t that satisfies $4^\kappa c_3\lambda^{1/20} \leq t \leq c_3^{-1}\lambda^{-1/20}e^{\lambda^{1/40}}$. Let $d_\infty(0,x) = r$, and L be a shortest path in $\mathbb{Z}^2$ between 0 and x. If $d_{\mathcal{U}}(0,x) \geq tr^\kappa$, then $d^S_{\mathcal{U}}(0,x) \geq c_3^{-1/\kappa}\lambda^{-1/20\kappa}t^{1/\kappa}r$, and so there exists a point y on $\gamma(0,x)$ with $d_\infty(0,y) \geq c_3^{-1/\kappa}\lambda^{-1/20\kappa}t^{1/\kappa}r$. Let y' be a dual point with $d_\infty(y,y') = \frac{1}{2}$ which is separated from infinity by $\gamma(0,x)\cup L$. The path in the dual tree $\gamma'(y',\infty)$ must pass through L at a point z', and the length of the section of $\gamma'(y',\infty)$ inside $B_\infty(z',r)$ will be of length at least $c_3^{-1}\lambda^{-1/20}r^\kappa$. Moreover, for each vertex z' of $\gamma'(y',\infty)$ inside $B_\infty(z',r)$, it must be the case that $A'_{z'}$ has d_∞-diameter greater than $d_\infty(0,y') - 2r$, and so $\mathrm{dep}(A'_{z'}) \geq 2^{-\kappa}c_3^{-1}\lambda^{-1/20}tr^\kappa$. Now, let $H = \{d_{\mathcal{U}}(0,x) \geq tr^\kappa\}$ and $F(z') = \{\mathrm{dep}(A'_{z'}) \geq 2^{-\kappa}c_3^{-1}\lambda^{-1/20}tr^\kappa\}$. We then have that

$$\mathbf{1}_{H\cap G(\lambda,r)}\sum_{z'\in B_\infty(0',2r)}\mathbf{1}_{F(z')} \geq \mathbf{1}_{H\cap G(\lambda,r)}c_3^{-1}\lambda^{-1/20}r^\kappa.$$

So, by (2.6),

$$\begin{aligned}\mathbf{P}(H\cap G(\lambda,r)) &\leq c_3r^{-\kappa}\lambda^{1/20}\mathbf{E}\left(\mathbf{1}_{H\cap G(\lambda,r)}\sum_{z'\in B_\infty(0',2r)}\mathbf{1}_{F(z')}\right)\\ &\leq c_3r^{-\kappa}\lambda^{1/20}\sum_{z'\in B_\infty(0',2r)}\mathbf{P}(F(z'))\\ &\leq cr^{2-\kappa}\lambda^{1/20}(2^{-\kappa}c_3^{-1}\lambda^{-1/20}tr^\kappa)^{-(2-\kappa)/\kappa}\\ &= c\lambda^{1/10\kappa}t^{-(2-\kappa)/\kappa}.\end{aligned}$$

Hence, taking $\lambda = (\log t)^{41}$, the result follows similarly to the end of the previous proof. □

3 Controlling paths

In this section, we provide a general technique for estimating from below the probability of seeing a particular path configuration in the UST. This will enable us to estimate from below the probability of seeing especially short paths between given points, and so prove the lower bound in Theorem 1.6(a). These estimates will also be a key ingredient in establishing volume and heat kernel fluctuations for the UST, as we do in the subsequent section.

Let $x \in \mathbb{Z}^2$, and $m \geq 1$. A *scale m path* from 0 to x, π say, is a sequence of distinct vertices $0 = x_0, x_1, \dots, x_N = x$ such that $x_i \in (m\mathbb{Z})^2$ and the Euclidean (i.e. ℓ^2) distance between x_{i-1} and x_i is equal to m for each $i = 1, \dots, N-1$, and also $x_N \in B_m(x_{N-1})$, where we define

$$B_r(x) := B_\infty(x, r/2).$$

We write $|\pi| = N$ for the length of the path. Now, fix a path π of length N. The rest of this section is devoted to defining an event $F_m(x,\pi)$ with $\mathbf{P}(F_m(x,\pi)) \geq e^{-cN}$ such that on this event the path from 0 to x in the UST is contained in $\cup_{i=0}^N B_m(x_i)$ and (up to constants) has length Nm^κ.

To complete the program described in the previous paragraph, we will again appeal to Wilson's algorithm. As at the start of Section 2, let $(S^x)_{x\in\mathbb{Z}^2}$ be a collection of independent simple random walks on $\mathbb{Z}^2$, where S^x is started from x. Slightly modifying the algorithm at (2.1), we use these to construct the part of the UST containing both 0 and x via an iterative procedure. In particular, let k be some integer that will be fixed later. We begin our construction by taking $\mathcal{U}_0 = \gamma_0$ to be the loop-erasure of $S^{(0,\lfloor m/k \rfloor)}$ run until it first hits the origin $x_0 = 0$. We then continue as at (2.1): for $i \geq 1$, let $\mathcal{U}_i = \mathcal{U}_{i-1} \cup \gamma_i$, where γ_i is the loop-erasure of S^{x_i} run until it first hits $\mathcal{U}_{i-1}$. We will later use the notation x_i' to represent the unique point in $\mathcal{U}_{i-1} \cap \gamma_i$. From Wilson's algorithm, we obtain that the path from 0 to x in the graph tree $\mathcal{U}_N$ is distributed identically to the path from 0 to x in $\mathcal{U}$. For convenience, we will henceforth assume that $\mathcal{U}$ has been constructed by continuing with Wilson's algorithm from $\mathcal{U}_N$, and so this equality is almost-sure.

We next define a sequence of 'good' events G_i. Given $\lambda \geq 2$, which will also be chosen later, set $G_0 := \{\gamma_0 \subseteq B_m(0)\} \cap \{|\gamma_0| \leq \lambda m^\kappa\}$, where $|\gamma_0|$ is the number of elements of the path γ_0. To define G_i for $i = 1, \dots, N-1$, first let R_i be the $m\lambda^{-2} \times 2m\lambda^{-2}$ rectangle consisting of $B_{m\lambda^{-2}}(x_i)$ and the adjacent square of side-length $m\lambda^{-2}$ that is closest to x_{i-1}; this is the rectangle about x_i with a solid border shown in Figure 1. Moreover, let Q_i be the union of the $m \times \frac{m(1-\lambda^{-2})}{2}$ rectangle contained in $B_m(x_i)$ that is closest to x_{i-1}, the $m \times \frac{m(1-\lambda^{-2})}{2}$ rectangle contained in $B_m(x_{i-1})$ that is furthest from x_{i-2} (if $i = 1$, take this to be the rectangle closest to x_i), and $B_{m\lambda^{-2}}(x_{i-1})$; this is the dotted region shown in Figure 1. Note in particular Q_i has essentially two forms, depending on whether x_{i-2}, x_{i-1}, x_i are co-linear or not; these are the two configurations are shown in Figure 1. For $i = 1, \dots N-1$, we then set $G_i = \cap_{j=1}^3 G_i^j$, where:

- G_i^1 is the event that S^{x_i} exits R_i on the side closest to x_{i-1} – call this exit time τ_i, and also $S^{x_i}_{\tau_i+\cdot}$ hits γ_{i-1} before exiting Q_i;
- G_i^2 is the event that $|\gamma_i| \in [\lambda^{-1} m^\kappa, \lambda m^\kappa]$;
- G_i^3 is the event that $|\gamma_i \cap B_{3m\lambda^{-2}}(x_i)| \leq \lambda(3m\lambda^{-2})^\kappa = 3^\kappa \lambda^{-3/2} m^\kappa$.

Finally, we take

$$G_N := \{\gamma_N \subseteq B_m(x_{N-1}) \cup B_m(x_N)\} \cap \{|\gamma_N| \leq \lambda m^\kappa\},$$
$$F_m(x,\pi) := \cap_{i=0}^N G_i.$$

To highlight the relevance of $F_m(x,\pi) = \cap_{i=0}^N G_i$ to controlling path lengths, we note that on this event we have that

$$d_{\mathcal{U}}(0,x) \leq \sum_{i=1}^N d_{\mathcal{U}}(x_{i-1}, x_i) \leq \sum_{i=0}^N |\gamma_i| \leq \lambda(N+1) m^\kappa \leq 2\lambda N m^\kappa. \tag{3.1}$$

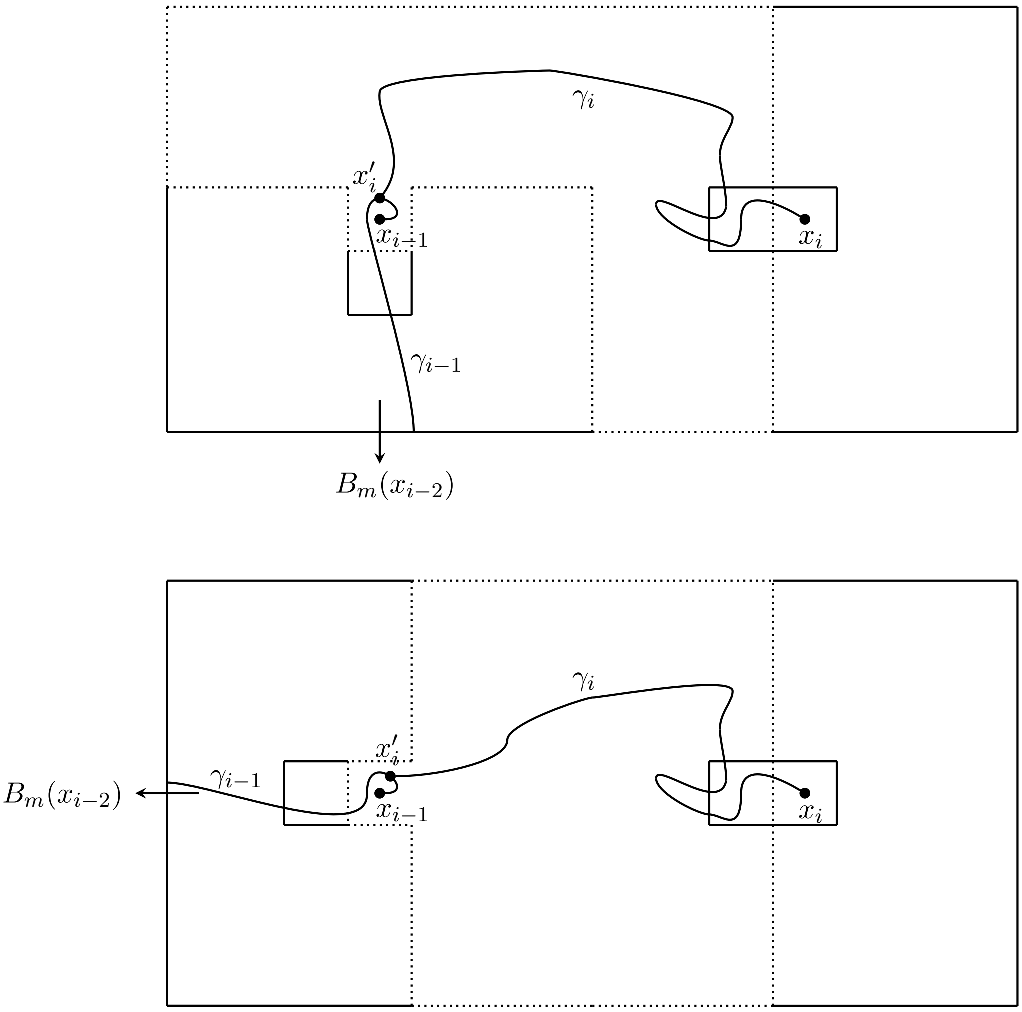

Figure 1: Configuration within $B_m(x_{i-1}) \cup B_m(x_i)$ on the event $G_0 \cap \cdots \cap G_i$.

Moreover, from the construction it is possible to deduce that, on $F_m(x, \pi)$,

$$\begin{aligned} d_{\mathcal{U}}(0,x) &\geq \sum_{i=1}^{N-1} d_{\mathcal{U}}(x_i', x_{i+1}') \\ &\geq \sum_{i=1}^{N-1} \left(d_{\mathcal{U}}(x_i', x_i) - d_{\mathcal{U}}(x_i, x_{i+1}') \right) \\ &\geq \sum_{i=1}^{N-1} \left(|\gamma_i| - |\gamma_i \cap B_{3m\lambda^{-2}}(x_i)| \right) \\ &\geq (N-1)\left(\lambda^{-1} - 3^{\kappa}\lambda^{-3/2}\right) m^{\kappa}, \end{aligned} \tag{3.2}$$

where the bound $d_{\mathcal{U}}(x_i, x_{i+1}') \leq |\gamma_i \cap B_{3m\lambda^{-2}}(x_i)|$ is a consequence of the fact that the random walk S^{x_i} does not backtrack to $B_{m\lambda^{-2}}(x_i)$ once it exits $R_i \subseteq B_{3m\lambda^{-2}}(x_i)$, which means that neither does γ_i, and so the path in $\mathcal{U}$ from x_i to x_{i+1}' must be contained within $B_{3m\lambda^{-2}}(x_i)$. In particular, for $N \geq 2$ and λ suitably large, this implies $d_{\mathcal{U}}(0,x) \geq \frac{1}{4\lambda} N m^\kappa$, and so we have the desired control over the lengths of paths.

The following is the key estimate of this section.

Proposition 3.1. *If k and λ are chosen large enough, then there exists a constant $c \in (0,\infty)$ and $m_0 \in \mathbb{N}$ such that*

$$\mathbf{P}\left(F_m(x,\pi)\right) \geq e^{-c|\pi|} \tag{3.3}$$

whenever $m \geq m_0$, $x \in \mathbb{Z}^2$, and π is a scale m path π from 0 to x.

Proof. It is enough to show that if k and λ are chosen large enough, then there exists a constant $c > 0$ such that

$$\mathbf{P}(G_0) \geq c \tag{3.4}$$

and also, for $i = 1, \dots, N$,

$$\mathbf{P}\left(G_i \middle| G_0, \dots, G_{i-1}\right) \geq c \tag{3.5}$$

uniformly over m, x and π.

To establish (3.4), we first note that [9, Lemma 2.6 and Proposition 2.7] imply

$$\mathbf{P}(G_0) \geq \mathbf{P}\left(\gamma_0 \subseteq B_m(0)\right) - \mathbf{P}\left(|\gamma_0| > \lambda m^\kappa\right) \geq 1 - ck^{-1/3} - c(\lambda k^\kappa)^{-1/5},$$

whenever $m \geq k \geq 1$. Taking any $\lambda \geq 1$ and k large yields a bound of the desired form.

For (3.5) when $i = 1, \dots, N-1$, we start by bounding $\mathbf{P}\left(G_i \middle| G_0, \dots, G_{i-1}\right)$ below by

$$\mathbf{P}\left(G_i^1 \middle| G_0, \dots, G_{i-1}\right) - \mathbf{P}\left(G_i^1 \cap (G_i^2)^c \middle| G_0, \dots, G_{i-1}\right) - \mathbf{P}\left(G_i^1 \cap (G_i^3)^c \middle| G_0, \dots, G_{i-1}\right).$$

Now, elementary random walk estimates yield that the first term here is bounded below by $c\lambda^{-4}$ whenever $\lambda^2 \geq k$ (this latter inequality is required for the case $i = 1$). Next, let $\tilde{\gamma}_i$ be the loop-erased random walk from x_i to the boundary of $(B_m(x_i) \cup Q_i) \backslash \gamma_{i-1}$. On G_i^1, we have that $\gamma_i = \tilde{\gamma}_i$. Hence, by [8, Theorem 5.8 and 6.7],

$$\mathbf{P}\left(G_i^1 \cap (G_i^2)^c \middle| G_0, \dots, G_{i-1}\right) \leq \mathbf{P}\left(|\tilde{\gamma}_i| \notin [\lambda^{-1} m^k, \lambda m^k] \middle| G_0, \dots, G_{i-1}\right) \leq c_1 e^{-c_2 \lambda^{3/5}}.$$

Similarly, let $\tilde{\tilde{\gamma}}_i$ be the loop-erased random walk from x_i to the boundary of $(R_i \cup Q_i) \backslash \gamma_{i-1}$. On G_i^1, we have that $\gamma_i = \tilde{\tilde{\gamma}}_i$. So, by applying [8, Theorem 5.8] again,

$$\begin{aligned} \mathbf{P}\left(G_i^1 \cap (G_i^3)^c \middle| G_0, \dots, G_{i-1}\right) &\leq \mathbf{P}\left(|\tilde{\tilde{\gamma}}_i \cap B_{3m\lambda^{-2}}(x_i)| > \lambda (3m\lambda^{-2})^\kappa \middle| G_0, \dots, G_{i-1}\right) \\ &\leq c_1 e^{-c_2 \lambda}. \end{aligned}$$

Combining these estimates we obtain

$$\mathbf{P}\left(G_i \middle| G_0, \dots, G_{i-1}\right) \geq c\lambda^{-4} - c_1 e^{-c_2 \lambda^{3/5}},$$

which is greater than $\frac{c}{2}\lambda^{-4} > 0$ for large λ.

The estimate (3.5) for $i = N$ is obtained similarly. $\square$

We can now conclude the proof of Theorem 1.6 – recall that (b) was proved at the end of Section 2, and the upper bound in (a) follows from Theorem 2.7.

Proof of the lower bound in Theorem 1.6(a). Without loss of generality, we may assume $y = 0$. Moreover, in view of Proposition 3.1, we can choose constants $c_1, c_2, c_3, c_4, c_5, c_6$ such that if

$k = c_1$, $\lambda = c_2$, then the estimate (3.3) holds with $c = c_3$ and $m_0 = c_4$, and also on $F_m(x, \pi)$ we have that

$$c_5 N m^\kappa \leq d_{\mathcal{U}}(0, x) \leq c_6 N m^\kappa.$$

Now, for any $m \geq 1$ and $x \in \mathbb{Z}^2$ with $d_\infty(0, x) \geq m$, one can choose a scale m path π from 0 to x such that $N \leq c_7 d_\infty(0, x)/m$. On $F_m(x, \pi)$, we therefore have that

$$d_{\mathcal{U}}(0, x) \leq c_6 c_7 d_\infty(0, x) m^{\kappa-1}.$$

It readily follows that if $d_\infty(0, x) \geq c_4(c_6c_7\lambda)^4$ and we choose $m = d_\infty(0, x)/(c_6c_7\lambda)^4$, which implies $m \geq c_4$ and $d_\infty(0, x) \geq m$, then on $F_m(x, \pi)$ it holds that $d_{\mathcal{U}}(0, x) \leq \lambda^{-1} d_\infty(0, x)^\kappa$. So we conclude that

$$\mathbf{P}\left(d_{\mathcal{U}}(0, x) \leq \lambda^{-1} d_\infty(0, x)^\kappa\right) \geq \mathbf{P}\left(F_m(x, \pi)\right) \geq e^{-c_3 N} \geq e^{-c_3 c_6^4 c_7^5 \lambda^4}.$$

□

Remark 3.2. Whilst it would be straightforward to apply our approach to construct a corresponding exponential estimate from below for the probability of seeing exceptionally long paths in the UST, a stronger polynomial bound for such an event is already known. Indeed, by considering that with polynomially large probability the loop-erased random walk from x to y exits $B_\infty(x, \lambda d_\infty(x, y))$, it was established in [9, Proposition 2.7] that

$$\mathbf{P}\left(d_{\mathcal{U}}(x, y) \geq \lambda d_\infty(x, y)^\kappa\right) \geq c\lambda^{-4/5-\varepsilon}$$

for $x, y \in \mathbb{Z}^2$, $\lambda \geq \lambda_0$ (cf. Proposition 2.11). The point of our approach is that it also gives control of the macroscopic shape of the long path.

4 Volume fluctuations

In this section, we prove Theorem 1.1. The main ingredient in the proofs of these results is the following lemma, which provides tail bounds for the volume of balls in the UST.

Lemma 4.1. *There exist constants c_1, c_2 such that, for all $r, \lambda \geq 1$,*

$$\mathbf{P}\left(|B_{\mathcal{U}}(0, r)| \geq \lambda r^{2/\kappa}\right) \geq c_1 e^{-c_2 \lambda^{\kappa/(\kappa-1)}} = c_1 e^{-c_2 \lambda^5}, \tag{4.1}$$

and also

$$\mathbf{P}\left(|B_{\mathcal{U}}(0, r)| \leq \lambda^{-1} r^{2/\kappa}\right) \geq c_1 e^{-c_2 \lambda^{\kappa/(2-\kappa)}} = c_1 e^{-c_2 \lambda^{5/3}}. \tag{4.2}$$

Remark 4.2. See [9, Theorem 1.2] for upper bounds of $\exp(-c\lambda^{1/3})$ for the probability in (4.1) and of $\exp(-c\lambda^{1/9})$ for the probability in (4.2).

Proof. Consider a square of $N \times N$ boxes, each of size $m \times m$, with the bottom left box centred on the origin. Let π be the scale m horizontal path from 0 to the point $x = ((N-1)m, 0)$, and suppose that the part of the UST containing 0 and x is constructed as in the event $F_m(x, \pi)$ of the previous section. Then, for each string of vertical boxes, assume that one has a similar construction, where at the bottom level we assume that the algorithm attaches to the horizontal

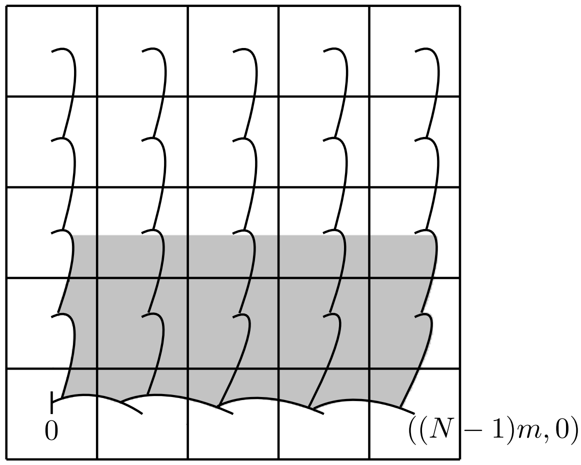

Figure 2: The tree $\mathcal{U}_{N,m}$ on the event $F(N,m)$.

part of the construction. If both such stages of this construction occur, we say that the event $F(N,m)$ holds. (See Figure 2.) Similarly to the proof of Proposition 3.1, we have that

$$\mathbf{P}\left(F(N,m)\right) \geq e^{-cN^2} \tag{4.3}$$

for all $N \geq 1$, $m \geq m_0$. Moreover, similarly to (3.1), we deduce that, on $F(N,m)$,

$$d_{\mathcal{U}}(0,x) \leq cNm^{\kappa} \tag{4.4}$$

for every $x \in \mathcal{U}_{N,m}$, where $\mathcal{U}_{N,m}$ is the subset of $\mathcal{U}$ built in defining the event $F(N,m)$. Now, on $F(N,m)$, we have that every vertex in the $Nm \times Nm$ region of boxes is within a d_∞-distance of m from a vertex in $\mathcal{U}_{N,m}$. Thus, conditioning on $\mathcal{U}_{N,m}$ and continuing to construct the remainder of $\mathcal{U}$ from this tree as the root, by a minor adaptation of the 'filling-in' argument of [9, Proposition 3.2], it is possible deduce that on an event of (conditional) probability greater than $1 - c_1 e^{-c_2 N^{1/3}}$ every vertex x contained in the bottom $Nm \times \frac{Nm}{2}$ squares that is inside the outer paths (i.e. the shaded region of Figure 2) satisfies

$$d_{\mathcal{U}}(x, \mathcal{U}_{N,m}) \leq (N^{1/2}m)^{\kappa} \leq Nm^{\kappa}. \tag{4.5}$$

In particular, putting the bounds at (4.4) and (4.5) together, we deduce that

$$\mathbf{P}\left(|B_{\mathcal{U}}(0, cNm^{\kappa})| \geq c(Nm)^2\right) \geq \left(1 - c_1 e^{-c_2 N^{1/3}}\right) e^{-cN^2} \geq c_3 e^{-c_4 N^2}.$$

Setting $r = cNm^{\kappa}$ and $\lambda = cN^{2(\kappa-1)/\kappa}/c^{2/\kappa}$ yields the result at (4.1).

For the result at (4.2), we argue similarly, though with a different initial tree configuration. We again consider a square of $N \times N$ boxes, each of size $m \times m$, centred on the origin. Let π be the scale m path that starts at 0 and spirals outwards around the boxes. Denoting the centre of the final box by x, we write $G(N,m) = F_m(x,\pi)$. (See Figure 3.) From Proposition 3.1, we have that

$$\mathbf{P}\left(G(N,m)\right) \geq e^{-cN^2}.$$

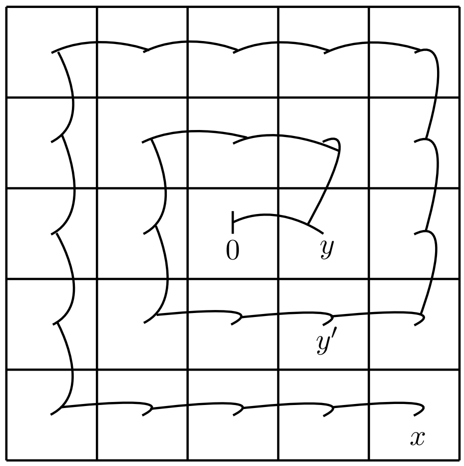

Figure 3: The tree $\tilde{\mathcal{U}}_{N,m}$ on the event $G(N,m)$.

Furthermore, let y and y' be centres of two adjacent boxes at a Euclidean distance approximately $Nm/3$ from the origin, but with y one circuit closer to the origin than y'. (See Figure 3.) By arguing as at (3.2), we have on $G(N,m)$ that $d_{\mathcal{U}}(0,y) \geq cN^2m^\kappa$.

Next, denote by $\tilde{\mathcal{U}}_{N,m}$ the tree constructed in the definition on $G(N,m)$, and note that every vertex in $B_{Nm}(0)$ is within d_∞-distance m of this set. Thus, similarly to the first part of the proof, we can again apply the 'filling-in' argument of [9, Proposition 3.2] to deduce that on an event of (conditional on $\tilde{\mathcal{U}}_{N,m}$) probability greater than $1 - c_1e^{-c_2N^{1/3}}$ every vertex x contained in $B_{Nm/2}(0)$ is within a $d_{\mathcal{U}}$-distance of Nm^κ. If this is the case and $G(N,m)$ occurs, it further holds that every point z on the straight line between y and y' satisfies

$$d_{\mathcal{U}}(0,z) \geq d_{\mathcal{U}}(0,y) - d_{\mathcal{U}}(y,z) \geq cN^2m^\kappa - Nm^\kappa \geq cN^2m^\kappa,$$

where we have applied the lower bound on $d_{\mathcal{U}}(0,y)$ from the previous paragraph. In particular, since by construction any path in $\mathcal{U}$ from $B_{Nm}(0)^c$ to 0 must pass through the line between y and y', it follows that $B_{Nm}(0)^c \subset B_{\mathcal{U}}(0, cN^2m^\kappa)^c$, which implies in turn that

$$\mathbf{P}\left(|B_{\mathcal{U}}(0, cN^2m^\kappa)| \leq (Nm)^2\right) \geq \left(1 - c_1e^{-c_2N^{1/3}}\right)e^{-cN^2} \geq c_3e^{-c_4N^2}.$$

Setting $r = cN^2m^\kappa$ and $\lambda = cN^{\frac{2(2-\kappa)}{\kappa}}$ yields (4.2). $\square$

Proof of Theorem 1.1. We start by showing large volumes occur almost-surely, i.e. (1.3). To this end, we define a sequence of scales:

$$D_i = e^{i^2}, \qquad m_i = e^{i^2}/\varepsilon(\log i)^{1/2}.$$

We now run Wilson's algorithm, using the family of independent SRW $(S^x, x \in \mathbb{Z}^2)$. At stage i we use all the vertices in $B_{2D_i}(0)$ which have not already been explored, in an order described in more detail below; write $\mathcal{U}_i$ for the tree obtained. Let $\mathcal{F}_i$ be the σ-field generated by the construction at the end of stage i.

By [9, Theorem 1.1], we have that

$$B_{2D_i}(0) \subseteq B_{\mathcal{U}}(0, \lambda^4 D_i^\kappa) \subseteq B_{\lambda^5 D_i}(0) \tag{4.6}$$

with probability greater than $1 - c\lambda^{-17/16}$.

Hence, if we run Wilson's algorithm from the vertices contained inside $B_{2D_i}(0)$ (in any order), taking 0 as the root, then the probability of seeing the part of the tree we generate, $\mathcal{U}_i$ say, leaving $B_{\lambda^5 D_i}(0)$ is less than $c\lambda^{-17/16}$. By applying a Borel-Cantelli argument we thus obtain that

$$\mathcal{U}_i \subseteq B_{i^5 D_i}(0) \subseteq B_{m_{i+1}}(0) \tag{4.7}$$

for large i, almost-surely. Moreover, from (4.6), we see that we may also assume that the $d_{\mathcal{U}}$-diameter of $\mathcal{U}_i$ is bounded above by $i^4 D_i^\kappa \leq m_{i+1}^\kappa$ for large i, almost-surely. Define the event $F(i)$ to be the event that (4.7) and the diameter estimate for $\mathcal{U}_i$ both hold.

Next, we define an event $G(i+1)$ as follows. In particular, we first suppose that it incorporates the event $F(i)$ holding. We then mimic the definition of the event

$$F(D_{i+1}/m_{i+1}, m_{i+1}) = F(c(\log{(i+1)})^{1/2}, m_{i+1})$$

from the proof of (4.1). However, we run the first random walk in the box $B_{m_{i+1}}(0)$ until it hits the root $\mathcal{U}_i \subseteq B_{m_{i+1}}(0)$, rather than the root 0. Let $\mathcal{U}'(i+1)$ be the part of the UST that is thus constructed. Next, extend $\mathcal{U}'(i+1)$ to a tree $\mathcal{U}(i+1)$ by running loop-erased random walks from each of the vertices contained in the bottom $\frac{D_{i+1}}{m_{i+1}} \times \frac{D_{i+1}}{2m_{i+1}}$ squares that is inside the outer paths until they hit the part of the tree already constructed as in Wilson's algorithm (again, we refer to the shaded region of Figure 2). We then complete the definition of $G(i+1)$ by supposing on this event that the $d_{\mathcal{U}}$-diameter of $\mathcal{U}(i+1)$ is bounded above by $cD_{i+1}m_{i+1}^{\kappa-1}$. (Since on $F(i)$ we also have an estimate for the $d_{\mathcal{U}}$-diameter of $\mathcal{U}(i)$ of m_{i+1}^κ, we can control the lengths of paths in the appropriate way.) Similarly to (4.3), this construction yields that, for large i,

$$\mathbf{P}\left(G(i+1)\,|\,\mathcal{F}_i\right) = \mathbf{P}\left(G(i+1)\,|\,\mathcal{F}_i\right)\mathbf{1}_{F(i)} \geq e^{-c(D_{i+1}/m_{i+1})^2} \geq i^{-c}$$

for some $c < 1$. Since it is clear that $G(i)$ is $\mathcal{F}_i$-measurable, then it follows from the conditional Borel-Cantelli lemma that $G(i)$ occurs infinitely often, almost-surely. Finally, we note that on $G(i)$ we have that $|B_{\mathcal{U}}(0, cD_{i+1}m_{i+1}^{\kappa-1})| \geq cD_{i+1}^2$. From this, the reparameterisation $r_i = cD_i m_i^{\kappa-1}$ yields the result.

To prove (1.4), we proceed in essentially in the same way. In particular, define an event $H(i+1)$ similarly to $G(i+1)$, but based on the event $G(D_{i+1}/m_{i+1}, m_{i+1})$ from the proof of (4.2) (i.e. using the spiral path of Figure 3, rather than the finger-like structure of Figure 2), and then 'filling-in' from all vertices in $B_{D_{i+1}/2}(0)$. Arguing as in the proof of (4.2), we deduce that $\mathbf{P}(H(i+1)\,|\,\mathcal{F}_i) \geq i^{-c}$ for some $c < 1$, $H(i)$ is $\mathcal{F}_i$-measurable, and moreover, on $H(i)$ we have that $|B_{\mathcal{U}}(0, cD_i m_i^{\kappa-2})| \leq D_i^2$. $\square$

5 Volume and resistance estimates on the UST

The aim of this section is to derive estimates for 'good events', on which we have control on the volume and resistance of the two-dimensional UST. These will be applied in the subsequent sections to deduce the heat kernel estimates and other results stated in the introduction. Much of what we do here will build on previous work from [4, 9]. As already noted, the main input for the averaged heat kernel upper bound was the adaptation of [4, Proposition 2.10] that was established in Proposition 2.9. A key difference in deriving the averaged heat kernel lower bound is that we will be need to understand the structure of the UST conditional on the presence of a given path, and deriving the relevant estimates requires substantial effort; our main result is Theorem 5.9.

Our first two lemmas relate to the following situation. Suppose we have begun the construction of the UST using Wilson's algorithm, and have constructed a tree $\mathcal{U}_0$. Write $\mathbf{P}_{\mathcal{T}}$ for the law of $\mathcal{U}$ conditional on the event $\{\mathcal{U}_0 = \mathcal{T}\}$. We wish to adapt the unconditioned results of [4, 9] to the law $\mathbf{P}_{\mathcal{T}}$. We begin with the following 'filling-in' lemma, based on [4, Lemma 2.3] and [9, Proposition 3.2]. If $\mathcal{T}$ is a tree contained in $\mathcal{U}$, then for each $x \in \mathbb{Z}^2$ there exists a unique self-avoiding path in $\mathcal{U}$ connecting x and $\mathcal{T}$; we denote this by $\gamma(x, \mathcal{T})$. We write $d_{\mathcal{U}}(x, \mathcal{T}) = \min\{d_{\mathcal{U}}(x, y), y \in \mathcal{T}\}$ for the length of this path.

Lemma 5.1. *Let $r \geq 1$, and $\mathcal{T}$ be a finite connected tree. There exist constants $c_i \in (0, \infty)$ not depending on r and $\mathcal{T}$ such that for each $\delta \leq \frac{1}{4}$ the following holds. Let $A_1 \subseteq A_2$ be subsets of $\mathbb{Z}^2$, with the property that any path in $\mathbb{Z}^2 \setminus \mathcal{T}$ between A_1 and A_2^c is of length greater than r. Suppose that $d_\infty(x, \mathcal{T}) \leq \delta r$ for all $x \in A_2$. Then there exists an event G such that*

$$\mathbf{P}_{\mathcal{T}}(G^c) \leq c_1 r^{-2} |A_2| \exp(-c_2 \delta^{-1/2}),$$

and on G we have that, for all $x \in A_1$:

$$d_{\mathcal{U}}(x, \mathcal{T}) \leq (\delta^{1/2} r)^\kappa; \qquad d_{\mathcal{U}}^{\mathcal{S}}(x, \mathcal{T}) \leq \delta^{1/2} r; \qquad \gamma(x, \mathcal{T}) \subseteq A_2.$$

Proof. If A_1 is a Euclidean ball of radius $r/2$, and A_2 is a Euclidean ball of radius r centred on the same point, then this is immediate from the proof of [4, Lemma 2.3]. (Checking the proof in [9] one sees that one can take the power of δ to be $\delta^{-1/2}$ rather than $\delta^{-1/3}$.) The proof for more general sets A_i is similar. □

Lemma 5.2 (Cf. [4, Lemma 2.5])**.** *Let $x \in \mathbb{Z}^2$, $r \geq 1$, $k \geq 2$, and $D_0 \subseteq \mathbb{Z}^2$ satisfy $B_{9r/8}(x) \subseteq \cup_{y \in D_0} B_{r/18k}(y)$. Let $\mathcal{T}$ be a finite connected tree such that $\mathcal{T} \subseteq B_{2r}(x)^c$, and write $\gamma = \gamma(x, \mathcal{T})$. There exists an event $F_1 = F_1(x, r, k)$ which satisfies*

$$\mathbf{P}_{\mathcal{T}}(F_1^c) \leq e^{-c_1 k^{1/8}},$$

and on $F_1(x, r, k)$ there exists $T \leq \tau_{x,r}(\gamma)$ such that, writing $W_x = \gamma[T]$:
(a) $k^{-1/4} r^\kappa \leq T \leq k^{1/4} r^\kappa$;
(b) $a^{-2} r \leq d_\infty(x, W_x) \leq r$;
(c) there exists $Y_x \in D_0$ such that $d_\infty(Y_x, W_x) \leq r/3k$, $d_{\mathcal{U}}^{\mathcal{S}}(Y_x, W_x) \leq 2r/3k$ and also $d_{\mathcal{U}}(Y_x, W_x) \leq c_1 (r/k)^\kappa$.

Proof. This follows as in [4]. The most delicate part of the argument is to verify that [4, Lemma 2.4] holds in this context. For this, we need to show that if $\gamma = \gamma(x, \mathcal{T})$, then γ does not make too many close returns to the segment $\gamma[x, \tau_{x,r}]$ after time $\tau_{x,(1+k^{-1/8})r}$. The argument in [4, Lemma 2.4] is for $\gamma(x, \infty)$, and the proof for $\gamma(x, \mathcal{T})$ is very similar. □

Towards stating Theorem 5.9, we now introduce an event similar to those of the kind considered in Sections 3 and 4, but incorporating the regularity of Definition 2.2. We now choose $N \in \mathbb{N}$ to be suitably large; in particular $N \geq 128$. Let $(\tilde{x}_i)_{i=0}^{N_1}$ be the path with an 'S shape' given by

$$((-N, -1), (-N+1, -1), \ldots, (N, -1),$$

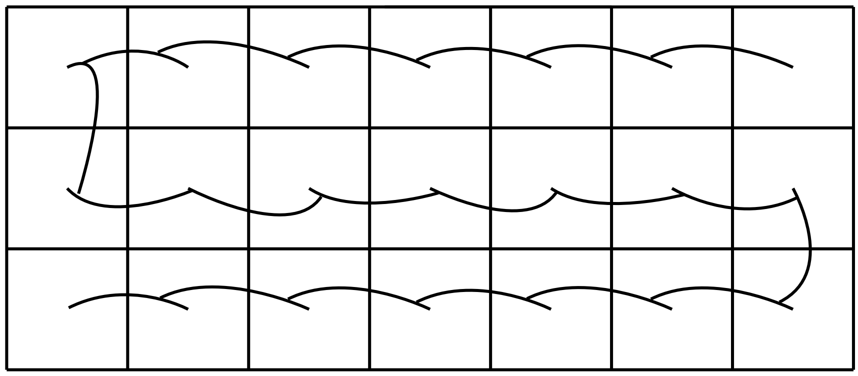

Figure 4: The part of $\mathcal{U}$ constructed on the event H_{N_1} (with $N_1 = 3$).

$$(N,0), (N-1,0), \dots, (-N,0),$$
$$(-N,1), (-N+1,1), \dots, (N,1));$$

note that $N_1 = 3(2N+1)-1$. Let $m \in \mathbb{N}$ with $m \geq 256$, we then let $(x_i)_{i=0}^{N_1}$ be the corresponding scale m path given by setting $x_i = m\tilde{x}_i$. (Ultimately we will only be interested in the situation when both N and m are very large.) Let $\Gamma(0)$ be the path in $\mathbb{R}^2$ which is the union of the line segments $[x_{i-1}, x_i]$ for $i = 1, \dots, N_1$. For $t > 0$ we write

$$\Gamma(t) = \{x \in \mathbb{R}^2 : d_\infty(x, \Gamma(0)) \leq mt\}.$$

We now use Wilson's algorithm to construct $\mathcal{U}$, and begin the construction using the points x_i, $0 \leq i \leq N_1$. We wish the tree constructed to be inside $\Gamma(1/8)$ – see Figure 4, and also to have some additional regularity properties given below. As above, let $(S_n^x)_{n\geq 0}$, $x \in \mathbb{Z}^2$, be independent SRW in $\mathbb{Z}^2$, where S^x is started from x. For $i = 1, \dots, N_1$ let R_i' be the rectangle, with sides $m/4$ and $5m/4$ which contains both $B_\infty(x_{i-1}, m/8)$ and $B_\infty(x_i, m/8)$. Let $R_i \subset R_i'$ be the rectangle with sides $m/4$ and $m/2$ which contains $B_\infty(x_i, m/8)$, and let $Q_i = R_i' \setminus B_\infty(x_i, m/8)$. Let $\mathcal{U}_0 = \gamma_0 = \{0\}$, and for $i \geq 1$ let

$$\gamma_i = \mathcal{L}(\mathcal{E}_{\mathcal{U}_{i-1}}(S^{x_i})), \qquad \mathcal{U}_i = \mathcal{U}_{i-1} \cup \gamma_i, \qquad \text{for } i = 1, \dots, N_1.$$

Define events G_i^j, $i = 1, \dots, N_1$, $j = 1, 2$, as follows.

- G_i^1 is the event that S^{x_i} first exits R_i on the side closest to x_{i-1}, within a d_∞-distance $m/16$ of the line segment between x_{i-1} and x_i – call this exit time τ_i, and also $S^{x_i}_{\tau_i+\cdot}$ hits γ_{i-1} before exiting Q_i;

- G_i^2 is the event that γ_i is $(\lambda, (3/2)me^{-c_1\lambda^{1/2}}, 3m/2)$-regular.

Let $G_i = G_i^1 \cap G_i^2$, and

$$H_i = \bigcap_{j=1}^{i} G_j, \qquad i = 1, \dots, N_1.$$

Similarly to Proposition 3.1, we then have the following.

Proposition 5.3. *There exist constants $c_1 \in (0,\infty)$ and $\lambda_0, m_0 \geq 2$ such that if $\lambda \geq \lambda_0$ and $m \geq m_0$, then*
$$\mathbf{P}(H_{N_1}) \geq e^{-c_1 N_1}. \tag{5.1}$$

Proof. Set $H_0 = \Omega$. Arguing as in the proof of Lemma 2.6, we have that $\mathbf{P}(H_1) = \mathbf{P}(G_1) \geq c_2 > 0$ for m sufficiently large. From this, the result at (5.1) will follow if we can prove that for some c_3 that
$$\mathbf{P}(H_i|H_{i-1}) \geq e^{-c_3}, \qquad i = 2, \dots, N_1. \tag{5.2}$$
We use induction. Suppose that (5.2) holds for $i-1$ for some $i \geq 2$. Since $\mathcal{U}_{i-1}$ contains a path from x_{i-1} to the boundary of $B_\infty(x_{i-1}, m/8)$, standard properties of the simple random walk S^{x_i} give us that
$$\mathbf{P}(G_i^1|H_{i-1}) \geq 2e^{-c_4}. \tag{5.3}$$
Now, let $\widetilde{\gamma}_i$ be the path in $\mathcal{U}_i$ from x_i to $\mathcal{U}_{i-2}$. Note that, if we condition on $\mathcal{U}_{i-2}$, then $\widetilde{\gamma}_i$ is equal in law to $\mathcal{L}(\mathcal{E}_{\mathcal{U}_{i-2}^c}(S^{x_i}))$. Thus, if we set

$$\widetilde{G}_i^2 = \{\mathcal{E}_{B_{3m}(x_i)}(\widetilde{\gamma}_i) \text{ is } (\lambda, (3/2)me^{-c_1\lambda^{1/2}}, 3m/2)\text{-regular}\},$$

then Lemma 2.5 yields that
$$\mathbf{P}\left((\widetilde{G}_i^2)^c|H_{i-2}\right) \leq c_5 e^{-c_6\lambda^{1/2}}. \tag{5.4}$$
It is also straightforward to verify that $G_i^1 \cap \widetilde{G}_i^2 \cap H_{i-1} \subseteq G_i^1 \cap G_i^2 \cap H_{i-1}$, and it therefore follows that

$$\mathbf{P}(H_i|H_{i-1}) = \mathbf{P}(G_i^1 \cap G_i^2|H_{i-1}) \geq \mathbf{P}(G_i^1 \cap \tilde{G}_i^2|H_{i-1}) \geq \mathbf{P}(G_i^1|H_{i-1}) - \mathbf{P}\left(\left(\widetilde{G}_i^2\right)^c|H_{i-1}\right).$$

Using (5.4) and the inductive hypothesis we have that

$$\mathbf{P}\left(\left(\widetilde{G}_i^2\right)^c|H_{i-1}\right) = \frac{\mathbf{P}\left(\left(\widetilde{G}_i^2\right)^c \cap H_{i-1}|H_{i-2}\right)}{\mathbf{P}\left(H_{i-1}|H_{i-2}\right)} \leq e^{c_3} \times c_5 e^{-c_6\lambda^{1/2}} \leq e^{-c_4}, \tag{5.5}$$

where the final inequality holds by taking λ_0 suitably large. Combining (5.3) and (5.5), we thus obtain that (5.2) holds for i. $\square$

We now fix $\lambda \geq \lambda_0$, where $\lambda_0 \geq 2$ is as in the previous proposition, and consider the uniform spanning tree obtained by conditioning on the event H_{N_1}. Let $\mathcal{T}$ be a fixed tree such that $\mathbf{P}(\mathcal{U}_{N_1} = \mathcal{T} \mid H_{N_1}) > 0$, and, as above, write $\mathbf{P}_\mathcal{T}(\cdot) = \mathbf{P}(\cdot|\mathcal{U}_{N_1} = \mathcal{T})$. We will derive volume and resistance bounds for balls $B_\mathcal{U}(x,r)$ where x is close to the middle section of $\mathcal{T}$ and r is of order m^κ that will hold with $\mathbf{P}_\mathcal{T}$ probability close to 1. To this end, we introduce some more notation. The tree $\mathcal{T}$ contains a path from $\tilde{x}_0$ to $\tilde{x}_{N_1}$; denote this by $\mathcal{T}_{trunk}$. For $a > 0$, let

$$\mathcal{T}_0 := \mathcal{T} \cap \left([-(N-1)m, (N-1)m] \times [-m/8, m/8]\right),$$
$$\mathcal{T}_{0,+} := \mathcal{T} \cap \left([-(N-1)m, (N-1)m] \times [7m/8, 9m/8]\right).$$

The following lemma, relating to distances on $\mathcal{T}$, follows easily from the definition of the event H_{N_1}.

Lemma 5.4. *Let $z_1, z_2 \in \mathcal{T}_0$ with $d_\infty(z_1, z_2) \geq 3m$. Then*

$$c_1\lambda^{-1}m^{\kappa-1}d_\infty(z_1,z_2) \leq d_\mathcal{U}(z_1,z_2) \leq c_2\lambda m^{\kappa-1}d_\infty(z_1,z_2).$$

Now, for $a \in (0,1)$, let

$$D(a) := [-aNm, aNm] \times [-9m/8, 9m/8]. \tag{5.6}$$

We wish to define the region in $D(a)$ which lies 'between' $\mathcal{T}_0 \cap \mathcal{T}_{trunk}$ and $\mathcal{T}_{0,+} \cap \mathcal{T}_{trunk}$. To do this precisely, write $\mathcal{T}^{\mathbb{R}}_{trunk}$ for the continuous piecewise linear self-avoiding path in $\mathbb{R}^2$ obtained by connecting neighbouring points in $\mathcal{T}_{trunk}$ by a line segment. Let $D^+_{\mathbb{R}}(a)$ be the closure of the connected component of $D(a) \setminus \mathcal{T}^{\mathbb{R}}_{trunk}$ which contains the point $(0, m/2)$, and define $D^+(a) = D^+_{\mathbb{R}}(a) \cap \mathbb{Z}^2$. To simplify our notation we will concentrate on the regions $D^+(a)$; exactly the same arguments apply to the corresponding region $D^-(a)$ lying 'below' $\mathcal{T}_0 \cap \mathcal{T}_{trunk}$.

Let $k \in \mathbb{N}$ satisfy $k \geq \lambda^4$ and $k \leq m$. We now choose a grid $\Lambda_1 \subset D^+(7/8)$ of points with separation of order $\frac{1}{4}k^{-1/4}m$ such that

$$D^+(3/4) \subseteq \bigcup_{z \in \Lambda_1} B_\infty\left(z, \tfrac{1}{4}k^{-1/4}m\right).$$

Since $|D(3/4)| \leq cNm^2$, we can choose this set so that $|\Lambda_1| \leq cNk^2$. Let $z \in \Lambda_1$ and set

$$\begin{aligned} G_{11}(z) &:= \left\{ S^z \text{ hits } \mathcal{T} \text{ before it leaves } B_\infty(z, k^{1/7}m) \right\}, \\ G_{12}(z) &:= \left\{ |\gamma(z, \mathcal{T})| \leq k^{1/7} m^\kappa \right\}, \\ G_{13}(z) &:= \left\{ |\gamma(z, \mathcal{T})| \geq k^{-1/7} d_\infty(z, \mathcal{T})^\kappa \right\}. \end{aligned}$$

Lemma 5.5. *If $\lambda^4 \leq k \leq m \wedge (N/8)^7$ and $z \in \Lambda_1$ then*

$$\mathbf{P}_{\mathcal{T}}(G_{1j}(z)^c) \leq ce^{-ck^{1/10}} \text{ for } j = 1, 2, 3.$$

Proof. Note that S^z can only leave $B_\infty(z, k^{1/7}m)$ without hitting $\mathcal{T}$ if it leaves horizontally at a distance of order $k^{1/7}m$ from z. Since every point in $D^+(3/4)$ is within a d_∞ distance $5m/8$ of $\mathcal{T}$ we obtain the bound on $\mathbf{P}(G_{11}(z)^c)$. The bound for G_{12} follows by part 1 of [9, Theorem 2.2] (with $D = D' = D(1)$ and $n = m$), and the bound for G_{13} follows by part 2 of the same theorem. □

Now let

$$F_2(k) = \bigcap_{z \in \Lambda_1} \left(G_{11}(z) \cap G_{12}(z) \cap G_{13}(z) \right);$$

by Lemma 5.5 we have

$$\mathbf{P}(F_2(k)^c) \leq cNk^2 e^{-ck^{1/10}} \leq cNe^{-c'k^{1/10}}. \tag{5.7}$$

Proposition 5.6. *There exists $\delta_1 > 0$ such that the following holds. Suppose that $\lambda^7 \leq k \leq m \wedge (\delta_1 N)^7$. There exists an event $F_3 = F_3(k)$ with*

$$\mathbf{P}_{\mathcal{T}}(F_3^c) \leq cNe^{-c_3 k^{1/10}} \tag{5.8}$$

such that on $F_3(k)$ the following properties hold.
(a) If $y \in D^+(5/8)$, then there exists $x \in \mathcal{T}$ with

$$d^S_{\mathcal{U}}(y, x) \leq 5k^{1/7}m, \quad d_{\mathcal{U}}(y, x) \leq 2k^{1/7}m^\kappa.$$

(b) If $y \in D^+(5/8)$ and $1 \le s \le \delta_1 \lambda^{-1} N$, then

$$B_{\mathcal{U}}(y, sm^\kappa) \subseteq B_\infty(y, c(\lambda s + k^{1/7})m) \cap D(3/4), \tag{5.9}$$

and thus

$$|B_{\mathcal{U}}(y, sm^\kappa)| \le c'(\lambda s + k^{1/7})^2 m^2. \tag{5.10}$$

Proof. We continue the construction of the UST from $\mathcal{U}_{N_1}$ by adding in the points in the grid $\Lambda_1 \cap D^+(3/4)$; write $\mathcal{U}_1^*$ for the tree thus obtained. We then complete the uniform spanning tree inside $D^+(5/8)$. We use the filling-in of Lemma 5.1 with $\delta = k^{-1/2}$, $r = m/8$, $A_1 = D^+(5/8)$, $A_2 = D^+(3/4)$, and write $\tilde{F}_3(k)$ for the 'good event' given by Lemma 5.1. Then

$$\mathbf{P}(\tilde{F}_3(k)^c) \le cNe^{-k^{1/4}}.$$

Now let $F_3(k) = F_2(k) \cap \tilde{F}_3(k)$; the bound (5.8) follows from (5.7) and the bound on $\mathbf{P}(\tilde{F}_3^c(k))$ given above.

In the remaining part of the proof, we assume $F_3(k)$ holds. Let $y_1 \in D^+(5/8)$. Then the event $\tilde{F}_3$ implies that there exists $w_1 \in \mathcal{U}_1^*$ with $d_{\mathcal{U}}^S(y_1, w_1) \le k^{-1/4}m$ and $d_{\mathcal{U}}(y_1, w_1) \le (k^{-1/4}m)^\kappa$. By the construction of $\mathcal{U}_1^*$ there exists a point $z_1 \in \Lambda_1$ such that $w_1 \in \gamma(z_1, \mathcal{T})$, and $\gamma(z_1, \mathcal{T}) \subset B_\infty(z_1, k^{1/7}m)$. Let x_1 be the point where $\gamma(z_1, \mathcal{T})$ meets $\mathcal{T}$. The events $G_{1i}(z_1)$ imply that $d_{\mathcal{U}}^S(w_1, x_1) \le 4k^{1/7}m$, and $d_{\mathcal{U}}(x_1, w_1) \le k^{1/7}m^\kappa$, and the bounds in (a) follow immediately.

For part (b), let $y_1, y_2 \in D^+(5/8)$ with $d_{\mathcal{U}}(y_1, y_2) \le sm^\kappa$. Let x_2 be the point where $\gamma(y_2, \mathcal{T})$ meets $\mathcal{T}$ – we may have $x_1 = x_2$. As $\mathcal{U}$ is a tree, we have $d_{\mathcal{U}}(x_1, x_2) \le d_{\mathcal{U}}(y_1, y_2) \le sm^\kappa$. Using Lemma 5.4, and taking λ_0 large enough so that $c_1^{-1}\lambda s \ge 3$, we obtain

$$d_\infty(x_1, x_2) \le c_1^{-1} \lambda m s.$$

Since $d_\infty(y_j, x_j) \le 5k^{1/7}m$, it follows that $d_\infty(y_1, y_2) \le c\lambda m s + 10mk^{1/7}$. This proves (5.9) and the volume upper bound (5.10) is then immediate. □

We now consider resistance bounds.

Proposition 5.7. *There exist $\delta_2 > 0$ and c_1 such that the following holds. Suppose that $\lambda^7 \le k \le m \wedge (\delta_2 N)^7$, and $c_1 k^{1/7} \le s \le \delta_2 N$. Let $F_3(k)$ be as in the previous proposition. On the event $F_3(k)$, we have*

$$sm^\kappa \ge R_{\text{eff}}(x, B_{\mathcal{U}}(x, sm^\kappa)^c) \ge \tfrac{1}{4} sm^\kappa \text{ for } y \in D^+(9/16). \tag{5.11}$$

Proof. The upper bound is immediate. For the lower bound let $y \in D^+(5/8)$, and write $B_1 = B_{\mathcal{U}}(y, \frac{1}{2}sm^\kappa)$, $B_2 = B_{\mathcal{U}}(y, sm^\kappa)$. It is sufficient to prove that there are exactly two points in ∂B_1 which are connected to ∂B_2 by a path outside B_1; a cut set argument then gives the bound (5.11).

Note first that by the construction of $\mathcal{T}$ there exists c such that each component $\mathcal{T}'$ of $\mathcal{T} \setminus \mathcal{T}_{trunk}$ satisfies

$$d_{\mathcal{U}}(z, z') \le c\lambda m^\kappa, \quad d_{\mathcal{U}}^S(z, z') \le cm, \text{ for } z, z' \in \mathcal{T}'.$$

By Proposition 5.6 and the observation above there is a point $x \in \mathcal{T}_{trunk}$ with $d_{\mathcal{U}}(x, y) \le c_1 k^{1/7} m^\kappa$ and $d_{\mathcal{U}}^S(x, y) \le c_1 k^{1/7} m$; we used here the fact that $k \ge \lambda^7$. Note that $x \in B_1$.

Let w_1, w_2 be the two points in $\mathcal{T}_{trunk} \cap \partial B_1$; it is clear that each of these is connected to ∂B_2 by a path outside B_1. Now let $z \in \partial B_2$ and suppose there exists $w \in \partial B_1$ with $w \neq w_1, w_2$ such that $\gamma(w, z)$ is disjoint from B_1. By Proposition 5.6 we have $d_\infty(y, z) \leq c(\lambda s + k^{1/7})m \leq c' sm$, so choosing δ_2 small enough we have that $z \in D^+(5/8)$. Let z' be the closest point in $\mathcal{T}$ to z. By Proposition 5.6 we have $d_{\mathcal{U}}(z, z') \leq 2k^{1/7} m^\kappa$, and it follows that the path $\gamma(z, y)$ must intersect $\mathcal{T}$. Let z'' be the closest point in $\mathcal{T}_{trunk}$ to z'. The definition of z implies that z'' must lie on $\mathcal{T}_{trunk}$ between w_1 and w_2, and hence $d_{\mathcal{U}}(y, z'') \leq \frac{1}{2} sm^\kappa$. Thus

$$d_{\mathcal{U}}(y, z) \leq d_{\mathcal{U}}(y, z'') + d_{\mathcal{U}}(z'', z') + d_{\mathcal{U}}(z', z) \leq \tfrac{1}{2} sm^\kappa + c\lambda m^\kappa + 2k^{1/7} m^\kappa,$$

which contradicts the fact that $z \in \partial B_2$ if c_1 is large enough. $\square$

Proposition 5.8. *There exist $\delta_3 > 0$ and c_1 such that the following holds. Suppose that $\lambda \leq k^{1/7} \leq \delta_3 N$ and $k \leq m^{1/2}$. There is an event $F_4(k)$ with $\mathbf{P}_{\mathcal{T}}(F_4^c) \leq cNe^{-c' k^{1/24}}$ such that on $F_4(k)$, if $c_1 k^{1/4} \leq s \leq \delta_3 N$, then*

$$|B_{\mathcal{U}}(x, s)| \geq c\lambda^{-1} k^{-5/2} sm^2 \quad \text{for all } x \in D^+(\tfrac{1}{2}).$$

Proof. We follow the general lines of [9, Theorem 3.4], but note that the event H_{N_1} means that the path $\mathcal{T}$ cannot loop back on itself too much. This means that the hardest part of the proof in [9], which uses [9, Lemma 3.7] is not needed.

We choose points $z_i \in \mathcal{T}$, $1 \leq i \leq N_2$, such that $B_\infty(z_i, m/2)$ are disjoint and $\mathcal{T} \subseteq \cup_i B_\infty(z_i, m)$. We have $cN \leq N_2 \leq c'N$. Write $m_1 = k^{-1/4} m$. For each i choose points $w_{ij} \in \mathcal{T} \cap B_\infty(z_i, m/2)$ with $1 \leq j \leq N_3$ such that $B_\infty(w_{ij}, m_1)$ are disjoint and $N_3 \geq ck^{1/4}$.

The event H_{N_1} implies that if $y_1, y_2 \in \mathcal{T} \cap B_\infty(z_i, m_1)$, then $d_{\mathcal{U}}(y_1, y_2) \leq c\lambda m_1^\kappa$. Choose also $a, b > 0$. Let $Q_1(w_{ij}) = B_\infty(w_{ij}, k^{-a} m_1)$ and $Q_2(w_{ij}) = B_\infty(w_{ij}, 2k^{-a} m_1)$. We cover $Q_2(w_{ij})$ by a grid Λ_3 of points with separation $k^{-b} k^{-a} m_1$, so that $|\Lambda_3| = 4k^{2b}$. We run Wilson's algorithm for the points in Λ_3, and declare this stage of the construction a success for w_{ij} if for all $y \in \Lambda_3$, the random walk S^y hits $\mathcal{T}$ before it leaves $B_\infty(w_{ij}, m_1)$. By the discrete Beurling estimate, [27], the probability of failure p_1 satisfies

$$p_1 \leq |\Lambda_3| ck^{-a/4} \leq ck^{2b - a/4}.$$

We choose $a = 1, b = 1/12$ and k large enough so that $p_1 < \frac{1}{2}$.

Using the 'stacks' construction in Wilson's algorithm, we can successively explore the UST in each box $B_\infty(w_{ij}, m_1)$, for $j = 1, \ldots, N_3$, and continue until we get a success. (See the argument in Theorem 5.4 of [5] for more details.) Conditional on previous failures, the probability of a failure at stage j is still bounded by p_1, and so since we have N_3 tries, the overall probability of failure is less than $ce^{-ck^{1/4}}$.

If this stage is a success for some j, we write $j_i = j$ and $w_i' = w_{ij_i}$. We then fill in the UST inside $Q_2(w_i')$. By the filling-in result of Lemma 5.1 (with $A_j = Q_j(w_i')$, $j = 1, 2$ and $r = k^{-1} m_1$, $\delta = k^{-1/12}$) the 'good' event $G = G(w_i')$ given there satisfies $\mathbf{P}_{\mathcal{T}}(G^c) \leq ce^{-ck^{1/24}}$. Moreover, if G holds, then every path from a point in $B_\infty(w_i', k^{-1} m_1)$ to $\mathcal{T}$ is contained in $B_\infty(w_i', m_1)$. Write G_i' for the event that both stages of the construction are successful.

Now set

$$\tilde{F}_4 = \bigcap_{i=1}^{N_2} G_i',$$

so that

$$\mathbf{P}_{\mathcal{T}}(\tilde{F}_4^c) \le cNe^{-ck^{1/24}}e^{-ck^{1/4}} \le cNe^{-c'k^{1/24}},$$

and define $F_4 := \tilde{F}_4 \cap F_3$, where F_3 is the event defined in Proposition 5.6.

Suppose now that F_4 holds. Let $y \in B_\infty(w_i', k^{-1}m_1)$. The event F_3 implies that $d_{\mathcal{U}}(y, \mathcal{T}) \le ck^{1/4}m^\kappa$, and the event G_i' implies that $\gamma(y, \mathcal{T}) \subseteq B_\infty(w_i', m_1)$. It follows that $d_{\mathcal{U}}(y, z_i) \le c_2(k^{1/4} + \lambda)m^\kappa$, and thus we obtain that

$$|B_{\mathcal{U}}(z_i, c_2(k^{1/4} + \lambda)m^\kappa)| \ge |B_\infty(w_i', k^{-1}m_1)| \ge m^2k^{-5/2}.$$

Next let $x \in D^+(1/2)$, and set $J = \{z_i : d_{\mathcal{U}}(z_i, x) \le \frac{1}{2}sm^\kappa\}$. By Lemma 5.4 we have $|J| \ge c\lambda^{-1}s$. Taking c_1 large enough we have $B_{\mathcal{U}}(z_i, c_2(k^{1/4} + \lambda)m^\kappa) \subset B_{\mathcal{U}}(x, sm^\kappa)$, and so

$$B_{\mathcal{U}}(x, sm^\kappa) \ge |J|m^2k^{-5/2} = c\lambda^{-1}sm^2k^{-5/2}.$$

□

We summarize the estimates of this section in the following theorem.

Theorem 5.9. *There exist constants c_i such that the following holds. Suppose that $\lambda^7 \le k \le m^{1/2} \wedge c_1N^4$. There exists an event $F_* = F_*(k)$ with $\mathbf{P}_{\mathcal{T}}(F_*^c) \le c_2Ne^{-c_3k^{1/24}}$, such that on $F_*(k)$ if $x \in D^+(1/2)$, and $c_4k^{1/4} \le s \le c_5N$ then*

$$c\lambda^{-1}k^{-5/2}m^2s \le |B_{\mathcal{U}}(x, sm^\kappa)| \le c\lambda m^2s, \tag{5.12}$$

$$\tfrac{1}{4}sm^\kappa \le R_{\text{eff}}(x, B_{\mathcal{U}}(x, sm^\kappa)^c) \le sm^\kappa. \tag{5.13}$$

Finally, we give a local version of the previous result. For $x \in \mathcal{T}$, and $s \in [c_4k^{1/4}, c_5N]$ define $F_*(x, k, s)$ to be the event that the estimates at (5.12) and (5.13) hold. By only considering boxes of size m within a distance csm of x, rather than the order N boxes considered in the previous result, one readily obtains the following.

Corollary 5.10. *Let $\lambda^7 \le k \le m^{1/2} \wedge c_1N^4$ and $c_4k^{1/4} \le s \le c_5N$. Then if $x \in \mathcal{T}_0 \cap D^+(1/2)$,*

$$\mathbf{P}_{\mathcal{T}}(F_*(x, k, s)^c) \le c\lambda se^{-c'k^{1/24}}.$$

6 Heat kernel bounds

In this section, we will obtain heat kernel bounds using the estimates given in the previous section, starting with the quenched fluctuations.

Proof of Corollary 1.2. By [7, Theorem 4.1], for any realisation of $\mathcal{U}$ we have that

$$p^{\mathcal{U}}_{2r|B_{\mathcal{U}}(0,r)|}(0,0) \le \frac{2}{|B_{\mathcal{U}}(0,r)|}. \tag{6.1}$$

Let $a_n = |B_{\mathcal{U}}(0,n)|n^{-d_f}$, and $t_n = n|B_{\mathcal{U}}(0,n)|$. Plugging these into (6.1), we have

$$p^{\mathcal{U}}_{2t_n}(0,0) \le \frac{2}{|B_{\mathcal{U}}(0,n)|} = 2t_n^{-d_f/d_w}a_n^{-(1+2/\kappa)} = 2t_n^{-d_w/d_f}a_n^{-5/13}.$$

By (1.3), $a_n > (\log\log n)^{1/5}$ infinitely often, almost-surely, giving the liminf statement.

We next prove the limsup statement. We use the construction of $\mathcal{U}$ given after the proof of Lemma 5.2 with a $3(2N+1)$ array of boxes of side m. As in Proposition 5.3, this has probability of success of at least $e^{-c_1 N}$. Given m, we define $R_* = m^\kappa$, $N = \frac{1}{2c_1}\log\log R_*$, $R = \frac{1}{2}Nm^\kappa$, and define

$$v(t) = \begin{cases} t^{d_f}, & \text{if } \ t \le R_*, \\ tR_*^{d_f-1}, & \text{if } \ t \ge R_*. \end{cases}$$

Let $\mathcal{T}$ be the tree given right after Proposition 5.3. Then by Theorem 5.9, there is an event $F_*(k)$ with $\mathbf{P}_{\mathcal{T}}(F_*(k)^c) \le cNe^{-c'k^{1/24}}$ such that the following hold on $F_*(k)$:

$$c_{1,k}v(R_0) \le |B_{\mathcal{U}}(0,R_0)| \le c_{2,k}v(R_0),$$
$$c_{3,k}R_0 \le R_{\text{eff}}(0, B_{\mathcal{U}}(0,R_0)^c) \le R_0,$$

where $R_0 := ck^\theta m^\kappa$ and $c_{i,k} := k^{q_i}$, where $q_i \in \mathbb{R}$. We now follow [22]. Let $r(t) = t$ and let $\mathcal{I}(t)$ be the inverse function of $r(t)\cdot v(t)$. After some calculations we get, noting $d_w = 1 + d_f$,

$$\mathcal{I}(t) = \begin{cases} t^{1/d_w}, & \text{if } \ t \le R_*^{d_w}, \\ t^{1/2}R_*^{-(d_f-1)/2}, & \text{if } \ t \ge R_*^{d_w}, \end{cases}$$

and the function $\tilde{k}(t) = v(\mathcal{I}(t))$ is

$$\tilde{k}(t) = \begin{cases} t^{d_f/(1+d_f)} = t^{d_f/d_w}, & \text{if } \ t \le R_*^{d_w}, \\ t^{1/2}R_*^{(d_f-1)/2}, & \text{if } \ t \ge R_*^{d_w}. \end{cases}$$

We can rewrite the final line as

$$\tilde{k}(t) = t^{d_f/d_w}\left(\frac{R_*^{d_w}}{t}\right)^{\alpha}, \quad \text{where} \quad \alpha = \frac{d_f-1}{2d_w} > 0.$$

One then finds from [22, Proposition 3.3] that

$$p_{2n}^{\mathcal{U}}(0,0) \ge \frac{c_{1,k}}{\tilde{k}(n)} \ \text{ for } \ \frac{c_{2,k}}{2}R^{d_w} \le n \le c_{2,k}R^{d_w},$$

for some $c_{i,k} = c_i k^{-q_i}$, $i = 1,2$, where $c_i, q_i > 0$. (Note that [22, Proposition 3.3] holds for $R \ge R_*$ if the assumption of the proposition holds for $R \ge R_*$.) So taking $T = c_{2,\lambda}R^{d_w}/2 = \frac{c_{2,\lambda}}{2(4c_1)^{d_w}}(R_*\log\log R_*)^{d_w}$, it holds that, given $\mathcal{T}$, with probability greater than $1 - cNe^{-c'k^{1/24}}$, we have

$$T^{d_f/d_w}\tilde{p}_T^{\mathcal{U}}(0,0) \ge c_{3,k}(\log\log R_*)^{\alpha d_w} \ge c'_{3,k}(\log\log T)^{\alpha d_w}.$$

In order to have $1 - cNe^{-c'k^{1/24}} \ge 1/2$, it is enough to take $k \asymp (\log N)^{24}$ which is comparable to $(\log\log\log R_*)^{24}$. (Note that this choice of k enjoys $k \le \sqrt{m} \wedge (N/8c\lambda)^{1/\theta}$ that is required in Theorem 5.9.) Hence we have

$$T^{d_f/d_w}\tilde{p}_T^{\mathcal{U}}(0,0) \ge c_6(\log\log T)^{\alpha d_w - \varepsilon},$$

for some $c_6 > 0$ and $\varepsilon > 0$ which is small.

The rest of the argument goes through similarly to the proof of Theorem 1.1 in Section 4. We choose $m(i) = e^{i^2/\kappa}$, so $R_*(i) = e^{i^2}$, $N(i) = (2c_1)^{-1}\log\log R_*(i)$, and $\sum_i e^{-c_1 N(i)} = \sum_i i^{-1} = \infty$. Similarly to (4.7), we have good separation of scales. Using the conditional Borel-Cantelli lemma, we obtain the desired lower bound. $\square$

6.1 Averaged heat kernel upper bound

To establish the upper bound of Theorem 1.4, we start by deducing upper estimates for the transition density that hold on the event $F_1(\lambda, n)$, which was defined at (2.4). In this subsection, we fix $\varepsilon_0 = 1/40$. Moreover, throughout this and the next subsection, we will write

$$\Phi(t,r) = \left(\frac{r^{d_w}}{t}\right)^{1/(d_w-1)},$$

where $d_w = 13/5$ was introduced at (1.5). We also set

$$\sigma_{x,r} = \inf\left\{n \geq 0 : d_{\mathcal{U}}(x, X_n^{\mathcal{U}}) = r\right\}, \qquad T_x = \inf\left\{n \geq 0 : X_n^{\mathcal{U}} = x\right\}. \tag{6.2}$$

Lemma 6.1 ([22, Proposition 3.3])**.** *There exist constants c_i and q_i such that if $B_{\mathcal{U}}(x,r)$ and $B_{\mathcal{U}}(x, c_1\lambda^{-q_1}r)$ are λ-good, then*

$$p_t^{\mathcal{U}}(x,x) \leq c_2\lambda^{q_2}t^{-d_f/d_w}, \qquad \text{if } \tfrac{1}{2}r^{d_w} \leq t \leq r^{d_w} \text{ and } t \in \mathbb{N},$$
$$P_x^{\mathcal{U}}\left(\sigma_{x,r} > c_3\lambda^{-q_3}r^{d_w}\right) \geq c_4\lambda^{-q_4}.$$

Lemma 6.2. *There exists λ_0 such that if $\lambda \geq \lambda_0$ then on the event $F_1(\lambda,n)$ it holds that*

$$p_t^{\mathcal{U}}(x,x) \leq c_2\lambda^{q_2}t^{-d_f/d_w}, \quad \text{for all } x \in B_\infty(0,n),\ n^{d_w\kappa}e^{-\lambda^{\varepsilon_0}/2} \leq t \leq n^{d_w\kappa} \text{ and } t \in \mathbb{N},$$
$$P_x^{\mathcal{U}}(\sigma_{x,r} > c_3\lambda^{-q_3}r^{d_w}) \geq c_4\lambda^{-q_4}, \quad \text{for all } x \in B_\infty(0,n) \text{ and } n^\kappa e^{-\lambda^{\varepsilon_0}/2} \leq r \leq n^\kappa. \tag{6.3}$$

Proof. This follows immediately from Lemma 6.1 and the definition of $F_1(\lambda, n)$. □

Lemma 6.3. *There exist λ_0, $q > 0$ such that the follows holds. Suppose $F_1(\lambda,n)$ holds with $\lambda \geq \lambda_0$. If $n/8 \leq d_\infty(0,x) \leq 7n/8$, then*

$$P_0^{\mathcal{U}}(T_x \leq t) \leq 2\exp(-c\lambda^{-q}\Phi(t,n^\kappa)) \quad \text{for } t \geq n^{\kappa d_w}e^{-\lambda^{\varepsilon_0}/2}. \tag{6.4}$$

Proof. Let y be the first point on the path $\gamma(0,x)$ with $d_\infty(0,y) \geq n/9$. By Lemma 2.10 we have $d_{\mathcal{U}}(0,y) \geq c\lambda^{-\kappa}n^\kappa$, and it is clear that $T_y \leq T_x$. Let $m \geq 1$, and set

$$R = c\lambda^{-\kappa}n^\kappa, \quad r = \frac{R}{m}, \quad s = \frac{t}{m}.$$

Moreover, let x_i be points on $\gamma(0,x)$ such that $x_0 = 0$ and $d_{\mathcal{U}}(x_{i-1}, x_i) = r$, and ξ_i be the duration from T_{x_i} until $X^{\mathcal{U}}$ leaves $B_{\mathcal{U}}(x_i, r)$. Using this notation, we have that

$$T_y \geq \sum_{i=0}^{m-1}\xi_i \geq s\sum_{i=0}^{m-1}\mathbf{1}_{\{\xi_i > s\}}.$$

We next choose $m \in \mathbb{N}$ so that $m \in [m_1, m_1+1]$ where

$$m_1^{d_w-1} = \lambda^{-b}\frac{n^{\kappa d_w}}{t},$$

and $b > 0$ will be chosen later. We set $q = (d_wq_4 + b)/(d_w - 1)$. If $t \geq \lambda^{-b}n^{\kappa d_w}$, then $\Phi(t,n^\kappa) \leq \lambda^{b/(d_w-1)}$, and so our choice of q ensures that the probability bound in (6.4) holds.

Thus we will assume that $m_1 \geq 1$, so that $m_1 \leq m \leq 2m_1$ and $r = R/m \geq R/2m_1$. Now, the condition on r in (6.3) holds if $R/2m_1 = c\lambda^{-\kappa}n^{\kappa}/2m_1 \geq e^{-\lambda^{\varepsilon_0}/2}n^{\kappa}$. This is equivalent to

$$2^{d_w-1}\lambda^{-b}\frac{n^{\kappa d_w}}{t} \leq (c\lambda^{-\kappa}e^{\lambda^{\varepsilon_0}/2})^{d_w-1},$$

i.e. $t \geq c\lambda^{\kappa(d_w-1)-b}e^{-\lambda^{\varepsilon_0}(d_w-1)/2}n^{\kappa d_w}$, and we observe that this holds if $t \geq n^{\kappa d_w}e^{-3\lambda^{\varepsilon_0}/5}$ and λ is large enough. To apply (6.3), we will also need that $s = t/m$ satisfies $s \leq c_3\lambda^{-q_3}r^{d_w}$. After some algebra we find this requires $\lambda^{-b} \leq c_3\lambda^{-q_3-\kappa d_w}$. So, taking $b > q_3 + \kappa d_w$ and λ_0 large enough, this condition is also satisfied.

With the choice of m in the previous paragraph, we can apply the bounds in (6.3) to deduce that $\sum_{i=0}^{m-1}\mathbf{1}_{\{\xi_i > s\}}$ stochastically dominates a binomial random variable with parameters m and $p = c_4\lambda^{-q_4}$. Applying the following general bound for a binomial random variable η,

$$\mathbf{P}\left(|\eta - \mathbf{E}\eta| > t\right) \leq 2\exp(-t^2/(2\mathbf{E}\eta + 2t/3)),$$

we thus deduce that

$$P_0^{\mathcal{U}}(T_y < \tfrac{1}{2}c_4\lambda^{-q_4}t) = P_0^{\mathcal{U}}(T_y < \tfrac{1}{2}smp) \leq 2e^{-cmp} = 2\exp(-c\lambda^{-q_4}m),$$

and the result follows by a reparameterisation of t. $\square$

The following result improves upon the corresponding bound in [9, Proposition 4.15] by obtaining an upper bound for $p_T^{\mathcal{U}}(x,y)$ on a set for which the probability has a uniform (in n) lower bound. Whilst the estimate holds for a more limited range of times, it is enough for our purposes. We take $\varepsilon_1 < \varepsilon_0 = 1/40$.

Proposition 6.4. *Suppose $F_1(\lambda, n)$ holds with $\lambda \geq \lambda_0$, then*

$$p_t^{\mathcal{U}}(0,x) \leq \lambda^{q_5}t^{-d_f/d_w}\exp\left(-\lambda^{-q_5}\Phi(t,n^{\kappa})\right)$$

whenever $x \in B_{3n/4}(0) \setminus B_{n/2}(0)$, $e^{-\lambda^{\varepsilon_1}}n^{\kappa d_w} \leq t \leq n^{\kappa d_w}$ and $t \in \mathbb{N}$.

Proof. Let z_1 be the first point on the path $\gamma(0,x)$ with $d_\infty(0,z_1) \geq n/8$, and z_2 be the first point on the path $\gamma(x,0)$ with $d_\infty(x,z_2) \geq n/8$. Let A_0 be the set of points y in $\mathbb{Z}^2$ such that the path $\gamma(0,y)$ does not contain z_1, and $A_x = \mathbb{Z}^2 \setminus A_0$. Then, as in the proof of [7, Theorem 4.9],

$$\begin{aligned} P_0^{\mathcal{U}}(X_t^{\mathcal{U}} = x) &= P_0^{\mathcal{U}}(X_t^{\mathcal{U}} = x, X_{[t/2]}^{\mathcal{U}} \in A_0) + P_0^{\mathcal{U}}(X_t^{\mathcal{U}} = x, X_{[t/2]}^{\mathcal{U}} \in A_x) \\ &\leq 4P_x^{\mathcal{U}}(X_t^{\mathcal{U}} = 0, X_{[t/2]}^{\mathcal{U}} \in A_0) + P_0^{\mathcal{U}}(X_t^{\mathcal{U}} = x, X_{[t/2]}^{\mathcal{U}} \in A_x) \\ &\leq 4P_x^{\mathcal{U}}(X_t^{\mathcal{U}} = 0, T_{z_2} < t/2) + P_0^{\mathcal{U}}(X_t^{\mathcal{U}} = x, T_{z_1} < t/2). \end{aligned} \tag{6.5}$$

For the second term above,

$$\begin{aligned} P_0^{\mathcal{U}}(X_t^{\mathcal{U}} = x, T_{z_1} < t/2) &= E_0^{\mathcal{U}}\left(\mathbf{1}_{\{T_{z_1} < t/2\}}P_{z_1}^{\mathcal{U}}(X_{t-T_{z_1}}^{\mathcal{U}} = x)\right) \\ &\leq P_0^{\mathcal{U}}(T_{z_1} < t/2)\sup_{t/2 \leq s \leq t, s \in \mathbb{N}} P_{z_1}^{\mathcal{U}}(X_s^{\mathcal{U}} = x) \\ &\leq 4P_0^{\mathcal{U}}(T_{z_1} < t/2)\sup_{t/2 \leq s \leq t, s \in \mathbb{N}} \sqrt{p_s^{\mathcal{U}}(z_1,z_1)p_s^{\mathcal{U}}(x,x)} \end{aligned}$$

$$\leq c\lambda^q \exp\left(-c'\lambda^{-q}\Phi(t, n^\kappa)\right) \lambda^q t^{-d_f/d_w}.$$

Here we used the Cauchy-Schwarz for the penultimate bound, and Lemmas 6.2 and 6.3 to obtain the final one. The first term of (6.5) is bounded in the same way. $\square$

We now have all the pieces in place, and the one remaining lemma we give provides the means to put these together.

Lemma 6.5. *Let G_k, $k \geq 1$, be a sequence of sets with $\mathbf{P}(G_k) \to 1$ and let $T \in \mathbb{N}$. If we have $p_T^{\mathcal{U}}(0, x) \leq a_k$ on G_k for each k, then*

$$\mathbf{E}p_T^{\mathcal{U}}(0, x) \leq a_1 + \sum_{k=2}^{\infty} a_k \mathbf{P}(G_{k-1}^c). \tag{6.6}$$

Proof. Set $A_1 = G_1$ and $A_k = G_k \backslash G_{k-1}$ for $k \geq 2$. Since $\mathbf{P}(\cup_k G_k) = 1$, we have $\mathbf{P}(\cup_k A_k) = 1$, and thus

$$\mathbf{E}p_T^{\mathcal{U}}(0, x) = \sum_{k=1}^{\infty} \mathbf{E}(p_T^{\mathcal{U}}(0, x)\mathbf{1}_{A_k}) \leq \sum_{k=1}^{\infty} a_k \mathbf{P}(A_k) \leq a_1 \mathbf{P}(A_1) + \sum_{k=2}^{\infty} a_k \mathbf{P}(G_{k-1}^c).$$

$\square$

Proof of the upper bound of Theorem 1.4. By [9, Theorem 4.4], we have that $\mathbf{E}p_{2T}^{\mathcal{U}}(0, 0) \leq cT^{-d_f/d_w}$. Hence, applying the Cauchy-Schwarz as in the proof of Proposition 6.4, we further have that, for all $x \in \mathbb{Z}^2$,

$$\mathbf{E}p_{2T}^{\mathcal{U}}(0, x) \leq \mathbf{E}[p_{2n}^{\mathcal{U}}(0, 0)^{1/2} p_{2T}^{\mathcal{U}}(x, x)^{1/2}] \leq \mathbf{E}(p_{2T}^{\mathcal{U}}(0, 0))^{1/2} \mathbf{E}(p_{2T}^{\mathcal{U}}(x, x))^{1/2} \leq cT^{-d_f/d_w}. \tag{6.7}$$

Hence if $d_\infty(0, x) \leq 16$, then the result follows.

Now let $d_\infty(0, x) \geq 16$. Choose n such that $x \in B_{3n/4}(0) \backslash B_{n/2}(0)$, and set $\Phi = \Phi(T, n^\kappa)$. Set for $k \geq 1$,

$$\lambda_k = k^{1/\varepsilon_0} \Phi^{1/(\varepsilon_0 + q_5)}.$$

Choose c_1 so that $\Phi \geq c_1$ implies that $\lambda_k^{-(d_w - 1)(\varepsilon_0 + q_5)} \geq e^{-\lambda_k^{\varepsilon_1}}$. If $\Phi \leq c_1$, then the estimate again follows from (6.7), so we assume that $\Phi > c_1$. We now use Lemma 6.5 with $G_k = F_1(\lambda_k, n)$. The definition of λ_k gives that $T \geq n^{\kappa d_w} \lambda_k^{-(d_w - 1)(\varepsilon_0 + q_5)} \geq n^{\kappa d_w} e^{-\lambda_k^{\varepsilon_1}}$, so Proposition 6.4 allows us to take

$$a_k = T^{-d_f/d_w} \lambda_k^{q_5} \exp\left(-\lambda_k^{-q_5} \Phi\right).$$

Thus the first term in the sum (6.6) is given by

$$a_1 = T^{-d_f/d_w} \Phi^{q_5/(q_5 + \varepsilon_0)} \exp(-\Phi^{\varepsilon_0/(q_5 + \varepsilon_0)}).$$

Appealing to Proposition 2.9, and for convenience replacing $\exp(-c\lambda^{1/16})$ with the weaker bound $\exp(-c\lambda^{\varepsilon_0})$, we see the kth term for $k \geq 2$ is bounded above by

$$T^{-d_f/d_w} k^{q_5/\varepsilon_0} \Phi^{q_5/(q_5 + \varepsilon_0)} \exp\left(-\Phi^{\varepsilon_0/(q_5 + \varepsilon_0)}(k^{-q_5/\varepsilon_0} + (k - 1))\right).$$

Summing this series, the bound follows with $\theta_2 = \varepsilon_0/(q_5 + \varepsilon_0)$. $\square$

6.2 Averaged heat kernel lower bound

In this subsection, we will use Theorem 5.9 to establish the averaged heat kernel lower bound from Theorem 1.4. The ideas of the following arguments are from [7, Section 4]. We first obtain deterministic diagonal and near-diagonal lower bounds that hold on realisations of $\mathcal{U}$ that occur with suitably high probability. We recall the notation $D^+(a)$ from 4 lines below (5.6), define $D^-(a)$ analogously for the corresponding part of the UST below $\mathcal{T}_0 \cap \mathcal{T}_{trunk}$, and set $D^\pm(a) := D^+(a) \cup D^-(a)$.

Lemma 6.6. *Let $\lambda \geq \lambda_0$, $m \geq m_0$ and $\lambda^7 \leq k \leq m^{1/2} \wedge c_1 N^4$. Moreover, let $\tilde{F}_*(k)$ be an event with the properties described in the statement of Theorem 5.9 for both $D^+(1/2)$ and $D^-(1/2)$, and in particular satisfies $\mathbf{P}_{\mathcal{T}}(\tilde{F}_*(k)^c) \leq cNe^{-c'k^{1/24}}$. Then there exist constants c_i, q_i such that on $\tilde{F}_*(k)$, if $x \in D^\pm(3/8)$ and $c_2 k^{1/4} \leq s \leq c_3 N$, then*

$$\tilde{p}_n^{\mathcal{U}}(x,x) \geq c_4 \lambda^{-3} k^{-5/2} s^{-1} m^{-2} \geq c_5 \lambda^{-q_1} k^{-q_2} n^{-d_f/d_w} \tag{6.8}$$

for $c_6 \lambda^{-1} k^{-5/2} s^2 m^{\kappa d_w} \leq n \leq c_7 \lambda^{-1} k^{-5/2} s^2 m^{\kappa d_w}$.

Proof. This can be obtained by modifying standard arguments. By a line-by-line modification of the proof of [21, Proposition 4.4.1, 4.4.3], for example, we have on $\tilde{F}_*(k)$ that

$$c' \lambda^{-1} k^{-5/2} s^2 m^{\kappa d_w} \leq E_x^{\mathcal{U}}\left(\sigma_{x, sm^\kappa}\right) \leq c\lambda s^2 m^{\kappa d_w}$$

for all $x \in D^\pm(3/8)$ and s in the given range. The above estimates and the Markov property (see [21, Proposition 4.4.3]) imply that the following holds on $\tilde{F}_*(k)$,

$$P_x^{\mathcal{U}}\left(\sigma_{x,s_k} > n\right) \quad \geq \quad \frac{c' \lambda^{-1} k^{-5/2} s^2 m^{\kappa d_w} - n}{c\lambda s^2 m^{\kappa d_w}},$$

for all $x \in D^\pm(3/8)$ and $n \geq 0$. Given this and the upper volume estimate that holds on $\tilde{F}_*(k)$, (6.8) can be proved as in [21, Proposition 4.4.4]. □

Lemma 6.7. *(a) Let $\lambda \geq \lambda_0$, $m \geq m_0$, $\lambda^7 \leq k \leq m^{1/2} \wedge c_1 N^4$ and $\alpha \in (0,1)$. Moreover, let $\tilde{F}_*(k)$ be an event as in Lemma 6.6 that satisfies $\mathbf{P}_{\mathcal{T}}(\tilde{F}_*(k)^c) \leq cNe^{-c'k^{1/24}}$. Then there exist constants c_i', q_i such that on $\tilde{F}_*(k)$, we have for $x \in D^\pm(3/8)$ and $y \in \mathcal{U}$ satisfying $d_{\mathcal{U}}(x,y) \leq 2s^\alpha m^\kappa$ for some $c_1' \lambda^{4/(1-\alpha)} k^{5/(1-\alpha)} \leq s \leq c_2' N$,*

$$\tilde{p}_n^{\mathcal{U}}(x,y) \geq c_3' \lambda^{-q_1} k^{-q_2} n^{-d_f/d_w} \text{ for } c_4' \lambda^{-1} k^{-5/2} s^2 m^{\kappa d_w} \leq n \leq c_5' \lambda^{-1} k^{-5/2} s^2 m^{\kappa d_w}. \tag{6.9}$$

(b) If $x_0 \in D(3/8) \cap \mathcal{T}_0$ and $x, y \in B_{\mathcal{U}}(x_0, s^{1/2} m^\kappa)$, then the same lower bound holds on an event $\tilde{F}_*(x_0,k,s)$ that satisfies $\mathbf{P}_{\mathcal{T}}(\tilde{F}(x_0,k,s)_*^c) \leq c\lambda s e^{-c'k^{1/24}}$.

Proof. By the discrete-time adaptation of [7, Lemma 4.3] (which can be obtained by applying estimates in [15, Section 4]) and Lemma 6.6, we have

$$\left| \frac{\tilde{p}_n^{\mathcal{U}}(x,y)}{\tilde{p}_n^{\mathcal{U}}(x,x)} - 1 \right|^2 \leq \frac{c d_{\mathcal{U}}(x,y)}{n \tilde{p}_n^{\mathcal{U}}(x,x)} \leq \frac{c' \lambda^4 k^5}{s^{1-\alpha}} \leq \frac{1}{4}.$$

Hence $|\tilde{p}_n^{\mathcal{U}}(x,y) - \tilde{p}_n^{\mathcal{U}}(x,x)| \leq \tilde{p}_n^{\mathcal{U}}(x,x)/2$, so we obtain

$$\tilde{p}_n^{\mathcal{U}}(x,y) \geq \tilde{p}_n^{\mathcal{U}}(x,x) - |\tilde{p}_n^{\mathcal{U}}(x,y) - \tilde{p}_n^{\mathcal{U}}(x,x)| \geq \tilde{p}_n^{\mathcal{U}}(x,x)/2 \geq c\lambda^{-q_1} k^{-q_2} n^{-d_f/d_w},$$

where we used Lemma 6.6 in the last inequality. This establishes part (a), and part (b) is obtained in the same way, but using Corollary 5.10 in place of Theorem 5.9. □

Definition 6.8. Let $M, N \in \mathbb{N}$, $\alpha = \frac{1}{2}$, $\lambda \geq \lambda_0$, $m \geq m_0$, $\mathcal{T}$ be a fixed tree as described after Proposition 5.3, and $x \in \mathcal{T}_0$ with $x \neq 0$. Set $r = d_{\mathcal{U}}(0,x)/N$ and let $z_0 = 0, z_1, \cdots, z_N = x$ be points on the path between 0 and x with $|d_{\mathcal{U}}(z_{i-1}, z_i) - r| \leq 1$ that are chosen in some fixed way. For $i = 1, \dots, N$, let ξ_i be the smallest integer k such that $\tilde{F}_*(z_i, k, k^{12})$, $\{|B_{\mathcal{U}}(z_{i-1}, \frac{1}{4}k^6 m^\kappa)| \geq m^2\}$ and $\{|B_{\mathcal{U}}(z_i, \frac{1}{4}k^6 m^\kappa)| \geq m^2\}$ hold. (Set $\xi_i = \infty$ if the requirements are not satisfied.) We then say that $G(q, x, N, M)$ holds if $\sum_{i=1}^N \xi_i^q \leq MN$.

Proposition 6.9. *It holds that*

$$\mathbf{P}_{\mathcal{T}}(G(q,x,N,M)) \geq 1 - \frac{c_q}{M} - cN\lambda \left(m^6 \wedge N\right) e^{-c' m^{1/48} \wedge (cN^{1/288})},$$

where c_q is a constant that depends on q.

Proof. By Corollary 5.10 and a simple union bound, it holds that

$$\mathbf{P}_{\mathcal{T}}\left(\xi_i > m^{1/2} \wedge (cN^{1/12}) \text{ for some } i\right) \leq N \times c\lambda \left(m^6 \wedge N\right) e^{-c' m^{1/48} \wedge (cN^{1/288})}.$$

Moreover, Corollary 5.10 and the Markov inequality yield that

$$\mathbf{P}_{\mathcal{T}}\left(\sum_{i=1}^N \xi_i^q \mathbf{1}_{\{\xi_i \leq m^{1/2} \wedge (cN^{1/12})\}} > MN\right) \leq \frac{c_q}{M}.$$

Putting these estimates together completes the proof. □

Proof of the lower bound of Theorem 1.4. For simplicity, we only consider the case when $x = (R, 0)$, where $R \in \mathbb{Z}_+$; see Remark 6.10 for the modifications necessary for the general case. Let $T \geq R$, and set $y = R^{\kappa d_w}/T$. We need to consider several cases. These will depend on constants $b, b' \geq 8$, which will be chosen below.

Case 1: $R \leq T \leq bR$. Let F be the event that the UST $\mathcal{U}$ contains the straight path along the x_1-axis between 0 and x. By considering the construction of $\mathcal{U}$ which starts by running S^x until it hits 0, we have $\mathbf{P}(F) \geq 4^{-R}$. Let z be the point adjacent to x on the path $\gamma(0,x)$. If the event F holds, then $P_0^{\mathcal{U}}(X_T \in \{x, z\}) \geq 4^{-T}$, and it follows immediately that $\mathbf{E}\tilde{p}_T^{\mathcal{U}}(0,x) \geq 4^{-R+T} \geq e^{-cT}$, which yields the desired lower bound for these values of R and T.

Case 2: $T \geq b' R^{\kappa d_w}$. To begin with, suppose that $R \geq 1$. We use Lemma 6.7, and take $\lambda = \lambda_0$. Given k we set $c_1 N^4 = k$, and choose $k = k_0 \geq \lambda_0$ large enough so that $\mathbf{P}_{\mathcal{T}}(\tilde{F}_*(k_0)^c) \leq \frac{1}{2}$. We take $s = c_1' \lambda_0^8 k_0^{10}$. As the constants λ, k, s do not depend on R or T, we can absorb them into constants c_i. The bound (6.9) holds for $T \in [c_1 m^{\kappa d_w}, c_2 m^{\kappa d_w}]$, so we choose m so that T is in this range; this gives that $m \geq cR$. (NB. By increasing the value of b' if necessary, we can further ensure that $m \geq m_0$ and $m^{1/2} \geq k_0$.) The construction of $\tilde{F}_*(k_0)$ in Section 5 implies that on this event $d_{\mathcal{U}}(0,x) \leq cs^{1/2} m^\kappa$, and thus we have the lower bound

$$\tilde{p}_T^{\mathcal{U}}(0,x) \geq cT^{-d_f/d_w}.$$

Since $\mathbf{P}(H_{N_1}) \geq \exp(-cN) \geq \exp(-c' k_0)$ and $\mathbf{P}_{\mathcal{T}}(\tilde{F}_*(k_0)) \geq \frac{1}{2}$, the averaged lower bound $\mathbf{E}\tilde{p}_T^{\mathcal{U}}(0,x) \geq cT^{-d_f/d_w}$ follows. If $R = 0$, then one can use the same event as for $R = 1$ to deduce the result, since one also has that $\tilde{p}_T^{\mathcal{U}}(0,0) \geq cT^{-d_f/d_w}$ on that event.

Case 3: $bR < T < b'R^{\kappa d_w}$. Choose $N, m \in \mathbb{N}$ to satisfy

$$N \le (b')^{\frac{\kappa d_w}{\kappa d_w - 1}} \left(\frac{|x|^{\kappa d_w}}{T} \right)^{1/(\kappa d_w - 1)} < N+1, \quad m \le R/N < m+1.$$

Note that $N + 1 \ge (b')^{\frac{\kappa d_w}{\kappa d_w - 1}} y^{1/(\kappa d_w - 1)} \ge 8$, and $R/N \ge b^{1/(\kappa d_w - 1)}$. Hence choosing b large enough we can ensure that $m \ge m_0$ and also that if $\mathcal{T}$ is a tree selected in the way described after Proposition 5.3, then $\mathbf{P}_{\mathcal{T}}(G(43/2, x, N, M)) \ge \frac{1}{2}$. The reason we take $q = 43/2$ is that this is the power of ξ_i that arises in the time range for the estimate (6.9). More precisely on $G(q, x, N, M)$, for each $i = 1, \dots, N$, it holds that

$$\tilde{p}^{\mathcal{U}}_{n_i}(z, y) \ge c_1 \lambda^{-q_1} \xi_i^{-q_2} n_i^{-d_f/d_w} \tag{6.10}$$

for $c_2 \lambda^{-1} \xi_i^{-5/2} (\xi_i^{12})^2 m^{\kappa d_w} \le n_i \le c_3 \lambda^{-1} \xi_i^{-5/2} (\xi_i^{12})^2 m^{\kappa d_w}$ and $z, y \in B_{\mathcal{U}}(z_i, \xi_i^6 m^\kappa)$. Since $24 - 5/2 = 43/2$, in the argument below, we will need to sum over the quantities $\xi_i^{43/2}$; restricting to the event $G(43/2, x, N, M)$ ensures that we can control this sum.

Now, since it holds that $d_{\mathcal{U}}(z_{i-1}, z_i) \le c\lambda m^\kappa$, the estimate (6.10) includes the case when $z \in B_{\mathcal{U}}(z_{i-1}, \frac{1}{4}\xi_i^6 m^\kappa)$ and $y \in B_{\mathcal{U}}(z_i, \frac{1}{4}\xi_i^6 m^\kappa)$. Setting $\tilde{T} := \sum_{i=1}^N n_i$, where n_i, $i = 1, \dots, N$, satisfy the previous constraints, we then have that

$$c_2 \lambda^{-1} \sum_i \xi_i^{43/2} m^{\kappa d_w} \le \tilde{T} \le c_3 \lambda^{-1} \sum_i \xi_i^{43/2} m^{\kappa d_w}. \tag{6.11}$$

Moreover, writing $B_i := B_{\mathcal{U}}(z_i, \frac{1}{4}(\xi_i^6 \wedge \xi_{i+1}^6) m^\kappa)$, we have that

$$\begin{aligned}
&\tilde{p}^{\mathcal{U}}_{\tilde{T}}(0, x) \\
&\quad \ge \frac{1}{2} \sum_{y_1 \in B_1} \cdots \sum_{y_{N-1} \in B_{N-1}} \tilde{p}^{\mathcal{U}}_{n_1}(0, y_1) \cdots \tilde{p}^{\mathcal{U}}_{n_N}(y_{N-1}, x) \mathbf{1}_{\{n_i - d_{\mathcal{U}}(y_{i-1}, y_i) \text{ is even for each } i\}} \\
&\quad \ge \prod_{j=1}^{N-1} \left(|B_j| \, c_1 \lambda^{-q_1} \xi_j^{-q_2} n_j^{-d_f/d_w} \right) c_1 \lambda^{-q_1} \xi_N^{-q_2} n_N^{-d_f/d_w} \\
&\quad \ge c_\lambda \tilde{T}^{-d_f/d_w} \exp\left(-c'_\lambda N - (q_2 + (43/2) d_f/d_w) \sum_{i=1}^N \log \xi_i - \frac{d_f}{d_w} \log \left(\sum_{i=1}^N \xi_i^{43/2} \right) \right) \\
&\quad \ge c_\lambda \tilde{T}^{-d_f/d_w} \exp(-c'_\lambda N),
\end{aligned}$$

where in the last inequality we used $\sum_{i=1}^N \log \xi_j \le \sum_{i=1}^N \xi^{43/2} \le MN$. Note that in (6.11), we may take $c_2 > 0$ as small as we like. (This is because $\tilde{p}^{\mathcal{U}}_n(x, x)$ is monotone decreasing, which means we can take c_6 in Lemma 6.6 as small as desired. Moreover, we can take c'_4 in Lemma 6.7 to match this). In particular, taking $c_2 \le 8^{-\kappa d_w} M^{-1}$, we obtain

$$c_2 \lambda^{-1} \sum_i \xi_i^{43/2} m^{\kappa d_w} \le 8^{-\kappa d_w} N m^{\kappa d_w} \le T.$$

Hence we may take $\tilde{T} \le T$. If $T \le c_3 \lambda^{-1} \sum_i \xi_i^{43/2} m^{\kappa d_w}$, then we can choose n_j so that $T = \tilde{T}$. If not, let $T' = T - \tilde{T} \le N r^{d_w}$. Let j_0 be such that ξ_{j_0} is minimal, and add N' extra steps between $B_{j_0 - 1}$ and B_{j_0} in the chaining argument above, each with time length satisfying the constraint of n_{j_0} and the total time of the additional steps is equal to T'. The latter constraints readily imply that $N' \leq cN$, and we further observe that each extra step contributes a factor of $c_\lambda \xi_{j_0}^{-(q_2+(43/2)d_f/d_w)}$ to the lower bound. Thus the total contribution is no less than $e^{-c_\lambda N}$. Taking the average over $G(43/2, x, N, M)$ and $\mathcal{T}$, we obtain the result. □

Remark 6.10. For a general $z = (z_1, z_2) \in \mathbb{Z}^2$ we need to replace the tree $\mathcal{T}$ defined in Section 5 by a tree which connects 0 and z. We replace the 'S-shaped' path $(\tilde{x}_i)_{i=0}^{N_1}$ defined just after Lemma 5.2 with a path for the which the central section has 'L' shape which connects 0 and $(z_1/m, z_2/m)$, and the rest of the path shields the central section from the remainder of $\mathbb{Z}^2$. The estimates of Section 5 and 6 all work for this path, and the proof of the lower bound on $\mathbf{E}\tilde{p}_n^{\mathcal{U}}(0, z)$ then follows.

7 Failure of the elliptic Harnack inequalities

The aim of this section is to make precise and prove Corollary 1.3. We start by giving the definition of the elliptic Harnack inequality that we consider, as well as a related metric doubling property.

Definition 7.1. Let $(X_\omega, d_\omega, \mu_\omega)$ be a weighted random graph.
(i) We say that the *large scale elliptic Harnack inequalities (LS-EHI)* hold (for the random walk associated with $(X_\omega, d_\omega, \mu_\omega)$) if there exists a deterministic constant $C > 1$ and, for each $x_0 \in X_\omega$, there exists an $R_{1,x_0}(\omega) > 0$ such that the following inequality is satisfied

$$\sup_{B_{d_\omega}(x_0,R)} u \leq C \inf_{B_{d_\omega}(x_0,R)} u.$$

for any $x_0 \in X_\omega$, $R \geq R_{1,x_0}(\omega)$ and any non-negative bounded harmonic function u on $B_{d_\omega}(x_0, 2R)$.
(ii) We say that the *large scale metric doubling property (LS-MD)* holds if there exists a deterministic constant $M \in \mathbb{N}$ and, for each $x_0 \in X_\omega$, there exists $R_{2,x_0}(\omega) > 0$ such that, for any $x_0 \in X_\omega$ and $R \geq R_{2,x_0}(\omega)$, $B_{d_\omega}(x_0, R)$ can be covered by M balls of radius $R/2$.

The main result in this section is the following.

Theorem 7.2. *(LS-EHI) does not hold for the random walk on $\mathcal{U}$.*

For the proof, we use the following proposition.

Proposition 7.3. *(LS-EHI) implies (LS-MD).*

Proof. The proof is a line-by-line modification of [10, Theorem 3.11]. Hence we omit it. □

The following lemma will be used to check that (LS-MD) is violated for $\mathcal{U}$.

Lemma 7.4. *There exists a constant $\delta > 0$ such that, $\mathbf{P}$-a.s., one can find a divergent sequence $(R_n)_{n\geq 1}$ for which there exist at least n disjoint $d_{\mathcal{U}}$-balls of radius δR_n contained in $B_{\mathcal{U}}(0, R_n)$.*

Proof. Let $(G(i))_{i\geq 1}$ be the events described in the proof of Theorem 1.1, where it was shown that $G(i)$ holds infinitely often, $\mathbf{P}$-a.s. Now, let $(z_j)_{j=1}^{\varepsilon(\log i)^{1/2}}$ be the vertices at the centres of the top row of boxes in the configuration shown in Figure 2 for $N = D_i/m_i$ and $m = m_i$. On $G(i)$, we have that

$$d_{\mathcal{U}}(0, z_j) \leq C \left(\frac{D_i}{m_i}\right) m_i^{\kappa}, \qquad \forall j = 1, 2, \ldots, \varepsilon(\log i)^{1/2},$$

and also

$$d_{\mathcal{U}}(z_j, L_i) \geq c \left(\frac{D_i}{m_i} - 2\right) m_i^{\kappa}, \qquad \forall j = 1, 2, \ldots, \varepsilon(\log i)^{1/2},$$

where L_i is the bottom row of boxes in the configuration shown in Figure 2. It readily follows that there exist at least $\varepsilon(\log i)^{1/2}$ disjoint $d_{\mathcal{U}}$-balls of radius $\frac{c}{2}\left(\frac{D_i}{m_i} - 2\right) m_i^{\kappa}$ contained in $B_{\mathcal{U}}(0, C\left(\frac{D_i}{m_i}\right) m_i^{\kappa})$. Hence taking

$$R_n = C \left(\frac{D_i}{m_i}\right) m_i^{\kappa}, \qquad \delta = \frac{c}{4C},$$

where $i = e^{(n/\varepsilon)^2}$ yields the result. □

Proof of Theorem 7.2. Let $\delta > 0$, and suppose that LS-MD holds. Then there exists a constant $M' = M'(M, \delta)$ such that: for each $x \in X_\omega$, there exists $R'_{x,\omega} < \infty$, such that if $R \geq R'_{x,\omega}$, then the ball $B_{\mathcal{U}}(x, R)$ can be covered by M' balls of radius δR. However, Lemma 7.4 shows that this fails for $\mathcal{U}$. Hence Proposition 7.3 yields the result. □

8 Scaling limits

In this section, we prove the results stated in the introduction concerning scaling limits of the random walk, namely Theorems 1.7 and 1.8, and Corollaries 1.10 and 1.12.

Proof of Theorem 1.7. By the separability of the Gromov-Hausdorff-vague topology (see, for example, [1, Proposition 5.12]), it is possible to suppose that we have a sequence $(\mathcal{U}_n)_{n\geq 1}$ of copies of $\mathcal{U}$, all built on the same probability space, so that

$$\left(\mathcal{U}_n, n^{-\kappa} d_{\mathcal{U}_n}, n^{-2}\mu_{\mathcal{U}_n}, 0\right) \to \left(\mathcal{T}, d_{\mathcal{T}}, \mu_{\mathcal{T}}, \rho_{\mathcal{T}}\right)$$

holds $\mathbf{P}$-a.s. (Note that for this part of the article, we do not need the spatial embeddings into $\mathbb{R}^2$.) It follows from [1, Proposition 5.9] that, $\mathbf{P}$-a.s., there exists a metric space (M, d_M) so that the spaces $(\mathcal{U}_n, n^{-\kappa} d_{\mathcal{U}_n})$, $n \geq 1$, and $(\mathcal{T}, d_{\mathcal{T}})$ can be isometrically embedded into (M, d_M) in such a way that: 0 and $\rho_{\mathcal{T}}$ are mapped to a common point, 0_M say; the embedded measures $n^{-2}\mu_{\mathcal{U}_n}$ converge vaguely to the embedded version of $\mu_{\mathcal{T}}$; and, for all but countably many r, the sets $\mathcal{U}_n \cap \bar{B}_M(0_M, r)$, where $\bar{B}_M(0_M, r)$ is the closed ball in M of radius r centred at 0_M, converge to $\mathcal{T} \cap \bar{B}_M(0_M, r)$ with respect to the Hausdorff distance between compact subsets of (M, d_M). As a consequence (see, [13, Theorem 7.1]), we moreover have that the laws of the random walks $(X^{\mathcal{U}_n}_{tn^{\kappa+2}})_{t\geq 0}$ converge weakly to the law of $(X^{\mathcal{T}}_t)_{t\geq 0}$, when these are considered as measures on $D(\mathbb{R}_+, M)$. Consequently, we have that the Assumptions 1 and 5 of [14] are satisfied (actually Assumption 1 requires the convergence of measures of balls under the various laws, but this condition is readily relaxed to the requirement that the balls in question are continuity sets for the limiting measure), and hence we can apply [14, Theorem 1 and Proposition 14] to deduce that the associated transition densities satisfy, $\mathbf{P}$-a.s.,

$$\left(n^2\tilde{p}^{\mathcal{U}_n}_{\lfloor tn^{2+\kappa}\rfloor}(0,0)\right)_{t>0}\rightarrow\left(p_t^{\mathcal{T}}(\rho_{\mathcal{T}},\rho_{\mathcal{T}})\right)_{t>0}. \tag{8.1}$$

Reparameterising this, the first part of the theorem follows.

In view of the distributional limit we have just proved, to prove the scaling limit at (1.12) it will suffice to check the following integrability condition: for any $p\geq 1$, there exists a constant $C\in(0,\infty)$ such that

$$\sup_{n\geq 1} n^{d_f/d_w}\left\|\tilde{p}^{\mathcal{U}}_n(0,0)\right\|_p\leq C, \tag{8.2}$$

where $\|\cdot\|_p$ is the L_p norm with respect to $\mathbf{P}$. Now, by Lemma 6.2, on the event $F_1(\lambda,n^{1/d_w\kappa})$, it holds that $\tilde{p}^{\mathcal{U}}_n(0,0)\leq c\lambda^q n^{-d_f/d_w}$. Hence, if $\Lambda_n:=\inf\{\lambda\geq 1:\ F_1(\lambda,n^{1/d_w\kappa})\text{ holds}\}$, then

$$n^{d_f/d_w}\left\|\tilde{p}^{\mathcal{U}}_n(0,0)\right\|_p\leq c\left\|\Lambda_n^q\right\|_p. \tag{8.3}$$

Since Proposition 2.9 yields that the right-hand side above is uniformly bounded in n, this completes the proof. □

In preparation for the proof of Theorem 1.8, we verify the equicontinuity of the averaged heat kernel under scaling.

Proposition 8.1. *There exists a constant $C\in(0,\infty)$ such that*

$$\sup_{n\geq 1} n^{d_f/d_w}\left|\mathbf{E}\tilde{p}^{\mathcal{U}}_{\lfloor tn\rfloor}\left(0,[xn^{\frac{1}{\kappa d_w}}]\right)-\mathbf{E}\tilde{p}^{\mathcal{U}}_{\lfloor tn\rfloor}\left(0,[yn^{\frac{1}{\kappa d_w}}]\right)\right|\leq Ct^{-d_f/2d_w}|x-y|^{\kappa/2},$$

for all $x,y\in\mathbb{R}^2$, $t>0$.

Proof. From [14, Lemmas 9 and 10], we have for every $x,y\in\mathbb{Z}^2$ and $n\geq 1$ that

$$\left(\tilde{p}^{\mathcal{U}}_n(0,x)-\tilde{p}^{\mathcal{U}}_n(0,y)\right)^2\leq\frac{2d_{\mathcal{U}}(x,y)\tilde{p}^{\mathcal{U}}_{2\lceil n/2\rceil}(0,0)}{n}. \tag{8.4}$$

Hence Jensen's and Hölder's inequalities yield that, for any $\varepsilon>0$,

$$\left|\mathbf{E}\tilde{p}^{\mathcal{U}}_n(0,x)-\mathbf{E}\tilde{p}^{\mathcal{U}}_n(0,y)\right|\leq\sqrt{\frac{2}{n}}\left\|d_{\mathcal{U}}(x,y)^{1/2}\right\|_{1+\varepsilon}\left\|\tilde{p}^{\mathcal{U}}_{2\lceil n/2\rceil}(0,0)^{1/2}\right\|_{\frac{1+\varepsilon}{\varepsilon}},$$

where we again write $\|\cdot\|_p$ for the L_p norm with respect to $\mathbf{P}$. Now, by Theorem 1.6, it holds that, for suitably small ε,

$$\left\|d_{\mathcal{U}}(x,y)^{1/2}\right\|_{1+\varepsilon}\leq C|x-y|^{\kappa/2}. \tag{8.5}$$

Moreover, from (8.3) (and Proposition 2.9), we have that

$$\left\|\tilde{p}^{\mathcal{U}}_{2\lceil n/2\rceil}(0,0)^{1/2}\right\|_{\frac{1+\varepsilon}{\varepsilon}}\leq Cn^{-d_f/2d_w}.$$

Since $n^{d_f/d_w}\times n^{-1/2}\times(n^{\frac{1}{\kappa d_w}})^{\kappa/2}\times n^{-d_f/2d_w}=1$, combining these estimates readily yields the result. □

We moreover note the following rerooting invariance property of the limiting tree.

Proposition 8.2. *(a) For any* $x \in \mathbb{R}^2$,

$$\mathbf{P}\left(\left|\phi_{\mathcal{T}}^{-1}(x)\right| > 1\right) = 0.$$

(b) For any $x \in \mathbb{R}^2$,

$$\left(\mathcal{T}, d_{\mathcal{T}}, \mu_{\mathcal{T}}, \phi_{\mathcal{T}} - x, \phi_{\mathcal{T}}^{-1}(x)\right) \stackrel{d}{=} \left(\mathcal{T}, d_{\mathcal{T}}, \mu_{\mathcal{T}}, \phi_{\mathcal{T}}, \rho_{\mathcal{T}}\right). \tag{8.6}$$

Proof. We first prove the result of part (a) for $x \in \mathbb{R}^2\backslash\{0\}$. In particular, by the scale and rotational invariance properties of (1.9) and (1.10), respectively, we have that $\mathbf{P}(|\phi_{\mathcal{T}}^{-1}(x)| > 1)$ is a constant for $x \in \mathbb{R}^2\backslash\{0\}$. Moreover, as was noted in the proof of [4, Theorem 1.3], we know that the Lebesgue measure of $\{x : |\phi_{\mathcal{T}}^{-1}(x)| > 1\}$ is zero, $\mathbf{P}$-a.s. Hence, it follows from Fubini's theorem that $\mathbf{P}(|\phi_{\mathcal{T}}^{-1}(x)| > 1) = 0$ for all $x \in \mathbb{R}^2\backslash\{0\}$.

We next prove part (b) for $x \in \mathbb{R}^2\backslash\{0\}$. To begin with, we note from part (a) that the left-hand side of (8.6) is a well-defined measured, rooted spatial tree, $\mathbf{P}$-a.s. Moreover, by the separability of the Gromov-Hausdorff-type topology that we are considering (see [4, Proposition 3.4]), it is possible to suppose that we have realisations of the relevant random objects built on a common probability space so that

$$\left(\mathcal{U}, \delta_n^{\kappa} d_{\mathcal{U}}, \delta_n^2 \mu_{\mathcal{U}}, \delta_n \phi_{\mathcal{U}}, 0\right) \to \left(\mathcal{T}, d_{\mathcal{T}}, \mu_{\mathcal{T}}, \phi_{\mathcal{T}}, \rho_{\mathcal{T}}\right),$$

almost-surely as $n \to \infty$ (cf. the proof of Theorem 1.7). It follows that it is almost-surely possible to choose a (random) $x_n^R \in \delta_n \mathbb{Z}^2$ such that

$$\left(\mathcal{U}, \delta_n^{\kappa} d_{\mathcal{U}}, \delta_n^2 \mu_{\mathcal{U}}, \delta_n \phi_{\mathcal{U}} - x_n^R, x_n^R\right) \to \left(\mathcal{T}, d_{\mathcal{T}}, \mu_{\mathcal{T}}, \phi_{\mathcal{T}} - x, \phi_{\mathcal{T}}^{-1}(x)\right).$$

In particular, this implies that $|x_n^R - x| \to 0$, almost-surely. Moreover, let $x_n \in \delta_n \mathbb{Z}^2$ be a deterministic sequence such that $|x_n - x| \to 0$. One can then deduce from [9, Theorem 1.1] (and the Borel-Cantelli lemma) that there exists a deterministic subsequence n_i along which $\delta_{n_i}^{\kappa} d_{\mathcal{U}}(x_{n_i}, x_{n_i}^R) \to 0$, almost-surely. Hence we find that, almost-surely,

$$\left(\mathcal{U}, \delta_{n_i}^{\kappa} d_{\mathcal{U}}, \delta_{n_i}^2 \mu_{\mathcal{U}}, \delta_{n_i} \phi_{\mathcal{U}} - x_{n_i}, x_{n_i}\right) \to \left(\mathcal{T}, d_{\mathcal{T}}, \mu_{\mathcal{T}}, \phi_{\mathcal{T}} - x, \phi_{\mathcal{T}}^{-1}(x)\right).$$

By the translation invariance of $\mathcal{U}$ (see [34, Theorem 2.3]), the left-hand side here has the same distribution as $(\mathcal{U}, \delta_{n_i}^{\kappa} d_{\mathcal{U}}, \delta_{n_i}^2 \mu_{\mathcal{U}}, \delta_{n_i} \phi_{\mathcal{U}}, 0)$, which we know converges in distribution to $(\mathcal{T}, d_{\mathcal{T}}, \mu_{\mathcal{T}}, \phi_{\mathcal{T}}, \rho_{\mathcal{T}})$, and so the result follows.

Finally, let $x \in \mathbb{R}^2\backslash\{0\}$. Then, from part (b) (for such x), we know that $|\phi_{\mathcal{T}}^{-1}(0)|$ is equal in distribution to $|\phi_{\mathcal{T}}^{-1}(x)|$. And, from part (a) (again, for such x), we know the latter is $\mathbf{P}$-a.s. equal to 1. In particular, we find that $\phi_{\mathcal{T}}^{-1}(0) = \{\rho_{\mathcal{T}}\}$, $\mathbf{P}$-a.s. Hence both part (a) and part (b) are readily extended to include the point $x = 0$. □

Proof of Theorem 1.8. From [4, Theorem 1.4], we know that, under the averaged law $\int P_0^{\mathcal{U}}(\cdot)\mathbf{P}(d\mathcal{U})$,

$$\left(n^{-1/d_w\kappa} X_{tn}^{\mathcal{U}}\right)_{t \geq 0} \xrightarrow{d} \left(\phi_{\mathcal{T}}\left(X_t^{\mathcal{T}}\right)\right)_{t \geq 0}. \tag{8.7}$$

Applying this in conjunction with Theorem 1.7 (specifically (1.12)) and Proposition 8.1, elementary analysis arguments yield that, for each fixed $t \in (0,\infty)$, $\phi_{\mathcal{T}}(X_t^{\mathcal{T}})$ admits a density $q_t(x) \in C(\mathbb{R}^2, \mathbb{R})$ satisfying the convergence result of part (c). From this, part (a) of the theorem is a simple consequence of Proposition 8.1. Moreover, given the continuity of the density

q_t in the spatial variable, part (b) follows from the scale and rotational invariance properties at (1.9) (1.10).

For part (d), we again recall from the proof of [4, Theorem 1.3] that the Lebesgue measure of $\{x : |\phi_{\mathcal{T}}^{-1}(x)| > 1\}$ is zero, and also from the latter result that $\mu_{\mathcal{T}} = \mathcal{L} \circ \phi_{\mathcal{T}}$, where $\mathcal{L}$ is two-dimensional Lebesgue measure. Putting these observations together yields

$$\begin{aligned}
\int_B q_t(x)dx &= \mathbf{E}\left(P_0^{\mathcal{U}}(X_t^{\mathcal{T}} \in \phi_{\mathcal{T}}^{-1}(B))\right) \\
&= \mathbf{E}\left(\int_{\phi_{\mathcal{T}}^{-1}(B)} p_t^{\mathcal{T}}(\rho_{\mathcal{T}}, x)\mu_{\mathcal{T}}(dx)\right) \\
&= \mathbf{E}\left(\int_B p_t^{\mathcal{T}}(\rho_{\mathcal{T}}, \phi_{\mathcal{T}}^{-1}(x))\mathbf{1}_{\{|\phi_{\mathcal{T}}^{-1}(x)|=1\}}dx\right) \\
&= \int_B \mathbf{E}\left(p_t^{\mathcal{T}}(\rho_{\mathcal{T}}, \phi_{\mathcal{T}}^{-1}(x))\right)dx,
\end{aligned}$$

for all Borel $B \subseteq \mathbb{R}^2$, where we have applied Fubini's theorem and Proposition 8.2(a) to obtain the final equality. It follows that the desired equality holds for Lebesgue almost-every x, and so to complete the proof, it will suffice to show that, for each fixed t, $p_t(x) := \mathbf{E}(p_t^{\mathcal{T}}(\rho_{\mathcal{T}}, \phi_{\mathcal{T}}^{-1}(x)))$ is continuous in x. Now, from the rotational invariance of (1.10), we have that $p_t(x)$ is constant on circles centred at the origin. And thus, to check continuity at $x \neq 0$, it will suffice to show that $p_t(\lambda x) \to p_t(x)$ as $\lambda \to 1$. Moreover, by the scale invariance property (1.9), this is equivalent to checking that $p_{\lambda t}(x) \to p_t(x)$ as $\lambda \to 1$, and doing this is our next aim. Arguing as in the proof of [20, Theorem 10.4], for example, and applying the monotonicity of the on-diagonal part of the heat kernel, one can deduce that, for $s, t > r$,

$$\left|p_s^{\mathcal{T}}(\rho_{\mathcal{T}}, \phi_{\mathcal{T}}^{-1}(x)) - p_t^{\mathcal{T}}(\rho_{\mathcal{T}}, \phi_{\mathcal{T}}^{-1}(x))\right| \leq 2r^{-1}|t-s|\sqrt{p_{r/2}^{\mathcal{T}}(\rho_{\mathcal{T}}, \rho_{\mathcal{T}})p_{r/2}^{\mathcal{T}}(\phi_{\mathcal{T}}^{-1}(x), \phi_{\mathcal{T}}^{-1}(x))}.$$

From this, the Cauchy-Schwarz inequality, the rerooting invariance of Proposition 8.2(b), and (1.12), we see that

$$|p_s(x) - p_t(x)| \leq 2r^{-1}|t-s|p_{r/2}(0) = Cr^{-1-d_f/d_w}|t-s|,$$

which implies that $p_{\lambda t}(x) \to p_t(x)$ as $\lambda \to 1$, as desired. To deal with the case $x = 0$, we again argue as in the proof of [20, Theorem 10.4], for example (cf. (8.4)), to deduce that

$$\left|p_t^{\mathcal{T}}(\rho_{\mathcal{T}}, \rho_{\mathcal{T}}) - p_t^{\mathcal{T}}(\rho_{\mathcal{T}}, \phi_{\mathcal{T}}^{-1}(x))\right| \leq t^{-1}\sqrt{d_{\mathcal{T}}(\rho_{\mathcal{T}}, \phi_{\mathcal{T}}^{-1}(x))p_t^{\mathcal{T}}(\rho_{\mathcal{T}}, \rho_{\mathcal{T}})}.$$

This implies

$$|p_t(0) - p_t(x)| \leq t^{-1}\left\|d_{\mathcal{T}}(\rho_{\mathcal{T}}, \phi_{\mathcal{T}}^{-1}(x))^{1/2}\right\|_{1+\varepsilon}\left\|p_t^{\mathcal{T}}(\rho_{\mathcal{T}}, \rho_{\mathcal{T}})^{1/2}\right\|_{\frac{1+\varepsilon}{\varepsilon}}.$$

From (8.1) and (8.2), we have that the term $\|p_t^{\mathcal{T}}(\rho_{\mathcal{T}}, \rho_{\mathcal{T}})^{1/2}\|_{\frac{1+\varepsilon}{\varepsilon}}$ is finite for any $\varepsilon > 0$. Moreover, arguing as in the proof of Proposition 8.2, for each x, one has that there exists a sequence (x_n) such that $|x_n - x| \to 0$ and, along a subsequence (n_i),

$$n_i^{-\kappa}d_{\mathcal{U}}(0, x_{n_i}) \xrightarrow{d} d_{\mathcal{T}}(\rho_{\mathcal{T}}, \phi_{\mathcal{T}}^{-1}(x)).$$

Hence from (8.5) we obtain that $\|d_{\mathcal{T}}(\rho_{\mathcal{T}},\phi^{-1}(x))^{1/2}\|_{1+\varepsilon} \leq C|x|^{\kappa/2}$, where the constant does not depend on x. In particular, these estimates imply that $p_t(x) \to p_t(0)$ as $|x| \to 0$, and so the proof is complete. $\square$

Proof of Corollary 1.10. This is an easy application of Theorems 1.4 and 1.8(c). $\square$

Proof of Corollary 1.11. We begin with the bounds for $|X_n^{\mathcal{U}}|$. Integrating the upper bound of Theorem 1.4, we find that

$$n^{-p/d_w\kappa}\mathbf{E}(E_0^{\mathcal{U}}|X_n^{\mathcal{U}}|^p) \leq n^{-p/d_w\kappa}\sum_{x\in\mathbb{Z}^2} c_1 n^{-d_f/d_w}|x|^p \exp\left\{-c_2\left(\frac{|x|^{\kappa d_w}}{n}\right)^{\frac{\theta_2}{d_w-1}}\right\} \leq c_3,$$

as required. The lower bound follows in a similar fashion.

For the upper bound on $d_{\mathcal{U}}(0,X_n^{\mathcal{U}})$ set $R_k = \lceil e^k n^{1/d_w}\rceil$, $B_k = B_{\mathcal{U}}(0,R_k)$ for $k \geq 0$, $B_{-1} = \emptyset$, and $D_k = B_k \setminus B_{k-1}$ for $k \geq 0$. Let $k_0 = ((d_w-1)/d_w)\log n$. Note that if $k > k_0$ then $R_k > n$, so that $P_0^{\mathcal{U}}(X_n^{\mathcal{U}} \in D_k) = 0$. (Recall we are looking at the discrete time walk.) Write $\sigma_k = \sigma_{0,R_k+1}$, where $\sigma_{x,r}$ was defined at (6.2). We then have

$$\begin{aligned}\mathbf{E}E_0^{\mathcal{U}} d_{\mathcal{U}}(0,X_n^{\mathcal{U}})^p &\leq \mathbf{E}\sum_{k=0}^{k_0} 2^p e^{pk} n^{p/d_w} P_0^{\mathcal{U}}(X_n^{\mathcal{U}} \in D_k)\\ &\leq 2^p n^{p/d_w}\sum_{k=0}^{k_0} e^{pk}\mathbf{E}P_0^{\mathcal{U}}(\sigma_k \leq n). \qquad (8.8)\end{aligned}$$

Now, by an almost identical argument to Lemma 6.3, it is possible to check that on the event $F_1(\lambda_k, \lambda_k R_k^{1/\kappa})$ with $\lambda_k := (4k)^{40}$ we have

$$P_0^{\mathcal{U}}(\sigma_k \leq n) \leq C\exp(-c\lambda_k^{-q_4} m_k) = C\exp(-c\lambda_k^{-q} e^{kd_w/(d_w-1)});$$

here, $m_k := (c_3\lambda_k^{-q_3})^{1/(d_w-1)}\Phi(R_k^{d_w}/n)$ represents the number of steps into which the stopping time is decomposed, where c_3, q_3, q_4 are as in (6.3). Hence, by Proposition 2.9,

$$\mathbf{E}P_0^{\mathcal{U}}(\sigma_k \leq n) \leq \mathbf{P}\left(F_1(\lambda_k, \lambda_k R_k^{1/\kappa})^c\right) + C\exp(-c\lambda_k^{-q} e^{kd_w/(d_w-1)}) \leq Ce^{-ck^{40/16}},$$

which implies that the sum in (8.8) is finite, and so establishes the upper bound. The lower bound is proved by the same argument as is used in [9, Theorem 4.4]. $\square$

Proof of Corollary 1.12. From (8.7) we have under the averaged law that

$$\left(n^{-1/d_w\kappa}\left|X_{tn}^{\mathcal{U}}\right|\right)_{t\geq 0} \xrightarrow{d} \left(\left|\phi_{\mathcal{T}}\left(X_t^{\mathcal{T}}\right)\right|\right)_{t\geq 0}.$$

Part (a) now follows using the uniform integrability given by Corollary 1.11.

For part (b), we start by noting that the convergence at (1.8) implies that the same result holds if $\delta\phi_{\mathcal{U}}$ is replaced by the map $x \mapsto \delta^\kappa d_{\mathcal{U}}(0,x)$, and $\phi_{\mathcal{T}}$ is replaced by the map $x \mapsto d_{\mathcal{T}}(\rho_{\mathcal{T}},x)$. As a consequence, in place of the random walk convergence result of (8.7), one obtains that

$$\left(n^{-1/d_w} d_{\mathcal{U}}(0,X_{tn}^{\mathcal{U}})\right)_{t\geq 0} \xrightarrow{d} \left(d_{\mathcal{T}}(\rho_{\mathcal{T}},X_t^{\mathcal{T}})\right)_{t\geq 0}.$$

(Concretely, apply [13, Theorem 7.2].) Part (b) then also follows from Corollary 1.11. $\square$

Remark 8.3. Let $\mathcal{R} := \{x : |\phi_{\mathcal{T}}^{-1}(\{x\})| = 1\}$. With $\mathbf{P}$-probability one, we have that $\mathcal{L}(\mathcal{R}^c) = 0$, where we again use $\mathcal{L}$ to denote Lebesgue measure on $\mathbb{R}^2$, and moreover $0 \in \mathcal{R}$ (see Proposition 8.2 and its proof). Since $\mu_{\mathcal{T}}(\phi_{\mathcal{T}}^{-1}(\mathcal{R}^c)) = \mathcal{L}(\mathcal{R}^c)$ (by [4, Theorem 1.3]), it follows that, $\mathbf{P}$-a.s., for any $x \in \mathcal{R}$ and $t \geq 0$,

$$P^{\mathcal{T}}_{\phi_{\mathcal{T}}^{-1}(x)}\left(\phi_{\mathcal{T}}(X^{\mathcal{T}}_t) \in \mathcal{R}\right) = \int_{\phi_{\mathcal{T}}^{-1}(\mathcal{R})} p^{\mathcal{T}}_t(\phi_{\mathcal{T}}^{-1}(x), y)\mu_{\mathcal{T}}(dy) = 1,$$

where $P^{\mathcal{T}}_{\phi_{\mathcal{T}}^{-1}(x)}$ is the quenched law of $X^{\mathcal{T}}$ started from $\phi_{\mathcal{T}}^{-1}(x)$. It readily follows that, when started from $x \in \mathcal{R}$ (including from $x = 0$), $\phi_{\mathcal{T}}(X^{\mathcal{T}})$ is a Markov process, and moreover has transition density that is determined by $(p^{\mathcal{T}}_t(\phi_{\mathcal{T}}^{-1}(y), \phi_{\mathcal{T}}^{-1}(z)))_{y,z\in\mathcal{R}}$ (and which is defined arbitrarily elsewhere). On the other hand, if τ is a stopping time for $\phi_{\mathcal{T}}(X^{\mathcal{T}})$ such that $P^{\mathcal{T}}_{\phi_{\mathcal{T}}^{-1}(x)}(\phi_{\mathcal{T}}(X^{\mathcal{T}}_\tau) \in \mathcal{R}^c) > 0$, then it is clear that the quenched law of $(\phi_{\mathcal{T}}(X^{\mathcal{T}}_{\tau+t}))_{t\geq 0}$ does not only depend on $\phi_{\mathcal{T}}(X^{\mathcal{T}}_\tau)$, and so $\phi_{\mathcal{T}}(X^{\mathcal{T}})$ is not strong Markov. Indeed, the situation is somewhat similar to that of reflecting Brownian motion in a planar domain with a slit removed (cf. comments in [12, Section 3]), though the slit is replaced in our case by the dense set $\mathcal{R}^c \subseteq \mathbb{R}^2$, which we note coincides with the 'dual trunk' studied in [35, Section 10].

A Appendix: Short LERW paths

In this section we improve the estimates in [8] to prove Theorem 2.7. We begin by considering the following situation, which is described in terms of parameters $m, n, N \in \mathbb{N}$ satisfying $4 \leq n \leq m \leq m + 2n \leq N$, cf. [8, Definition 1.4]. Let $B_m = B_\infty(0, m)$, $B_N = B_\infty(0, N)$, and $x \in \partial_R B_m$, where for a square B we write $\partial_R B$ for the right-hand side of the interior boundary of B. Moreover, let $x_1 = x + (\frac{n}{2}, 0)$, and define $A_n(x) = B_\infty(x, n/4)$. Finally, we also suppose we are given a subset $K \subseteq B_m$ that contains a path in B_m from 0 to x. Importantly, we note that the latter assumption was not made in [8]; it is the key to removing the terms in $\log(N/n)$ in [8, Lemmas 4.6 and 6.1, and Propositions 6.2 and 6.3]. We also remark that in [8] the balls B_n and B_N were in the ℓ_2 norm on $\mathbb{Z}^2$ rather than the ℓ_∞ norm, but this makes no essential difference to the arguments.

The first result of the section concerns the Green's function G of a simple random walk S on $\mathbb{Z}^2$. Given a subset $A \subsetneq \mathbb{Z}^2$, we write $G_A(y, z)$ for the expected number of visits that S makes to z when it starts at y up until it exits A. In the proof, we write $\mathbf{P}_x$ for the law of the random walk started from x, and $\mathbf{E}_x$ for the corresponding expectation.

Lemma A.1. *There exist constants c_i such that, for $y, z \in A_n(x)$,*

$$c_1 \log\left(\frac{n}{1 \vee |y-z|}\right) \leq G_{B_N \setminus K}(y, z) \leq c_2 \log\left(\frac{n}{1 \vee |y-z|}\right). \tag{A.1}$$

Proof. Set $A_1 = B_{5n/16}(x_1)$ and $A_2 = B_{3n/8}(x_1)$. We note that

$$c_1 \log\left(\frac{n}{1 \vee |y-z|}\right) \leq G_{A_1}(y, z) \leq G_{A_2}(y, z) \leq c_2 \log\left(\frac{n}{1 \vee |y-z|}\right).$$

(Cf. The applications of results from [28, Chapter 6] that appear as [8, Proposition 2.4].) Hence, since $G_{B_N\setminus K}(y, z) \geq G_{A_1}(y, z)$, the lower bound is immediate. For the upper bound, writing

T_A and τ_A for the hitting and exit time of a subset $A \subseteq \mathbb{Z}^2$ by the simple random walk S, respectively, we have

$$\begin{aligned} G_{B_N \setminus K}(y,z) &= G_{A_2}(y,z) + \mathbf{E}_y \left(G_{B_N \setminus K}(S_{\tau_{A_2}}, z) \right) \\ &\leq c_2 \log \left(\frac{n}{1 \vee |y-z|} \right) + \max_{w \in \partial A_2} \mathbf{P}_w(T_{A_1} < T_K) \max_{w' \in \partial A_1} G_{B_N \setminus K}(w', z). \end{aligned}$$

By the discrete Harnack inequality (see [28, Theorem 6.3.9], for example) and the fact that K contains a path from x to 0, we have that $\mathbf{P}_w(T_{A_1} < T_K) \leq 1 - c_3$ for all $w \in \partial A_2$. Further, for $w' \in \partial A_1$ we have

$$\mathbf{P}_{w'}(T_z < \tau_{A_2}) \leq 1 \wedge \frac{c_4}{\log n}.$$

(Again, cf. [8, Proposition 2.4].) Combining these estimates gives $G_{B_N \setminus K}(z,z) \leq c_2 \log(n) + (1-c_3) G_{B_N \setminus K}(z,z)$, and thus $G_{B_N \setminus K}(z,z) \leq \frac{c_2}{c_3} \log(n)$. Hence

$$G_{B_N \setminus K}(y,z) \leq c_2 \log \left(\frac{n}{1 \vee |y-z|} \right) + c_5,$$

which yields the bound (A.1). □

Next, let $\tilde{S}$ be a random walk started at x and conditioned to leave B_N before its first return to K. We write $\tilde{G}(\cdot,\cdot)$ for the Green's function of $\tilde{S}$.

Lemma A.2 (Cf. [8, Lemma 4.6])**.** *There exist constants c_i such that, for $z \in A_n(x)$ we have $c_1 \leq \tilde{G}(x,z) \leq c_2$.*

Proof. We follow the proof in [8]. Taking $y = z$ in (A.1) we can improve the upper bound on $G_{B_N \setminus K}(z,z)$ in [8, (4.10)] to $c \log n$. Using Lemma A.1 again, we can improve the upper bound in the equation above [8, (4.11)], and hence improve the upper bound in [8, (4.11)] from $c \log(N/n)/\log N$ to $c/\log n$. With these new bounds the argument of [8, Lemma 4.6] gives that $\tilde{G}(x,z) \leq c_2$. □

The following two results refine some conditional hitting time estimates from [8].

Lemma A.3 (cf. [8, (6.1)])**.** *There exists a constant c_1 such that if $D_1 = \partial_R B_\infty(x, n/16)$ and $K' = K \setminus \{x\}$, then, for $v \in D_1$,*

$$\mathbf{P}_v \left(T_x < \tau_{B_\infty(x,n/8)} \mid T_x < T_{K'} \wedge \tau_{B_N} \right) \geq c_1 > 0.$$

Proof. Write $B' = B_{n/8}(x)$. The second displayed equation on [8, p. 2409] gives

$$\mathbf{P}_v \left(T_x < \tau_{B_{n/8}(x)} \mid T_x < T_{K'} \wedge \tau_{B_N} \right) = \frac{G_{B' \setminus K}(v,v)}{G_{B_N \setminus K}(v,v)} \times \frac{\mathbf{P}_x(T_v < \tau_{B'} \wedge T_K^+)}{\mathbf{P}_x(T_v < \tau_{B_N} \wedge T_K^+)}, \tag{A.2}$$

where $T_K^+ = \min\{j \geq 1 : S_j \in K\}$. As in Lemma A.1 we have that $G_{B_N \setminus K}(v,v) \leq c \log n$, and so the ratio of Green's functions in (A.2) is bounded below by a constant $c > 0$. Using the strong Markov property at $\tau_{B'}$ we obtain

$$\mathbf{P}_x(T_v < \tau_{B_N} \wedge T_K^+) \leq \mathbf{P}_x(T_v < \tau_{B'} \wedge T_K^+) + \mathbf{P}_x(\tau_{B'} \leq T_K^+) \max_{y \in \partial B'} \mathbf{P}_y(T_v < \tau_{B_N} \wedge T_K^+).$$

The argument at the top of [8, p. 2410] gives that

$$\mathbf{P}_x(\tau_{B'} \le T_K^+) \le c(\log n)\,\mathbf{P}_x(T_v < \tau_{B'} \wedge T_K^+).$$

Moreover, for $y \in \partial B'$,

$$\mathbf{P}_y(T_v < \tau_{B_N} \wedge T_K^+) \le \frac{G_{\mathbb{Z}^2\setminus K}(y,v)}{G_{\mathbb{Z}^2\setminus K}(v,v)},$$

and as in Lemma A.1 we have $G_{\mathbb{Z}^2\setminus K}(y,v) \le c$, $G_{\mathbb{Z}^2\setminus K}(v,v) \ge c\log n$. Combining these estimates concludes the proof. □

Lemma A.4 (cf. [8, (6.2)]). *There exists a constant $c > 0$ such that if $w \in \partial_R B_\infty(x,n)$, then*

$$\mathbf{P}_w\left(\tau_{B_N} < T_{B_\infty(x,7n/8)} \,\middle|\, \tau_{B_N} < T_K\right) \ge c. \tag{A.3}$$

Proof. As on [8, p. 2410], we let y_0 be the point in $B_n(x)$ that maximises $\mathbf{P}_y(\tau_{B_N} < T_K)$. Writing $B_7 = B_\infty(x,7n/8)$, $T_7 = T_{B_7}$, we have

$$\begin{aligned}\mathbf{P}_{y_0}(\tau_{B_N} < T_K) &= \mathbf{P}_{y_0}(\tau_{B_N} < T_K \wedge T_7) + \mathbf{E}^{y_0}(\mathbf{1}_{\{T_7 < \tau_{B_N}\wedge T_K\}}\mathbf{P}_{S_{T_7}}(\tau_{B_N} < T_K))\\ &\le \mathbf{P}_{y_0}(\tau_{B_N} < T_K \wedge T_7) + \max_{v\in\partial B_7}\mathbf{P}_v(\tau_{B_N} < T_K).\end{aligned}$$

Since K contains a path from 0 to x, the discrete Harnack inequality (again, see [28, Theorem 6.3.9], for example) gives us that there exists a constant $p_1 > 0$ such that

$$\mathbf{P}_v(\tau_{B_\infty(x,n)} < T_K) \le 1 - p_1, \qquad \text{for all } v \in \partial B_7.$$

Thus

$$\mathbf{P}_{y_0}(\tau_{B_N} < T_K) \le \mathbf{P}_{y_0}(\tau_{B_N} < T_K \wedge T_7) + (1-p_1)\mathbf{P}_{y_0}(\tau_{B_N} < T_K),$$

which proves (A.3) in the case $w = y_0$. We can now use a reflection argument as on [8, p. 2410-2411] to obtain the general case. □

These estimates now lead to an improved lower bound on the length of a LERW. Recall the definition of the conditioned r.w. $\tilde{S}$, and set $L_1 = \mathcal{L}(\mathcal{E}_{B_N}(\tilde{S}))$, $L_2 = \mathcal{E}_{B_n(x)}(L_1)$.

Lemma A.5 (cf. [8, Lemma 6.1]). *There exists a constant $c > 0$ such that, for any $z \in A_n(x)$,*

$$\mathbf{P}(z \in L_2) \ge cn^{\kappa-2}.$$

Proof. Using Lemmas A.4 and A.3 to replace [8, (6.1),(6.2)], this follows as in [8]. □

Proposition A.6 (cf. [8, Proposition 6.2 and 6.3]). *There exist constants c_1, c_2 and $p > 0$ such that*

$$\begin{aligned}&c_1 n^\kappa \le \mathbf{E}M \le c_2 n^\kappa,\\ &\mathbf{E}(M^2) \le c_2 n^{2\kappa},\\ &\mathbf{P}(M \le c_3 n^\kappa) \le 1-p.\end{aligned} \tag{A.4}$$

Proof. Given Lemmas A.5 and A.2 the bounds on $\mathbf{E}(M)$ and $\mathbf{E}(M^2)$ follow as in [8]. The final inequality is then immediate from a second moment bound. $\square$

Proof of Theorem 2.7. We follow the proof of [8, Proposition 6.6], first proving the result in the case when $D = B_N(0)$, where $N/2 \le nk \le 3N/4$ for some $k \ge 4$. Set $L = \mathcal{L}(\mathcal{E}_{B_N(0)}(S^0))$, and, for $j = 1, \dots k$, let $\gamma_j = \mathcal{E}_{B_{jn}(0)}(L)$. Let x_j be the point where L first exits $B_{jn}(0)$, and $B_j = B_n(x_j)$. Let α_j be the path L from x_{j-1} to its first exit from B_{j-1}, and V_j be the number of hits by α_j on the set B_{j-1}. Let $\mathcal{F}_j$ be the σ-field generated by γ_j. Using the domain Markov property for the LERW (see [24]) and then (A.4), we have

$$\mathbf{P}\left(V_j \le c_3 n^\kappa | \mathcal{F}_{j-1}\right) = \mathbf{P}\left(M^{\gamma_{j-1}}_{(j-1)n,n,N,x_{j-1}} \le c_3 n^\kappa\right) \le 1 - p. \tag{A.5}$$

Let $\eta_j = \mathbf{1}_{\{V_j \le c_3 n^\kappa\}}$. By (A.5), $\sum_{j=1}^k \eta_j$ stochastically dominates a binomial random variable with parameters k and p, and so there exists a constant $c > 0$ such that

$$\mathbf{P}\left(\sum_{j=1}^k \eta_j < \tfrac{1}{2} pk\right) \le e^{-ck}.$$

Setting $L' = \mathcal{E}_{B_{nk}(0)}(L)$, we have $|L'| \ge c_3 n^\kappa \sum_{j=1}^k \eta_j$, and thus as $N/2 \le nk \le 3N/4$ we obtain

$$\mathbf{P}\left(|L'| < ck^{-1/4} N^\kappa\right) \le e^{-ck};$$

taking $k = c\lambda^{1/(\kappa-1)} = c\lambda^4$ this gives the result when $D = B_N$. Note that the proof above actually gives the lower bound for the length of L' rather than L, so we can use Lemma 2.1 with $D_1 = B_N$, $D_2 = D$ to obtain a lower bound of the same form for $|\mathcal{E}_{B_N(0)}(\mathcal{L}(\mathcal{E}_D(S^0)))|$. $\square$

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Martin T. Barlow
Department of Mathematics, University of British Columbia, Vancouver, B.C., V6T 1Z2, Canada.
E-mail: `barlow@math.ubc.ca`

David A. Croydon
Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606-8502, Japan.
E-mail: `croydon@kurims.kyoto-u.ac.jp`

Takashi Kumagai
Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606-8502, Japan.
E-mail: `kumagai@kurims.kyoto-u.ac.jp`