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Stability of standing waves for L^2-critical nonlinear Schrödinger equations with attractive inverse-power potential

By

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Abstract

We consider stability of standing waves for L^2-critical nonlinear Schrödinger equations with an attractive inverse-power potential. We prove that if the frequency of ground-state standing wave is sufficiently large, then it is orbitally stable. Our results are extensions of the results of Fukuizumi (2005), in which similar results were proven for L^2-critical nonlinear Schrödinger equations with smooth potentials such as harmonic potential.

§ 1. Introduction

We consider the following nonlinear Schrödinger equations with the L^2-critical nonlinearity and an inverse-power potential:

$$i\partial_t u = -\Delta u - \frac{\gamma}{|x|^\alpha} u - |u|^{p-1}u, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^N, \tag{1.1}$$

where $N \in \mathbb{N}$, $\gamma > 0$, $0 < \alpha < \min\{2, N\}$, and $p = 1 + 4/N$. It is known that the Cauchy problem for (1.1) is locally well-posed in $H^1(\mathbb{R}^N, \mathbb{C})$ if $1 < p < 2^* - 1$ (see [4]), where 2^* is the Sobolev critical exponent defined by

$$2^* := \begin{cases} \infty & \text{if } N = 1, 2, \\ \dfrac{2N}{N-2} & \text{if } N \geq 3. \end{cases}$$

The L^2-norm and the energy

$$E(v) := \frac{1}{2}\|\nabla v\|_{L^2}^2 - \frac{\gamma}{2}\int_{\mathbb{R}^N} \frac{|v|^2}{|x|^\alpha} - \frac{1}{p+1}\|v\|_{L^{p+1}}^{p+1}$$

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are conserved quantities of (1.1).

The aim of this note is to prove stability of ground-state standing waves for (1.1) with sufficiently large frequency. Our results are inspired by Fukuizumi [7], where the similar results were proven for L^2-critical nonlinear Schrödinger equations with smooth potentials such as harmonic potential. However, Fukuizumi [7] does not treat the case of potentials having singularities such as inverse-power potentials. In this note we extend the results of [7] to the case of inverse-power potentials.

Eq. (1.1) has standing wave solutions with the form

$$u_\omega(t,x) = e^{i\omega t}\phi(x),$$

where $\omega \in \mathbb{R}$ is the frequency, and $\phi \in H^1(\mathbb{R}^N, \mathbb{C}) \setminus \{0\}$ is a nontrivial solution of the elliptic equation

$$-\Delta\phi + \omega\phi - \frac{\gamma}{|x|^\alpha}\phi - |\phi|^{p-1}\phi = 0, \quad x \in \mathbb{R}^N. \tag{1.2}$$

Eq. (1.2) is written as $S'_\omega(\phi) = 0$, where S_ω is the action defined by

$$\begin{aligned} S_\omega(v) &:= E(v) + \frac{\omega}{2}\|v\|_{L^2}^2 \\ &= \frac{1}{2}\|\nabla v\|_{L^2}^2 + \frac{\omega}{2}\|v\|_{L^2}^2 - \frac{\gamma}{2}\int_{\mathbb{R}^N} \frac{|v|^2}{|x|^\alpha} - \frac{1}{p+1}\|v\|_{L^{p+1}}^{p+1}. \end{aligned}$$

We define the set of all ground states by

$$\mathcal{G}_\omega := \{\, \phi \in \mathcal{A}_\omega \mid S_\omega(\phi) \le S_\omega(\psi) \text{ for all } \psi \in \mathcal{A}_\omega \,\},$$

where $\mathcal{A}_\omega$ is the set of all nontrivial solutions:

$$\mathcal{A}_\omega := \{\, \phi \in H^1(\mathbb{R}^N, \mathbb{C}) \mid \phi \ne 0,\ S'_\omega(\phi) = 0 \,\}.$$

The following results on existence and uniqueness of ground states are known (see e.g. [6, 9] for existence and see [5] for uniqueness).

Proposition 1.1. *Let $1 < p < 2^* - 1$ and $\omega > -e_0$, where*

$$e_0 := \inf\left\{ \|\nabla v\|_{L^2}^2 - \gamma\int_{\mathbb{R}^N} \frac{|v|^2}{|x|^\alpha} \;\middle|\; v \in H^1(\mathbb{R}^N, \mathbb{C}),\ \|v\|_{L^2} = 1 \right\} < 0$$

is the smallest eigenvalue of the operator $-\Delta - \gamma|x|^{-\alpha}$. Then $\mathcal{G}_\omega$ is not empty.

Moreover, there exists the positive, radial, and decreasing function $\phi_\omega \in \mathcal{A}_\omega$ such that

$$\mathcal{G}_\omega = \{\, e^{i\theta}\phi_\omega \mid \theta \in \mathbb{R} \,\}.$$

In particular, $\omega \mapsto \phi_\omega$ is a C^1-mapping from $]-e_0, \infty[$ to $H^1_{\mathrm{rad}}(\mathbb{R}^N, \mathbb{R})$.

Hereafter, we denote the unique positive radial ground state by ϕ_ω. To state our results, we define orbital stability of standing waves.

Definition 1.2. We say that a standing wave solution $u(t) = e^{i\omega t}\phi$ is orbitally *stable* if for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $u_0 \in H^1(\mathbb{R}^N, \mathbb{C})$ satisfies $\|u_0 - \phi\|_{H^1} < \delta$, then the solution $u(t)$ of (1.1) with $u(0) = u_0$ exists globally in time and satisfies

$$\inf_{\theta\in\mathbb{R}} \|u(t) - e^{i\theta}\phi\|_{H^1} < \varepsilon$$

for all $t \in \mathbb{R}$.

We say that a standing wave solution is orbitally *unstable* if it is not stable.

We review some known results related to our works. First, we recall the case of $\gamma = 0$. Cazenave and Lions [3] proved that if $1 < p < 1 + 4/N$, then the ground-state standing wave $e^{i\omega t}\phi_\omega$ is stable for any $\omega > 0$. On the other hand, Berestycki and Cazenave [2] proved that if $1 + 4/N \le p < 2^* - 1$, then the standing wave is unstable for any $\omega > 0$ (see also [16] for the case $p = 1 + 4/N$).

Next, we recall the cases of $\gamma > 0$. The following results are known. For any $1 < p < 2^* - 1$, if ω is sufficiently close to $-e_0$, then the standing wave $e^{i\omega t}\phi_\omega$ is stable [9]. When $1 < p < 1 + 4/N$, if ω is sufficiently large, then the standing wave is stable [9]. When $1+4/N < p < 2^*-1$, if ω is sufficiently large, then the standing wave $e^{i\omega t}\phi_\omega$ is unstable [8] (see also [6, 14] for strong instability).

For the nonlinear Schrödinger equation with a suitable attractive potential

$$i\partial_t u = -\Delta u + V(x)u - |u|^{p-1}u, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^N,$$

Fukuizumi [7] proved that even when $p = 1 + 4/N$, the standing wave is stable for sufficiently large ω. These phenomena are different from the case of $\gamma = 0$. We can find similar results in [13] for double power nonlinear Schrödinger equations and in [10] for nonlinear Schrödinger equations with an attractive delta potential.

In this note we prove the similar results as Fukuizumi [7] for (1.1). The following is the main result of this note.

Theorem 1.3. *Let $p = 1+4/N$. Then there exists $\omega_* > -e_0$ such that if $\omega > \omega_*$, the standing wave solution $e^{i\omega t}\phi_\omega$ of* (1.1) *is stable.*

Remark. In the case $N \ge 3$ and $\alpha = 2$, the equation (1.1) has the scaling invariance, that is, if $u(t)$ is a solution of (1.1), then $\lambda^{2/(p-1)}u(\lambda^2 t, \lambda x)$ with $\lambda > 0$ is also a solution of (1.1). Therefore, when $p = 1 + 4/N$, we can show strong instability of ground states for any $\omega > 0$ by using the same argument of Berestycki and Cazenave [2] as in the case without potential.

The proof of Theorem 1.3 is based on the argument of Fukuizumi [7]. We use the following sufficient conditions for stability of standing waves.

Proposition 1.4 ([11, 15])**.** *Let* $1 < p < 2^* - 1$*. If* $\partial_\omega \|\phi_\omega\|_{L^2}^2 > 0$ *at* $\omega = \omega_0$*, then the standing wave* $e^{i\omega_0 t}\phi_{\omega_0}(x)$ *is stable.*

In [7], the assumption in Proposition 1.4 is verified for sufficiently large ω. One of the key of the proof in [7] is the uniform L^2_{rad}-boundedness in ω of the linearized inverse operators around the ground states. We can show the uniform boundedness by using regularity of the potential. However, in the case of (1.1), even when $v \in C_c(\mathbb{R}^N)$, the function $|x|^{-\alpha}v$ is not L^2-function if $\alpha \geq N/2$ and $v(0) > 0$. From this observation, the L^2-boundedness seems to be not effective, and we cannot establish the L^2-boundedness. Instead here, by investigating the properties of linearized operators in more details and by using the homogeneity of the potential $-\gamma|x|^{-\alpha}$, we establish the uniform H^1_{rad}-H^{-1}_{rad} boundedness of the inverse operators (Lemma 2.3) and use it to verify the positivity of the derivative $\partial_\omega \|\phi_\omega\|_{L^2}^2$.

Remark. We seem to generalize our results to the case with more general potentials $V(x)$, but we do not pursue this further in this note.

Remark. The results of [7, 8, 9] and ours are summarized as follows. When $\omega \searrow -e_0$, the ground state of (1.2) converges to that of the linear equation $-\Delta\phi + V(x)\phi = 0$ up to some scaling (see [9, Section 4]). Therefore, the stability can be understood by regarding (1.1) as a perturbation of the linear equation. On the other hand, when $\omega \to \infty$, the rescaled ground state of (1.2) converges to that of the nonlinear equation with $\omega = 1$ and $\gamma = 0$ (see Section 2 below). Since the ground state without potential is stable if $p < 1 + 4/N$ and unstable if $p \geq 1 + 4/N$, we might expect that the ground state of (1.2) is unstable for large ω when $p \geq 1 + 4/N$. However, in the L^2-critical case $p = 1 + 4/N$, the effect of attractive potential for large ω is very weak but sufficient to contribute to stability of ground states.

We can expect that the ground state is stable for the whole range $]-e_0, \infty[$ of ω, but the stability for a middle range of ω is still not known.

The rest of this note is organized as follows: In section 2, we investigate properties of ground states and their linearized operators. In section 3, we prove Theorem 1.3 by using Proposition 1.4.

§ 2. Properties of ground states and their linearized operators

In this section we prepare some lemmas to prove Theorem 1.3. Hereafter we only consider real-valued and radial functions. We denote

$$L^2_{\mathrm{rad}}(\mathbb{R}^N) := L^2_{\mathrm{rad}}(\mathbb{R}^N, \mathbb{R}), \quad H^1_{\mathrm{rad}}(\mathbb{R}^N) := H^1_{\mathrm{rad}}(\mathbb{R}^N, \mathbb{R}), \quad H^{-1}_{\mathrm{rad}}(\mathbb{R}^N) := H^1_{\mathrm{rad}}(\mathbb{R}^N, \mathbb{R})^*.$$

We put

$$\tilde{\phi}_\omega(x) := \omega^{-1/(p-1)}\phi_\omega(x/\sqrt{\omega}).$$

Then we have $\phi_\omega(x) = \omega^{1/(p-1)}\tilde{\phi}_\omega(\sqrt{\omega}\,x)$, and $\tilde{\phi}_\omega$ satisfies

$$-\Delta\tilde{\phi}_\omega + \tilde{\phi}_\omega - \omega^{-1+\alpha/2}\frac{\gamma}{|x|^\alpha}\tilde{\phi}_\omega - \tilde{\phi}_\omega^p = 0, \quad x \in \mathbb{R}^N. \tag{2.1}$$

Let $\tilde{\phi}_\infty \in H^1_{\mathrm{rad}}(\mathbb{R})$ be the unique positive radial solution of

$$-\Delta\phi + \phi - \phi^p = 0, \quad x \in \mathbb{R}^N$$

(see e.g., [1] for existence and [12] for uniqueness). Then we have the following convergence results.

Lemma 2.1. *$\tilde{\phi}_\omega \to \tilde{\phi}_\infty$ in $H^1_{\mathrm{rad}}(\mathbb{R}^N)$ as $\omega \to \infty$.*

Proof. See [8]. □

By differentiating (2.1) with respect to ω, we get

$$\tilde{L}_\omega(\partial_\omega\tilde{\phi}_\omega) = -\omega^{-2+\alpha/2}\left(1 - \frac{\alpha}{2}\right)\frac{\gamma}{|x|^\alpha}\tilde{\phi}_\omega, \tag{2.2}$$

where

$$\tilde{L}_\omega := -\Delta + 1 - \omega^{-1+\alpha/2}\frac{\gamma}{|x|^\alpha} - p\tilde{\phi}_\omega^{p-1}.$$

Let

$$\tilde{L}_\infty := -\Delta + 1 - p\tilde{\phi}_\infty^{p-1}.$$

Then we have

$$\tilde{L}_\omega = \tilde{L}_\infty - \omega^{-1+\alpha/2}\frac{\gamma}{|x|^\alpha} - p(\tilde{\phi}_\omega^{p-1} - \tilde{\phi}_\infty^{p-1}). \tag{2.3}$$

The following nondegeneracy results are important for our analysis.

Lemma 2.2. *For $\omega \in\,]-e_0, \infty]$, the operator $\tilde{L}_\omega\colon H^1_{\mathrm{rad}}(\mathbb{R}^N) \to H^{-1}_{\mathrm{rad}}(\mathbb{R}^N)$ is injective. In particular, the range $\tilde{L}_\omega(H^1_{\mathrm{rad}}(\mathbb{R}^N))$ is dense in $H^{-1}_{\mathrm{rad}}(\mathbb{R}^N)$.*

Proof. The injectivity (i.e. nondegeneracy) were proven by [12, 17] in the case of $\omega = \infty$ and by [5] in the case of $-e_0 < \omega < \infty$. We show the density of the range $\tilde{L}_\omega(H^1_{\mathrm{rad}}(\mathbb{R}^N))$ in $H^{-1}_{\mathrm{rad}}(\mathbb{R}^N)$. We regard $\tilde{L}_\omega\colon D(\tilde{L}_\omega) \to L^2_{\mathrm{rad}}(\mathbb{R}^N)$ as the operator on $L^2_{\mathrm{rad}}(\mathbb{R}^N)$. Then since $\ker(\tilde{L}_\omega) = \{0\}$, it follows that $L^2_{\mathrm{rad}}(\mathbb{R}^N) = \tilde{L}_\omega(D(\tilde{L}_\omega))^- \oplus \ker(\tilde{L}_\omega) = \tilde{L}_\omega(D(\tilde{L}_\omega))^-$. Since $L^2_{\mathrm{rad}}(\mathbb{R}^N)$ is dense in $H^{-1}_{\mathrm{rad}}(\mathbb{R}^N)$, the range $\tilde{L}_\omega(D(\tilde{L}_\omega))$ is also dense in $H^{-1}_{\mathrm{rad}}(\mathbb{R}^N)$. Noting that $D(\tilde{L}_\omega) \subset H^1_{\mathrm{rad}}(\mathbb{R}^N)$, we have the conclusion. □

Since $\tilde{L}_\omega \colon H^1_{\mathrm{rad}}(\mathbb{R}^N) \to H^{-1}_{\mathrm{rad}}(\mathbb{R}^N)$ is injective, its inverse operator

$$\tilde{L}_\omega^{-1} \colon \tilde{L}_\omega(H^1_{\mathrm{rad}}(\mathbb{R}^N)) \to H^1_{\mathrm{rad}}(\mathbb{R}^N)$$

is defined and surjective. The following estimate is the key of our proof.

Lemma 2.3. *There exist $C_0 > 0$ and $\omega_0 > -e_0$ such that*

$$\|\tilde{L}_\omega^{-1} f\|_{H^1} \le C_0 \|f\|_{H^{-1}}$$

for all $\omega > \omega_0$ and $f \in \tilde{L}_\omega(H^1_{\mathrm{rad}}(\mathbb{R}^N))$.

Proof. It suffices to show that there exist $C_0 > 0$ and $\omega_0 > -e_0$ such that

$$\|v\|_{H^1} \le C_0 \|\tilde{L}_\omega v\|_{H^{-1}} \tag{2.4}$$

for all $v \in H^1_{\mathrm{rad}}(\mathbb{R}^N)$ and $\omega > \omega_0$. First, we show that there exists $C_1 > 0$ such that

$$\|v\|_{H^1} \le C_1 \|\tilde{L}_\infty v\|_{H^{-1}} \tag{2.5}$$

for all $v \in H^1_{\mathrm{rad}}(\mathbb{R}^N)$. If not, for any $n \in \mathbb{N}$ there exists $v_n \in H^1_{\mathrm{rad}}(\mathbb{R}^N)$ such that $\|v_n\|_{H^1} = 1$ and $\|\tilde{L}_\infty v_n\|_{H^{-1}} < 1/n$. This implies that

$$\begin{aligned} &\|\tilde{L}_\infty v_n\|_{H^{-1}} \to 0, \\ &|\langle \tilde{L}_\infty v_n, v_n \rangle| \le \|\tilde{L}_\infty v_n\|_{H^{-1}} \|v_n\|_{H^1} \to 0 \end{aligned} \tag{2.6}$$

as $n \to \infty$.

On the other hand, for any $g \in \tilde{L}_\infty(H^1_{\mathrm{rad}}(\mathbb{R}^N))$, since there exists $w \in H^1_{\mathrm{rad}}(\mathbb{R}^N)$ such that $\tilde{L}_\infty w = g$, we have

$$|\langle g, v_n \rangle| = |\langle \tilde{L}_\infty w, v_n \rangle| = |\langle \tilde{L}_\infty v_n, w \rangle| \le \|\tilde{L}_\infty v_n\|_{H^{-1}} \|w\|_{H^1} \to 0$$

as $n \to \infty$. Since $\tilde{L}_\infty(H^1_{\mathrm{rad}}(\mathbb{R}^N))$ is dense in $H^{-1}_{\mathrm{rad}}(\mathbb{R}^N)$ by Lemma 2.2, we see that $v_n \rightharpoonup 0$ weakly in $H^1_{\mathrm{rad}}(\mathbb{R}^N)$. Therefore, by $\|v_n\|_{H^1} = 1$, we obtain

$$\langle \tilde{L}_\infty v_n, v_n \rangle = \|v_n\|_{H^1}^2 - p \int_{\mathbb{R}^N} \tilde{\phi}_\infty^{p-1} |v_n|^2 = 1 - p \int_{\mathbb{R}^N} \tilde{\phi}_\infty^{p-1} |v_n|^2 \to 1$$

as $n \to \infty$. This contradicts (2.6). Thus, the inequality (2.5) holds.

Next, we show (2.4). Let $v \in H^1_{\mathrm{rad}}(\mathbb{R}^N)$. By (2.3) and (2.5) we have

$$\begin{aligned} \|\tilde{L}_\omega v\|_{H^{-1}} &\ge \|\tilde{L}_\infty v\|_{H^{-1}} - \omega^{-1+\alpha/2} \gamma \||x|^{-\alpha} v\|_{H^{-1}} - \|(\tilde{\phi}_\omega^{p-1} - \tilde{\phi}_\infty^{p-1}) v\|_{H^{-1}} \\ &\ge \frac{1}{C_1} \|v\|_{H^1} - \omega^{-1+\alpha/2} \gamma \||x|^{-\alpha} v\|_{H^{-1}} - \|(\tilde{\phi}_\omega^{p-1} - \tilde{\phi}_\infty^{p-1}) v\|_{H^{-1}}. \end{aligned}$$

The second and third terms in the right hand side are estimated as follows:

$$\||x|^{-\alpha}v\|_{H^{-1}} \lesssim \|v\|_{H^1},$$
$$\|(\tilde{\phi}_\omega^{p-1} - \tilde{\phi}_\infty^{p-1})v\|_{H^{-1}} \lesssim \|\tilde{\phi}_\omega - \tilde{\phi}_\infty\|_{H^1}\|v\|_{H^1}.$$

Therefore, since $\alpha < 2$ and $\tilde{\phi}_\omega \to \tilde{\phi}_\infty$ in $H^1(\mathbb{R}^N)$, if ω is sufficiently large, we obtain

$$\|\tilde{L}_\omega v\|_{H^{-1}} \geq \frac{1}{2C_1}\|v\|_{H^1}.$$

This completes the proof. □

§ 3. Proof of Theorem 1.3

In this section, we prove Theorem 1.3 by using Proposition 1.4.

Lemma 3.1. *For $\omega > -e_0$,*

$$(p-1)\int_{\mathbb{R}^N} \tilde{\phi}_\omega^p \partial_\omega \tilde{\phi}_\omega = \left(1 - \frac{\alpha}{2}\right)\omega^{-2+\alpha/2}\gamma \int_{\mathbb{R}^N} \frac{\tilde{\phi}_\omega^2}{|x|^\alpha}, \tag{3.1}$$

$$\|\tilde{\phi}_\omega\|_{L^2}^2 = \omega^{-1+\alpha/2}\left(1 - \frac{\alpha}{2}\right)\gamma \int_{\mathbb{R}^N} \frac{\tilde{\phi}_\omega^2}{|x|^\alpha} - \left\{\frac{(p+1)(N-2)}{2} - N\right\}\frac{1}{p+1}\|\tilde{\phi}_\omega\|_{L^{p+1}}^{p+1}. \tag{3.2}$$

Proof. First, we show (3.1). By multiplying $\partial_\omega \tilde{\phi}_\omega$ with (2.1) and integrating it, we have

$$\int_{\mathbb{R}^N} \nabla\tilde{\phi}_\omega \cdot \nabla\partial_\omega\tilde{\phi}_\omega + \int_{\mathbb{R}^N} \tilde{\phi}_\omega \partial_\omega \tilde{\phi}_\omega - \omega^{-1+\alpha/2}\gamma \int_{\mathbb{R}^N} \frac{\tilde{\phi}_\omega \partial_\omega \tilde{\phi}_\omega}{|x|^\alpha} - \int_{\mathbb{R}^N} \tilde{\phi}_\omega^p \partial_\omega \tilde{\phi}_\omega = 0. \tag{3.3}$$

By multiplying $\tilde{\phi}_\omega$ with (2.2) and integrating it, we have

$$\int_{\mathbb{R}^N} \nabla\tilde{\phi}_\omega \cdot \nabla\partial_\omega\tilde{\phi}_\omega + \int_{\mathbb{R}^N} \tilde{\phi}_\omega \partial_\omega \tilde{\phi}_\omega - \omega^{-1+\alpha/2}\gamma \int_{\mathbb{R}^N} \frac{\tilde{\phi}_\omega \partial_\omega \tilde{\phi}_\omega}{|x|^\alpha} - p\int_{\mathbb{R}^N} \tilde{\phi}_\omega^p \partial_\omega \tilde{\phi}_\omega = -\left(1 - \frac{\alpha}{2}\right)\omega^{-2+\alpha/2}\gamma \int_{\mathbb{R}^N} \frac{\tilde{\phi}_\omega^2}{|x|^\alpha}. \tag{3.4}$$

By (3.3) and (3.4), we obtain

$$(p-1)\int_{\mathbb{R}^N} \tilde{\phi}_\omega^p \partial_\omega \tilde{\phi}_\omega = \left(1 - \frac{\alpha}{2}\right)\omega^{-2+\alpha/2}\gamma \int_{\mathbb{R}^N} \frac{\tilde{\phi}_\omega^2}{|x|^\alpha}.$$

Next, we show (3.2). The action corresponding to the equation (2.1) is given by

$$\tilde{S}_\omega(v) = \frac{1}{2}\|\nabla v\|_{L^2}^2 + \frac{1}{2}\|v\|_{L^2}^2 - \omega^{-1+\alpha/2}\frac{\gamma}{2}\int_{\mathbb{R}^N}\frac{|v|^2}{|x|^\alpha} - \frac{1}{p+1}\|v\|_{L^{p+1}}^{p+1}.$$

We compute

$$\tilde{S}_\omega(\lambda^\delta\tilde{\phi}_\omega(\lambda\cdot)) = \frac{\lambda^{2\delta+2-N}}{2}\|\nabla\tilde{\phi}_\omega\|_{L^2}^2 + \frac{\lambda^{2\delta-N}}{2}\|\tilde{\phi}_\omega\|_{L^2}^2 - \lambda^{2\delta+\alpha-N}\omega^{\alpha/2-1}\frac{\gamma}{2}\int_{\mathbb{R}^N}\frac{\tilde{\phi}_\omega^2}{|x|^\alpha} - \frac{\lambda^{(p+1)\delta-N}}{p+1}\|\tilde{\phi}_\omega\|_{L^{p+1}}^{p+1}. \tag{3.5}$$

We take $\delta = (N-2)/2$. Then we have $2\delta+2-N = 0$, $2\delta-N = -2$, $2\delta+\alpha-N = -(2-\alpha)$, and $(p+1)\delta - N = (p+1)(N-2)/2 - N$. By differentiating (3.5) at $\lambda = 1$, since $\tilde{S}'_\omega(\tilde{\phi}_\omega) = 0$, we get

$$0 = -\|\tilde{\phi}_\omega\|_{L^2}^2 + \omega^{-1+\alpha/2}\left(1-\frac{\alpha}{2}\right)\gamma\int_{\mathbb{R}^N}\frac{\tilde{\phi}_\omega^2}{|x|^\alpha} - \left\{\frac{(p+1)(N-2)}{2} - N\right\}\frac{1}{p+1}\|\tilde{\phi}_\omega\|_{L^{p+1}}^{p+1}.$$

This completes the proof. □

Lemma 3.2. *If $p = 1+4/N$, then there exists $\omega_* > -e_0$ such that $\partial_\omega\|\phi_\omega\|_{L^2}^2 > 0$ for all $\omega > \omega_*$.*

Proof. Note that since $p = 1+4/N$, we have $\|\phi_\omega\|_{L^2} = \|\tilde{\phi}_\omega\|_{L^2}$. By differentiating (3.2) with respect to ω, we have

$$\begin{aligned}\partial_\omega\|\phi_\omega\|_{L^2}^2 &= \partial_\omega\|\tilde{\phi}_\omega\|_{L^2}^2 \\ &= -\omega^{-2+\alpha/2}\left(1-\frac{\alpha}{2}\right)^2\gamma\int_{\mathbb{R}^N}\frac{\tilde{\phi}_\omega^2}{|x|^\alpha} + 2\omega^{-1+\alpha/2}\left(1-\frac{\alpha}{2}\right)\gamma\int_{\mathbb{R}^N}\frac{\tilde{\phi}_\omega\partial_\omega\tilde{\phi}_\omega}{|x|^\alpha} \\ &\quad - \left\{\frac{(p+1)(N-2)}{2} - N\right\}\int_{\mathbb{R}^N}\tilde{\phi}_\omega^p\partial_\omega\tilde{\phi}_\omega.\end{aligned} \tag{3.6}$$

We note that

$$\partial_\omega\tilde{\phi}_\omega = -\omega^{-2+\alpha/2}\left(1-\frac{\alpha}{2}\right)\gamma\tilde{L}_\omega^{-1}\Big(\frac{\tilde{\phi}_\omega}{|x|^\alpha}\Big)$$

by (2.2), and that

$$\left.\frac{(p+1)(N-2)}{2} - N\right|_{p=1+4/N} = -\frac{4}{N} = -\left.(p-1)\right|_{p=1+4/N}.$$

Therefore, combining (3.6) and (3.1), we have

$$\begin{aligned}&\partial_\omega\|\phi_\omega\|_{L^2}^2 \\ &= \omega^{-2+\alpha/2}\frac{\alpha}{2}\left(1-\frac{\alpha}{2}\right)\gamma\int_{\mathbb{R}^N}\frac{\tilde{\phi}_\omega^2}{|x|^\alpha} + 2\omega^{-1+\alpha/2}\left(1-\frac{\alpha}{2}\right)\gamma\int_{\mathbb{R}^N}\frac{\tilde{\phi}_\omega\partial_\omega\tilde{\phi}_\omega}{|x|^\alpha} \\ &= \omega^{-2+\alpha/2}\frac{\alpha}{2}\left(1-\frac{\alpha}{2}\right)\gamma\int_{\mathbb{R}^N}\frac{\tilde{\phi}_\omega^2}{|x|^\alpha} - 2\omega^{-3+\alpha}\left(1-\frac{\alpha}{2}\right)^2\gamma^2\int_{\mathbb{R}^N}\frac{\tilde{\phi}_\omega}{|x|^\alpha}\tilde{L}_\omega^{-1}\Big(\frac{\tilde{\phi}_\omega}{|x|^\alpha}\Big).\end{aligned}$$

Since $\tilde{\phi}_\omega \to \tilde{\phi}_\infty$ in $H^1(\mathbb{R}^N)$ as $\omega \to \infty$, we have

$$\int_{\mathbb{R}^N} \frac{\tilde{\phi}_\omega^2}{|x|^\alpha} \to \int_{\mathbb{R}^N} \frac{\tilde{\phi}_\infty^2}{|x|^\alpha} > 0$$

as $\omega \to \infty$. Moreover, since $|x|^{-\alpha}\tilde{\phi}_\omega \in \tilde{L}_\omega(H^1_{\mathrm{rad}}(\mathbb{R}^N))$ by (2.2), we can use Lemma 2.3 to get

$$\begin{aligned}\left|\int_{\mathbb{R}^N} \frac{\tilde{\phi}_\omega}{|x|^\alpha} \tilde{L}_\omega^{-1}\Big(\frac{\tilde{\phi}_\omega}{|x|^\alpha}\Big)\right| &\le \left\|\frac{\tilde{\phi}_\omega}{|x|^\alpha}\right\|_{H^{-1}} \left\|\tilde{L}_\omega^{-1}\Big(\frac{\tilde{\phi}_\omega}{|x|^\alpha}\Big)\right\|_{H^1} \\ &\lesssim \left\|\frac{\tilde{\phi}_\omega}{|x|^\alpha}\right\|_{H^{-1}}^2 \lesssim \|\tilde{\phi}_\omega\|_{H^1}^2 \le 2\|\tilde{\phi}_\infty\|_{H^1}^2\end{aligned}$$

for sufficiently large ω. We note that since $0 < \alpha < 2$, we get $-2 + \alpha/2 > -3 + \alpha$. Therefore, we obtain

$$\partial_\omega \|\phi_\omega\|_{L^2}^2 \gtrsim \omega^{-2+\alpha/2} + o(\omega^{-2+\alpha/2})$$

as $\omega \to \infty$. This means that $\partial_\omega \|\phi_\omega\|_{L^2}^2 > 0$ if ω is sufficiently large. This completes the proof. □

Proof of Theorem 1.3. Theorem 1.3 follows from Proposition 1.4 and Lemma 3.2. □

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