On branching laws of Speh representations

Nozomi Ito

Abstract

In this paper, we consider the branching law of the Speh representation $\mathrm{Sp}(\pi, n+l)$ of GL_{2n+2l} with respect to the block diagonal subgroup $\mathrm{GL}_n \times \mathrm{GL}_{n+2l}$ for any generic representation π of GL_2 over any p-adic field. We use the Shalika model of $\mathrm{Sp}(\pi, n)$ to construct certain zeta integrals, which were constructed by Ginzburg and Kaplan independently, and study them. Finally, using these zeta integrals, we obtain a nonzero $\mathrm{GL}_n \times \mathrm{GL}_{n+2l}$-map from $\mathrm{Sp}(\pi, n+l)$ to $\tau \boxtimes \tau^\vee \chi_\pi \times \mathrm{Sp}(\pi, l)$ for any irreducible representation τ of GL_n. These results form part of the local theory of the Miyawaki lifting for unitary groups.

Contents

0 Introduction

For a long time, the lifting problem has been one of the most important problems in the theory of automorphic representations. With the advent of Arthur's endoscopic classification, we can now understand half of this problem, namely the existence of liftings, in many cases. However, we still do not know much about the other half, namely the construction of liftings. If an automorphic representation is constructed explicitly, it is possible to obtain additional data that is important for application in number theory, such as the data of Fourier coefficients, which cannot be obtained by only considering it as an abstract representation. Thus, the construction of liftings is still important.

One method to construct liftings is using global periods. In other words, it is a method to pullback an automorphic form to the product of two groups and consider its inner product with another automorphic form on the first factor. Of course, the product group and the automorphic representation to be pullbacked need to be taken properly (roughly speaking, the product group needs to be 'sufficiently large' and the automorphic representation needs to be 'sufficiently small'). This type of construction is interesting because its nonvanishing is often related to some special L-values (e.g. the theta lifting).

The Miyawaki lifting, which is the object of our interest, is one such construction. This is a construction defined by using the pullbacks of Ikeda lifts to block diagonal subgroups as kernel functions, introduced by Ikeda and modified and generalized by some researchers.

The purpose of this paper is to study the branching laws of Speh representations associated to generic representations of GL_2 over any p-adic field with respect to any block diagonal subgroup. It is essentially to study the theory of the local Miyawaki lifting for split unitary groups over p-adic fields.

In the following subsections, we explain the above in more detail. Moreover, in the last subsection, we will state the main results of this paper.

0.1 The Miyawaki lifting: The Siegel case

First, we recall the result of Ikeda [Ike06], the origin of the theory of the Miyawaki lifting.

0.1.1 Siegel modular forms

We introduce some notations for Siegel modular forms.

Let us denote the adele ring of $\mathbb{Q}$ by $\mathbb{A}_{\mathbb{Q}}$. For any $m \in \mathbb{Z}_{\geq 0}$, we define $\mathrm{Sp}_{2m}(R)$ by

$$\mathrm{Sp}_{2m}(R) = \left\{ g \in \mathrm{GL}_{2m}(R) \;\middle|\; g \begin{pmatrix} & -1_m \\ 1_m & \end{pmatrix} {}^t g = \begin{pmatrix} & -1_m \\ 1_m & \end{pmatrix} \right\}$$

for any ring R and Siegel upper-half plane $\mathfrak{h}_m$ by

$$\mathfrak{h}_m = \{ Z \in \mathrm{M}_m(\mathbb{C}) \mid Z = {}^t Z,\ \mathrm{Im} Z : \text{positive definite} \}.$$

For $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}_{2m}(\mathbb{R})$ $(A, B, C, D \in \mathrm{M}_m(\mathbb{R}))$ and $Z \in \mathfrak{h}_m$, we define a group action of $\mathrm{Sp}_{2m}(\mathbb{R})$ on $\mathfrak{h}_m$ by

$$gZ = (AZ + B)(CZ + D)^{-1}$$

and put

$$j(g, Z) = \det(CZ + D).$$

Let k be an integer and $m \in \mathbb{Z}_{>0}$. We consider a holomorphic function f on $\mathfrak{h}_m$ such that

$$f(gZ) = j(g, Z)^k f(Z)$$

for any $g \in \mathrm{Sp}_{2m}(\mathbb{Z})$ and $Z \in \mathfrak{h}_m$. Then, since

$$f(Z + B) = \det B^k f(Z)$$

for any symmetric matrix B in $\mathrm{M}_m(\mathbb{Z})$, $f = 0$ if k is odd and f has the Fourier expansion

$$f(Z) = \sum_S A_f(S) \exp(2\pi\sqrt{-1}\mathrm{tr}(SZ))$$

if k is even, where S runs over all the half-integral symmetric matrices of size m. We note that $A_f(S) = 0$ if S is not positive semi-definite and $m > 1$ (Koecher's principle). We call f a Siegel modular form of degree m and weight k if $m > 1$. For $m = 1$, we call f a Siegel modular form of degree 1 and weight k if f is a modular form of weight k, i.e. $A_f(S) = 0$ if $S < 0$. We denote by $M_k(\mathrm{Sp}_{2m}(\mathbb{Z}))$ the space of Siegel modular forms of degree m and weight k. Moreover, we call $f \in M_k(\mathrm{Sp}_{2m}(\mathbb{Z}))$ a Siegel cusp form if f lies in the kernel of the Φ-operator, i.e., the function

$$Z \mapsto \lim_{t \to +\infty} f(\mathrm{diag}(Z, \sqrt{-1}t)$$

on $\mathfrak{h}_{m-1}$ is zero. Note that $f \in M_k(\mathrm{Sp}_{2m}(\mathbb{Z}))$ is a Siegel cusp form if and only if $A_f(S) = 0$ unless S is positive definite. We denote by $S_k(\mathrm{Sp}_{2m}(\mathbb{Z}))$ the space of all Siegel cusp forms in $M_k(\mathrm{Sp}_{2m}(\mathbb{Z}))$ (formally, we define $M_k(\mathrm{Sp}_0(\mathbb{Z})) = S_k(\mathrm{Sp}_0(\mathbb{Z})) = \mathbb{C}$).

For any $f \in M_k(\mathrm{Sp}_{2m}(\mathbb{Z}))$, we define an automorphic form φ_f on $\mathrm{Sp}_{2m}(\mathbb{A}_\mathbb{Q})$ by

$$\varphi_f(\gamma g_\infty k) = j(g_\infty, \sqrt{-1}1_m)^{-k} f(g_\infty(\sqrt{-1}1_m))$$

for $\gamma \in \mathrm{Sp}_{2m}(\mathbb{Q}), g_\infty \in \mathrm{Sp}_{2m}(\mathbb{R})$, and $k \in K_0 = \prod_{p:\mathrm{prime}} \mathrm{Sp}_{2m}(\mathbb{Z}_p)$ (recall the strong approximation theorem $\mathrm{Sp}_{2m}(\mathbb{A}_\mathbb{Q}) = \mathrm{Sp}_{2m}(\mathbb{Q})(\mathrm{Sp}_{2m}(\mathbb{R}) \times K_0))$. f is called a Hecke eigenform if φ_f is a Hecke eigenform. We note that if $f \in S_k(\mathrm{Sp}_{2m}(\mathbb{Z}))$, then φ_f is a cusp from.

For any Hecke eigenform $f \in S_k(\mathrm{Sp}_{2m}(\mathbb{Z}))$, we denote by π_f the cuspidal representation of $\mathrm{Sp}_{2m}(\mathbb{A}_\mathbb{Q})$ generated by φ_f and define the standard L-function $L(s, f, \mathrm{st})$ of f by

$$L(s, f, \mathrm{st}) = \prod_{p:\mathrm{prime}} L(s, (\pi_f)_p, \mathrm{st}),$$

where $L(s, (\pi_f)_p, \mathrm{st})$ is the local L-function of the p-th component $(\pi_f)_p$ of π_f associated to the standard embedding

$$\widehat{\mathrm{Sp}_{2m}} = \mathrm{SO}_{2m+1}(\mathbb{C}) \hookrightarrow \mathrm{GL}_{2m+1}(\mathbb{C}).$$

0.1.2 The Ikeda lifting

The Miyawaki lifting is defined by using the block diagonal restrictions of Ikeda lifts. Ikeda lifts are the Siegel modular forms given by the following result of Ikeda.

Theorem 0.1 ([Ike01, Theorem 3.2])**.** Let k, m be nonnegative integers such that $k+m$ is even and f a normalized Hecke eigenform of $S_{2k}(\mathrm{SL}_2(\mathbb{Z}))$. Then, there is an explicitly-constructed Hecke eigenform $\mathcal{F} \in S_{k+m}(Sp_{4m}(\mathbb{Z}))$, which we call the Ikeda lift of f to $S_{k+m}(Sp_{4m}(\mathbb{Z}))$, such that

$$L(s, \mathcal{F}, \mathrm{st}) = \zeta(s) \prod_{i=1}^{2m} L(s+k+m-i, f),$$

where $L(s, f)$ is the Hecke L-function of f.

0.1.3 The Miyawaki lifting

We explain the 2006 work of Ikeda [Ike06].

Let k, n, and r be nonnegative integers such that $k+n+r$ is even. Let f be a normalized Hecke eigenform of $S_{2k}(\mathrm{SL}_2(\mathbb{Z}))$ and $g \in S_{k+n+r}(\mathrm{Sp}_{2r}(\mathbb{Z}))$. We denote the Ikeda lift of f to $S_{k+n+r}(\mathrm{Sp}_{4n+4r}(\mathbb{Z}))$ by $\mathcal{F}$. Then, the Miyawaki lift $\mathcal{M}_{\mathcal{F}}(g)$ of g with respect to $\mathcal{F}$ is defined by

$$\mathcal{M}_{\mathcal{F}}(g)(Z) = \int_{\mathrm{Sp}_{2r}(\mathbb{Z})\backslash \mathfrak{h}_r} \mathcal{F}(\mathrm{diag}(Z, W))\overline{g^c(W)}(\det \mathrm{Im} W)^{k+n-1} dW$$

for any $Z \in \mathfrak{h}_{2n+r}$, where $g^c = \overline{g(-\overline{(\cdot)})} \in S_{k+n+r}(\mathrm{Sp}_{2r}(\mathbb{Z}))$.

For this $\mathcal{M}_{\mathcal{F}}(g)$, Ikeda proved the following:

Theorem 0.2 ([Ike06, Theorem 1.1])**.** We have $\mathcal{M}_{\mathcal{F}}(g) \in S_{k+n+r}(\mathrm{Sp}_{4n+2r}(\mathbb{Z}))$. Moreover, if g is a Hecke eigenform and $\mathcal{M}_{\mathcal{F}}(g) \neq 0$, then $\mathcal{M}_{\mathcal{F}}(g)$ is a Hecke eigenform whose standard L-function is equal to

$$L(s, g, \mathrm{st}) \prod_{i=1}^{2n} L(s+k+n+r-i, f).$$

The reason why this lifting is called the 'Miyawaki' lifting is that Ikeda constructed it to approach the following conjecture of Miyawaki in 1992:

Conjecture 0.3 ([Miy92])**.** Given normalized Hecke eigenforms $f \in S_{2k-4}(\mathrm{SL}_2(\mathbb{Z}))$ and $g \in S_k(\mathrm{SL}_2(\mathbb{Z}))$, there should be a Hecke eigenform $F_{f,g} \in S_k(\mathrm{Sp}_6(\mathbb{Z}))$ whose standard L-function is equal to

$$L(s, g, \mathrm{st})L(s+k-2, f)L(s+k-3, f).$$

Indeed, Theorem 0.1 reduces the above conjecture to the nonvanishing of Miyawaki lifts. However, we do not know the nonvanishing of them in general.

0.2 The Miyawaki lifting: The hermitian case

In 2018, the hermitian analogue of Theorem 0.2 was shown by Atobe and Kojima [AK18]. In the same year, Kim and Yamauchi defined the analogue of the Miyawaki lifting for exceptional groups and obtained the similar result [KY18]. Here, we recall the former result. It is completely parallel to §0.1.

0.2.1 Hermitian modular forms

We introduce some notations for hermitian modular forms.

Let $K = \mathbb{Q}(\sqrt{-D})$ be an imaginary quadratic field with discriminant $-D$ and Galois conjugation c. We denote the Dirichlet character which corresponds to $K/\mathbb{Q}$ by $\chi_{K/\mathbb{Q}}$ and the adele ring of K by $\mathbb{A}_K$.

For any $m \in \mathbb{Z}_{\geq 0}$, we define $\mathrm{U}(m, m)(R)$ by

$$\mathrm{U}(m,m)(R) = \left\{ g \in \mathrm{GL}_{2m}(R \otimes_{\mathbb{Q}} K) \;\middle|\; g \begin{pmatrix} & -1_m \\ 1_m & \end{pmatrix} {}^t g^c = \begin{pmatrix} & -1_m \\ 1_m & \end{pmatrix} \right\}$$

for any $\mathbb{Q}$-algebra R and hermitian upper-half plane $\mathcal{H}_m$ by

$$\mathcal{H}_m = \{Z \in \mathrm{M}_m(\mathbb{C}) \mid \sqrt{-1}^{-1}(Z - {}^t\overline{Z}) : \text{positive definite}\}.$$

For $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{U}(m,m)(\mathbb{R})$ and $Z \in \mathcal{H}_m$, we define a group action of $\mathrm{U}(m,m)(\mathbb{R})$ on $\mathcal{H}_m$ by

$$gZ = (AZ + B)(CZ + D)^{-1}$$

and put

$$j(g, Z) = \det(CZ + D),$$

which is similar to the Siegel case.

For simplicity, we assume that the class number of K is equal to one. Put

$$\Gamma^m = \mathrm{U}(m,m)(\mathbb{Q}) \cap \mathrm{GL}_{2m}(\mathcal{O}_K),$$

where $\mathcal{O}_K$ is the ring of integers of K.

Let k be an integer, $m \in \mathbb{Z}_{>0}$, and σ a character of Γ^m. We consider a holomorphic function f on $\mathcal{H}_m$ such that

$$f(gZ) = \sigma(g) j(g, Z)^k f(Z)$$

for any $g \in \Gamma^m$, $Z \in \mathcal{H}_m$. Then, similar to the Siegel case, $f = 0$ if k is odd and f has the Fourier expansion

$$f(Z) = \sum_H A_f(H) \exp(2\pi\sqrt{-1}\mathrm{tr}(HZ))$$

if k is even, where H runs over all the half-integral hermitian matrices in $\mathrm{M}_m(K)$. We note that $A_f(H) = 0$ if H is not positive semi-definite and $m > 1$. We call f a hermitian modular form of degree m and weight k with character σ if $m > 1$. For $m = 1$, we call f a hermitian modular form of degree 1 and weight k with character σ if $A_f(H) = 0$ for any $H < 0$. We denote by $M_k(\Gamma^m, \sigma)$ the space of hermitian modular forms of degree m and weight k with character σ. Moreover, we call $f \in M_k(\Gamma^m, \sigma)$ a hermitian cusp form if the function

$$Z \mapsto \lim_{t \to +\infty} f(\mathrm{diag}(Z, \sqrt{-1}t)$$

on $\mathcal{H}_{m-1}$ is zero. We denote by $S_k(\Gamma^m, \sigma)$ the space of all hermitian cusp forms in $M_k(\Gamma^m, \sigma)$ (formally, we put $M_k(\Gamma^0, \sigma) = S_k(\Gamma^0, \sigma) = \mathbb{C}$).

Assume that k is even and $\sigma = \det^{-k/2}$. Then, for any $f \in M_k(\Gamma^m, \det^{-k/2})$, we can define an automorphic form φ_f on $\mathrm{U}(m,m)(\mathbb{A}_\mathbb{Q})$ by

$$\varphi_f(\gamma g_\infty k) = j(g_\infty, \sqrt{-1}1_m)^{-k} f(g_\infty(\sqrt{-1}1_m)) \det(g_\infty)^{k/2}$$

for $\gamma \in \mathrm{U}(m,m)(\mathbb{Q}), g_\infty \in \mathrm{U}(m,m)(\mathbb{R})$, and $k \in K_0 = \prod_{p:\mathrm{prime}} \mathrm{U}(m,m)(\mathbb{Q}_p) \cap \mathrm{GL}_{2m}(\mathcal{O}_{K_p})$, where $\mathcal{O}_{K_p}$ is the ring of integers of $\mathbb{Q}_p \otimes_\mathbb{Q} K$ (if p splits in K, define $\mathcal{O}_{K_p} = \mathbb{Z}_p \times \mathbb{Z}_p$ formally). Since

$$\mathrm{U}(m,m)(\mathbb{A}_\mathbb{Q}) = \mathrm{U}(m,m)(\mathbb{Q})(\mathrm{U}(m,m)(\mathbb{R}) \times K_0)$$

(note that the cardinality of $\mathrm{U}(m,m)(\mathbb{Q})\backslash\mathrm{U}(m,m)(\mathbb{A}_\mathbb{Q})/(\mathrm{U}(m,m)(\mathbb{R}) \times K_0)$ is equal to the class number of K in general), this is well-defined. f is called a Hecke eigenform if φ_f is a Hecke eigenform. We note that if $f \in S_k(\Gamma^m, \det^{-k/2})$ then φ_f is a cusp form.

For any Hecke eigenform $f \in S_k(\Gamma^m, \det^{-k/2})$, we denote by π_f the cuspidal representation of $\mathrm{U}(m,m)(\mathbb{A}_\mathbb{Q})$ generated by φ_f and we define the standard L-function $L(s, f, \mathrm{st})$ by

$$L(s, f, \mathrm{st}) = \prod_{\mathfrak{p} < \infty} L(s, \mathrm{BC}(\pi_f)_\mathfrak{p}),$$

where $\mathfrak{p}$ runs over all finite places of K and $\mathrm{BC}(\pi_f)_\mathfrak{p}$ is the $\mathfrak{p}$-th component of the standard base change $\mathrm{BC}(\pi_f)$ of π_f to $\mathrm{GL}_{2m}(\mathbb{A}_K)$ (we recall the definition in §0.3.1) and $L(s, \mathrm{BC}(\pi_f)_\mathfrak{p})$ is the local L-function of $\mathrm{BC}(\pi_f)_\mathfrak{p}$ associated to the identity map of $\mathrm{GL}_{2m}(\mathbb{C})$.

0.2.2 The hermitian Ikeda lifting

To define the Miyawaki lifting, we need the Ikeda lifting. The following result of Ikeda is the hermitian analogue of Theorem 0.1.

Theorem 0.4 ([Ike08, Theorem 5.1, 5.2])**.** Let k, m be nonnegative integers and put $l = 2k+m$ (resp. $l = 2k+m-1$) if m is even (resp. odd). Let f be a normalized Hecke eigenform of $S_{2k}(\mathrm{SL}_2(\mathbb{Z}))$ (resp. $S_{2k+1}(\Gamma_0(D), \chi_{K/\mathbb{Q}})$) if m is even (resp. odd). Then, there is an explicitly-constructed Hecke eigenform $\mathcal{F} \in S_l(\Gamma^m, \det^{-l/2})$, which we call the (hermitian) Ikeda lift of f to $S_l(\Gamma^m, \det^{-l/2})$, such that

$$L(s, \mathcal{F}, \mathrm{st}) = \prod_{i=1}^{m} \prod_{\mathfrak{p}<\infty} L(s + m/2 + 1/2 - i, (\pi_f^{\mathrm{GL}})_{\mathfrak{p}}^K),$$

where π_f^{GL} is the cuspidal representation of $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$ corresponding to f and $(\pi_f^{\mathrm{GL}})^K$ is the base change lift of π_f^{GL} to $\mathrm{GL}_2(\mathbb{A}_K)$.

0.2.3 The hermitian Miyawaki lifting

We state the result of Atobe and Kojima [AK18].

Let k, n, and r be nonnegative integers and put $l = 2k+n+2r$ (resp. $l = 2k+n+2r-1$) if n is even (resp. odd). Let f be a normalized Hecke eigenform of $S_{2k}(\mathrm{SL}_2(\mathbb{Z}))$ (resp. $S_{2k+1}(\Gamma_0(D), \chi_{K/\mathbb{Q}})$) if n is even (resp. odd) and $g \in S_l(\Gamma^r, \det^{-l/2})$. We denote the Ikeda lift of f to $S_l(\Gamma^{2n+r}, \det^{-l/2})$ by $\mathcal{F}$. Then, the (hermitian) Miyawaki lift $\mathcal{M}_{\mathcal{F}}(g)$ of g with respect to $\mathcal{F}$ is defined by

$$\mathcal{M}_{\mathcal{F}}(g)(Z) = \int_{\Gamma^m \backslash \mathcal{H}_r} \mathcal{F}(\mathrm{diag}(Z, W)) \overline{g(W)} (\det \mathrm{Im} W)^{l-2r} dW$$

for any $Z \in \mathcal{H}_{n+r}$.

For this $\mathcal{M}_{\mathcal{F}}(g)$, they proved the hermitian analogue of Theorem 0.2:

Theorem 0.5 ([AK18, Theorem 1.1], see also Theorem 5.3 in *loc. cit.*)**.** We have $\mathcal{M}_{\mathcal{F}}(g) \in S_l(\Gamma^{n+r}, \det^{-l/2})$. Moreover, if g is a Hecke eigenform and $\mathcal{M}_{\mathcal{F}}(g) \neq 0$, then $\mathcal{M}_{\mathcal{F}}(g)$ is a Hecke eigenform whose standard L-function is equal to

$$L(s, g, \mathrm{st}) \prod_{i=1}^{n} \prod_{\mathfrak{p}<\infty} L(s + n/2 + 1/2 - i, (\pi_f^{\mathrm{GL}})_{\mathfrak{p}}^K).$$

More precisely speaking, by the calculation of the infinite component, they obtained the following equation of the complete L-function

$$L(s, \mathrm{BC}(\pi_{\mathcal{M}_{\mathcal{F}}(g)})) = L(s, \mathrm{BC}(\pi_g)) \prod_{i=1}^{n} L(s + n/2 + 1/2 - i, (\pi_f^{\mathrm{GL}})^K)$$

if $k \geq n/2$.

0.3 The representation-theoretical Miyawaki lifting

We can generalize the theory of the Miyawaki lifting representation-theoretically. There are two styles, namely,

- the generalization using representation-theoretical Ikeda lifts which is explicitly constructed (cf. [IY20, Yam20]) and
- the generalization using Arthur's endoscopic classification ([Art13] and others), which is independent of the theory of the representation-theoretical Ikeda lifting.

Atobe studied the representation-theoretical Miyawaki lifting for Symplectic/Metaplectic groups under the former style [Ato20] using the representation-theoretical Ikeda lifting given by Ikeda and Yamana [IY20]. We recall the representation-theoretical Miyawaki lifting for unitary groups under the latter style defined by the author [Ito] (it is basically defined in the same way for other groups).

Let F be a number field and E a quadratic extension of F with the nontrivial element c of $\mathrm{Gal}(E/F)$. We denote the ring of adeles of F and E by $\mathbb{A}_F$ and $\mathbb{A}_E$, respectively. For any nondegenerate hermitian spaces V over E, we denote its isometry group by $\mathrm{U}(V)$, which is an algebraic group over F.

0.3.1 The endoscopic classification

First, we review the necessary parts of Arthur's endoscopic classification for unitary groups, which is completed by Mok in the quasi-split case [Mok15] and almost completed by Kaletha, Minguez, Shin, and White in the general case [KMSW].

Consider the formal commutative sum:

$$\psi = \boxplus_{i=1}^{l} k_i \mu_i \boxtimes [m_i].$$

Here,

- k_i, n_i, m_i are positive integers,
- μ_i is an irreducible unitary cuspidal representation of $\mathrm{GL}_{n_i}(\mathbb{A}_E) = \mathrm{GL}_{n_i}(E \otimes_F \mathbb{A}_F)$,
- $[m_i]$ is the unique irreducible m_i-dimensional algebraic representation of $\mathrm{SL}_2(\mathbb{C})$, and
- $\mu_i \boxtimes [m_i]$ (we often suppress $\boxtimes$ and write $\mu_i[m_i]$ for short) is a formal tensor product of μ_i and $[m_i]$ such that if $\mu_i = \mu_j$ and $m_i = m_j$, then $i = j$.

Put $N = \sum_{i=1}^{l} k_i n_i m_i$. We denote the isobaric automorphic representation

$$\boxplus_{i=1}^{l} (\boxplus_{j=1}^{m_i} \mu_i |\cdot|_{\mathbb{A}_E}^{m_i/2+1/2-i})^{\boxplus k_i}$$

of $\mathrm{GL}_N(\mathbb{A}_E)$ by ϕ_ψ, where $|\cdot|_{\mathbb{A}_E}$ is the idele norm of $\mathbb{A}_E^\times$. Moreover, for any place v of F, we define ψ_v by

$$\psi_v = \oplus_{i=1}^{l} \boxtimes_{w|v} (\mu_{i,w} \boxtimes [m_i]^{\oplus k_i}),$$

where w runs over all places of E on v. Here, we identify the w-th component $\mu_{i,w}$ of μ_i with the representation of the Langlands group of E_w which corresponds to $\mu_{i,w}$ under the local Langlands classification for general linear groups. Then, ψ_v is a representation of $\prod_{w|v}(L_{E_w} \times \mathrm{SL}_2(\mathbb{C}))$, where W_{E_w} is the Weil group of E_w and

$$L_{E_w} = \begin{cases} W_{E_w} & w : \text{archimedean;} \\ W_{E_w} \times \mathrm{SL}_2(\mathbb{C}) & w : \text{nonarchimedean.} \end{cases}$$

We say that ψ is a global discrete A-parameter of degree N if

- $k_i = 1$ for any i and
- for any i, μ_i is conjugate selfdual with parity $(-1)^{m_i+N}$, i.e. the Asai L-function $L(s, \mu_i, \mathrm{As}^{(-1)^{m_i+n}})$ (see [GGP12, §7]) has a simple pole at $s = 1$.

We denote by $\Psi_2(N)$ the set of global discrete A-parameters of degree N. We note that for each place v of F which does not split in E, ψ_v for any $\psi \in \Psi_2(N)$ is conjugate selfdual with parity $(-1)^{N-1}$, i.e. there is a nondegenerate bilinear form $B\colon \mathbb{C}^N \times \mathbb{C}^N \to \mathbb{C}$ such that

$$B(\psi_v(w)x, \psi_v(w_c^{-1} w w_c)y) = B(x, y),$$
$$B(y, x) = (-1)^{N-1} B(x, \psi_v(w_c^2)y)$$

for any $w \in L_{E_v} \times \mathrm{SL}_2(\mathbb{C})$ and $x, y \in \mathbb{C}^N$, where w_c is a fixed element of $W_{F_v} \setminus W_{E_v}$ (the definition does not depend on the choice of w_c). If v is a place of F which splits into two places w_1, w_2 in E, then $\psi_v = \psi_{w_1} \boxtimes \psi_{w_2}$ satisfies $\psi_{w_1}^\vee = \psi_{w_2}$, where we identify $W_{E_{w_1}}$ with $W_{E_{w_2}}$ naturally and $\psi_{w_1}^\vee$ is the dual of ψ_{w_1}.

Let V be a nondegenerate N-dimensional hermitian space over E and $\psi \in \Psi_2(N)$. Then, for each place v of F, the Local A-packet $\Pi_{\psi_v}(V_v)$, which is a multiset of irreducible representations of $\mathrm{U}(V)(F_v)$, is defined (it depends only on ψ_v and the hermitian space $V_v = V \otimes_F F_v$ over E_v). We put

$$\Pi_\psi(V) = \{\otimes_v \pi_v \in \otimes_v \Pi_{\psi_v}(V_v) \mid \pi_v \text{ is unramified for almost all } v\}.$$

We note that

- if ψ_v is trivial on the inertia of W_{E_w} for any $w|v$ and $\mathrm{U}(V)$ is unramified over F_v, then $\Pi_{\psi_v}(V_v)$ contains a unique unramified representation, which corresponds to the L-parameter $(\phi_\psi)_v$ and
- if v splits into two places w_1, w_2 in E, then $\Pi_{\psi_v}(V_v)$ is singleton and it consists of the representation which corresponds to $(\phi_\psi)_{w_1}$, where we identify $\mathrm{U}(V)(F_v)$ with $\mathrm{GL}_N(E_{w_1})$ by the composition of the projection to the first factor

  $$\mathrm{GL}_N(E_{w_1}) \times \mathrm{GL}_N(E_{w_2}) \twoheadrightarrow \mathrm{GL}_N(E_w)$$

  and the natural embedding

  $$\mathrm{U}(V)(F_v) \hookrightarrow \mathrm{GL}_N(E_{w_1}) \times \mathrm{GL}_N(E_{w_2}).$$

Then, the claim of the endoscopic classification is as follows:

Theorem 0.6 ([Mok15, Theorem 2.5.2], [KMSW, Theorem 1.7.1])**.** Let

$$L^2_\psi = \bigoplus_\pi \pi,$$

where π runs over the representations in $\Pi_\psi(V)$ which satisfies a certain condition (we do not discuss it here). Then, the discrete spectrum $L^2_{\mathrm{disc}}(\mathrm{U}(V)(F)\backslash \mathrm{U}(V)(\mathbb{A}_F))$ of $L^2(\mathrm{U}(V)(F)\backslash \mathrm{U}(V)(\mathbb{A}_F))$ is decomposed as

$$L^2_{\mathrm{disc}}(\mathrm{U}(V)(F)\backslash \mathrm{U}(V)(\mathbb{A}_F)) = \bigoplus_{\psi \in \Psi_2(N)} L^2_\psi.$$

If an irreducible discrete automorphic representation π of $\mathrm{U}(V)(\mathbb{A}_F)$ is a subspace of L^2_ψ, then we say that 'π has the A-parameter ψ' and we denote ϕ_ψ by $\mathrm{BC}(\pi)$ (the standard base change). Note that two irreducible discrete automorphic representations π and π' of $\mathrm{U}(V)(\mathbb{A}_F)$ are nearly-equivalent if and only if $\mathrm{BC}(\pi) = \mathrm{BC}(\pi')$.

0.3.2 The definition of representation-theoretical Ikeda lifts

We introduce the representation-theoretical Ikeda lifts. Let V be as above and assume $N = 2m$ is even.

Definition 0.7. Let π be an irreducible discrete automorphic representation of $\mathrm{U}(V)(\mathbb{A}_F)$. π is called an Ikeda lift if the A-parameter of π is equal to $\phi[m]$ for some ϕ, where ϕ is a conjugate self-dual cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_E)$ with parity $(-1)^m$ or equal to $\chi \boxplus \chi'$ for some distinct conjugate self-dual automorphic characters χ, χ' of $\mathbb{A}_E^\times$ with parity $(-1)^m$.

We note that any automorphic representation generated by some hermitian Ikeda lift (see §0.2.2) or representation-theoretical hermitian Ikeda lift constructed in [Yam20] is an Ikeda lift in this sense.

The following is the most important property of Ikeda liftings.

Proposition 0.8. Let π be an Ikeda lifting of $\mathrm{U}(V)(\mathbb{A}_F)$ and v a place of F. Assume $\mathrm{U}(V)$ is unramified over F_v and π_v is unramified. Then, π_v is isomorphic to a quotient of some degenerate principal series representation of $\mathrm{U}(V)(F_v) \simeq \mathrm{U}(m,m)(F_v)$, i.e. a parabolic induction

$$I_m(\chi) := \mathrm{Ind}_P^{\mathrm{U}(m,m)(F_v)} \delta_P^{1/2} \otimes \chi \circ \det,$$

where P is the block upper triangular parabolic subgroup of $\mathrm{U}(m,m)(F_v)$ whose Levi subgroup is isomorphic to $\mathrm{GL}_m(E \otimes_F F_v)$, χ is a character of $(E \otimes_F F_v)^\times$, and δ_P is the modulus character of P.

0.3.3 The definition of representation-theoretical Miyawaki lifts

We define the representation-theoretical Miyawaki lifting.

Let V_1, V_2 be nondegenerate hermitian spaces over E such that $\dim V_1 = n \in \mathbb{Z}_{>0}$, $\dim V_2 = n+2l$, for $l \in \mathbb{Z}_{\geq 0}$. Put $V = V_1 \perp V_2$ and think of $\mathrm{U}(V_1) \times \mathrm{U}(V_2)$ as a subgroup of $\mathrm{U}(V)$ by the natural embedding. Let Π be an Ikeda lifting of $\mathrm{U}(V)(\mathbb{A}_F)$ with A-parameter $\phi[n+l]$ and τ a discrete automorphic representation of $\mathrm{U}(V_1)(\mathbb{A}_F)$ with A-parameter ψ.

Definition 0.9 (cf. §0.2.3)**.** For any $f \in \tau$ and $\mathcal{F} \in \Pi$, define an automorphic form $\mathcal{M}_{\mathcal{F}}(f)$ of $\mathrm{U}(V_2)$ by

$$\mathcal{M}_{\mathcal{F}}(f)(g) = \int_{\mathrm{U}(V_1)(F)\backslash \mathrm{U}(V_1)(\mathbb{A}_F)} \mathcal{F}|_{\mathrm{U}(V_1)(\mathbb{A}_F)\times \mathrm{U}(V_2)(\mathbb{A}_F)}(h,g)\overline{f(h)}dh$$

as long as it converges. We call the representation of $\mathrm{U}(V_2)(\mathbb{A}_F)$ generated by all $\mathcal{M}_{\mathcal{F}}(f)$ the Miyawaki lift of τ with respect to Π and denote it by $\mathcal{M}_\Pi(\tau)$.

Here, we recall that unramified degenerate principal series representations have simple branching laws on unramified representations as follows:

Proposition 0.10 ([AK18, Proposition 2.2, 2.3])**.** Assume $\mathrm{U}(V)$, $\mathrm{U}(V_1)$ and $\mathrm{U}(V_2)$ are unramified over F_v. Let τ_1 (resp. τ_2) be an irreducible unramified representation of $\mathrm{U}(V_1)(F_v)$ (resp. $\mathrm{U}(V_2)(F_v)$) with L-parameter ϕ_1 (resp. ϕ_2). Then, if there is a surjective $\mathrm{U}(V_1)(F_v) \times \mathrm{U}(V_2)(F_v)$-map

$$I_{n+l}(\chi) \twoheadrightarrow \tau_1 \boxtimes \tau_2$$

for a character χ of $(E \otimes_F F_v)^\times$, we have

$$\phi_2 = \phi_1^\vee \chi(\chi^c)^{-1} \bigoplus_{i=1}^{l} (\chi \oplus (\chi^c)^{-1})|\cdot|_{F_v}^{l/2+1/2-i}.$$

This proposition and Proposition 0.8 determine the near equivalence classes of Miyawaki lifts. Namely, the following holds:

Corollary 0.11. Assume $\mathcal{M}_\Pi(\tau)$ is a subspace of $L^2(\mathrm{U}(V)(F)\backslash \mathrm{U}(V)(\mathbb{A}_F))$ and

$$\mathcal{M}_{\phi[n+l]}(\psi) := \psi^\vee \chi_\phi \boxplus \phi[l]$$

is a discrete A-parameter (namely, multiplicity-free), where χ_ϕ is the central character of ϕ and $\psi^\vee$ is the dual of ψ. Then, we have

$$\mathcal{M}_\Pi(\tau) \subset L^2_{\mathcal{M}_{\phi[n+l]}(\psi)}.$$

0.4 The nonvanishing of Miyawaki lifts

For application, it is important to determine the nonvanishing of Miyawaki lifts. Some results are already known (e.g. [KY19] and [Ato20, §5]), however, we do not know the complete determination of it, either classically or representation-theoretically.

However, similarly to other problems for global periods, we can divide this problem into two parts:

- (local problem) Determine the branching laws of the local components of Ikeda liftings with respect to block diagonal subgroups.
- (global problem) Give a relationship between the nonvanishing of the Miyawaki lifting and some special L-value.

In this paper, we give a partial answer of the local problem for split unitary groups, i.e., general linear groups, over p-adic fields.

0.5 The work of Lapid and Mao

This paper is highly related to the work of Lapid and Mao [LM20] (see also §1.2, 1.3). Let us quote their result here.

Let F be a p-adic field with absolute value $|\cdot|$ and ring of integers $\mathcal{O}$. Let q be the cardinality of the residue field of F and ψ a nontrivial additive character of F. For $k', n' \in \mathbb{Z}_{>0}$, define $w_{(k',n')} \in \mathrm{GL}_{n'k'}(F)$ by

$$(w_{(k',n')})_{i,j} = \delta_{k'(i-pn'+n'-1)+p,j} \quad \text{if } pn'-n'+1 \leq i \leq pn'.$$

Let π^1, π^2 be irreducible generic representations of $\mathrm{GL}_k(F)$ ($k \in \mathbb{Z}_{>0}$) and π^1_n, π^2_n their Speh representations of $\mathrm{GL}_{nk}(F)$ for $n \in \mathbb{Z}_{>0}$, respectively. Let $\mathcal{W}^{\psi}_{\mathrm{Sh}}(\pi^i_n)$ be the Shalika model of π^i_n, i.e., the unique realization of π^i_n in

$$\mathrm{Ind}_U^{\mathrm{GL}_{nk}(F)} \Psi_{(k,n)},$$

where a parabolic subgroup P of $\mathrm{GL}_{nk}(F)$ with unipotent radical U is defined by

$$P = MU,\ M = w_{(k,n)}^{-1} \begin{pmatrix} \mathrm{GL}_n & & & & \\ & \mathrm{GL}_n & & & \\ & & \ddots & & \\ & & & \mathrm{GL}_n & \\ & & & & \mathrm{GL}_n \end{pmatrix} w_{(k,n)},\ U = w_{(k,n)}^{-1} \begin{pmatrix} 1_n & & & & \\ & 1_n & & * & \\ & & \ddots & & \\ & & & 1_n & \\ & & & & 1_n \end{pmatrix} w_{(k,n)}$$

and a character $\Psi_{(k,n)}$ of U is defined by

$$\Psi_{(k,n)} \left(w_{(k,n)}^{-1} \begin{pmatrix} 1_n & X_1 & & & \\ & 1_n & X_2 & * & \\ & & \ddots & \ddots & \\ & & & 1_n & X_{k-1} \\ & & & & 1_n \end{pmatrix} w_{(k,n)} \right) = \psi \left(\sum_{1 \leq i \leq k-1} \mathrm{tr} X_i \right),$$

where $X_i \in \mathrm{M}_k(F)$.

For any $W^1_{\mathrm{Sh}} \in \mathcal{W}^{\psi}_{\mathrm{Sh}}(\pi^1_n), W^2_{\mathrm{Sh}} \in \mathcal{W}^{\psi}_{\mathrm{Sh}}(\pi^2_n), \Phi \in \mathcal{S}(\mathrm{M}_{n,nk}(F))$ and $s \in \mathbb{C}$, put

$$Z(W^1_{\mathrm{Sh}}, W^2_{\mathrm{Sh}}, \Phi, s) = \int_{U \backslash \mathrm{GL}_{nk}(F)} W^1_{\mathrm{Sh}}(g) W^2_{\mathrm{Sh}}(g) \Phi(\eta g) |\det g|^s dg,$$

where $\eta \in \mathrm{M}_{n,nk}(F)$ is defined by

$$\eta_{i,j} = \begin{cases} 1 & \text{if } j = ki; \\ 0 & \text{otherwise.} \end{cases}$$

This zeta integral is an analogue of the Rankin-Selberg zeta integral defined in [JPSS83] for equal-rank two representations.

They proved the following:

Theorem 0.12 ([LM20, Theorem 5.1])**.** The integral $Z(W^1_{\mathrm{Sh}}, W^2_{\mathrm{Sh}}, \Phi, s)$ has the following properties.

(i) If $\mathrm{Re}(s)$ is sufficiently large, then the integral defining $Z(W^1_{\mathrm{Sh}}, W^2_{\mathrm{Sh}}, \Phi, s)$ is absolutely convergent for any $W^1_{\mathrm{Sh}}, W^2_{\mathrm{Sh}}$, and Φ.

(ii) For any $W^1_{\mathrm{Sh}}, W^2_{\mathrm{Sh}}$, and Φ, $Z(W^1_{\mathrm{Sh}}, W^2_{\mathrm{Sh}}, \Phi, s)$ admits a meromorphic continuation to $\mathbb{C}$ and

$$\prod_{i=0}^{n-1} L(s-i, \pi^1 \boxtimes \pi^2)^{-1} Z(W^1_{\mathrm{Sh}}, W^2_{\mathrm{Sh}}, \Phi, s)$$

defines an element of $\mathbb{C}[q^{-s}, q^s]$, where $L(s, \pi^1 \boxtimes \pi^2)$ is the Rankin-Selberg local L-function defined in [JPSS83].

(iii) if π_n^1 and π_n^2 are unramified, then

$$Z(W_{\mathrm{Sh}}^1, W_{\mathrm{Sh}}^2, \Phi, s) = \prod_{i=0}^{n-1} L(s-i, \pi^1 \boxtimes \pi^2)$$

up to a constant if W_{Sh}^1 and W_{Sh}^2 are unramified vectors and Φ is the characteristic function of $\mathrm{M}_{n,nk}(\mathcal{O})$.

(iv) We have a local functional equation

$$Z(\widehat{W_{\mathrm{Sh}}^1}, \widehat{W_{\mathrm{Sh}}^2}, \widehat{\Phi}, n-s) = \chi_{\pi^2}(-1)^{n(k-1)} \prod_{i=0}^{n-1} \gamma(s-i, \pi^1 \boxtimes \pi^2, \psi) Z(W_{\mathrm{Sh}}^1, W_{\mathrm{Sh}}^2, \Phi, n-s)$$

for any $W_{\mathrm{Sh}}^1, W_{\mathrm{Sh}}^2$, and Φ, where χ_{π^2} is the central character of π^2, $\gamma(s, \pi^1 \boxtimes \pi^2)$ is the γ-factor for $L(s, \pi^1 \boxtimes \pi^2)$, $\widehat{W_{\mathrm{Sh}}^i}$ is defined by

$$\widehat{W_{\mathrm{Sh}}^i}(g) = W_{\mathrm{Sh}}^i \left(w_{(nk,1)} {}^t g^{-1} \right),$$

and $\widehat{\Phi}$ is the Fourier transform

$$\widehat{\Phi}(X) = \int_{\mathrm{M}_{n,nk}(F)} \Phi\left(\mathrm{tr}({}^t Y w_{(n,1)} X) \right) dY.$$

0.6 The main results

We state the main results of this paper. Let F and ψ be as above. Let π be a generic irreducible representation of $\mathrm{GL}_2(F)$ with central character χ_π. We assume that π is approximately tempered (see §1.1, note that if π is unitary, then it is approximately tempered). Let π_n be the Speh representation $\mathrm{Sp}(\pi, n)$. We note that the local components of any Ikeda lifts at the finite places where the unitary group splits are always of this form (see Remark 1.2). We denote by $\mathcal{W}_{\mathrm{Sh}}^\psi(\pi_n)$ the Shalika model of π_n. Let $\tau \neq 0$ be a smooth representation of $\mathrm{GL}_n(F)$ which is realized as a subrepresentation of a (normalized) parabolic induction $\tau_1 \times \cdots \times \tau_m$ for some irreducible representations τ_i. For $W_{\mathrm{Sh}} \in \mathcal{W}_{\mathrm{Sh}}(\pi_n)$, $s \in \mathbb{C}$, and a matrix coefficient f of τ, we put

$$Z(W_{\mathrm{Sh}}, s, f) := \int_{\mathrm{GL}_n} \Phi_{W_{\mathrm{Sh}}}(g) f(g) |\det g|^{s-\frac{1}{2}} dg,$$

where $\Phi_{W_{\mathrm{Sh}}}$ is the restriction of W_{Sh} to $w_{(2,n)}^{-1} \mathrm{diag}(\mathrm{GL}_n(F), 1_n) w_{(2,n)} \simeq \mathrm{GL}_n(F)$. Then the following holds:

Theorem 0.13 (Theorem 2.1)**.** (i) If $\mathrm{Re}(s)$ is sufficiently large, then the integral defining $Z(W_{\mathrm{Sh}}, s, f)$ converges absolutely for any W_{Sh} and f. Moreover, $Z(W_{\mathrm{Sh}}, s, f)$ admits meromorphic continuation to all of $\mathbb{C}$ and there is a (unique) polynomial $P(X) \in \mathbb{C}[X]$ such that $P(0) = 1$ and

$$\langle Z(W_{\mathrm{Sh}}, s, f) \mid W_{\mathrm{Sh}} \in \mathcal{W}_{\mathrm{Sh}}^\psi(\pi_n),\ f : \text{a matrix coefficient of } \tau \rangle_{\mathbb{C}} = P(q^{-s})^{-1} \mathbb{C}[q^{-s}, q^s],$$

where q is the cardinality of the residue field of F; denote $P(q^{-s})^{-1}$ by $L(\pi; s, \tau)$.

(ii) There is a function $\gamma(s) \in \mathbb{C}(q^{-s})$ such that

$$Z(\widetilde{W_{\mathrm{Sh}}}, 1-s, f(\cdot^{-1})) = \gamma(s) Z(W_{\mathrm{Sh}}, s, f)$$

for any W_{Sh} and f, where $\widetilde{W_{\mathrm{Sh}}} := \chi_\pi^{-1}(\det) W_{\mathrm{Sh}}(\left(\begin{smallmatrix} 1_n & 0 \\ 0 & -1_n \end{smallmatrix} \right) \cdot \left(\begin{smallmatrix} 0 & 1_n \\ -1_n & 0 \end{smallmatrix} \right))(\in \mathcal{W}_{\mathrm{Sh}}^{\psi^{-1}}(\mathrm{Sp}(\pi^\vee, n))$; denote $\gamma(s)$ by $\gamma(\pi; s, \tau, \psi)$.

(iii) If τ is irreducible and generic, then

$$L(\pi; s, \tau) = L(s, \pi \boxtimes \tau),$$

where the right-hand side is the local L-factor of $\pi \boxtimes \tau$ defined by Jacquet, Piatetski-Shapiro and Shalika [JPSS83].

Remark 0.14.

- We constructed the above zeta integral based on the work of Lapid and Mao [LM20], but in fact the integral had already been defined by Ginzburg ([Gin]) and Kaplan (Appendix C of [CFK]) in a more general setting, namely for an irreducible generic representation π of $\mathrm{GL}_k(F)$ for any k and any local field F of characteristic zero (either archimedean or p-adic), independently. Then, the really new result in the above theorem is (iii) only (see Remark 2.2).
- In recent years, some analogues of Rankin-Selberg zeta integrals using Speh representations were given by some researchers:
  - Zeta integrals for Speh representations of type $(n, k) \times (n, k)$ were defined in [LM20].
  - We can find zeta integrals for Speh representations of type $(n, k) \times (n-1, k)$ in the recent work of Atobe, Kondo, and Yasuda ([AKY]).
  - The zeta integrals defined in Appendix C of [CFK] (and [Gin]) are (essentially) for Speh representations of type $(n, 1) \times (k, n)$.

  Here, we say that the Speh representation $\mathrm{Sp}(\pi', l)$ is of type (m, l) if π' is a (generic, irreducible) representation of GL_m. Then, in all three of the above papers, the determination of the L-factors was left unsolved (some partial results, such as [CFK, Proposition C.10], were given, see Remark 2.2). In contrast, our result (iii) is a fortunate example solving this problem, albeit only for Speh representations of type $(n, 1) \times (2, n)$.

Next we consider the general rank case. Since

$$\Phi_{\pi_n(\mathrm{diag}(g_1,g_2))W_{\mathrm{Sh}}} = \Phi_{W_{\mathrm{Sh}}}(g_2^{-1} \cdot g_1)\chi_\pi(\det g_2)$$

for any $W_{\mathrm{Sh}} \in \mathcal{W}^{\psi}_{\mathrm{Sh}}(\pi_n)$ and $g_1, g_2 \in \mathrm{GL}_n$, the linear extension of $L(\pi; s, \tau^\vee)^{-1} Z(\cdot, s, \cdot)|_{s=\frac{1}{2}}$ on $W_{\mathrm{Sh}}(\pi_n) \otimes \tau^\vee \otimes \tau$ defines a nonzero element of

$$\mathrm{Hom}_{\mathrm{GL}_n \times \mathrm{GL}_n}(\pi_n \otimes (\tau^\vee \boxtimes \tau\chi_\pi^{-1}), \mathbb{C}) \simeq \mathrm{Hom}_{\mathrm{GL}_n \times \mathrm{GL}_n}(\pi_n, \tau \boxtimes \tau^\vee \chi_\pi).$$

Then, by simple consideration (see §4), the following holds:

Theorem 0.15 (Theorem 4.1)**.** The space

$$\mathrm{Hom}_{\mathrm{GL}_n \times \mathrm{GL}_{n+2l}}(\pi_{n+l}, \tau \boxtimes \tau^\vee \chi_\pi \times \pi_l)$$

is nonzero.

Remark 0.16. The above theorem only means that local Miyawaki lifts for split unitary groups are always nonvanishing. To complete our purpose, there remain two problems, namely, uniqueness and multiplicity at most one (see Conjecture 4.2).

We now give the organization of this paper. In §1, we introduce the notations, Speh representations, and the models of Speh representations according to [LM20]. In §2, we introduce and study the above zeta integral. We show some properties which zeta integrals should have in general, and give the proof of Theorem 0.13. However, we postpone the proof of the functional equation, which is necessary to show Theorem 0.13(iii), to the next section. In §3, we show the functional equation as just announced. The essential of the section is the inequality $\dim_{\mathbb{C}} \mathrm{Hom}_{\mathrm{GL}_n \times \mathrm{GL}_n}(\pi_n, \tau \boxtimes \tau^\vee \chi_\pi) \leq 1$ for supercuspidal τ. In §4, we make some remarks about the branching laws of Speh representations with respect to block diagonal subgroups of general size, including the proof of Theorem 0.15.

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1 Preliminaries

1.1 Notation

Throughout this paper, fix a p-adic field F with absolute value $|\cdot|$ and ring of integers $\mathcal{O}$. Let q be the cardinality of the residue field of F. If G is an algebraic group over F, we also use G to denote $G(F)$. The term 'representation' is used to refer to a smooth, complex representation of an algebraic group over F.

We denote by $\mathrm{Irr}\mathrm{GL}_m$ the set of equivalence classes of irreducible representations of GL_m and put $\mathrm{Irr} = \bigcup_{0\leq m} \mathrm{Irr}\mathrm{GL}_m$. We denote by $\mathrm{Irr}_{\mathrm{gen}}\mathrm{GL}_m$ (resp. $\mathrm{Irr}_{\mathrm{sc}}\mathrm{GL}_m$) the subset consisting of all generic (resp. supercuspidal) elements of $\mathrm{Irr}\mathrm{GL}_m$ and put $\mathrm{Irr}_{\mathrm{gen}} = \bigcup_{0\leq m} \mathrm{Irr}_{\mathrm{gen}}\mathrm{GL}_m$, $\mathrm{Irr}_{\mathrm{sc}} = \bigcup_{0\leq m} \mathrm{Irr}_{\mathrm{sc}}\mathrm{GL}_m$.

For $\boldsymbol{m} = (m_1, \dots, m_l) \in (\mathbb{Z}_{>0})^l$, we denote the block upper triangular parabolic subgroup of type $\boldsymbol{m}$ with unipotent radical $U_{\boldsymbol{m}}$ by $P_{\boldsymbol{m}}$, a Levi subgroup $\mathrm{diag}(\mathrm{GL}_{m_1}, \dots, \mathrm{GL}_{m_l}) \simeq \mathrm{GL}_{m_1} \times \cdots \times \mathrm{GL}_{m_l}$ of $P_{\boldsymbol{m}}$ by $M_{\boldsymbol{m}}$, and the modulus character of $P_{\boldsymbol{m}}$ by $\delta_{P_{\boldsymbol{m}}}$.

We use the notation Ind_H^G and ind_H^G to denote induction and induction with compact support (both *unnormalized*) from a subgroup H of G. If $\pi_1, \dots, \pi_l$ are representations of $\mathrm{GL}_{m_1}, ..., \mathrm{GL}_{m_l}$ respectively, then we denote the parabolically induced representations

$$\mathrm{Ind}_{P_{\boldsymbol{m}}}^{\mathrm{GL}_m} \delta_{P_{\boldsymbol{m}}}^{\frac{1}{2}} \otimes \pi_1 \boxtimes \cdots \boxtimes \pi_l \text{ and } \mathrm{Ind}_{P_{\boldsymbol{m}}}^{\mathrm{GL}_m} \pi_1 \boxtimes \cdots \boxtimes \pi_l$$

by $\pi_1 \times \cdots \times \pi_l$ (normalized induction) and $\pi_1 * \cdots * \pi_l$ (unnormalized induction), respectively, where $m = \sum_i m_i$ and $\boldsymbol{m} = (m_1, \dots, m_l)$.

If there is no confusion, we often denote $(\overbrace{m_1, m_1, \dots, m_1}^{l_1}, \overbrace{m_2, m_2, \dots, m_2}^{l_2}, \dots)$ by $({m_1}^{l_1}, {m_2}^{l_2}, \dots)$.

Let π be a representation of GL_m. We denote the contragradient representation of π by $\pi^\vee$ and $\pi \otimes \chi \circ \det$ by $\pi\chi$ for any $\chi \in \mathrm{Irr}\mathrm{GL}_1$.

For $\pi \in \mathrm{Irr}_{\mathrm{sc}}$, we denote the unique irreducible subrepresentation of $\pi|\cdot|^{\frac{m-1}{2}} \times \pi|\cdot|^{\frac{m-3}{2}} \times \cdots \times \pi|\cdot|^{-\frac{m-1}{2}}$ by $\mathrm{St}(\pi, m)$ (generalized Steinberg representation).

We denote by $\mathrm{Alg}'\mathrm{GL}_m$ the set of equivalence classes of representations $\pi \neq 0$ of GL_m such that

$$\pi \subset \pi_1 \times \cdots \times \pi_l$$

for some $\pi_1, \dots, \pi_l \in \mathrm{Irr}$ (equivalently, $\pi_1, \dots, \pi_l \in \mathrm{Irr}_{\mathrm{sc}}$) and put $\mathrm{Alg}' = \bigcup_{0\leq m} \mathrm{Alg}'\mathrm{GL}_m$. We note that Alg' is closed under parabolic induction. For any $\pi \in \mathrm{Alg}'$, we denote the central character of π by χ_π.

Let $\pi \in \mathrm{Irr}_{\mathrm{gen}}$. Then, π can be written uniquely (up to permutation) as

$$\pi = \mathrm{St}(\rho_1, m_1)|\cdot|^{r_1} \times \cdots \times \mathrm{St}(\rho_l, m_l)|\cdot|^{r_l}$$

with cuspidal unitary representations ρ_i and $r_i \in \mathbb{R}$. We say that π is approximately tempered if $r_i - r_j < 1$ for any i and j, following [LM20]. We note that if $\pi \in \mathrm{Irr}_{\mathrm{gen}}$ is essentially unitary, then it is approximately tempered.

1.2 Speh representations (cf. [LM20, §2])

For the rest of this section, fix $\pi \in \mathrm{Irr}_{\mathrm{gen}}\mathrm{GL}_k$.

Let $\{\rho_1, \dots, \rho_r\}$ ($\rho_i \in \mathrm{Irr}_{\mathrm{sc}}$) be the cuspidal support of π, which is a multiset of $\mathrm{Irr}_{\mathrm{sc}}$. For each $n \in \mathbb{Z}_{>0}$, we define the Speh representation $\mathrm{Sp}(\pi, n)$ as the representation corresponding to the multisegment

$$\sum_{i=1}^{r} \{\rho_i|\cdot|^{-\frac{n-1}{2}}, \dots, \rho_i|\cdot|^{\frac{n-1}{2}}\}$$

under the Zelevinsky classification [Zel80]. By rearranging the indices, we assume that $\rho_1, \dots, \rho_{i-1} \neq \rho_i|\cdot|^{-m}$ for all i and all $m \in \mathbb{Z}_{>0}$. Then, $\mathrm{Sp}(\pi, n)$ is the unique irreducible subrepresentation of

$$\mathrm{Sp}(\rho_1, n) \times \cdots \times \mathrm{Sp}(\rho_r, n),$$

where $\mathrm{Sp}(\rho_i, n)$ is the unique irreducible quotient of $\rho_i|\cdot|^{\frac{n-1}{2}} \times \rho_i|\cdot|^{\frac{n-3}{2}} \times \cdots \times \rho_i|\cdot|^{-\frac{n-1}{2}}$.

Other realizations of $\mathrm{Sp}(\pi, n)$ are known:

Proposition 1.1 ([LM20, Corollary 2.11])**.** $\mathrm{Sp}(\pi, n)$ is a unique irreducible subrepresentation of

$$\mathrm{Sp}(\pi, n-1)|\cdot|^{-\frac{1}{2}} \times \pi|\cdot|^{\frac{n-1}{2}}.$$

In particular, $\mathrm{Sp}(\pi, n)$ is both a subrepresentation of

$$\Pi = \pi|\cdot|^{-\frac{n-1}{2}} \times \pi|\cdot|^{-\frac{n-3}{2}} \times \cdots \times \pi|\cdot|^{\frac{n-1}{2}}$$

and a quotient of

$$\tilde{\Pi} = \pi|\cdot|^{\frac{n-1}{2}} \times \pi|\cdot|^{\frac{n-3}{2}} \times \cdots \times \pi|\cdot|^{-\frac{n-1}{2}}.$$

Remark 1.2 ([LM20, Remark 2.12])**.** Assume π is approximately tempered. Then, $\mathrm{Sp}(\pi, n)$ is the Langlands quotient of $\tilde{\Pi}$. In particular, $\mathrm{Sp}(\pi, n)$ is the unique subrepresentation of $\mathrm{Sp}(\pi, n_1)|\cdot|^{-\frac{n_2}{2}} \times \mathrm{Sp}(\pi, n_2)|\cdot|^{\frac{n_1}{2}}$ for any $n_1, n_2 \in \mathbb{Z}_{>0}$ such that $n_1 + n_2 = n$.

From now on, we will denote $\mathrm{Sp}(\pi, n)$ by π_n for short.

1.3 The models (cf. [LM20, §3])

For the rest of this section, fix a nontrivial character ψ of F.

For each $n \in \mathbb{Z}_{>0}$, we define $w_{(k,n)} \in \mathrm{GL}_{kn}$ by

$$(w_{(k,n)})_{i,j} = \delta_{k(i-pn+n-1)+p,j} \quad \text{if } pn - n + 1 \leq i \leq pn$$

for $p \in \{1, \ldots, k\}$ and a function $\Psi_{(k,n)} : \mathrm{GL}_{kn} \to \mathbb{C}$ by

$$\Psi_{(k,n)}(g) = \psi\left(\sum_{k \nmid i} g_{i,i+1}\right).$$

Then, the restrictions of $\Psi_{(k,n)}$ to $U_{(1^{kn})}$ and $U_{(n^k)}^{w_{(k,n)}} = w_{(k,n)}^{-1} U_{(n^k)} w_{(k,n)}$ are both characters. We note that

$$\Psi_{(k,n)}\left(\begin{pmatrix} 1_n & X_1 & & & * \\ & 1_n & X_2 & & \\ & & \ddots & \ddots & \\ & & & 1_n & X_{k-1} \\ & & & & 1_n \end{pmatrix}^{w_{(k,n)}}\right) = \psi\left(\sum_{1 \leq i \leq k-1} \mathrm{tr} X_i\right),$$

where $X_i \in \mathrm{M}_k(F)$. We put

$$\mathcal{W}^{\psi}_{\mathrm{Ze},k,n} = \mathrm{Ind}^{\mathrm{GL}_{kn}}_{U_{(1^{kn})}} \Psi_{(k,n)}|_{U_{(1^{kn})}} \quad \text{and} \quad \mathcal{W}^{\psi}_{\mathrm{Sh},k,n} = \mathrm{Ind}^{\mathrm{GL}_{kn}}_{U^{w_{(k,n)}}_{(n^k)}} \Psi_{(k,n)}|_{U^{w_{(k,n)}}_{(n^k)}}.$$

Then, it is known that $\dim_{\mathbb{C}}(\pi_n, \mathcal{W}^{\psi}_{\mathrm{Ze},k,n}) = \dim_{\mathbb{C}}(\pi_n, \mathcal{W}^{\psi}_{\mathrm{Sh},k,n}) = 1$ ([LM20, Theorem 3.1]) . We denote the images of π_n on $\mathcal{W}^{\psi}_{\mathrm{Ze},k,n}$ and $\mathcal{W}^{\psi}_{\mathrm{Sh},k,n}$ by $\mathcal{W}^{\psi}_{\mathrm{Ze}}(\pi_n)$ (Zelevinsky model) and $\mathcal{W}^{\psi}_{\mathrm{Sh}}(\pi_n)$ (Shalika model), respectively. We note that

$$W_{\mathrm{Sh}}(\mathrm{diag}(\overbrace{g, \ldots, g}^{k})^{w_{(k,n)}} \cdot) = \chi_{\pi}(\det g) W_{\mathrm{Sh}}, \ g \in \mathrm{GL}_n$$

for any $W_{\mathrm{Sh}} \in \mathcal{W}^{\psi}_{\mathrm{Sh}}(\pi_n)$.

The relation between $\mathcal{W}^{\psi}_{\mathrm{Ze}}(\pi_n)$ and $\mathcal{W}^{\psi}_{\mathrm{Sh}}(\pi_n)$ is as follows:

Proposition 1.3 ([LM20, Lemma 3.8, 3.11])**.** Let $\mathfrak{n}_n$ (resp. $\bar{\mathfrak{n}}_n = {}^t\mathfrak{n}_n$) be the set of upper (resp. lower) triangular nilpotent matrices in $\mathrm{M}_n(F)$ and

$$N_n = w_{(k,n)}^{-1} \begin{pmatrix} 1_n & \bar{\mathfrak{n}}_n & \bar{\mathfrak{n}}_n & \dots & \bar{\mathfrak{n}}_n \\ & 1_n & \bar{\mathfrak{n}}_n & \dots & \bar{\mathfrak{n}}_n \\ & & \ddots & \ddots & \vdots \\ & & & 1_n & \bar{\mathfrak{n}}_n \\ & & & & 1_n \end{pmatrix} w_{(k,n)},\ N'_n = 1_{kn} + w_{(k,n)}^{-1} \begin{pmatrix} \mathfrak{n}_n & & & & \\ \mathfrak{n}_n & \mathfrak{n}_n & & & \\ \vdots & \vdots & \ddots & & \\ \mathfrak{n}_n & \mathfrak{n}_n & \dots & \mathfrak{n}_n & \\ 0_n & 0_n & \dots & \dots & 0_n \end{pmatrix} w_{(k,n)}.$$

Then, the isomorphism $\mathcal{T}_n : \mathcal{W}^{\psi}_{\mathrm{Ze}}(\pi_n) \xrightarrow{\sim} \mathcal{W}^{\psi}_{\mathrm{Sh}}(\pi_n)$ and its inverse $\mathcal{T}_n^{-1} : \mathcal{W}^{\psi}_{\mathrm{Sh}}(\pi_n) \xrightarrow{\sim} \mathcal{W}^{\psi}_{\mathrm{Ze}}(\pi_n)$ are given by

$$\mathcal{T}_n W_{\mathrm{Ze}} = \int_{N_n} W_{\mathrm{Ze}}(u\cdot)du,\ \mathcal{T}_n^{-1} W_{\mathrm{Sh}} = \int_{N'_n} W_{\mathrm{Sh}}(u'\cdot)du'$$

for $W_{\mathrm{Ze}} \in \mathcal{W}^{\psi}_{\mathrm{Ze}}(\pi_n)$ and $W_{\mathrm{Sh}} \in \mathcal{W}^{\psi}_{\mathrm{Sh}}(\pi_n)$, where these integrands are (pointwise) compactly supported.

Now we consider 'intermediate' models between $\mathcal{W}^{\psi}_{\mathrm{Ze}}(\pi_n)$ and $\mathcal{W}^{\psi}_{\mathrm{Sh}}(\pi_n)$. Take

$$\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_{nk}) = (\overbrace{\overbrace{k-1, k-2, \dots, 0}, \overbrace{k-1, k-2, \dots, 0}, \dots, \overbrace{k-1, k-2, \dots, 0}}^{n}) + (\lambda_k^k, \lambda_{2k}^k, \dots, \lambda_{nk}^k) \in \mathbb{Z}^{nk}$$

and define a parabolic subgroup $P = MU$ of GL_{kn} by $P = \left\{ g \in \mathrm{GL}_{kn} \;\middle|\; g_{i,j} = 0 \text{ if } \lambda_i < \lambda_j \right\}$. Then, the restriction of $\Psi_{(k,n)}$ to U is a character and $\dim_{\mathbb{C}}(\pi_n, \mathrm{Ind}_U^{\mathrm{GL}_{nk}} \Psi_{(k,n)}|_U) = 1$ in general ([LM20, Theorem 3.1]). Take $\boldsymbol{n} = (n_1, \dots, n_l) \in (\mathbb{Z}_{>0})^l$ such that $\sum_i n_i = n$ and assume that

$$\lambda_{ik} = -jk \ \text{ if } \ \sum_{p=1}^{j} n_p < i \leq \sum_{p=1}^{j+1} n_p$$

for $j \in \{0, \dots, l-1\}$. Then, we have

$$P = \mathrm{diag}(P^{w_{(k,n_1)}}_{(n_1^k)}, \dots, P^{w_{(k,n_l)}}_{(n_l^k)}) U_{\boldsymbol{n}}$$

and

$$\Psi|_U = (\Psi_{(k,n_1)}|_{U^{w_{(k,n_1)}}_{(n_1^k)}} \boxtimes \dots \boxtimes \Psi_{(k,n_l)}|_{U^{w_{(k,n_l)}}_{(n_l^k)}}) \otimes 1_{U_{\boldsymbol{n}}}.$$

For this $\boldsymbol{n}$, we put

$$\mathcal{W}^{\psi}_{\mathrm{k},\boldsymbol{n}} = \mathrm{Ind}_U^{\mathrm{GL}_{kn}} \Psi_{(k,n)}|_U \simeq \mathcal{W}^{\psi}_{\mathrm{Sh},k,n_1} * \dots * \mathcal{W}^{\psi}_{\mathrm{Sh},k,n_l}$$

and denote the image of π_n on $\mathcal{W}^{\psi}_{k,\boldsymbol{n}}$ by $\mathcal{W}^{\psi}_{\boldsymbol{n}}(\pi_n)$. Then, using Proposition 1.3, we obtain the relations between $\mathcal{W}^{\psi}_{\boldsymbol{n}}(\pi_n)$ and $\mathcal{W}^{\psi}_{\mathrm{Ze}}(\pi_n)$, $\mathcal{W}^{\psi}_{\mathrm{Sh}}(\pi_n)$ as follows:

Corollary 1.4. Let $\mathfrak{n}_{\boldsymbol{n}} = U_{\boldsymbol{n}} - 1_n, \bar{\mathfrak{n}}_{\boldsymbol{n}} = {}^t\mathfrak{n}_{\boldsymbol{n}}$ and

$$N_{\boldsymbol{n},1} = \mathrm{diag}(N_{n_1}, \dots, N_{n_l}),\ N'_{\boldsymbol{n},1} = \mathrm{diag}(N'_{n_1}, \dots, N'_{n_l}),$$

$$N_{\boldsymbol{n},2} = w_{(k,n)}^{-1} \begin{pmatrix} 1_n & \bar{\mathfrak{n}}_{\boldsymbol{n}} & \bar{\mathfrak{n}}_{\boldsymbol{n}} & \dots & \bar{\mathfrak{n}}_{\boldsymbol{n}} \\ & 1_n & \bar{\mathfrak{n}}_{\boldsymbol{n}} & \dots & \bar{\mathfrak{n}}_{\boldsymbol{n}} \\ & & \ddots & \ddots & \vdots \\ & & & 1_n & \bar{\mathfrak{n}}_{\boldsymbol{n}} \\ & & & & 1_n \end{pmatrix} w_{(k,n)},\ N'_{\boldsymbol{n},2} = 1_{kn} + w_{(k,n)}^{-1} \begin{pmatrix} \mathfrak{n}_{\boldsymbol{n}} & & & & \\ \mathfrak{n}_{\boldsymbol{n}} & \mathfrak{n}_{\boldsymbol{n}} & & & \\ \vdots & \vdots & \ddots & & \\ \mathfrak{n}_{\boldsymbol{n}} & \mathfrak{n}_{\boldsymbol{n}} & \dots & \mathfrak{n}_{\boldsymbol{n}} & \\ 0_n & 0_n & \dots & \dots & 0_n \end{pmatrix} w_{(k,n)}.$$

Then, the isomorphisms

$$\mathcal{T}_{\boldsymbol{n},1} : \mathcal{W}^{\psi}_{\mathrm{Ze}}(\pi_n) \xrightarrow{\sim} \mathcal{W}^{\psi}_{\boldsymbol{n}}(\pi_n),\ \mathcal{T}_{\boldsymbol{n},2} : \mathcal{W}^{\psi}_{\boldsymbol{n}}(\pi_n) \xrightarrow{\sim} \mathcal{W}^{\psi}_{\mathrm{Sh}}(\pi_n)$$

and their inverses

$$\mathcal{T}_{\boldsymbol{n},1}^{-1} : \mathcal{W}_{\boldsymbol{n}}^{\psi}(\pi_n) \xrightarrow{\sim} \mathcal{W}_{\mathrm{Ze}}^{\psi}(\pi_n),\ \mathcal{T}_{\boldsymbol{n},2}^{-1} : \mathcal{W}_{\mathrm{Sh}}^{\psi}(\pi_n) \xrightarrow{\sim} \mathcal{W}_{\boldsymbol{n}}^{\psi}(\pi_n)$$

are given by

$$\mathcal{T}_{\boldsymbol{n},1} W_{\mathrm{Ze}} = \int_{N_{\boldsymbol{n},1}} W_{\mathrm{Ze}}(u\cdot)du,\ \mathcal{T}_{\boldsymbol{n},2} W = \int_{N_{\boldsymbol{n},2}} W(u\cdot)du,$$
$$\mathcal{T}_{\boldsymbol{n},1}^{-1} W = \int_{N'_{\boldsymbol{n},1}} W(u'\cdot)du',\ \mathcal{T}_{\boldsymbol{n},2}^{-1} W_{\mathrm{Sh}} = \int_{N'_{\boldsymbol{n},2}} W_{\mathrm{Sh}}(u'\cdot)du'$$

for $W_{\mathrm{Ze}} \in \mathcal{W}_{\mathrm{Ze}}^{\psi}(\pi_n)$, $W \in \mathcal{W}_{\boldsymbol{n}}^{\psi}(\pi_n)$, and $W_{\mathrm{Sh}} \in \mathcal{W}_{\mathrm{Sh}}^{\psi}(\pi_n)$.

Remark 1.5. By proposition 1.1, we have

$$W_{\mathrm{Ze}}|_{M_{(k(n-1),k)}} \in \mathcal{W}_{\mathrm{Ze}}^{\psi}(\pi_{n-1})|\cdot|^{\frac{1}{2}(k-1)} \otimes \mathcal{W}_{\mathrm{Ze}}^{\psi}(\pi)|\cdot|^{-\frac{1}{2}(n-1)(k-1)}$$

and

$$W_{\mathrm{Ze}}|_{M_{(k^n)}} \in \mathcal{W}_{\mathrm{Ze}}^{\psi}(\pi)|\cdot|^{\frac{1}{2}(n-1)(k-1)} \otimes \mathcal{W}_{\mathrm{Ze}}^{\psi}(\pi)|\cdot|^{\frac{1}{2}(n-3)(k-1)} \otimes \cdots \otimes \mathcal{W}_{\mathrm{Ze}}^{\psi}(\pi)|\cdot|^{-\frac{1}{2}(n-1)(k-1)}$$

for any $W_{\mathrm{Ze}} \in \mathcal{W}_{\mathrm{Ze}}^{\psi}(\pi_n)$ (see [LM20, §3.1]). Similarly, by Remark 1.2, we have

$$W|_{M_{(kn_1,kn_2)}} \in {\mathcal{W}_{\mathrm{Sh}}}^{\psi}(\pi_{n_1})|\cdot|^{\frac{1}{2}n_2(k-1)} \otimes {\mathcal{W}_{\mathrm{Sh}}}^{\psi}(\pi_{n_2})|\cdot|^{-\frac{1}{2}n_1(k-1)}$$

for any $W \in \mathcal{W}_{(n_1,n_2)}^{\psi}(\pi_n)$ if π is approximately tempered, where $n_1 + n_2 = n$. However, we do not know whether this holds in general.

2 The local zeta integral

For the rest of this paper, fix $\pi \in \mathrm{Irr}_{\mathrm{gen}}\mathrm{GL}_2$, $n \in \mathbb{Z}_{>0}$ and a nontrivial character ψ of F. Moreover, for each $m \in \mathbb{Z}_{>0}$, we write $w_{(2,m)} = w_m$, $\mathcal{W}_{\mathrm{Ze},2,m}^{\psi} = \mathcal{W}_{\mathrm{Ze},m}^{\psi}$, $\mathcal{W}_{\mathrm{Sh},2,m}^{\psi} = \mathcal{W}_{\mathrm{Sh},m}^{\psi}$, and $\mathcal{W}_{2,\boldsymbol{m}}^{\psi} = \mathcal{W}_{\boldsymbol{m}}^{\psi}$ for short.

2.1 The main results

For any $W_{\mathrm{Sh}} \in \mathcal{W}_{\mathrm{Sh}}^{\psi}(\pi_n)$, define a function $\Phi_{W_{\mathrm{Sh}}}$ on GL_n by

$$\Phi_{W_{\mathrm{Sh}}}(g) = W_{\mathrm{Sh}}(\left(\begin{smallmatrix} g & 0 \\ 0 & 1_n \end{smallmatrix}\right)^{w_n}).$$

We note that

$$\Phi_{\pi_n(\mathrm{diag}(g_1,g_2)^{w_n})W_{\mathrm{Sh}}} = \Phi_{W_{\mathrm{Sh}}}(g_2^{-1}\cdot g_1)\chi_2(\det g_2),\ \Phi_{\pi_n\left(\left(\begin{smallmatrix} 1_n & X \\ & 1_n \end{smallmatrix}\right)^{w_n}\right)W_{\mathrm{Sh}}} = \psi(\mathrm{tr}(X\cdot))\Phi_{W_{\mathrm{Sh}}}$$

for $g_1, g_2 \in \mathrm{GL}_n$, and $X \in \mathrm{M}_n(F)$ (see §1.3). In particular, $\Phi_{W_{\mathrm{Sh}}}$ is bi-K-invariant for some open compact subgroup K of GL_n. Moreover, the set $\{\Phi_{W_{\mathrm{Sh}}} \mid W_{\mathrm{Sh}} \in \mathcal{W}_{\mathrm{Sh}}^{\psi}(\pi_n)\}$ does not depend on ψ since for any additive character $\psi' = \psi(a\cdot)$ of F, where $a \in F^{\times}$, the isomorphism $\mathcal{W}_{\mathrm{Sh}}^{\psi'}(\pi_n) \xrightarrow{\sim} \mathcal{W}_{\mathrm{Sh}}^{\psi}(\pi_n)$ is given by

$$W'_{\mathrm{Sh}} \mapsto W'_{\mathrm{Sh}}(\mathrm{diag}(1_n, a1_n)^{w_n}\cdot)$$

for $W'_{\mathrm{Sh}} \in \mathcal{W}_{\mathrm{Sh}}^{\psi'}(\pi_n)$.

For $W_{\mathrm{Sh}} \in \mathcal{W}_{\mathrm{Sh}}^{\psi}(\pi_n)$, define $\widetilde{W_{\mathrm{Sh}}} \in \mathcal{W}_{\mathrm{Sh}}^{\psi^{-1}}(\pi_n^{\vee})$ by

$$\widetilde{W_{\mathrm{Sh}}}(g^{w_n}) = \chi_{\pi}^{-1}(\det g)W_{\mathrm{Sh}}(\left(\left(\begin{smallmatrix} 1_n & 0 \\ 0 & -1_n \end{smallmatrix}\right) g \left(\begin{smallmatrix} 0 & 1_n \\ -1_n & 0 \end{smallmatrix}\right)\right)^{w_n})$$

(note that $\pi_n\chi_\pi^{-1} \simeq \mathrm{Sp}(\pi\chi_\pi^{-1}, n) \simeq \mathrm{Sp}(\pi^\vee, n) \simeq \mathrm{Sp}(\pi, n)^\vee = \pi_n^\vee$). We note that

$$\Phi_{\widetilde{W_{\mathrm{Sh}}}}(g) = W_{\mathrm{Sh}}\left(\left(\left(\begin{smallmatrix} g & 0 \\ 0 & 1_n \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & 1_n \\ 1_n & 0 \end{smallmatrix}\right)\right)^{w_n}\right).$$

Let $\tau \in \mathrm{Alg}'\mathrm{GL}_n$. For any matrix coefficient f of τ, denote a matrix coefficient $f(\cdot^{-1})$ of $\tau^\vee$ by $f^\vee$.

Finally, for $W_{\mathrm{Sh}} \in \mathcal{W}_{\mathrm{Sh}}^{\psi}(\pi_n)$ and a matrix coefficient f of any $\tau \in \mathrm{Alg}'\mathrm{GL}_n$, define the zeta integral $Z(W_{\mathrm{Sh}}, s, f)$ with complex variable s by

$$Z(W_{\mathrm{Sh}}, s, f) = \int_{\mathrm{GL}_n} \Phi_{W_{\mathrm{Sh}}}(g) f(g) |\det g|^{s-\frac{1}{2}} dg.$$

The main result of this paper is as follows:

Theorem 2.1. Let $\tau \in \mathrm{Alg}'\mathrm{GL}_n$. Assume π is approximately tempered except (i).

(i) If $\mathrm{Re}(s)$ is sufficiently large, then the integral defining $Z(W_{\mathrm{Sh}}, s, f)$ converges absolutely for any $W_{\mathrm{Sh}} \in \mathcal{W}_{\mathrm{Sh}}^{\psi}(\pi_n)$ and matrix coefficient f of τ.

(ii) $Z(W_{\mathrm{Sh}}, s, f)$ admits a meromorphic continuation to all of $\mathbb{C}$ and there is a (unique) polynomial $P(X) \in \mathbb{C}[X]$ such that $P(0) = 1$ and

$$I(\pi, \tau) := \langle Z(W_{\mathrm{Sh}}, s, f) \mid W_{\mathrm{Sh}} \in \mathcal{W}_{\mathrm{Sh}}^{\psi}(\pi_n),\ f : \text{a matrix coefficient of } \tau \rangle_{\mathbb{C}} = P(q^{-s})^{-1}\mathbb{C}[q^{-s}, q^s];$$

denote $P(q^{-s})^{-1}$ by $L(\pi; s, \tau)$.

(iii) If $\tau = \tau_1 \times \tau_2$ for $\tau_1, \tau_2 \in \mathrm{Alg}'$, then

$$L(\pi; s, \tau) = L(\pi; s, \tau_1) L(\pi; s, \tau_2).$$

(iv) There is a function $\gamma(s) \in \mathbb{C}(q^{-s})$ such that

$$Z(\widetilde{W_{\mathrm{Sh}}}, 1-s, f^\vee) = \gamma(s) Z(W_{\mathrm{Sh}}, s, f)$$

for any W_{Sh} and f; denote $\gamma(s)$ by $\gamma(\pi; s, \tau, \psi)$.

(v) If τ is a subrepresentation of $\tau_1 \times \tau_2$ for $\tau_1, \tau_2 \in \mathrm{Alg}'$, then

$$\gamma(\pi; s, \tau, \psi) = \gamma(\pi; s, \tau_1, \psi)\gamma(\pi; s, \tau_2, \psi).$$

(vi) If $\tau \in \mathrm{Irr}_{\mathrm{gen}}\mathrm{GL}_n$, then

$$L(\pi; s, \tau) = L(s, \pi \boxtimes \tau),$$

where the right-hand side is the local L-factor defined by Jacquet, Piatetski-Shapiro and Shalika [JPSS83].

Remark 2.2. As mentioned in Remark 0.14, some of our results are known. Specifically, they are as follows:

- (i) follows from the discussion at the beginning of Appendix C of [CFK].
- (ii) is a specialized version of [CFK, Theorem C.6].
- (iv) is a conclusion of [CFK, Theorem C.1].
- (v) is a specialized version of [CFK, Theorem C.2].

On the other hand:

- (iii) is a partial refinement of [CFK, Theorem C.6 (2)], which says that
$$L(\pi; s, \tau) \in L(\pi; s, \tau_1)L(\pi; s, \tau_2)\mathbb{C}[q^{-s}, q^s].$$
- (vi) is a partial refinement of [CFK, Proposition C.10], which says that
$$L(s, \pi \boxtimes \tau) \in L(\pi; s, \tau)\mathbb{C}[q^{-s}, q^s].$$

The proofs of (iii) and (vi) (essentially, Proposition 2.7, Proposition 2.10, and Proposition 2.11) depend on the facts in §1.3. Only if $k = 2$, then $N'_{\boldsymbol{n},2}$ (in §1.3) coincides with the unipotent radical of some upper triangular parabolic subgroup of $\mathrm{GL}_n \subset \mathrm{GL}_{2n}$, so that, by partial integration, we can obtain other representations of our zeta integral using intermediate models. Our method does not seem to be (immediately) applicable to the general case in [CFK] if $k > 2$.

Remark 2.3. (i) By a simple computation, one can see that the ψ-dependence of $\gamma(\pi; s, \tau, \psi)$ is given by
$$\gamma(\pi; s, \tau, \psi(a\cdot)) = \chi_\pi^n(a)\chi_\tau^2(a)|a|^{n(2s-1)}\gamma(\pi; s, \tau, \psi)$$
for any $a \in F^\times$.

(ii) We put
$$\epsilon(\pi; s, \tau, \psi) := \gamma(\pi; s, \tau, \psi)L(\pi; s, \tau)/L(\pi^\vee; 1 - s, \tau^\vee).$$
Then, we have $\epsilon(\pi; s, \tau, \psi), \epsilon(\pi^\vee; 1 - s, \tau^\vee, \psi) \in \mathbb{C}[q^{-s}, q^s]$ by Theorem 2.1(ii), (iv) and
$$\epsilon(\pi; s, \tau, \psi)\epsilon(\pi^\vee; 1 - s, \tau^\vee, \psi) = 1$$
by the above (i) and using Theorem 2.1(iv) twice. In particular, we can write $\epsilon(\pi; s, \tau, \psi) = cq^{ls}$ for some $c \in \mathbb{C}$ and $l \in \mathbb{Z}$.

We prove Theorem 2.1 in several parts: The proof of (ii) is given in §2.3 by reduction to the generic case. (ii) for generic representations is proved in §2.2 (Corollary 2.8). The proofs of (iii) and (v) are given in §2.3 as byproducts of the proof of (ii). The proof of (iv) is a bit technical, so we postpone it to the next section. (vi) is proved using the entire §2.4.

Finally, (i) is proved here. To prove this, we need the following two lemmas.

Lemma 2.4. (i) For any $W_{\mathrm{Sh}} \in \mathcal{W}^\psi_{\mathrm{Sh}}(\pi_n)$, there is an open compact subset C of $\mathrm{M}_n(F)$ such that $\mathrm{supp}\Phi_{W_{\mathrm{Sh}}} \subset C$.

(ii) If $r \in \mathbb{R}$ is sufficiently large, then $\Phi_{W_{\mathrm{Sh}}}|\cdot|^r$ is L^2-function for any $W_{\mathrm{Sh}} \in \mathcal{W}^\psi_{\mathrm{Sh}}(\pi_n)$.

Proof. (i) and (ii) immediately follow from [LM20, Lemma 3.2] and [LM20, Proposition 4.2], respectively. □

Lemma 2.5. Let G be an algebraic group over F and K an open compact subgroup of G. Let $\mathcal{F}$ be a right K-invariant function on G. If $\mathcal{F}$ is an L^1-function, then $\mathcal{F}$ is an L^2-function.

Proof. We have the lemma as follows:
$$\int_G |\mathcal{F}(g)|^2 dg = \int_{G\times K} |\mathcal{F}(g)\mathcal{F}(gk)|dgdk \leq \int_{G\times G} |\mathcal{F}(g)\mathcal{F}(gg')|dgdg' = \left(\int_G |\mathcal{F}(g)|dg\right)^2 < \infty.$$
□

proof of Theorem 2.1 (i). By Lemma 2.4 (i), we have
$$\Phi_{W_{\mathrm{Sh}}}(g)f(g)|\det g|^{s-\frac{1}{2}} = \Phi_{W_{\mathrm{Sh}}}(g)|\det g|^{s_1} \times 1_{C\cap\mathrm{GL}_n}(g)f(g)|\det g|^{s_2}$$
for any W_{Sh}, f, and $g \in \mathrm{GL}_n$, where C is some open compact subset of $\mathrm{M}_n(F)$ and $s - 1/2 = s_1 + s_2$. By Lemma 2.4 (ii), $\Phi_{W_{\mathrm{Sh}}}|\cdot|^{s_1}$ is an L^2-function for any W_{Sh} if $\mathrm{Re}(s_1)$ is sufficiently large. On the other hand, $1_{C\cap\mathrm{GL}_n}f|\cdot|^{s_2}$ is an L^2-function for any f by Lemma 2.5 and the convergence of the zeta integral of Godement and Jacquet [GJ72, Theorem 3.3] if $\mathrm{Re}(s_2)$ is sufficiently large. Thus the integral defining $Z(W_{\mathrm{Sh}}, s, f)$ converges absolutely for any W_{Sh} and f if $\mathrm{Re}(s)$ is sufficiently large. □

2.2 The generic case

In this subsection, we assume $\tau \in \mathrm{Irr}_{\mathrm{gen}}\mathrm{GL}_n$ (we do not need to assume that π is approximately tempered here). We let $\mathcal{W}^\psi(\tau') = \mathcal{W}^\psi_{\mathrm{Ze}}(\tau')$ for any $\tau' \in \mathrm{Irr}_{\mathrm{gen}}$.

For any open compact subgroup K of GL_n and $h \in \mathrm{GL}_n$, define $L_{h,K} \in \mathcal{W}^\psi(\tau)^\vee$ by

$$\langle W, L_{h,K}\rangle = \int_K W(hk)dk$$

for $W \in \mathcal{W}^\psi(\tau)$. We can obtain $\mathcal{W}^\psi(\tau)^\vee = \langle L_{h,K}\rangle_{\mathbb{C}}$ easily.

For any $W_{\mathrm{Sh}} \in \mathcal{W}^\psi_{\mathrm{Sh}}(\pi_n)$ and $W \in \mathcal{W}^\psi(\tau)$, we consider the following zeta integral

$$Z(W_{\mathrm{Sh}}, s, W) = \int_{\mathrm{GL}_n} \Phi_{W_{\mathrm{Sh}}}(g)W(g)|\det g|^{s-\frac{1}{2}}dg.$$

Proposition 2.6. (i) If $\mathrm{Re}(s)$ is sufficiently large, then the integral defining $Z(W_{\mathrm{Sh}}, s, W)$ converges absolutely for any $W_{\mathrm{Sh}} \in \mathcal{W}^\psi_{\mathrm{Sh}}(\pi_n)$ and $W \in \mathcal{W}^\psi(\tau)$.

(ii) For any matrix coefficient f of τ and $W_{\mathrm{Sh}} \in \mathcal{W}^\psi_{\mathrm{Sh}}(\pi_n)$, there are $W^i_{\mathrm{Sh}} \in \mathcal{W}^\psi_{\mathrm{Sh}}(\pi_n)$ and $W^i \in \mathcal{W}^\psi(\tau)$ $(i = 1, \dots, l)$ such that

$$Z(W_{\mathrm{Sh}}, s, f) = \sum_{1\le i\le l} Z(W^i_{\mathrm{Sh}}, s, W^i)$$

if $\mathrm{Re}(s)$ is sufficiently large.

(iii) For any $W \in \mathcal{W}^\psi(\tau)$ and $W_{\mathrm{Sh}} \in \mathcal{W}^\psi_{\mathrm{Sh}}(\pi_n)$, there is a matrix coefficient f of τ such that

$$Z(W_{\mathrm{Sh}}, s, W) = Z(W_{\mathrm{Sh}}, s, f)$$

if $\mathrm{Re}(s)$ is sufficiently large.

Proof. (i) For any sufficiently large r, $1_{C\cap \mathrm{GL}_n} W|\cdot|^r$ is an L^1-function of GL_n for any open compact subset C of $\mathrm{M}_n(F)$ and $W \in \mathcal{W}^\psi(\tau)$ (see [JPSS79, (2.3.6), (3.1)]). Thus it immediately holds by Lemma 2.4, 2.5.

(ii) We can assume that $f(g) = \langle W(\cdot g), L_{h,K}\rangle$ for some $W \in \mathcal{W}^\psi(\tau)$ and open compact subgroup K of GL_n such that $\Phi_{W_{\mathrm{Sh}}}$ is bi-K-invariant. Then, by (i), we have

$$Z(W_{\mathrm{Sh}}, s, f) = \int_{\mathrm{GL}_n} \Phi_{W_{\mathrm{Sh}}}(h^{-1}gh)W(gh)|\det g|^{s-\frac{1}{2}}dg = Z(\chi(\det h)^{-1}W_{\mathrm{Sh}}(\cdot \mathrm{diag}(h,h)^{w_n}), s, W(\cdot h))$$

if $\mathrm{Re}(s)$ is sufficiently large.

(iii) Take sufficiently small K and put $f(g) = \langle W(\cdot g), L_{1,K}\rangle$. ☐

Proposition 2.7. Let $W_{\mathrm{Sh}} \in \mathcal{W}^\psi_{\mathrm{Sh}}(\pi_n)$ and $W \in \mathcal{W}^\psi(\tau)$. Then, $L(s, \pi \boxtimes \tau)^{-1}Z(W_{\mathrm{Sh}}, s, W)$ defines an entire function of s and $Z(W_{\mathrm{Sh}}, s, W) \in L(s, \pi \boxtimes \tau)\mathbb{C}[q^{-s}, q^s]$.

Proof. First we remark that for any $W' \in \mathcal{W}^\psi(\tau')$ ($\tau' \in \mathrm{Irr}_{\mathrm{gen}}\mathrm{GL}_m$), there is a constant $C' > 0$ such that

$$W'(u\mathrm{diag}(a_1, \dots, a_{m'})k) \neq 0 \Rightarrow |a_1| \le C'|a_2| \le \cdots \le C'^{m-1}|a_m|$$

for any $u \in U_{(1^m)}, a_i \in F^\times$ and $k \in \mathrm{GL}_m(\mathcal{O})$ in general ([LM20, Lemma 3.2]).

If $n = 1$, then $Z(W_{\mathrm{Sh}}, s, W)$ coincides with a Rankin-Selberg zeta integral for $\pi \boxtimes \tau$. In this case, $Z(W_{\mathrm{Sh}}, s, W) \in L(s, \pi \boxtimes \tau)\mathbb{C}[q^{-s}, q^s]$ is trivial.

Assume that $n \ge 2$. Then, for any $u \in U_{(1^n)}, h \in \mathrm{GL}_n$, and $h' \in \mathrm{GL}_{2n}$, we have

$$W_{\mathrm{Sh}}\left(\begin{pmatrix} u & 0 \\ 0 & 1_n\end{pmatrix}^{w_n} h'\right) W(uh) = W_{\mathrm{Sh}}\left(\begin{pmatrix} u & 0 \\ 0 & 1_n\end{pmatrix}^{w_n} x_n h'\right) W(h),$$

where we put

$$x_m := \begin{pmatrix} 1_m & \begin{smallmatrix} 0 & 0 \\ 1_{m-1} & 0 \end{smallmatrix} \\ & 1_m \end{pmatrix}^{w_m} = \mathrm{diag}(1, \overbrace{\left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right), \dots, \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)}^{m-1}, 1).$$

Thus we have

$$Z(W_{\mathrm{Sh}}, s, W) = \int_{U_{(1^n)}\backslash \mathrm{GL}_n} W_{\mathrm{Ze}} \left(x_n \begin{pmatrix} g_n & \\ & 1_n \end{pmatrix}^{w_n} \right) W(g_n) |\det g_n|^{s-\frac{1}{2}} d\overline{g}_n$$

if $\mathrm{Re}(s)$ is sufficiently large, where $W_{\mathrm{Ze}} := \mathcal{T}_n^{-1} W_{\mathrm{Sh}} \in \mathcal{W}^{\psi}_{\mathrm{Ze}}(\pi_n)$. We can write

$$\begin{aligned} Z(W_{\mathrm{Sh}}, s, W) = &\int_{U_{(1^2)}\backslash \mathrm{GL}_2 \times (F^\times)^{n-2} \times \mathrm{GL}_n(\mathcal{O})} W_{\mathrm{Ze}} \left(x_n \begin{pmatrix} \mathrm{diag}(g_2, a_3, \dots, a_n)k & \\ & 1_n \end{pmatrix}^{w_n} \right) \\ & W(\mathrm{diag}(g_2, a_2, \dots, a_n)k) |\det g_2|^{s-n+2-\frac{1}{2}} |a_3|^{s-n+5-\frac{1}{2}} \dots |a_n|^{s+n-1-\frac{1}{2}} d\overline{g}_2 d^\times a_3 \dots d^\times a_n dk \\ = &\int_{(F^\times)^n \times \mathrm{GL}_2(\mathcal{O}) \times \mathrm{GL}_n(\mathcal{O})} W_{\mathrm{Ze}} \left(x_n \begin{pmatrix} \mathrm{diag}(\left(\begin{smallmatrix} a_1 & \\ & a_2 \end{smallmatrix}\right))k_2, a_3, \dots, a_n)k & \\ & 1_n \end{pmatrix}^{w_n} \right) \\ & W(\mathrm{diag}(\left(\begin{smallmatrix} a_1 & \\ & a_2 \end{smallmatrix}\right))k_2, a_2, \dots, a_n)k) |a_1|^{s-n+1-\frac{1}{2}} \dots |a_n|^{s+n-1-\frac{1}{2}} d\overline{g}_2 d^\times a_3 \dots d^\times a_n dk_2 dk. \end{aligned}$$

Let $b_i \in F^\times$ $(i = 1, \dots, n)$ such that $|b_1| \le C|b_2| \le \dots \le C^{n-1}|b_n|$, where C is the constant in the above remark for W. Consider when the inequality

$$\begin{aligned} \mathcal{F}(b_1, \dots, b_n, k') := &W_{\mathrm{Ze}} \left(x_n \begin{pmatrix} \mathrm{diag}(b_1, b_2, \dots, b_n)k' & \\ & 1_n \end{pmatrix}^{w_n} \right) \\ &= W_{\mathrm{Ze}} \left(\left(\begin{pmatrix} 1 &&&&&&&& \\ & 1 &&&&&&& \\ & 1 & 1 &&&&&& \\ &&& 1 &&&&& \\ &&& 1 & 1 &&&& \\ &&&&& \ddots &&& \\ &&&&&& 1 && \\ &&&&&& 1 & 1 & \\ &&&&&&&& 1 \end{pmatrix} \begin{pmatrix} b_1 &&&&&&& \\ & 1 &&&&&& \\ && b_2 &&&&& \\ &&& 1 &&&& \\ &&&& \ddots &&& \\ &&&&& b_n && \\ &&&&&& 1 & \end{pmatrix} \begin{pmatrix} k' & \\ & 1_m \end{pmatrix}^{w_n} \right) \right) \\ &\neq 0 \end{aligned}$$

holds for some $k' \in \mathrm{GL}_n(\mathcal{O})$. If $|b_n|$ is sufficiently large, then

$$\begin{aligned} &\mathcal{F}(b_1, \dots, b_n, k') \\ =&W_{\mathrm{Ze}} \left(\begin{pmatrix} 1 &&&&&& \\ & 1 &&&&& \\ & 1 & 1 &&&& \\ &&& \ddots &&& \\ &&&& 1 && \\ &&&& 1 & 1 & \\ &&&&&& 1_3 \end{pmatrix} \begin{pmatrix} b_1 &&&&&& \\ & 1 &&&&& \\ && b_2 &&&& \\ &&& 1 &&& \\ &&&& \ddots && \\ &&&&& b_n & \\ &&&&&& 1 \end{pmatrix} \begin{pmatrix} 1_{2n-2} && \\ & 1 & b_n^{-1} \\ && 1 \end{pmatrix} \begin{pmatrix} k' & \\ & 1_n \end{pmatrix}^{w_m} \right) \\ =&W_{\mathrm{Ze}} \left(\begin{pmatrix} 1 &&&&&& \\ & 1 &&&&& \\ & 1 & 1 &&&& \\ &&& \ddots &&& \\ &&&& 1 && \\ &&&& 1 & 1 & \\ &&&&&& 1_3 \end{pmatrix} \begin{pmatrix} b_1 &&&&&& \\ & 1 &&&&& \\ && b_2 &&&& \\ &&& 1 &&& \\ &&&& \ddots && \\ &&&&& b_n & \\ &&&&&& 1 \end{pmatrix} \begin{pmatrix} k' & \\ & 1_n \end{pmatrix}^{w_n} \right) = 0 \end{aligned}$$

by Remark 1.5 and the above remark. If $|b_n|$ is sufficiently small, then, since

$$\begin{pmatrix} 1 &&&&&&& \\ & 1 &&&&&& \\ & 1 & 1 &&&&& \\ &&& \ddots &&&& \\ &&&& 1 &&& \\ &&&& 1 & 1 && \\ &&&&&& 1 \end{pmatrix} = \begin{pmatrix} 1 &&&&&&& \\ & -1 & 1 &&&&& \\ && 1 &&&&& \\ &&& \ddots &&&& \\ &&&& -1 & 1 && \\ &&&&& 1 && \\ &&&&&& 1 \end{pmatrix} w' \begin{pmatrix} 1 &&&&&&& \\ & 1 & 1 &&&&& \\ && 1 &&&&& \\ &&& \ddots &&&& \\ &&&& 1 & 1 && \\ &&&&& 1 && \\ &&&&&& 1 \end{pmatrix}$$

($w' := \mathrm{diag}(1 \overbrace{\left(\begin{smallmatrix} & 1 \\ 1 & \end{smallmatrix}\right), \ldots, \left(\begin{smallmatrix} & 1 \\ 1 & \end{smallmatrix}\right)}^{n-1}, 1))$, we have

$$\begin{aligned}
&\mathcal{F}(b_1, \ldots, b_n, k') \\
=& W_{\mathrm{Ze}}\left(\begin{pmatrix} b_1 &&&&&&& \\ & -b_2 &&&&&& \\ && 1 &&&&& \\ &&& -b_3 &&&& \\ &&&& \ddots &&& \\ &&&&& 1 && \\ &&&&&& -b_n & \\ &&&&&&& 1_2 \end{pmatrix} w' \begin{pmatrix} 1 &&&&&&& \\ & 1 & b_2 &&&&& \\ && 1 &&&&& \\ &&& \ddots &&&& \\ &&&& 1 & b_n && \\ &&&&& 1 && \\ &&&&&& 1 \end{pmatrix} \begin{pmatrix} k' & \\ & 1_n \end{pmatrix}^{w_n} \right) \\
=& W_{\mathrm{Ze}}\left(\begin{pmatrix} b_1 &&&&&&& \\ & -b_2 &&&&&& \\ && 1 &&&&& \\ &&& -b_3 &&&& \\ &&&& \ddots &&& \\ &&&&& 1 && \\ &&&&&& -b_n & \\ &&&&&&& 1_2 \end{pmatrix} w' \begin{pmatrix} k' & \\ & 1_n \end{pmatrix}^{w_n} \right) = 0
\end{aligned}$$

also by Remark 1.5 and the above remark. Then, by repeating a similar argument, we have that there is a constant $c > 0$ such that if $\mathcal{F}(b_1, \ldots, b_n, k')W(\mathrm{diag}(b_1, \ldots, b_n)k') \neq 0$ for some $k' \in \mathrm{GL}_n(\mathcal{O})$, then $|b_2|, |b_3|^{\pm 1}, |b_4|^{\pm 1}, \ldots, |b_n|^{\pm 1} < c$.

Then, integrating in a_i ($i > 2$) and k, and dividing the integral interval with respect to a_2, we have

$$\begin{aligned}
Z(W_{\mathrm{Sh}}, s, W) = \sum_i F_i \int_{U_{(1^2)}\backslash \mathrm{GL}_2} & W'_{\pi i}(g_2) W'_i(\mathrm{diag}(g_2, 1_{n-2})) |\det g_2|^{s - \frac{n-2}{2}} 1_{x(\mathcal{O},\mathcal{O})}((\begin{smallmatrix} 0 & 1 \end{smallmatrix}) g_2) d\overline{g}_2 \\
&+ \sum_j F'_j \int_{F^\times} W''_{\pi j}(\mathrm{diag}(a,1)) W''_j(\mathrm{diag}(a, 1_{n-1})) |a|^{s - \frac{n}{2}} d^\times a
\end{aligned}$$

for some $F_i, F'_j \in \mathbb{C}[q^{-s}, q^s], W'_{\pi i}, W''_{\pi j} \in \mathcal{W}^{\psi^{-1}}_{\mathrm{Ze}}(\pi), W'_i, W''_j \in \mathcal{W}^\psi(\tau)$ $(i = 1, \ldots, l_1, j = 1, \ldots, l_2)$ and $x \in F^\times$ such that $|x| \ll c$ by Remark 1.5. Since the set $\{\Phi_{W_\pi} \mid W_\pi \in \mathcal{W}^\psi(\pi)\}$ has already been calculated explicitly (see [Bum97, Theorem 4.7.2, 4.7.3]), it is easy to check that the latter sum is an element of $L(s, \pi \boxtimes \tau)\mathbb{C}[q^{-s}, q^s]$. If $n = 2$, then the integrals in the former sum are Rankin-Selberg zeta integrals for $\pi \boxtimes \tau$. If $n > 2$, then for each i, we have

$$\int_{U_{(1^2)}\backslash \mathrm{GL}_2} W'_{\pi i}(g_2)W'_i(\mathrm{diag}(g_2,1_{n-2}))|\det g_2|^{s-\frac{n-2}{2}}1_{x(\mathcal{O},\mathcal{O})}((\,0\ 1\,)g_2)d\overline{g}_2$$
$$=\int_{U_{(1^2)}\backslash \mathrm{GL}_2} W'_{\pi i}(g_2)W'_i(\mathrm{diag}(g_2,1_{n-2}))|\det g_2|^{s-\frac{n-2}{2}}d\overline{g}_2$$
$$-\int_{F^\times}\int_{|x|<|b|}\int_{\mathrm{GL}_2(\mathcal{O})} W'_{\pi i}(\mathrm{diag}(a,b)k_2)W'_i(\mathrm{diag}(\mathrm{diag}(a,b)k_2,1_{n-2}))|a|^{s-\frac{n}{2}}|b|^{s-\frac{n}{2}-2}dkd^\times bd^\times a.$$

The former integral is a Rankin-Selberg zeta integral for $\pi\boxtimes\tau$. On the other hand, if $|b|$ is sufficiently large, then $W'(\mathrm{diag}(\mathrm{diag}(a,b)k_2,1_{n-2})=0$ for any a and k_2. Therefore, integrating in b and k_2, we can see that the latter integral is an element of $L(s,\pi\boxtimes\tau)\mathbb{C}[q^{-s},q^s]$.

Consequently, we have $Z(W_{\mathrm{Sh}},s,W)\in L(s,\pi\boxtimes\tau)\mathbb{C}[q^{-s},q^s]$. □

We can see that $I(\pi,\tau)(=\langle Z(W_{\mathrm{Sh}},s,f)\mid W_{\mathrm{Sh}}\in\mathcal{W}^\psi_{\mathrm{Sh}}(\pi_n),\ f:\text{a matrix coefficient of }\tau\rangle_{\mathbb{C}})$ is a nonzero fractional ideal of $\mathbb{C}[q^{-s},q^s]$, in the same way as for the space generated by the zeta integrals of Godement and Jacquet for any admissible representation of any general linear group (see the discussion below Theorem 3.3 of [GJ72]). Then, by Proposition 2.6 and 2.7, we have the following:

Corollary 2.8. We have

$$\begin{aligned}I(\pi,\tau)&=\langle Z(W_{\mathrm{Sh}},s,W)\mid W_{\mathrm{Sh}}\in\mathcal{W}_{\mathrm{Sh}}(\pi_n),\ W\in\mathcal{W}^\psi(\tau)\rangle_{\mathbb{C}}\\&\subset L(s,\pi\boxtimes\tau)\mathbb{C}[q^{-s},q^s]\end{aligned}$$

In particular, Theorem 2.1 (ii) holds for irreducible generic τ without assuming that π is approximately tempered.

For the proof of Theorem 2.1 (vi), we end this subsection with the following lemma.

Lemma 2.9. If $n=1,2$ and τ is irreducible and supercuspidal, then we have $L(\pi;s,\tau)=L(s,\pi\boxtimes\tau)$.

Proof. If $n=1$, then it is trivial.

Assume $n=2$. Then, for any $W_{\mathrm{Sh}}\in\mathcal{W}^\psi_{\mathrm{Sh}}(\pi_2)$ and $W\in\mathcal{W}^\psi(\tau)$, similar to Proposition 2.6, we have

$$\begin{aligned}Z(W_{\mathrm{Sh}},s,W)&=\int_{U_{(1^2)}\backslash\mathrm{GL}_2} W_{\mathrm{Ze}}\left(\begin{pmatrix}\mathrm{diag}(1,-1)g_2&\\&1_2\end{pmatrix}w_2\right)W(g_2)|\det g_2|^{s-\frac{1}{2}}1_{x(\mathcal{O},\mathcal{O})}((\,0\ 1\,)g_2)d\overline{g}_2\\&+\int_{F^\times}\int_{|x|<|b|}\int_{\mathrm{GL}_2(\mathcal{O})}W_{\mathrm{Ze}}\left(x_2\begin{pmatrix}\mathrm{diag}(a,b)k&\\&1_2\end{pmatrix}^{w_2}\right)W(\mathrm{diag}(a,b)k_2)|a|^{s-1}|b|^{s-3}dkd^\times bd^\times a.\end{aligned}$$

for some $x\in F^\times$. Then, since τ is supercuspidal, the latter integral is an element of $\mathbb{C}[q^{-s},q^s]$. Thus, clearly the functions $Z(W_{\mathrm{Sh}},s,W)$ generate $L(s,\pi\boxtimes\tau)\mathbb{C}[q^{-s},q^s]$. □

2.3 More results in the case that π is approximately tempered

For the rest of this section, we assume that π is approximately tempered. Then, for any $W\in\mathcal{W}^\psi_{(n_1,n_2)}(\pi_n)$,

$$W|_{\mathrm{diag}(\mathrm{GL}_{2n_1},\mathrm{GL}_{2n_2})}\in{\mathcal{W}_{\mathrm{Sh}}}^\psi(\pi_{n_1})|\cdot|^{\frac{1}{2}n_2}\otimes{\mathcal{W}_{\mathrm{Sh}}}^\psi(\pi_{n_2})|\cdot|^{-\frac{1}{2}n_1}\qquad(1)$$

holds, where $n_1+n_2=n$ (see Remark 1.5). We fix $\tau_i\in\mathrm{Alg}'\mathrm{GL}_{n_i}$ $(i=1,2)$ and put $\tau_0=\tau_1\times\tau_2$.

Similar to $L_{h,K}$ in the previous subsection, for any open compact subgroup K of GL_n, $h\in\mathrm{GL}_n$, and $v_i^\vee\in\tau_i$ $(i=1,2)$, we define $L_{h,K,v_1,v_2}\in\tau_0^\vee\simeq\tau_1^\vee\times\tau_2^\vee$ by

$$\langle v,L_{h,K,v_1^\vee,v_2^\vee}\rangle=\int_K\langle v(hk),v_1^\vee\otimes v_2^\vee\rangle dk$$

for any $v\in\tau_1\times\tau_2$. Then, it is easy to verify that $\tau_0^\vee=\langle L_{h,K,v_1^\vee,v_2^\vee}\rangle_{\mathbb{C}}$.

Proposition 2.10. Let $W_{\mathrm{Sh}} \in \mathcal{W}^{\psi}_{\mathrm{Sh}}(\pi_n)$ and f a matrix coefficient of τ_0.

(i) There are $W^j_{\mathrm{Sh},i} \in \mathcal{W}^{\psi}_{\mathrm{Sh}}(\pi_{n_j})$ and matrix coefficients f^j_i of τ_i $(i = 1, \ldots, l,\ j = 1, 2)$ such that

$$Z(W_{\mathrm{Sh}}, s, f) = \sum_{1 \leq i \leq l} Z(W^1_{\mathrm{Sh},i}, s, f^1_i) Z(W^2_{\mathrm{Sh},i}, s, f^2_i)$$

if $\mathrm{Re}(s)$ is sufficiently large.

(ii) For the above $W^j_{\mathrm{Sh},i}$ and f^j_i, we have

$$Z(\widetilde{W_{\mathrm{Sh}}}, 1-s, f^\vee) = \sum_{1 \leq i \leq l} Z(\widetilde{W^1_{\mathrm{Sh},i}}, 1-s, (f^1_i)^\vee) Z(\widetilde{W^2_{\mathrm{Sh},i}}, 1-s, (f^2_i)^\vee)$$

if $\mathrm{Re}(s)$ is sufficiently small.

Proof. (i) We can assume that $f(g) = \langle v(\cdot g), L_{h,K,v_1^\vee,v_2^\vee} \rangle$, where K is an open compact subgroup of GL_n such that $\Phi_{W_{\mathrm{Sh}}}$ is bi-K-invariant and $v \in \tau_0$. Since $1_{C \cap \mathrm{GL}_n} \langle v(\cdot), v_1^\vee \otimes v_2^\vee \rangle |\cdot|^r$ is an L^1-function of GL_n for any compact subset C of $\mathrm{M}_n(F)$ and sufficiently large r by Iwasawa decomposition and [GJ72, Theorem 3.3], if $\mathrm{Re}(s)$ is sufficiently large, then we have

$$\begin{aligned} Z(W_{\mathrm{Sh}}, s, f) &= \int_{\mathrm{GL}_n \times K} \Phi_{W_{\mathrm{Sh}}}(g) \langle v(hkg), v_1^\vee \otimes v_2^\vee \rangle |\det g|^{s-\frac{1}{2}} dg dk \\ &= \int_{\mathrm{GL}_n} \Phi_{W_{\mathrm{Sh}}}(h^{-1}gh) \langle v(gh), v_1^\vee \otimes v_2^\vee \rangle |\det g|^{s-\frac{1}{2}} dg \end{aligned}$$

by Lemma 2.4, 2.5. By Iwasawa decomposition $\mathrm{GL}_n = U_{(n_1,n_2)} M_{(n_1,n_2)} \mathrm{GL}_n(\mathcal{O})$, the above integral can be written as

$$\begin{aligned} \int_{\mathrm{GL}_{n_1} \times \mathrm{GL}_{n_2} \times \mathrm{GL}_n(\mathcal{O})} W\left(\begin{pmatrix} \left(\begin{smallmatrix} g_1 & \\ & 1_{n_1} \end{smallmatrix}\right)^{w_{n_1}} & \\ & \left(\begin{smallmatrix} g_2 & \\ & 1_{n_2} \end{smallmatrix}\right)^{w_{n_2}} \end{pmatrix} \begin{pmatrix} kh & \\ & h \end{pmatrix}^{w_n} \right) \delta_{P_{(n_1,n_2)}}\left(\left(\begin{smallmatrix} g_1 & \\ & g_2 \end{smallmatrix}\right)\right)^{-\frac{1}{2}} \\ \langle \tau_1(g_1) \otimes \tau_2(g_2) v(kh), v_1^\vee \otimes v_2^\vee \rangle |\det g|^{s-\frac{1}{2}} dg_1 dg_2 dk \times \chi_\pi(\det h)^{-1}, \end{aligned}$$

where

$$W = \int_{\mathrm{M}_{n_1,n_2}(F)} W_{\mathrm{Sh}}\left(\begin{pmatrix} 1_{n_1} & X & \\ & 1_{n_2} & \\ & & 1_n \end{pmatrix}^{w_n} \cdot \right) dX \in \mathcal{W}^{\psi}_{(n_1,n_2)}(\pi_n)$$

(Corollary 1.4). Since the integration over $\mathrm{GL}_n(\mathcal{O})$ becomes a finite sum, we have (i) by (1).

(ii) We have

$$\begin{aligned} Z(\widetilde{W_{\mathrm{Sh}}}, 1-s, f) &= \int_{\mathrm{GL}_n \times K} \Phi_{\widetilde{W_{\mathrm{Sh}}}}(g) \langle v(hkg^{-1}), v_1^\vee \otimes v_2^\vee \rangle |\det g|^{-s+\frac{1}{2}} dg dk \\ &= \int_{\mathrm{GL}_n} \Phi_{\widetilde{W_{\mathrm{Sh}}}}(h^{-1}gh) \langle v(g^{-1}h), v_1^\vee \otimes v_2^\vee \rangle |\det g|^{-s+\frac{1}{2}} dg. \end{aligned}$$

if $\mathrm{Re}(s)$ is sufficiently small. By Iwasawa decomposition $\mathrm{GL}_n = \mathrm{GL}_n(\mathcal{O}) U_{(n_1,n_2)} M_{(n_1,n_2)}$, the above integral can be written as

$$\begin{aligned} &\int_{\mathrm{GL}_{n_1} \times \mathrm{GL}_{n_2} \times \mathrm{GL}_n(\mathcal{O})} W\left(\begin{pmatrix} \left(\begin{smallmatrix} g_1 & \\ & 1_{n_1} \end{smallmatrix}\right)^{w_{n_1}} & \\ & \left(\begin{smallmatrix} g_2 & \\ & 1_{n_2} \end{smallmatrix}\right)^{w_{n_2}} \end{pmatrix} \left(\begin{pmatrix} h & \\ & k^{-1}h \end{pmatrix} \begin{pmatrix} & 1_n \\ 1_n & \end{pmatrix} \right)^{w_n} \right) \delta_{P_{(n_1,n_2)}}\left(\left(\begin{smallmatrix} g_1 & \\ & g_2 \end{smallmatrix}\right)\right)^{-\frac{1}{2}} \\ &\chi_\pi(\det g_1)^{-1} \chi_\pi(\det g_2)^{-1} \langle \tau_1(g_1^{-1}) \otimes \tau_2(g_2^{-1}) v(k^{-1}h), v_1^\vee \otimes v_2^\vee \rangle |\det g_1 \det g_2|^{-s+\frac{1}{2}} dg_1 dg_2 dk \times \chi_\pi(\det h)^{-1} \\ &= \int_{\mathrm{GL}_{n_1} \times \mathrm{GL}_{n_2} \times \mathrm{GL}_n(\mathcal{O})} W\left(\begin{pmatrix} \left(\left(\begin{smallmatrix} g_1 & \\ & 1_{n_1} \end{smallmatrix}\right)\left(\begin{smallmatrix} & 1_{n_1} \\ 1_{n_1} & \end{smallmatrix}\right)\right)^{w_{n_1}} & \\ & \left(\left(\begin{smallmatrix} g_2 & \\ & 1_{n_2} \end{smallmatrix}\right)\left(\begin{smallmatrix} & 1_{n_2} \\ 1_{n_2} & \end{smallmatrix}\right)\right)^{w_{n_2}} \end{pmatrix} \begin{pmatrix} kh & \\ & h \end{pmatrix}^{w_n} \right) \delta_{P_{(n_1,n_2)}}\left(\left(\begin{smallmatrix} g_1 & \\ & g_2 \end{smallmatrix}\right)\right)^{-\frac{1}{2}} \\ &\chi_\pi(\det g_1)^{-1} \chi_\pi(\det g_2)^{-1} \langle \tau_1(g_1^{-1}) \otimes \tau_2(g_2^{-1}) v(kh), v_1^\vee \otimes v_2^\vee \rangle |\det g_1 \det g_2|^{-s+\frac{1}{2}} dg_1 dg_2 dk \times \chi_\pi(\det h)^{-1}. \end{aligned}$$

Comparing the above integral and the integral in the proof of (i), we have (ii). $\square$

Proposition 2.11. Let $W_{\mathrm{Sh}}^j \in \mathcal{W}_{\mathrm{Sh}}^{\psi}(\pi_{n_j})$ and f^j a matrix coefficient of τ_j $(j = 1, 2)$. Then, there are $W_{\mathrm{Sh}} \in \mathcal{W}_{\mathrm{Sh}}^{\psi}(\pi_n)$ and a matrix coefficient f of τ_0 such that

$$Z(W_{\mathrm{Sh}}^1, s, f^1)Z(W_{\mathrm{Sh}}^2, s, f^2) = Z(W_{\mathrm{Sh}}, s, f)$$

if $\mathrm{Re}(s)$ is sufficiently large.

Proof. Since π_m is irreducible for any m, we can take $W_{\mathrm{Sh}} \in \mathcal{W}_{\mathrm{Sh}}^{\psi}(\pi_n)$ such that

$$\left(\int_{\mathrm{M}_{n_1,n_2}(F)} W_{\mathrm{Sh}}\left(\begin{pmatrix} 1_{n_1} & X & \\ & 1_{n_2} & \\ & & 1_n \end{pmatrix}^{w_n} \cdot\right) dX\right)\Bigg|_{\mathrm{diag}(\mathrm{GL}_{2n_1}, \mathrm{GL}_{2n_2})} = W_{\mathrm{Sh}}^1 |\cdot|^{\frac{n_2}{2}} \otimes W_{\mathrm{Sh}}^2 |\cdot|^{-\frac{n_1}{2}}$$

by Corollary 1.4 and (1). Write $f^i = \langle \tau_i(\cdot)v_i, v_i^{\vee}\rangle$ $(v_i \in \tau_i, v_i^{\vee} \in \tau_i^{\vee})$ and define $v \in \tau_0$ by

$$v(g) = \int_{P_{(n_1,n_2)}} 1_K(pg)\delta_{P_{(n_1,n_2)}}(\mathrm{diag}(m_1, m_2))^{-\frac{1}{2}}\tau_1(m_1^{-1})v_1 \otimes \tau_2(m_2^{-1})v_2 d_r p,$$

where $p = \begin{pmatrix} m_1 & * \\ & m_2 \end{pmatrix}$, K is an open compact subgroup of GL_n such that $\Phi_{W_{\mathrm{Sh}}}$ is bi-K-invariant, and $d_r p$ is the right Haar measure of $P_{(n_1,n_2)}$. Then, for $f = \langle \tau_0(\cdot)v, L_{h,K,v_1^{\vee},v_2^{\vee}}\rangle$, we have

$$\begin{aligned}
Z(W_{\mathrm{Sh}}, s, f) &= \int_{\mathrm{GL}_n} \Phi_{W_{\mathrm{Sh}}}(g)\langle v(g), v_1^{\vee} \otimes v_2^{\vee}\rangle |\det g|^{s-\frac{1}{2}} dg \\
&= \int_{\mathrm{GL}_n \times P_{(n_1,n_2)}} \Phi_{W_{\mathrm{Sh}}}(p^{-1}g)1_K(g)\delta_{P_{(n_1,n_2)}}(\mathrm{diag}(m_1, m_2))^{-\frac{1}{2}} f^1(m_1^{-1})f^2(m_2^{-1})|\det p^{-1}g|^{s-\frac{1}{2}} d_r p dg \\
&= \int_{\mathrm{GL}_{n_1} \times \mathrm{GL}_{n_2}} \Phi_{W_{\mathrm{Sh}}^1}(m_1)\Phi_{W_{\mathrm{Sh}}^2}(m_2)f^1(m_1)f^2(m_2)|\det m_1 \det m_2|^{s-\frac{1}{2}} dm_1 dm_2 \\
&= Z(W_{\mathrm{Sh}}^1, s, f^1)Z(W_{\mathrm{Sh}}^2, s, f^2)
\end{aligned}$$

if $\mathrm{Re}(s)$ is sufficiently large. □

We give the proofs of Theorem 2.1 (ii), (iii), and (v):

proof of Theorem 2.1 (ii), (iii), (v). (ii) Embedding τ to the parabolic induction of some irreducible generic representation and using Proposition 2.10 (i) repeatedly, we can assume that τ is generic and irreducible. However, we have already proved (ii) for generic representations (Corollary 2.8).

(iii) It follows from Proposition 2.10 (i) and Proposition 2.11.

(v) It follows from Proposition 2.10 (ii). □

2.4 The proof of Theorem 2.1 (vi)

We have already shown Theorem 2.1 (i), (ii), (iii), and (v) ((iv) will be shown in the next section). At the end of this section, we prove (vi). We suppose that Theorem 2.1 (iv) holds here.

proof of Theorem 2.1 (vi). We can assume that $\tau = \mathrm{St}(\rho, m)$ for some $\rho \in \mathrm{Irr}_{\mathrm{sc}}$ and m by Theorem 2.1 (iii). If $L(s, \pi \boxtimes \tau) = 1$, then $L(\pi, s, \tau) = L(s, \pi \boxtimes \tau) = 1$ by Corollary 2.8. Thus we only have to consider the case where

1. $\rho \in \mathrm{Irr}\mathrm{GL}_1$, $\pi = \rho' \times \rho^{-1}|\cdot|^t$ or $\mathrm{St}(\rho^{-1}|\cdot|^t, 2)$ $(\rho' \in \mathrm{Irr}\mathrm{GL}_1, t \in \mathbb{C}$ s.t. $\rho'\rho|\cdot|^{-t} \neq |\cdot|^{\pm 1})$ or
2. $\rho \in \mathrm{Irr}_{\mathrm{sc}}\mathrm{GL}_2$, $\pi = \rho^{\vee}|\cdot|^t$ $(t \in \mathbb{C})$.

By Corollary 2.8, we have

$$L(\pi; s, \tau) = Q(q^{-s}, q^{s}) L(s, \pi \boxtimes \tau)$$

and

$$L(\pi^{\vee}; 1-s, \tau^{\vee}) = \tilde{Q}(q^{-s}, q^{s}) L(1-s, \pi^{\vee} \boxtimes \tau^{\vee})$$

for some $Q(X,Y), \tilde{Q}(X,Y) \in \mathbb{C}[X,Y]$. Dividing the second equation by the first equation, we have that $\gamma(\pi; s, \tau)$ coincides with $Q(q^{-s}, q^{s})\tilde{Q}(q^{-s}, q^{s})^{-1}\gamma(s, \pi \boxtimes \tau)$ up to a unit. By Theorem 2.1 (v), we have

$$\gamma(\pi; s, \tau) = \prod_{i=1}^{m} \gamma(\pi; s + m/2 + 1/2 - i, \rho).$$

By Lemma 2.9, the equation $L(\pi; s, \rho) = L(s, \pi \boxtimes \rho)$ holds. Thus $\gamma(\pi; s, \tau)$ coincides with $\gamma(s, \pi \boxtimes \tau)$ up to a unit. Consequently, we have

$$Q(q^{-s}, q^{s}) = cq^{ls}\tilde{Q}(q^{-s}, q^{s})$$

for some $c \in F^{\times}$ and $l \in \mathbb{Z}$. Namely, $Q(q^{-s}, q^{s}) \in \mathbb{C}[q^{-s}, q^{s}]$ is a common factor of $L(s, \pi \boxtimes \tau)^{-1}$ and $L(1-s, \pi^{\vee} \boxtimes \tau^{\vee})^{-1}$.

Assume that π and τ satisfy the conditions in 2. Then, we have

$$L(s, \pi \boxtimes \tau)^{-1} = 1 - q^{-s-t-\frac{m-1}{2}}$$

and

$$L(1-s, \pi^{\vee} \boxtimes \tau^{\vee})^{-1} = 1 - q^{-1+s+t-\frac{m-1}{2}} = -q^{-1+s+t-\frac{m-1}{2}}(1 - q^{-s-t+\frac{m+1}{2}})$$

by Theorem of [JPSS83, §(8.2)]. Thus $L(s, \pi \boxtimes \tau)^{-1}$ and $L(1-s, \pi^{\vee} \boxtimes \tau^{\vee})^{-1}$ are relatively prime, and we have $L(\pi; s, \tau) = L(s, \pi \boxtimes \tau)$.

Assume that π and τ satisfy the conditions in 1. We also assume $n > 1$ since we have already shown that $L(\pi; s, \tau) = L(s, \pi \boxtimes \tau)$ for $n = 1$ (Lemma 2.9). Then, we have

$$L(s, \pi \boxtimes \tau)^{-1} = \begin{cases} (1 - q^{-s-t-\frac{n}{2}})(1 - q^{-s-t-\frac{n}{2}+1}) & \text{if } \pi = \mathrm{St}(\rho^{-1}|\cdot|^{t}, 2); \\ (1 - q^{-s-t'-\frac{n-1}{2}})(1 - q^{-s-t-\frac{n-1}{2}}) & \text{if } \pi = \rho^{-1}|\cdot|^{t'} \times \rho^{-1}|\cdot|^{t} \ (|\cdot|^{t-t'} \neq |\cdot|^{\pm 1}); \\ (1 - q^{-s-t-\frac{n-1}{2}}) & \text{otherwise} \end{cases}$$

and

$$L(1-s, \pi^{\vee} \boxtimes \tau^{\vee})^{-1} = \begin{cases} (1 - q^{-s-t+\frac{n}{2}+1})(1 - q^{-s-t+\frac{n}{2}}) & \text{if } \pi = \mathrm{St}(\rho^{-1}|\cdot|^{t}, 2); \\ (1 - q^{-s-t'+\frac{n+1}{2}})(1 - q^{-s-t+\frac{n+1}{2}}) & \text{if } \pi = \rho^{-1}|\cdot|^{t'} \times \rho^{-1}|\cdot|^{t} \ (|\cdot|^{t-t'} \neq |\cdot|^{\pm 1}); \\ (1 - q^{-s-t+\frac{n+1}{2}}) & \text{otherwise} \end{cases}$$

up to a unit. Therefore, we have $L(\pi; s, \tau) = L(s, \pi \boxtimes \tau)$ unless $\pi = \rho^{-1}|\cdot|^{t \pm n} \times \rho^{-1}|\cdot|^{t}$ for some $t \in \mathbb{C}$. However, the case $\pi = \rho^{-1}|\cdot|^{t \pm n} \times \rho^{-1}|\cdot|^{t}$ does not occur since π is approximately tempered. $\square$

Remark 2.12. Using [CFK, Proposition C.10], Theorem 2.1 (vi) follows from Theorem 2.1 (iii) and Corollary 2.8 immediately.

3 The functional equation

In this section, we prove the functional equation (Theorem 2.1 (iv)). As mentioned in Remark 2.2, it has already been proven. However, for the convenience of the readers, we prove it without the results in [CFK].

The key to the proof is the following proposition.

Proposition 3.1. Let $\tau \in \mathrm{Irr}_{\mathrm{sc}}\mathrm{GL}_n$ and $\tau' \in \mathrm{Irr}\mathrm{GL}_n$. Then, we have

$$\dim_{\mathbb{C}} \mathrm{Hom}_{\mathrm{GL}_n \times \mathrm{GL}_n}(\pi_n, \tau \boxtimes \tau') \leq 1$$

if $\tau' = \tau^\vee \chi_\pi$ and

$$\dim_{\mathbb{C}} \mathrm{Hom}_{\mathrm{GL}_n \times \mathrm{GL}_n}(\pi_n, \tau \boxtimes \tau') = 0$$

otherwise. (Here, we think of $\mathrm{GL}_n \times \mathrm{GL}_n$ as the diagonal subgroup $\mathrm{diag}(\mathrm{GL}_n, \mathrm{GL}_n)$ of GL_{2n}.)

We give the proof in §3.4.

By this proposition, we obtain Theorem 2.1 (iv) as follows:

proof of Theorem 2.1 (iv). By Proposition 2.10 (ii), we can assume $\tau \in \mathrm{Irr}_{\mathrm{sc}}\mathrm{GL}_n$.

For any $s \in \mathbb{C}$, the map

$$(W_{\mathrm{Sh}}, v, v^\vee) \mapsto Z(W_{\mathrm{Sh}}, s, \langle \tau(\cdot)v, v^\vee \rangle)/L(\pi; s, \tau),\ W_{\mathrm{Sh}} \in \mathcal{W}^{\psi}_{\mathrm{Sh}}(\pi_n),\ v \in \tau,\ v^\vee \in \tau^\vee$$

is well-defined by Corollary 2.8 and its linear extension on $\mathcal{W}^{\psi}_{\mathrm{Sh}}(\pi_n) \otimes \tau \otimes \tau^\vee$ defines a nonzero element of

$$\begin{aligned}\mathrm{Hom}_{\mathrm{diag}(\mathrm{GL}_n,\mathrm{GL}_n)^{w_n}}(\pi_n \otimes (\tau|\cdot|^{s-\frac{1}{2}} \boxtimes \tau^\vee \chi_\pi^{-1}|\cdot|^{-s+\frac{1}{2}}), \mathbb{C}) &\simeq \mathrm{Hom}_{\mathrm{diag}(\mathrm{GL}_n,\mathrm{GL}_n)^{w_n}}(\pi_n, \tau^\vee|\cdot|^{-s+\frac{1}{2}} \boxtimes \tau\chi_\pi|\cdot|^{s-\frac{1}{2}}) \\ &\simeq \mathrm{Hom}_{\mathrm{GL}_n \times \mathrm{GL}_n}(\pi_n, \tau^\vee|\cdot|^{-s+\frac{1}{2}} \boxtimes \tau\chi_\pi|\cdot|^{s-\frac{1}{2}}).\end{aligned}$$

On the other hand, it is easy to check that the linear extension of

$$(W_{\mathrm{Sh}}, v, v^\vee) \mapsto Z(\widetilde{W_{\mathrm{Sh}}}, 1-s, \langle v, \tau^\vee(\cdot)v^\vee \rangle)/L(\pi^\vee; 1-s, \tau^\vee),\ W_{\mathrm{Sh}} \in \mathcal{W}^{\psi}_{\mathrm{Sh}}(\pi_n),\ v \in \tau,\ v^\vee \in \tau^\vee$$

on $\mathcal{W}^{\psi}_{\mathrm{Sh}}(\pi_n) \otimes \tau \otimes \tau^\vee$ defines a nonzero element of the same space. By Proposition 3.1, these two maps coincide up to a constant. Thus, there is a function ϵ on $\mathbb{C}$ such that

$$Z(\widetilde{W_{\mathrm{Sh}}}, 1-s, \langle v, \tau^\vee(\cdot)v^\vee \rangle)/L(\pi^\vee; 1-s, \tau^\vee) = \epsilon(s) Z(W_{\mathrm{Sh}}, s, \langle \tau(\cdot)v, v^\vee \rangle)/L(\pi; s, \tau)$$

for any $W_{\mathrm{Sh}} \in \mathcal{W}^{\psi}_{\mathrm{Sh}}(\pi_n),\ v \in \tau,\ v^\vee \in \tau^\vee$. By Theorem 2.1(ii), we obtain

$$\gamma(s) = L(\pi^\vee; 1-s, \tau^\vee)\epsilon(s)/L(\pi; s, \tau) \in \mathbb{C}(q^{-s})$$

as required. □

We introduce some additional notations.

Let k, l be positive integers such that $k \geq l$. Then, we often identify GL_{k-l} with $\mathrm{diag}(\mathrm{GL}_{k-l}, 1_l) \subset \mathrm{GL}_k$. Define

$$D^k_l = \begin{pmatrix} \mathrm{GL}_{k-l} & * \\ & 1_l \end{pmatrix} = \mathrm{GL}_{k-l} U_{(k-l,l)} \subset \mathrm{GL}_k$$

and

$$D^k_l \rtimes D^l_m = \begin{pmatrix} \mathrm{GL}_{k-l} & * & * \\ & \mathrm{GL}_{l-m} & * \\ & & 1_m \end{pmatrix} = D^k_l \begin{pmatrix} 1_{k-l} & \\ & D^l_m \end{pmatrix} \subset D^k_m$$

for $m \in \mathbb{Z}_{>0}$ such that $l \geq m$. Moreover, for representations μ and σ of GL_{k-l} and D^l_m, respectively, we denote

$$\mathrm{Ind}^{D^k_m}_{D^k_l \rtimes D^l_m} \mu \boxtimes \sigma$$

by $\mu \rtimes \sigma$, where $\mu \boxtimes \sigma$ is regarded as a representation of $D^k_l \rtimes D^l_m$ by

$$U_{(k-l,l)} \backslash D^k_l \rtimes D^l_m \simeq \mathrm{GL}_{k-l} \times D^l_m.$$

We note that $(\mu \rtimes \sigma)|_{\mathrm{GL}_{k-m}} = \mu * \sigma|_{\mathrm{GL}_{l-m}}$.

Let i be a positive integer such that $i \leq n$. We define

$$N_i = \left\{ \begin{pmatrix} X & Y \\ & 1_i \end{pmatrix} \middle| \; X \in U_{(1^i)}, Y \in \mathrm{M}_i(F) : \text{upper triangular} \right\} \subset D_i^{2i}$$

and a character Ψ_{N_i} of N_i by

$$\Psi_{N_i}\left(\begin{pmatrix} X & Y \\ & 1_i \end{pmatrix}\right) = \psi(\mathrm{tr} Y)$$

and put

$$I_i = \mathrm{ind}_{N_i}^{D_i^{2i}} \Psi_{N_i}.$$

3.1 The Kirillov-Shalika model

We start with the following lemma:

Lemma 3.2. We regard $\psi \circ \mathrm{tr}$ as a character of $U_{(n,n)}$ by $U_{(n,n)} \simeq \mathrm{M}_n(F)$.

(i) Any nonzero subrepresentation of $\mathrm{Ind}_{U_{(n,n)}}^{D_n^{2n}} \psi \circ \mathrm{tr}$ contains $\mathrm{ind}_{U_{(n,n)}}^{D_n^{2n}} \psi \circ \mathrm{tr}$. In particular, $\mathrm{ind}_{U_{(n,n)}}^{D_n^{2n}} \psi \circ \mathrm{tr}$ is the unique irreducible subrepresentation of $\mathrm{Ind}_{U_{(n,n)}}^{D_n^{2n}} \psi \circ \mathrm{tr}$.

(ii) We have $I_n \simeq \mathrm{ind}_{U_{(n,n)}}^{D_n^{2n}} \psi \circ \mathrm{tr}$.

(i) and (ii) are special cases of more general statements [LM20, Lemma 3.12] and [LM20, Lemma 3.14], respectively.

Let χ be a character of $F^\times$. Then, we can extend the action of D_n^{2n} on $\mathrm{Ind}_{U_{(n,n)}}^{D_n^{2n}} \psi \circ \mathrm{tr}$ to $P_{(n,n)}$ by

$$(\mathrm{diag}(1_n, g))f := \chi(\det g) f\left(\mathrm{diag}(g^{-1}, 1_n) \cdot\right)$$

for any $g \in \mathrm{GL}_n$. We denote the extended representation by $\tilde{I}_n^\chi$ and the $P_{(n,n)}$-submodule $\mathrm{ind}_{U_{(n,n)}}^{D_n^{2n}} \psi \circ \mathrm{tr}$ of $\tilde{I}_n^\chi$ by I_n^χ.

Let $\mathcal{W}'^{\psi}_{\mathrm{Sh}}(\pi_n)$ be the 'ordinal' Shalika model of π_n i.e., a (unique) subspace of $\mathrm{Ind}_{U_{(n,n)}}^{\mathrm{GL}_{2n}} \psi \circ \mathrm{tr}$ which realizes π_n. We note that the isomorphism from $\mathcal{W}^{\psi}_{\mathrm{Sh}}(\pi_n)$ to $\mathcal{W}'^{\psi}_{\mathrm{Sh}}(\pi_n)$ is given by $W_{\mathrm{Sh}} \mapsto W_{\mathrm{Sh}}(w_n \cdot)$ for $W_{\mathrm{Sh}} \in \mathcal{W}^{\psi}_{\mathrm{Sh}}(\pi_n)$.

We define the Kirillov-Shalika model $\mathcal{K}_\psi(\pi_n)$ of π_n by

$$\mathcal{K}_\psi(\pi_n) = \{W'_{\mathrm{Sh}}|_{D_n^{2n}} \mid W'_{\mathrm{Sh}} \in \mathcal{W}^{\psi}_{\mathrm{Sh}}(\pi_n)\}.$$

Then, the kernel of $\mathcal{W}'^{\psi}_{\mathrm{Sh}}(\pi_n) \twoheadrightarrow \mathcal{K}_\psi(\pi_n)$ is a $P_{(n,n)}$-module. Therefore, as a space with $P_{(n,n)}$-action induced by $\pi_n|_{P_{(n,n)}}$, $\mathcal{K}_\psi(\pi_n)$ is a subrepresentation of $\tilde{I}_n^{\chi_\pi}$.

If π is approximately tempered, then $\mathcal{W}'^{\psi}_{\mathrm{Sh}}(\pi_n) \twoheadrightarrow \mathcal{K}_\psi(\pi_n)$ is bijective ([LM20, Corollary 4.4]). Thus, we have the following proposition.

Proposition 3.3. Assume π is approximately tempered. Then, π_n has a unique irreducible D_n^{2n}-submodule. Moreover, this submodule is a $P_{(n,n)}$-submodule and is isomorphic to $I_n^{\chi_\pi}$ as a $P_{(n,n)}$-representation.

3.2 Special cases

We consider Proposition 3.1 for some special cases.

Lemma 3.4. Assume π is not supercuspidal. Then, Proposition 3.1 holds.

Proof. Since π is not supercuspidal, π_n is a quotient of the degenerate principal series $\mathbf{1}_{\mathrm{GL}_n}\chi_1 \times \mathbf{1}_{\mathrm{GL}_n}\chi_2$ for some $\chi_1, \chi_2 \in \mathrm{Irr}\mathrm{GL}_1$. Then we have

$$\dim_{\mathbb{C}} \mathrm{Hom}_{\mathrm{GL}_n \times \mathrm{GL}_n}(\pi_n, \tau \boxtimes \tau') \leq \dim_{\mathbb{C}} \mathrm{Hom}_{\mathrm{GL}_n \times \mathrm{GL}_n}(\mathbf{1}_{\mathrm{GL}_n}\chi_1 \times \mathbf{1}_{\mathrm{GL}_n}\chi_2, \tau \boxtimes \tau').$$

Similar to [HKS96, Theorem 4.3], the equality

$$\dim_{\mathbb{C}} \mathrm{Hom}_{\mathrm{GL}_n \times \mathrm{GL}_n}(\mathbf{1}_{\mathrm{GL}_n}\chi_1 \times \mathbf{1}_{\mathrm{GL}_n}\chi_2, \tau \boxtimes \tau') = \begin{cases} 1 & \text{if } \tau' = \tau^\vee \chi_\pi; \\ 0 & \text{otherwise} \end{cases}$$

is easy to verify by the filtration of $\mathbf{1}_{\mathrm{GL}_n}\chi_1 \times \mathbf{1}_{\mathrm{GL}_n}\chi_2$ in [AK18, Lemma 2.5]. □

Lemma 3.5. Assume π is supercuspidal. Then, Proposition 3.1 holds if $n = 1, 2$.

Proof. If $n = 1$, then it follows from $\{\Phi_W \mid W \in \mathcal{W}^\psi(\pi)\} = \mathcal{S}(F^\times)$.

Assume $n = 2$. Then

$$\dim_{\mathbb{C}} \mathrm{Hom}_{\mathrm{GL}_2 \times \mathrm{GL}_2}(\pi_2, \tau \boxtimes \tau') = 0$$

if $\tau' \neq \tau^\vee \chi_\pi$ and

$$\dim_{\mathbb{C}} \mathrm{Hom}_{\mathrm{GL}_2 \times \mathrm{GL}_2}(\pi_2, \tau \boxtimes \tau^\vee \chi_\pi) \leq 1$$

unless $\tau = \pi |\cdot|^{\frac{1}{2}}$ by [LM20, Proposition 7.1]. Since $\mathrm{Hom}_{\mathrm{GL}_n \times \mathrm{GL}_n}(\pi_n, \tau \boxtimes \tau') \simeq \mathrm{Hom}_{\mathrm{GL}_n \times \mathrm{GL}_n}(\pi_n, \tau' \boxtimes \tau)$ in general, we have $\dim_{\mathbb{C}} \mathrm{Hom}_{\mathrm{GL}_2 \times \mathrm{GL}_2}(\pi_2, \tau \boxtimes \tau^\vee \chi_\pi) \leq 1$ in general. □

3.3 The filtration

In this subsection, we give a good filtration of π_n. Assume that π is supercuspidal here.

The following fact is essential:

Proposition 3.6. For any $\pi' \in \mathrm{Irr}_{\mathrm{sc}}\mathrm{GL}_2$, we have

$$\mathrm{Sp}(\pi', n)|_{D_1^{2n}} \simeq \mathrm{Sp}(\pi', n-1)|\cdot|^{1/2} \rtimes I_1.$$

This is a special case of a more general result [Zel80, Theorem 3.5].

Using this, we obtain the following:

Lemma 3.7. Let i, k be positive integers such that $k < 2i < 2n$ and $\sigma = (\sigma, V)$ be a representation of D_k^{2i}. Then, the following D_k^{2n}-module

$$\Sigma = \pi_{n-i}|\cdot|^{\frac{i}{2}} \rtimes \sigma$$

has a D_{k+1}^{2n}-submodule J such that

$$\Sigma|_{D_{k+1}^{2n}}/J \simeq \pi_{n-i}|\cdot|^{\frac{i}{2}} \rtimes \sigma|_{D_{k+1}^{2i}}, \ J \simeq \pi_{n-i-1}|\cdot|^{\frac{i+1}{2}} \rtimes \mathrm{ind}_G^{D_{k+1}^{2i+2}} \boldsymbol{\sigma},$$

where

$$G = \begin{pmatrix} 1 & * & * & * \\ & \mathrm{GL}_{2i-k} & & * \\ & & 1 & \\ & & & 1_k \end{pmatrix}$$

and $\boldsymbol{\sigma} = (\boldsymbol{\sigma}, V)$ is defined by

$$\boldsymbol{\sigma}\left(\begin{pmatrix} 1 & * & a & * \\ & b & & c \\ & & 1 & \\ & & & 1_k \end{pmatrix}\right) = \psi(a)\sigma\left(\begin{pmatrix} b & c \\ & 1_k \end{pmatrix}\right).$$

Proof. Put $H = D^{2n}_{2i} \rtimes D^{2i}_k$. Since

$$H\backslash D^{2n}_k / D^{2n}_{k+1} \simeq P_{(2n-2i,2i-k)}\backslash \mathrm{GL}_{2n-k}/D^{2n-k}_1,$$

we have

$$D^{2n}_i = HD^{2n}_{k+1} \sqcup HwD^{2n}_{k+1},$$

where $w = \begin{pmatrix} 1_{2n-2i-1} & & & \\ & & 1 & \\ & 1_{2i-k} & & \\ & & & 1_k \end{pmatrix}$. We note that $H \cap D^{2n}_{k+1} = D^{2n}_{2i} \rtimes D^{2i}_{k+1}$ and

$$w^{-1}Hw \cap D^{2n}_{k+1} = \begin{pmatrix} \mathrm{GL}_{2n-2i-1} & * & * & * \\ & \mathrm{GL}_{2i-k} & & * \\ & & 1 & \\ & & & 1_k \end{pmatrix}.$$

Put $J = \{f \in \Sigma \mid \mathrm{supp} f \subset HwD^{2n}_{i+1}\}$. Then, J is D^{2n}_{k+1}-module and we have

$$\Sigma|_{D^{2n}_{k+1}}/J \simeq \pi_{n-i}|\cdot|^{\frac{i}{2}} \rtimes \sigma|_{D^{2i}_{k+1}},\ J \simeq \mathrm{ind}^{D^{2n}_{k+1}}_{w^{-1}Hw\cap D^{2n}_{k+1}}\sigma',$$

where $\pi_{n-i}|\cdot|^{i/2} = (\pi_{n-i}|\cdot|^{i/2}, V')$ and $\sigma' = (\sigma', V' \otimes V)$ is defined by

$$\sigma'\left(\begin{pmatrix} a & * & b & * \\ & c & & d \\ & & 1 & \\ & & & 1_k \end{pmatrix}\right) = \pi_{n-i}|\cdot|^{i/2}\left(\begin{pmatrix} a & b \\ & 1 \end{pmatrix}\right) \otimes \sigma\left(\begin{pmatrix} c & d \\ & 1_k \end{pmatrix}\right).$$

Then, since $\pi_{n-i}|\cdot|^{i/2}|_{D^{2n-2i-1}_1} \simeq \pi_{n-i-1}|\cdot|^{\frac{i+1}{2}} \rtimes I_1$ by Proposition 3.6, we have

$$\sigma' \simeq \mathrm{ind}^{w^{-1}Hw\cap D^{2n}_{k+1}}_{G'}\pi_{n-i-1}|\cdot|^{\frac{i+1}{2}} \boxtimes \boldsymbol{\sigma},$$

where $G' = D^{2n}_{2i+2}\mathrm{diag}(1_{2n-2i-2}, G)$ and $\pi_{n-i-1}|\cdot|^{\frac{i+1}{2}} \boxtimes \boldsymbol{\sigma}$ is regarded as a representation of G' by

$$U_{(2n-2i-2,2i+2)}\backslash G' \simeq \mathrm{GL}_{2n-2i-2} \times G.$$

Thus we have

$$J \simeq \pi_{n-i-1}|\cdot|^{\frac{i+1}{2}} \rtimes \mathrm{ind}^{D^{2i+2}_{k+1}}_G \boldsymbol{\sigma}.$$

$\square$

We obtain the following:

Proposition 3.8. There is a sequence

$$\pi_n = J'_1 \supset J'_2 \cdots \supset J'_n$$

of subspaces of π_n such that

- J'_i is isomorphic to $J_i = J_i(\pi, n) := \pi_{n-i}|\cdot|^{i/2} \rtimes I_i$ as D^{2n}_i-representation ($i = 1, \ldots, n$) and
- $J'_i|_{D^{2n}_{i+1}}/J'_{i+1} \simeq K_i = K_i(\pi, n) := \pi_{n-i}|\cdot|^{\frac{i}{2}} \rtimes I_i|_{D^{2i}_{i+1}}$ ($i = 1, \ldots, n-1$).

Proof. We have $\pi_n|_{D^{2n}_1} \simeq J_1$ by proposition 3.6. Then, using Lemma 3.7 repeatedly, we have this proposition immediately (note that if $\sigma = I_i$, we have $\mathrm{ind}^{D^{2i+2}_{i+1}}_G \boldsymbol{\sigma} \simeq I_{i+1}$ under the notations in Lemma 3.7). $\square$

3.4 The proof of Proposition 3.1

Finally, we give the proof of Proposition 3.1.

proof of Proposition 3.1. By Lemma 3.4 and 3.5, we can assume that π is supercuspidal and $n > 2$.

By Proposition 3.3 and 3.8, the restriction map

$$\mathrm{Hom}_{\mathrm{GL}_n\times\mathrm{GL}_n}(\pi_n, \tau\times\tau') \to \mathrm{Hom}_{\mathrm{GL}_n\times\mathrm{GL}_n}(J'_n, \tau\times\tau') \simeq \mathrm{Hom}_{\mathrm{GL}_n\times\mathrm{GL}_n}(I_n^{\chi_\pi}, \tau\times\tau')$$

is well-defined. We show that this map is injective. Suppose $\mathrm{Hom}_{\mathrm{GL}_n\times\mathrm{GL}_n}(\pi_n/J'_n, \tau\times\tau') \neq 0$ for the sake of contradiction. Then, we have $\mathrm{Hom}_{\mathrm{GL}_n}(\pi_n/J'_n, \tau) \neq 0$. By Proposition 3.8, we have $\mathrm{Hom}_{\mathrm{GL}_n}(K_i, \tau) \neq 0$ for some $i \in \{1, \ldots, n-1\}$. Finally, using Proposition 3.6 and Lemma 3.7 repeatedly, we have

$$\mathrm{Hom}_{\mathrm{GL}_n}(\pi_{n-j}|\cdot|^{j/2} \rtimes \sigma, \tau) = \mathrm{Hom}_{\mathrm{GL}_n}(\pi_{n-j}|\cdot|^{j/2} * \sigma|_{\mathrm{GL}_{2j-n}}, \tau) \neq 0$$

for some $j \in \mathbb{Z}_{>0}$ such that $j < n \leq 2j$ and representation σ of D_n^{2j}. This contradicts $\tau \in \mathrm{Irr}_{\mathrm{sc}}$ and $n/2 > 1$.

Therefore, we have

$$\dim_{\mathbb{C}} \mathrm{Hom}_{\mathrm{GL}_n\times\mathrm{GL}_n}(I_n^{\chi_\pi}, \tau\times\tau') \geq \dim_{\mathbb{C}} \mathrm{Hom}_{\mathrm{GL}_n\times\mathrm{GL}_n}(\pi_n, \tau\times\tau').$$

On the other hand, the equality

$$\dim_{\mathbb{C}} \mathrm{Hom}_{\mathrm{GL}_n\times\mathrm{GL}_n}(I_n^{\chi_\pi}, \tau\times\tau') = \begin{cases} 1 & \text{if } \tau' = \tau^\vee\chi_\pi; \\ 0 & \text{otherwise} \end{cases}$$

holds since $I_n^{\chi_\pi}|_{\mathrm{GL}_n\times\mathrm{GL}_n} \simeq \mathrm{ind}_{\Delta\mathrm{GL}_n}^{\mathrm{GL}_n\times\mathrm{GL}_n}\chi_\pi(\det)$, where $\Delta\mathrm{GL}_n$ is the diagonal embedding of GL_n to $\mathrm{GL}_n\times\mathrm{GL}_n$. Thus we get the required statement. □

4 Some remarks for general rank case

By Theorem 2.1 (ii), we have

$$\mathrm{Hom}_{\mathrm{GL}_n\times\mathrm{GL}_n}(\pi_n, \tau\boxtimes\tau^\vee\chi_\pi) \neq 0$$

for any $\tau \in \mathrm{Irr}\mathrm{GL}_n$ if π is approximately tempered. To conclude this paper, let us consider the branching laws of the Speh representations with respect to general block diagonal subgroups.

Theorem 4.1. Assume π is approximately tempered. Then, we have

$$\mathrm{Hom}_{\mathrm{GL}_n\times\mathrm{GL}_{n+2l}}(\pi_{n+l}, \tau\boxtimes\tau^\vee\chi_\pi\times\pi_l) \neq 0$$

for any $\tau \in \mathrm{Irr}\mathrm{GL}_n$.

Proof. Let L be a nonzero element of $\mathrm{Hom}_{\mathrm{GL}_n\times\mathrm{GL}_n}(\pi_n|\cdot|^{\frac{l}{2}}, \tau\boxtimes\tau^\vee\chi_\pi|\cdot|^l)$. For $f \in \pi_n|\cdot|^{-\frac{l}{2}}\times\pi_l|\cdot|^{\frac{n}{2}} = \pi_n|\cdot|^{\frac{l}{2}} * \pi_l|\cdot|^{-\frac{n}{2}}$, define $\tilde{L}f(g) = L\otimes\mathrm{id}_{\pi_l|\cdot|^{-\frac{n}{2}}}(f(g))$ for any $g \in \mathrm{GL}_{n+2l}$. Then, $\tilde{L}$ is a nonzero element of

$$\mathrm{Hom}_{\mathrm{GL}_n\times\mathrm{GL}_{n+2l}}(\pi_n|\cdot|^{-\frac{l}{2}}\times\pi_l|\cdot|^{\frac{n}{2}}, \tau\boxtimes\tau^\vee\chi_\pi\times\pi_l)$$

and the following diagram

$$\begin{array}{ccc}
\pi_n|\cdot|^{-\frac{l}{2}}\times\pi_l|\cdot|^{\frac{n}{2}} & \xrightarrow{f\mapsto f(1_{2n+2l})} & \pi_n|\cdot|^{\frac{l}{2}}\boxtimes\pi_l|\cdot|^{-\frac{n}{2}} \\
\tilde{L}\downarrow & & L\otimes\mathrm{id}\downarrow \\
\tau\boxtimes\tau^\vee\chi_\pi\times\pi_l & \xrightarrow{f'\mapsto f'(1_{n+2l})} & \tau\boxtimes\tau^\vee\chi_\pi|\cdot|^l\boxtimes\pi_l|\cdot|^{-\frac{n}{2}}
\end{array}$$

is commutative.

Since π is approximately tempered, π_{n+l} is a subrepresentation of $\pi_n|\cdot|^{-\frac{l}{2}}\times\pi_l|\cdot|^{\frac{n}{2}}$. Since $\pi_n|\cdot|^{\frac{l}{2}}\boxtimes\pi_l|\cdot|^{-\frac{n}{2}}$ is irreducible, π_{n+l} is mapped onto $\pi_n|\cdot|^{\frac{l}{2}}\boxtimes\pi_l|\cdot|^{-\frac{n}{2}}$ by substituting the unit. Then, by the above commutative diagram, we have $\tilde{L}|_{\pi_{n+l}}$ is a nonzero element of $\mathrm{Hom}_{\mathrm{GL}_n\times\mathrm{GL}_{n+2l}}(\pi_{n+l}, \tau\boxtimes\tau^\vee\chi_\pi\times\pi_l)$. □

Let $\tau' \in \mathrm{Irr}\mathrm{GL}_{n+2l}$. According to [AK18, Proposition 2.3], τ' and $\tau^\vee\chi_\pi \times \pi_l$ have the same cuspidal support if π, τ, and τ' are unramified and $\mathrm{Hom}_{\mathrm{GL}_n \times \mathrm{GL}_{n+2l}}(\pi_{n+l}, \tau \boxtimes \tau') \neq 0$ (we note that if π and τ are unramified and unitary, then $\tau^\vee\chi_\pi \times \pi_l$ is unramified and unitary). This fact is important for determining the near equivalence classes of global Miyawaki lifts (see [Ito]). Furthermore, considering Proposition 3.1, it is natural to expect the following conjecture:

Conjecture 4.2. Assume π, τ, and τ' are unitary. Then, the following should hold:

(i) (uniqueness) $\mathrm{Hom}_{\mathrm{GL}_n \times \mathrm{GL}_{n+2l}}(\pi_{n+l}, \tau \boxtimes \tau') \neq 0 \Rightarrow \tau' = \tau^\vee\chi_\pi \times \pi_l$.

(ii) (multiplicity at most one) $\dim_{\mathbb{C}} \mathrm{Hom}_{\mathrm{GL}_n \times \mathrm{GL}_{n+2l}}(\pi_{n+l}, \tau \boxtimes \tau') \leq 1$.

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