RIMS Kôkyûroku Bessatsu
B88 (2021), 105–118

Gevrey well-posedness and ill-posedness of third-order nonlinear Schrödinger equations on the torus

By

Nobu Kishimoto*

Abstract

Tsutsumi and the author recently proved unique existence of real analytic solutions and non-existence of Gevrey solutions for certain nonlinear dispersive equations posed on the torus. In this note, we revisit these results and prove them in a slightly more general setting.

§ 1. Introduction

In [6, 7], Tsutsumi and the author investigated the Cauchy problem associated with the following third-order nonlinear Schrödinger equations:

$$\partial_t u = \alpha_1 \partial_x^3 u + i\alpha_2 \partial_x^2 u + i\gamma_1 |u|^2 u + \gamma_2 \partial_x\big(|u|^2 u\big) - i\Gamma u \partial_x\big(|u|^2\big), \quad (t,x) \in (-T,T)\times\mathbb{T}, \tag{1.1}$$

$$u(0,x) = u_0(x), \qquad x \in \mathbb{T}, \tag{1.2}$$

where $\mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$ is the one-dimensional torus, α_j, γ_j $(j = 1, 2)$ are real constants, Γ is a complex constant and T is a positive constant. The equation (1.1), which formally conserves the L^2-norm, has a background in physics as a model for the signal propagation in a crystal optical fiber; see, e.g., [1]. It was mentioned in [6] that for any real analytic initial data u_0 the solution to (1.1)–(1.2) exists uniquely in a certain class of functions

Received January 31, 2020. Revised January 19, 2021.
2020 Mathematics Subject Classification(s): 35Q53, 35Q55, 35A01.
Key Words: *third-order nonlinear Schrödinger equation, well-posedness, smoothing effect, non-existence, Gevrey class.*
Partially supported by JSPS KAKENHI Grant Number JP16K17626 and by Research Institute for Mathematical Sciences, an International Joint Usage/Research Center located in Kyoto University.
*Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606-8502, Japan.
e-mail: `nobu@kurims.kyoto-u.ac.jp`

that are analytic in x and continuous in t. This existence result is classical and does not require any further assumption on α_j, γ_j, and Γ. In fact, the main result in [6] was non-existence of solutions with Sobolev initial data under the assumption

$$\alpha_1 \neq 0, \quad \mathrm{Re}(\Gamma) \neq 0, \quad \frac{2\alpha_2}{3\alpha_1} \notin \mathbb{Z}. \tag{1.3}$$

It is then natural to ask whether the Cauchy problem (1.1)–(1.2) has a solution for initial data in Gevrey classes, which are intermediate classes of Sobolev spaces and the class of analytic functions. In [7], this problem was answered negatively, as naturally expected from the observation made in [6] that the resonant nonlinear interactions in (1.1) give rise to a Cauchy-Riemann type operator. However, the proof was far from trivial because of the nonlinear setting, and moreover, the third-order dispersion was needed to overcome the derivative loss in the nonlinearity.

The aim of this note is to revisit these existence and non-existence results in analytic and Gevrey classes obtained in [6, 7] and prove them in a slightly more general setting for future use. The results on the particular Cauchy problem (1.1)–(1.2) are stated as Corollaries 3.2 and 4.3 below. In particular, for analytic initial data we prove local *well-posedness*. Here, well-posedness in a space X means unique existence of a solution in C_tX, the space of continuous functions in t with values in X, and continuity of the data-to-solution mapping from X to C_tX.

We will take analytic or Gevrey class as the data space X, but in practice we will estimate the solutions in certain Banach spaces of analytic or Gevrey functions (which are strictly smaller than the entire analytic or Gevrey classes). This is because Gevrey class is defined by suitable limiting procedure with a sequence of such Banach spaces and is not Banach nor even metrizable in itself. For instance, the space of all analytic functions $\mathcal{A}$ can be defined as the inductive limit $\varinjlim_{r\downarrow 0}\mathcal{A}(r)$ of Banach spaces $\mathcal{A}(r)$, each of which is a certain subspace of the class of all analytic functions with radius of analyticity r.

Let us recall the existence result in [6], which asserts unique existence of solutions u in $C([-T,T];\mathcal{A}(r/2))$ for initial data u_0 in $\mathcal{A}(r)$ with existence time $T = T(\|u_0\|_{\mathcal{A}(r)}) > 0$, for any $r > 0$. Due to the derivative loss in the nonlinearity, it is reasonable to construct solutions in a space of reduced radius of analyticity as above. However, in [6] it was not shown (in fact, not likely) that the solution remains in the same space $\mathcal{A}(r)$ as the initial data, and hence the result was not referred to as *well-posedness*. In this article, we see that the Cauchy problem is well-posed in $\mathcal{A}$, that is, for any $u_0 \in \mathcal{A}$ the solution exists uniquely in $C([-T,T];\mathcal{A})$ and depends continuously on initial data in the $\mathcal{A}$-topology. Note that this is different from well-posedness in $\mathcal{A}(r)$ for each $r > 0$; indeed, it is possible that the solution immediately loses radius of analyticity but remains analytic for a while, in which case the problem is ill-posed in each $\mathcal{A}(r)$ but can be well-posed in $\mathcal{A}$. Well-posedness in each $\mathcal{A}(r)$ basically implies that in $\mathcal{A}$, whereas the converse is not true in general.

Clearly, the relation is opposite for non-existence (or ill-posedness) results; non-existence in $\mathcal{A}$ implies that in each $\mathcal{A}(r)$ but the converse does not necessarily hold. In fact, for a Gevrey class $\mathcal{G}$ and a suitable defining sequence $\{\mathcal{G}(r)\}_{r>0}$, non-existence in each $\mathcal{G}(r)$ can be shown essentially in the same argument as for the Sobolev case [6], whereas more elaborate analysis is required for proving non-existence in $\mathcal{G}$, which was the main contribution of [7].

There are several papers which study the ill-posedness nature of other nonlinear evolution equations within the framework of regular function spaces; e.g., the degenerate Zakharov equations by Colin and Métivier [3], the Prandtl equations by Gérard-Varet and Dormy [4], and the incompressible Hall- and electron-MHD equations by Jeong and Oh [5]. These papers concern either Gevrey-ill-posedness for a linearized equation around a specific solution or Sobolev-ill-posedness for the full nonlinear equation, leaving Gevrey-ill-posedness of the full equation as challenging open problems.

The plan of this note is as follows. In the next preliminary section we introduce Gevrey classes and give fundamental nonlinear estimates. In Section 3 we prove local well-posedness in Gevrey classes for general equations with derivative-type nonlinearities. In Section 4 a Gevrey smoothing effect of certain elliptic-type equations is established, and as a corollary, non-existence of solution in Gevrey classes is deduced.

§ 2. Preliminaries

§ 2.1. Gevrey classes on $\mathbb{T}^d$

The standard definition of Gevrey class of order σ, denoted by G^σ, is the set of all C^∞ functions whose n-th derivative has magnitude of growth order at most $C^n(n!)^\sigma$ for some $C>0$. Hence, it can be defined as the inductive limit ($b\to 0$) of Banach spaces

$$\tilde{G}^\sigma_b := \{f\in C^\infty(\mathbb{T}^d) : \|f\|_{\tilde{G}^\sigma_b} := \sup_{n\geq 0} b^n (n!)^{-\sigma} \max_{|\alpha|\leq n} \|\partial^\alpha f\|_{L^\infty(\mathbb{T}^d)} < \infty\}, \qquad b>0,$$

that is, the union $G^\sigma = \bigcup_{b>0}\tilde{G}^\sigma_b$ equipped with the inductive limit topology. See, e.g., [8, Definition 1.4.1]. Note that G^1 is the space of real analytic functions on $\mathbb{T}^d$.

Here, we use a different definition which is well suited for our analysis. It turns out that our definition of G^σ gives the same topological space as that defined above.

Definition 2.1. For $\sigma\geq 1$, $a>0$, and $s\in\mathbb{R}$, define the Banach space $G^\sigma_{a,s}$ by

$$G^\sigma_{a,s} := \{f\in C^\infty(\mathbb{T}^d) : \|f\|_{G^\sigma_{a,s}} := \sup_{k\in\mathbb{Z}^d} \langle k\rangle^s e^{a|k|^{1/\sigma}} |\hat{f}_k| < \infty\},$$

where $\langle k\rangle := \max\{1,|k|\}$, and

$$\hat{f}_k := \frac{1}{(2\pi)^{d/2}}\int_{\mathbb{T}^d} f(x)e^{-ik\cdot x}\,dx, \qquad k\in\mathbb{Z}^d.$$

Define G^σ, Gevrey class of order σ, as the inductive limit ($a\to 0$) of $\{G^\sigma_{a,0}\}_{a>0}$.

It is easy to see the following inclusion relations:

- $G^1 \subsetneq G^{\sigma_1} \subsetneq G^{\sigma_2} \subsetneq \bigcup_{\sigma\geq 1} G^\sigma \subsetneq C^\infty \qquad (1<\sigma_1<\sigma_2)$
- $G^{\sigma_1} \subsetneq \bigcap_{a>0} G^{\sigma_2}_{a,0} \subsetneq G^{\sigma_2}_{a_1,0} \subsetneq G^{\sigma_2}_{a_2,0} \subsetneq G^{\sigma_2} \qquad (1\leq\sigma_1<\sigma_2,\ a_1>a_2>0)$
- $G^\sigma_{a_1,0} \subsetneq G^\sigma_{a_1,-s_2} \subsetneq G^\sigma_{a_1,-s_1} \subsetneq \bigcup_{s\leq 0} G^\sigma_{a_1,s} \subsetneq \bigcap_{s\geq 0} G^\sigma_{a_2,s} \subsetneq G^\sigma_{a_2,s_1} \subsetneq G^\sigma_{a_2,s_2} \subsetneq G^\sigma_{a_2,0}$
  $(\sigma\geq 1,\ a_1>a_2>0,\ s_1>s_2>0)$

We recall some fundamental properties of Gevrey classes. For the proof, see [7, Appendix] and references therein.

Lemma 2.2. *Let $\sigma\geq 1$. The following holds.*
(i) G^σ is a complete Montel space (in particular, every bounded set is precompact).
(ii) The above embeddings are all continuous (in fact, compact).
(iii) A set $A\subset G^\sigma$ is bounded if and only if $A\subset G^\sigma_{a,0}$ for some $a>0$ and it is bounded in $G^\sigma_{a,0}$.
(iv) Let $I\subset\mathbb{R}$ be a compact interval. Then, $u\in C(I;G^\sigma)$ if and only if $u\in C(I;G^\sigma_{a,0})$ for some $a>0$.

§ 2.2. Multilinear estimates in Gevrey spaces

Definition 2.3. Let $\beta\in\mathbb{R}$. We say $N[u]$ is a nonlinearity of β-derivative type if it is of the form

$$N[u] = \frac{1}{(2\pi)^{d/2}}\sum_{k\in\mathbb{Z}^d} e^{ik\cdot x}\sum_{\substack{k_1,k_2,\dots,k_p\in\mathbb{Z}^d\\ k=k_1+k_2+\cdots+k_p}} M(k_1,k_2,\dots,k_p)\omega^{(1)}_{k_1}\omega^{(2)}_{k_2}\cdots\omega^{(p)}_{k_p} \tag{2.1}$$

for some $p\geq 1$ (degree of nonlinearity, possibly linear), $\omega^{(j)}\in\{\hat{u},\,\hat{\bar{u}}\}$ $(1\leq j\leq p)$, and a function $M:(\mathbb{Z}^d)^p\to\mathbb{C}$ satisfying

$$|M(k_1,\dots,k_p)|\leq C\Big(\frac{\langle k_1\rangle\langle k_2\rangle\cdots\langle k_p\rangle}{\langle k_{\max}\rangle}\Big)^{s_0}\langle k_{\max}\rangle^\beta, \qquad k_1,\dots,k_p\in\mathbb{Z}^d$$

for some $C>0$ and $s_0\geq 0$, where $k_{\max} := \max_{1\leq j\leq p}|k_j|$.

Namely, a nonlinearity of β-derivative type is a power-type nonlinearity with loss of at most β derivatives. The following lemma is easily verified:

Lemma 2.4. *If $N[u]$ is a nonlinearity of β-derivative type, then for any $\sigma \geq 1$, $a > 0$ and $s \geq 0$ it holds that*

$$\|N[u]\|_{G^{\sigma}_{a,s}} \leq Cp^s \|u\|^{p-1}_{G^{\sigma}_{a,s_0+d+1}} \|u\|_{G^{\sigma}_{a,s+\beta}}, \tag{2.2}$$

where $C > 0$ is a constant independent of s, a and σ. Moreover, for any $\beta' \in \mathbb{R}$, $\langle \nabla \rangle^{\beta'} N[u]$ is of $(\beta + \beta')$-derivative type.

Proof. The estimate (2.2) follows from the elementary inequality

$$\langle k_1 + \cdots + k_p \rangle^s e^{a|k_1+\cdots+k_p|^{1/\sigma}} \leq p^s \langle k_{\max} \rangle^s \prod_{j=1}^{p} e^{a|k_j|^{1/\sigma}} \qquad (s \geq 0,\ a > 0,\ \sigma \geq 1)$$

and the embedding $\langle \cdot \rangle^{-d-1} \ell^\infty \hookrightarrow \ell^1$. The latter claim is trivial when $\beta' \geq 0$ or the frequency interaction is restricted to the case $\langle k_1 + \cdots + k_p \rangle \sim \langle k_{\max} \rangle$. Otherwise, the claim follows from

$$\langle k_{\max} \rangle^{|\beta'|} \leq C \Big(\frac{\langle k_1 \rangle \langle k_2 \rangle \cdots \langle k_p \rangle}{\langle k_{\max} \rangle} \Big)^{|\beta'|}.$$

□

For $N \geq 0$, let $P_{\geq N}$ (resp. $P_{<N}$) denote the Fourier projection onto the set $\{n \in \mathbb{Z}^d : |n| \geq N\}$ (resp. $\{n \in \mathbb{Z}^d : |n| < N\}$).

The next lemma will play a crucial role in deriving a Gevrey smoothing effect (Proposition 4.1). This is an improvement of the crude estimate (2.2) in the following two respects: The constant can be made s-independent, and the index a can be replaced by the strictly smaller one θa except for one function. Note also that the assumption $\sigma > 1$ is essentially used in the proof.

Lemma 2.5. *Let $\sigma > 1$ and $a > 0$. Assume that $N[u]$ is a nonlinearity of degree $p \geq 2$ and of β-derivative type for some $\beta \in \mathbb{R}$. Then, there exists $C, C_0 > 0$ such that for any $s \geq 0$ it holds*

$$\|P_{\geq C_0 s^\sigma} N[u]\|_{G^{\sigma}_{a,s}} \leq C \|u\|^{p-1}_{G^{\sigma}_{\theta a,s_0+d+1}} \|u\|_{G^{\sigma}_{a,s+\beta}},$$

where $\theta := (1 - 1/p)^{1-1/\sigma} \in (0,1)$.

Proof. The claim is a consequence of the following inequality ([7, Lemma 2]):

$$\langle k_1 + \cdots + k_p \rangle^s e^{a|k_1+\cdots+k_p|^{1/\sigma}} \leq \max_{1 \leq q \leq p} \langle k_q \rangle^s e^{a|k_q|^{1/\sigma}} \prod_{\substack{1 \leq j \leq p \\ j \neq q}} e^{\theta a |k_j|^{1/\sigma}},$$

which is valid if $\sigma > 1$ and $|k_1 + \cdots + k_p| \geq C(\sigma, a, p)s^\sigma$. □

§ 3. Local well-posedness in Gevrey classes

In this section we consider an abstract nonlinear evolution equation of the form

$$\partial_t u = i\psi(D_x)u + N[u], \qquad (t,x) \in (-T,T) \times \mathbb{T}^d, \tag{3.1}$$

where

$$\psi(D_x)f := \mathcal{F}_k^{-1}[\psi_k \hat{f}_k], \qquad \psi = (\psi_k)_{k\in\mathbb{Z}^d} : \mathbb{Z}^d \to \mathbb{R}.$$

Proposition 3.1. *Assume that $N[u]$ is a (sum of) nonlinearity of $(1/\sigma)$-derivative type in the sense of Definition 2.3 for some $\sigma \geq 1$.*

Then, for any $a > 0$ and $r > 0$, there exists $T_0 = T_0(a,r) > 0$ such that the following holds: For any $0 < T \leq T_0$ and any $u_0 \in B(a,r) := \{f \in G^\sigma_{a,s_0+d+1} : \|f\|_{G^\sigma_{a,s_0+d+1}} \leq r\}$, there exists a unique solution $u \in C([-T,T]; G^\sigma_{a/2,0})$ to (3.1) on $[-T,T]$ with initial condition $u(0,x) = u_0(x)$ satisfying

$$\|u\|_{X_T} := \sup_{|t|\leq T} \|u(t)\|_{G^\sigma_{a(1-|t|/2T),s_0+d+1}} \leq 2\|u_0\|_{G^\sigma_{a,s_0+d+1}}.$$

Moreover, the mapping $B(a,r) \ni u_0 \mapsto u \in C([-T,T]; G^\sigma_{a/2,0})$ is Lipschitz continuous.

Proof. We only consider the case where $N[u]$ consists of a single nonlinearity of $(1/\sigma)$-derivative type. We will show, for any $u_0 \in G^\sigma_{a,s_0+d+1}$, that

$$\Psi_{u_0}[u](t) := e^{it\psi(D_x)}u_0 + \int_0^t e^{i(t-t')\psi(D_x)} N[u(t')]\, dt'$$

is a contraction mapping on $B := \{u \in C([-T,T]; G^\sigma_{a/2,0}) : \|u\|_{X_T} \leq 2\|u_0\|_{G^\sigma_{a,s_0+d+1}}\}$ (which is a complete metric space with metric induced by the X_T-norm) if T is sufficiently small according to a and $\|u_0\|_{G^\sigma_{a,s_0+d+1}}$.

For the linear part, we have

$$e^{it\psi(D_x)}u_0 \in C([-T,T]; G^\sigma_{a,s_0+d+1}), \qquad \|e^{it\psi(D_x)}u_0\|_{X_T} = \|u_0\|_{G^\sigma_{a,s_0+d+1}}.$$

For the Duhamel integral, we first observe, using (2.2), that $N[u] \in L^\infty([-T,T]; G^\sigma_{a/2,0})$ if $u \in C([-T,T]; G^\sigma_{a/2,0})$ and $\|u\|_{X_T} < \infty$. This implies that $\int_0^t e^{-it'\psi(D_x)} N[u(t')]\, dt'$

belongs to $C([-T,T];G^\sigma_{a/2,0})$, and so does the Duhamel integral. Moreover, we have

$$\begin{aligned}
&\Big\| \int_0^t e^{i(t-t')\psi(D_x)} N[u(t')]\,dt' \Big\|_{G^\sigma_{a(1-|t|/2T),s_0+d+1}} \\
&= \Big\| \langle k\rangle^{s_0+d+1} e^{a(1-\frac{|t|}{2T})|k|^{1/\sigma}} \int_0^t e^{i(t-t')\psi_k} \sum_{k=k_1+\cdots+k_p} M(k_1,\dots,k_p)\hat{u}_{k_1}(t')\cdots\hat{u}_{k_p}(t')\,dt' \Big\|_{\ell^\infty} \\
&\le C \sup_k \sum_{k=k_1+\cdots+k_p} \langle k_{\max}\rangle^{s_0+d+1} e^{a(1-\frac{|t|}{2T})\sum_{j=1}^p |k_j|^{\frac{1}{\sigma}}} \\
&\qquad\qquad \times \int_0^t \Big(\frac{\langle k_1\rangle\langle k_2\rangle\cdots\langle k_p\rangle}{\langle k_{\max}\rangle}\Big)^{s_0} \langle k_{\max}\rangle^{\frac{1}{\sigma}} \prod_{j=1}^p |\hat{u}_{k_j}(t')|\,dt' \\
&\le C \sup_k \sum_{k=k_1+\cdots+k_p} \Big(\frac{\langle k_1\rangle\langle k_2\rangle\cdots\langle k_p\rangle}{\langle k_{\max}\rangle}\Big)^{-(d+1)} \\
&\qquad\qquad \times \int_0^t \langle k_{\max}\rangle^{\frac{1}{\sigma}} e^{-a\frac{|t|-|t'|}{2T}\sum_{j=1}^p |k_j|^{\frac{1}{\sigma}}} \prod_{j=1}^p \|u(t')\|_{G^\sigma_{a(1-|t'|/2T),s_0+d+1}}\,dt' \\
&\le C\|u\|_{X_T}^p \sup_{k_1,\dots,k_p} \int_0^{|t|} \langle k_{\max}\rangle^{\frac{1}{\sigma}} e^{-a\frac{|t|-t'}{2T}\sum_{j=1}^p |k_j|^{\frac{1}{\sigma}}}\,dt', \qquad |t|\le T.
\end{aligned}$$

(Note that some of $\hat{u}_{k_j}$'s may be replaced with $\hat{\bar{u}}_{k_j}$ in the above computation.) The integral in the last line is evaluated by

$$|t| + \sup_{(k_1,\dots,k_p)\neq(0,\dots,0)} \langle k_{\max}\rangle^{\frac{1}{\sigma}} \Big(\frac{a}{2T}\sum_{j=1}^p |k_j|^{\frac{1}{\sigma}}\Big)^{-1} \le (1+2a^{-1})T, \qquad |t|\le T,$$

so that we obtain

$$\Big\| \int_0^t e^{i(t-t')\psi(D_x)} N[u(t')]\,dt' \Big\|_{X_T} \le C(1+a^{-1})T\|u\|_{X_T}^p.$$

A similar argument shows the corresponding difference estimate. As a consequence, Ψ_{u_0} is a contraction on B if $(1+a^{-1})T\|u_0\|^{p-1}_{G^\sigma_{a,s_0+d+1}} \ll 1$. By Banach's fixed point theorem, there is a unique solution of the Cauchy problem in B for such a T. The Lipschitz continuity of the data-to-solution mapping is an immediate consequence of the above estimates. □

Corollary 3.2. *Assume that $N[u]$ is a (sum of) nonlinearity of $(1/\sigma)$-derivative type in the sense of Definition 2.3 for some $\sigma \ge 1$. Then, the Cauchy problem associated with (3.1) is locally well-posed in G^σ in the following sense.*

(i) For any bounded set $A \subset G^\sigma$, there exists $T = T(A) > 0$ such that, for any $u_0 \in A$, there is a solution $u \in C([-T,T];G^\sigma)$ on the time interval $[-T,T]$ satisfying $u(0) = u_0$.

(ii) If $u_1, u_2 \in C(I; G^\sigma)$ are solutions on some time interval I containing 0 and $u_1(0) = u_2(0)$, then $u_1 \equiv u_2$ on I.

(iii) The mapping $A \ni u_0 \mapsto u \in C([-T(A), T(A)]; G^\sigma)$ from the initial datum to the unique solution given above is continuous. Here, A is given the relative topology as a subset of G^σ and $C([-T, T]; G^\sigma)$ is given the compact-open topology.

In particular, the Cauchy problem (1.1)–(1.2) is locally well-posed in G^1 for any $\alpha_1, \alpha_2 \in \mathbb{R}$ and $\gamma_1, \gamma_2, \Gamma \in \mathbb{C}$.

Proof. (i) From Lemma 2.2 (iii) and (ii), a bounded subset A of G^σ is a bounded subset of $G^\sigma_{2a,0}$ for some $a > 0$, and then of G^σ_{a,s_0+d+1}, i.e., $A \subset B(a, r)$ for some $a, r > 0$. Hence, we can take $T(A)$ to be $T_0(a, r)$ given in Proposition 3.1 and obtain a solution $u \in C([-T, T]; G^\sigma_{a/2,0})$ for any $u_0 \in A$. Again by Lemma 2.2 (ii), we see $C([-T, T]; G^\sigma_{a/2,0}) \subset C([-T, T]; G^\sigma)$.

(ii) We may assume that $I = [0, T']$ for some $T' > 0$. Let $u_1, u_2 \in C([0, T']; G^\sigma)$ be two solutions with $u_1(0) = u_2(0) =: u_0$. By Lemma 2.2 (iv) and (ii), there exists $a > 0$ such that $u_1, u_2 \in C([0, T']; G^\sigma_{a,s_0+d+1})$. Choose $r > 0$ and $0 < T'' \le \min\{T', T_0(a, r)\}$ such that

$$\max_{0 \le t \le T''} \|u_j(t)\|_{G^\sigma_{a,s_0+d+1}} \le 2\|u_0\|_{G^\sigma_{a,s_0+d+1}} \le 2r, \qquad j = 1, 2.$$

Since we have $u_j \in C([0, T'']; G^\sigma_{a/2,0})$ and $\|u_j\|_{X_{T''}} \le 2\|u_0\|_{G^\sigma_{a,s_0+d+1}}$ for $j = 1, 2$, the uniqueness assertion in Proposition 3.1 (suitably modified for solutions forward in time) shows that $u_1(t) = u_2(t)$ for $t \in [0, T'']$. Repeating this procedure (if necessary) gives uniqueness on $[0, T']$.

(iii) We first observe that a bounded set in G^σ is precompact, by Lemma 2.2 (i), and hence metrizable, since G^σ is the inductive limit of an increasing sequence of Banach spaces (see, e.g., [2, Theorem 2 and Examples 1.2]). It then suffices to show the sequential continuity of the mapping.

Let A be a bounded set in G^σ and, as in (i), choose $a, r > 0$ such that A is a bounded set in $G^\sigma_{2a,0}$ and $A \subset B(a, r)$. Assume that a sequence $\{u_{0,n}\}_{n \ge 1} \subset A$ converges to $u_0 \in A$. Then, since $\{u_{0,n}\}$ is bounded in $G^\sigma_{2a,0}$ and the embedding $G^\sigma_{2a,0} \hookrightarrow G^\sigma_{a,s_0+d+1}$ is compact by Lemma 2.2 (ii), $\{u_{0,n}\}$ converges to u_0 in G^σ_{a,s_0+d+1}. The continuity assertion in Proposition 3.1 shows the convergence of the corresponding solutions $u_n \to u$ in $C([-T(A), T(A)]; G^\sigma_{a/2,0})$. Finally, we note that the inclusion $C([-T, T]; G^\sigma_{a/2,0}) \subset C([-T, T]; G^\sigma)$ (the latter space given the compact-open topology) is continuous since $G^\sigma_{a/2,0} \subset G^\sigma$ is continuous. Therefore, u_n converges to u in $C([-T, T]; G^\sigma)$, and the mapping has been shown to be sequentially continuous.

Finally, we notice that the equation (1.1) is of the form (3.1) with $\psi_k = -\alpha_1 k^3 - \alpha_2 k^2$ and a sum of cubic nonlinearities of 1-derivative type, so that the above result can be applied. □

§ 4. Smoothing effect and non-existence of solutions in Gevrey classes

In this section, we shall prove a Gevrey smoothing effect on a function u satisfying an equation of the following type:

$$\partial_t\big(u - N_1[u]\big) = \psi(t, D_x)u + N_2[u], \qquad (t,x) \in [-T,T] \times \mathbb{Z}^d. \tag{4.1}$$

We assume that

$$\begin{aligned} &\psi(t,D_x) = \mathcal{F}_k^{-1}\psi_k(t)\mathcal{F}_x \qquad \text{with } \{\psi_k\}_{k\in\mathbb{Z}^d} \subset C([-T,T];\mathbb{R}) \text{ satisfying} \\ &A := \inf_{|k|\geq K} |k|^{-\frac{1}{\rho}} \min_{|t|\leq T} |\psi_k(t)| > 0 \quad \text{for some } K > 0 \text{ and } \rho \geq 1, \end{aligned} \tag{4.2}$$

$$\begin{aligned} &N_1[u] \text{ is a (sum of) nonlinearity of } (\beta - \tfrac{1}{\rho})\text{-derivative type and} \\ &N_2[u] \text{ is a (sum of) nonlinearity of } \beta\text{-derivative type, for some } \beta \in [0, \tfrac{1}{\rho}). \end{aligned} \tag{4.3}$$

The assumption (4.2) means that the differential operator $\psi(t, D_x)$ is of elliptic type of order $1/\rho$ uniformly in t.

Proposition 4.1. *Let $\rho \geq 1$, and assume (4.2)–(4.3). Let u be a function in $C([-T,T]; G^\sigma_{a,0})$ for some $T > 0$, $\sigma > \rho$, $a > 0$ and satisfy the equation (4.1) on the time interval $[-T,T]$. Then, there exists $\varepsilon > 0$ depending on $\sigma, a, \rho, A, K, N_1, N_2$, and $\|u\|_{C([-T,T];G^\sigma_{\theta a, s_0+d+1})}$, but not on T, $\|u\|_{C([-T,T];G^\sigma_{a,0})}$, such that $u \in C((-T,T); G^\sigma_{a+\varepsilon,0})$. Here, $\theta \in (0,1)$ is the maximum of the constants given in Lemma 2.5 for $N_1[u]$, $N_2[u]$.*

Proof. For $t_0, t \in [-T,T]$, we integrate the equation (4.1) in the Fourier side to have

$$\begin{aligned} \hat{u}_k(t_0) &= e^{-\int_{t_0}^t \psi_k} \hat{u}_k(t) + \Big(\hat{N}_1[u(t_0)]_k - e^{-\int_{t_0}^t \psi_k} \hat{N}_1[u(t)]_k\Big) \\ &\quad - \int_{t_0}^t e^{-\int_{t_0}^{t'} \psi_k} \Big(\psi_k(t')\hat{N}_1[u(t')]_k + \hat{N}_2[u(t')]_k\Big)\, dt', \qquad k \in \mathbb{Z}^d. \end{aligned}$$

By the assumption (4.2), for each $k \in \mathbb{Z}^d$ with $|k| \geq K$ it holds

$$\begin{aligned} &\text{either} \quad \min_{|t|\leq T} \psi_k(t) \geq A|k|^{\frac{1}{\rho}} \qquad \Big(\Rightarrow\ e^{-\int_{t_0}^t \psi_k} \leq e^{-A(t-t_0)|k|^{\frac{1}{\rho}}}, \quad t > t_0\Big) \\ &\text{or} \quad \max_{|t|\leq T} \psi_k(t) \leq -A|k|^{\frac{1}{\rho}} \qquad \Big(\Rightarrow\ e^{-\int_{t_0}^t \psi_k} \leq e^{-A(t_0-t)|k|^{\frac{1}{\rho}}}, \quad t < t_0\Big). \end{aligned}$$

Let t_0 and $(1 \geq)\ \delta > 0$ satisfy $t_0 \pm \delta \in [-T,T]$. For k satisfying the former,

$$\begin{aligned} |\hat{u}_k(t_0)| &\leq e^{-A\delta|k|^{\frac{1}{\rho}}} |\hat{u}_k(t_0+\delta)| + \Big(|\hat{N}_1[u(t_0)]_k| + |\hat{N}_1[u(t_0+\delta)]_k|\Big) \\ &\quad + \int_{t_0}^{t_0+\delta} \Big(e^{-\int_{t_0}^{t'} |\psi_k|} |\psi_k(t')||\hat{N}_1[u(t')]_k| + e^{-A(t'-t_0)|k|^{\frac{1}{\rho}}} |\hat{N}_2[u(t')]_k|\Big)\, dt', \end{aligned}$$

while for the latter,

$$|\hat{u}_k(t_0)| \le e^{-A\delta|k|^{\frac{1}{\rho}}}|\hat{u}_k(t_0-\delta)| + \Big(|\hat{N}_1[u(t_0)]_k| + |\hat{N}_1[u(t_0-\delta)]_k|\Big) + \int_{t_0-\delta}^{t_0}\Big(e^{-\int_{t'}^{t_0}|\psi_k|}|\psi_k(t')||\hat{N}_1[u(t')]_k| + e^{-A(t_0-t')|k|^{\frac{1}{\rho}}}|\hat{N}_2[u(t')]_k|\Big)\,dt'.$$

In both cases, we have

$$|k|^{\frac{1}{\rho}-\beta}|\hat{u}_k(t_0)| \le |k|^{\frac{1}{\rho}-\beta}e^{-A\delta|k|^{\frac{1}{\rho}}}\max_{|t-t_0|\le\delta}|\hat{u}_k(t)| + 3|k|^{\frac{1}{\rho}-\beta}\max_{|t-t_0|\le\delta}|\hat{N}_1[u(t)]_k| + A^{-1}|k|^{-\beta}\max_{|t-t_0|\le\delta}|\hat{N}_2[u(t)]_k|.$$

Now, we observe that

$$\sup_k |k|^{\frac{1}{\rho}-\beta}e^{-A\delta|k|^{\frac{1}{\rho}}} = (\tfrac{1-\rho\beta}{A\delta})^{1-\rho\beta}\sup_k\Big(\tfrac{A\delta}{1-\rho\beta}|k|^{\frac{1}{\rho}}e^{-\frac{A\delta}{1-\rho\beta}|k|^{\frac{1}{\rho}}}\Big)^{1-\rho\beta} \le C(\rho,\beta,A)\delta^{-(1-\rho\beta)},$$

and that $\langle\nabla\rangle^{\frac{1}{\rho}-\beta}N_1[u]$ and $\langle\nabla\rangle^{-\beta}N_2[u]$ are nonlinearities of 0-derivative type, by the assumption (4.3) and Lemma 2.4. Applying Lemma 2.5, for any $s\ge 0$ we have

$$\|P_{\ge\max\{C_0s^\sigma,K\}}u(t_0)\|_{G^\sigma_{a,s+(\frac{1}{\rho}-\beta)}} \le C_1\delta^{-(1-\rho\beta)}\|u\|_{C([t_0-\delta,t_0+\delta];G^\sigma_{a,s})},$$

where (and hereafter) C_1 denotes any positive constant depending on ρ, σ, a, A, K, N_1, N_2, and $\|u\|_{C([-T,T];G^\sigma_{\theta a,s_0+d+1})}$, but not on s and δ. On the other hand,

$$\|P_{<\max\{C_0s^\sigma,K\}}u(t_0)\|_{G^\sigma_{a,s+(\frac{1}{\rho}-\beta)}} \le \|u(t_0)\|_{G^\sigma_{\theta a,\frac{1}{\rho}-\beta}}\sup_{|k|<\max\{C_0s^\sigma,K\}}e^{(1-\theta)a|k|^{\frac{1}{\sigma}}}\langle k\rangle^s \le C_1\big(C_1s^\sigma\big)^s.$$

Hence, we have

$$\|u(t_0)\|_{G^\sigma_{a,s+\eta}} \le \max\Big\{C_1\big(C_1s^\sigma\big)^s,\, C_1\delta^{-\rho\eta}\|u\|_{C([t_0-\delta,t_0+\delta];G^\sigma_{a,s})}\Big\},\qquad s\ge 0,$$

where we have set $\eta := \frac{1}{\rho}-\beta\in(0,1]$. This estimate shows a Sobolev (i.e., polynomial order) smoothing effect in the interior of the time interval.

To obtain Gevrey (i.e., exponential order) smoothing, we iterate the above estimate, paying particular attention to the s-dependence. For any sufficiently large positive integer N (satisfying $N\ge T$ and $T^\rho N^{\sigma-\rho}\eta^\sigma\ge 1$), we apply the above estimate with $\delta = T/N(\le 1)$ repeatedly. Noticing $\delta^{-\rho}\le(N\eta)^\sigma$, we obtain

$$\begin{aligned}&\|u(0)\|_{G^\sigma_{a,N\eta}}\\ &\le \max\Big\{\max_{0\le n\le N-1}\big(C_1\delta^{-\rho\eta}\big)^{N-1-n}C_1\big(C_1(n\eta)^\sigma\big)^{n\eta},\,\big(C_1\delta^{-\rho\eta}\big)^N\|u\|_{C([-T,T];G^\sigma_{a,0})}\Big\}\\ &\le C_1^N(N\eta)^{\sigma N\eta}\max\big\{1,\,\|u\|_{C([-T,T];G^\sigma_{a,0})}\big\}.\end{aligned}$$

Since $m^m \le e^m m!$ for $m \in \mathbb{N}$, this estimate roughly means that the m-th derivative of the function $e^{a|\nabla|^{\frac{1}{\sigma}}}u(0)$ has growth of order at most $C_1^m(m!)^\sigma$ when $m \to \infty$, and hence $e^{a|\nabla|^{\frac{1}{\sigma}}}u(0) \in G^\sigma_{\varepsilon,0}$, or equivalently $u(0) \in G^\sigma_{a+\varepsilon,0}$, for some $\varepsilon > 0$ depending on C_1. For a precise argument, see the last part of the proof of [7, Lemma 3].

We remark that the increment ε does not depend on the size of interval T, nor the $C([-T,T];G^\sigma_{a,0})$-norm of u. In particular, the above argument implies $u(t) \in G^\sigma_{a+\varepsilon,0}$ for any $t \in (-T,T)$. While $\|u(t)\|_{G^\sigma_{a+\varepsilon,0}}$ may blow up as $t \to \pm T$, we see that it is bounded on any compact subinterval of $(-T,T)$. In fact, T-dependence of the norm $\|u(0)\|_{G^\sigma_{a+\varepsilon,0}}$ evaluated above comes from the contribution of $\|u(0)\|_{G^\sigma_{a,N\eta}}$ for N relatively small (depending on T), which we have neglected. Hence, $\|u(0)\|_{G^\sigma_{a+\varepsilon,0}}$ stays bounded as long as T does not approach to zero. Finally, we reduce ε by half and obtain the continuity of the mapping $t \mapsto u(t) \in G^\sigma_{a+\varepsilon/2,0}$ on any compact subinterval of $(-T,T)$ using the continuity in $G^\sigma_{a,0}$ and the interpolation inequality $\|f\|_{G^\sigma_{a+\varepsilon/2,0}} \le \|f\|^{1/2}_{G^\sigma_{a+\varepsilon,0}}\|f\|^{1/2}_{G^\sigma_{a,0}}$. □

Corollary 4.2. *Let $\rho \ge 1$, and assume (4.2)–(4.3). Let $u \in C([-T,T];G^\sigma)$, with $\sigma > \rho$ and $T > 0$, be a solution to (4.1) on the time interval $[-T,T]$. Then, $u \in \bigcap_{a>0} C((-T,T);G^\sigma_{a,0})$.*

Proof. The claim follows from the preceding proposition and the argument in the proof of [7, Theorem 1], so we give only an outline of the proof. By Lemma 2.2 (iv), u belongs to $C([-T,T];G^\sigma_{a_0,0})$ for some $a_0 > 0$. Define the function $\tilde{a} : (0,T] \to [a_0,\infty]$ by

$$\tilde{a}(t) := \sup\{a > 0 : u \in C([-t,t];G^\sigma_{a,0})\}.$$

On one hand, $\tilde{a}$ is monotone decreasing by definition. On the other hand, Proposition 4.1 implies that $\tilde{a}$ has to be discontinuous at any point where it is finite. Since the cardinality of discontinuous points of a monotone function with finite values is at most countable, we conclude that $\tilde{a} \equiv \infty$ on $(0,T)$. □

Corollary 4.3. *Let $\sigma > 1$, and assume that the condition (1.3) holds. Let $u \in C([-T,T];G^\sigma)$ be a non-trivial solution to (1.1) on $[-T,T]$ for some $T > 0$. Then, $u(t) \in \bigcap_{a>0} G^\sigma_{a,0}$ for all $t \in (-T,T)$.*

In particular, for any $u_0 \in G^\sigma \setminus \bigcap_{a>0} G^\sigma_{a,0}$ there exists no $T > 0$ for which the Cauchy problem (1.1)–(1.2) has a solution in $C([-T,T];G^\sigma)$.

Proof. Let $u \in C([-T,T];G^\sigma)$ be a non-trivial solution to (1.1). Since the L^2 norm is conserved (see [6, Lemma 2.5]), we see that $\|u(t)\|_{L^2} \equiv \|u(0)\|_{L^2} > 0$.

Let us recall how to deduce an equation of the form (4.1) from (1.1) (see [6, pp. 10008–10010] for details). We begin with converting the linear dispersive terms

$(\alpha_1\partial_x^3 + i\alpha_2\partial_x^2)u$ into time oscillation in the nonlinear terms by the transformation

$$u \quad\mapsto\quad v(t) := e^{-t(\alpha_1\partial_x^3 + i\alpha_2\partial_x^2)}u(t).$$

The Fourier coefficients $\hat{v}_k(t)$ then satisfy the following equation:

$$\partial_t \hat{v}_k = \sum_{\substack{k_1,k_2,k_3\in\mathbb{Z}\\ k=k_1+k_2+k_3}} \frac{i\gamma_1 + i\gamma_2 k + \Gamma(k_1+k_2)}{2\pi} e^{it\Phi}\hat{v}_{k_1}\hat{\bar{v}}_{k_2}\hat{v}_{k_3},$$

$$\Phi = \Phi(k_1,k_2,k_3) := 3\alpha_1(k_1+k_2)(k_2+k_3)\Big(k_1+k_3+\frac{2\alpha_2}{3\alpha_1}\Big).$$

The first term in the sum (with coefficient γ_1) is a cubic nonlinearity of 0-derivative type with $s_0 = 0$. (Although the multiplier M depends also on t, the bound on $|M|$ is uniform in t and thus the argument so far can be applied.)

We next divide the remaining part of the sum into the resonant and the non-resonant terms:

$$\begin{aligned}
&\sum_{k=k_1+k_2+k_3} \frac{i\gamma_2 k + \Gamma(k_1+k_2)}{2\pi} e^{it\Phi}\hat{v}_{k_1}\hat{\bar{v}}_{k_2}\hat{v}_{k_3} \\
&= \Bigg[\sum_{\substack{k=k_1+k_2+k_3\\ (k_1+k_2)(k_2+k_3)=0}} + \sum_{\substack{k=k_1+k_2+k_3\\ \Phi\neq 0}}\Bigg] \frac{i\gamma_2 k + \Gamma(k_1+k_2)}{2\pi} e^{it\Phi}\hat{v}_{k_1}\hat{\bar{v}}_{k_2}\hat{v}_{k_3} \\
&= \frac{2i\gamma_2 + \Gamma}{2\pi}\|u(0)\|_{L^2}^2 k\hat{v}_k - \frac{i\gamma_2}{2\pi}k|\hat{v}_k|^2\hat{v}_k - \frac{\Gamma}{2\pi}\Big(\sum_{k_3\in\mathbb{Z}} k_3|\hat{v}_{k_3}|^2\Big)\hat{v}_k \\
&\quad + \sum_{\substack{k=k_1+k_2+k_3\\ \Phi\neq 0}} \frac{i\gamma_2 k + \Gamma(k_1+k_2)}{2\pi} e^{it\Phi}\hat{v}_{k_1}\hat{\bar{v}}_{k_2}\hat{v}_{k_3}.
\end{aligned}$$

Note that the assumption $\frac{2\alpha_2}{3\alpha_1} \notin \mathbb{Z}$ (i.e., the fact $\Phi = 0 \Leftrightarrow (k_1+k_2)(k_2+k_3) = 0$) is essentially used here. We have also applied the L^2 conservation $\|v(t)\|_{L^2} = \|u(t)\|_{L^2} \equiv \|u(0)\|_{L^2}$. Observe that the second and the third terms on the right-hand side are cubic nonlinearities of 0-derivative type with $s_0 = 1/2$ and 1, respectively, since the corresponding multipliers are $M(k_1,k_2,k_3) = c_1 k_3 \chi_{k_1=k_2=k_3}$ and $c_2 k_3 \chi_{k_2=k_3}$.

The last term, which corresponds to the non-resonant part, is in itself a nonlinearity of 1-derivative type, which however can be dealt with by the third-order dispersion of the equation. We make a further decomposition into two sums over $(k_1,k_2,k_3) \in D := \{|k_1| \sim |k_2| \sim |k_3|\}$ and D^c, then the former is a cubic nonlinearity of 0-derivative type with $s_0 = 1/2$. For the latter, we apply differentiation by parts in t and substitute the (original) equation for $\hat{v}$:

$$\begin{aligned}
&\sum_{\substack{k=k_1+k_2+k_3 \\ \Phi\neq 0,\, D^c}} \frac{i\gamma_2 k + \Gamma(k_1+k_2)}{2\pi} e^{it\Phi} \hat{v}_{k_1} \bar{\hat{v}}_{k_2} \hat{v}_{k_3} \\
&= \partial_t \sum_{\substack{k=k_1+k_2+k_3 \\ \Phi\neq 0,\, D^c}} \frac{i\gamma_2 k + \Gamma(k_1+k_2)}{2\pi} \frac{e^{it\Phi}}{i\Phi} \hat{v}_{k_1} \bar{\hat{v}}_{k_2} \hat{v}_{k_3} \\
&\quad - \sum_{\substack{k=k_1+k_2+k_3 \\ \Phi\neq 0,\, D^c}} \frac{i\gamma_2 k + \Gamma(k_1+k_2)}{2\pi} \frac{e^{it\Phi}}{i\Phi} \sum_{k_1=k_{11}+k_{12}+k_{13}} \frac{i\gamma_1 + ik_1\gamma_2 + \Gamma(k_{11}+k_{12})}{2\pi} \\
&\qquad\qquad \times \hat{v}_{k_{11}} \bar{\hat{v}}_{k_{12}} \hat{v}_{k_{13}} \bar{\hat{v}}_{k_2} \hat{v}_{k_3} \\
&\quad - \text{(two similar terms)}.
\end{aligned}$$

Here, all the formal calculations are easily justified, because we consider a smooth solution. Noticing $|\Phi| \geq ck_{\max}^2$ in D^c (here we essentially use the third-order dispersion; i.e., the assumption $\alpha_1 \neq 0$), the first sum is the time derivative of a cubic nonlinearity of (-1)-derivative type with $s_0 = 0$, while we see ([7, Lemma 1]) that the other sums are quintic nonlinearities of 0-derivative type with $s_0 = 1$.

To treat the remaining term

$$\frac{2i\gamma_2 + \Gamma}{2\pi} \|u(0)\|_{L^2}^2 k\hat{v}_k = i\Big(\frac{2\gamma_2 + \mathrm{Im}(\Gamma)}{2\pi} \|u(0)\|_{L^2}^2 k\Big)\hat{v}_k + \frac{\mathrm{Re}(\Gamma)}{2\pi} \|u(0)\|_{L^2}^2 k\hat{v}_k,$$

we apply the second transformation

$$\hat{v}_k \quad \mapsto \quad \hat{w}_k(t) := \exp\Big(-i\frac{2\gamma_2 + \mathrm{Im}(\Gamma)}{2\pi}\|u(0)\|_{L^2}^2 kt\Big)\hat{v}_k(t)$$

to eliminate the first term on the right-hand side. The second term will be the one that is responsible for Gevrey smoothing effect.

We have so far obtained the equation for w of the form (4.1):

$$\partial_t w = \frac{\mathrm{Re}(\Gamma)}{2\pi} \|u(0)\|_{L^2}^2 (-i\partial_x) w + \partial_t N_1[w] + N_2[w],$$

where $N_1[w]$ is a cubic nonlinearity of (-1)-derivative type and $N_2[w]$ is a sum of cubic and quintic nonlinearities of 0-derivative type, and hence (4.3) is satisfied with $\rho = 1$ and $\beta = 0$. Finally, by the assumptions $\mathrm{Re}(\Gamma) \neq 0$ and $\|u(0)\|_{L^2} > 0$, the condition (4.2) is also satisfied with $\rho = 1$, $K = 1$ and $A = (2\pi)^{-1}|\mathrm{Re}(\Gamma)|\|u(0)\|_{L^2}^2$. Applying Corollary 4.2 (with $\sigma > 1 = \rho$) to w and noticing $|\hat{u}_k(t)| = |\hat{w}_k(t)|$, we have the desired Gevrey smoothing property for u, concluding the proof of Corollary 4.3. □

References

[1] Agrawal, G., Nonlinear Fiber Optics, 4th Edition, *Elsevier/Academic press, Burlington,* 2007.

[2] Cascales, B. and Orihuela, J., On compactness in locally convex spaces, *Math. Z.*, **195** (1987), 365–381.

[3] Colin, T. and Métivier, G., Instabilities in Zakharov equations for laser propagation in a plasma, *Phase space analysis of partial differential equations*, *Progr. Nonlinear Differential Equations Appl.*, **69**, Birkhäuser Boston, Boston, MA, 2006, pp. 63–81.

[4] Gérard-Varet, D. and Dormy, E., On the ill-posedness of the Prandtl equation, *J. Amer. Math. Soc.* **23** (2010), 591–609.

[5] Jeong, I.-J. and Oh, S.-J., On the Cauchy problem for the Hall and electron magnetohydrodynamic equations without resistivity I: Illposedness near degenerate stationary solutions, *preprint*, 2019.

[6] Kishimoto, N. and Tsutsumi, Y., Ill-posedness of the third order NLS equation with Raman scattering term, *Math. Res. Lett.*, **25** (2018), 1447–1484.

[7] Kishimoto, N. and Tsutsumi, Y., Ill-posedness of the Third Order NLS with Raman Scattering Term in Gevrey Spaces, *Mathematics of Wave Phenomena*, *Trends in Mathematics*, Birkhäuser Basel, 2020, pp. 219–233.

[8] Rodino, L., Linear partial differential operators in Gevrey spaces, *World Scientific Publishing Co., Inc., River Edge, NJ,* 1993.