Arakelov geometry over an adelic curve and dynamical systems

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Contents

Introduction

Let X be a smooth projective variety over an algebraically closed field K and $f : X \dashrightarrow X$ be a dominant rational map. The (first) dynamical degree δ_f of f is an invariant for estimating the geometric complexity of the iterations f^n of f. There are several equivalent definitions of the dynamical degree, and here we introduce the method in [10] using the linear map of f^* on the Néron-Severi group of X induced by f.

Let $N^1(X)$ be the group of all Cartier divisors on X modulo numerical equivalence. It is a free $\mathbb{Z}$-module of finite rank. We set $N^1(X)_{\mathbb{R}} = N^1(X) \otimes_{\mathbb{Z}} \mathbb{R}$. Let $p : Y \to X$ be the resolution of f such that $g = f \circ p$ is a morphism and Y is projective:

$$\begin{array}{ccc} Y & & \\ \downarrow{\scriptstyle p} & \searrow^{g} & \\ X & \overset{f}{\dashrightarrow} & X. \end{array}$$

Then the pull-back morphism $f^* : N^1(X) \to N^1(X)$ is defined by $f^*D := p_*(g^*D)$ for $D \in N^1(X)$. Note that this definition is independent of the choice of (Y, p). By abuse of notation, we also denote by f^* the linear map on $N^1(X)_{\mathbb{R}}$ induced by f^*. The *(first) dynamical degree* δ_f of f is defined by

$$\delta_f := \lim_{n\to\infty} (\rho((f^n)^* : N^1(X)_{\mathbb{R}} \to N^1(X)_{\mathbb{R}}))^{\frac{1}{n}},$$

where $\rho(\cdot)$ is the spectral radius. By definition, we have $\delta_f \geq 1$.

In arithmetic dynamical systems, one of invariants related to the dynamical degree δ_f is the arithmetic degree $\alpha_f(P)$, which estimates the asymptotic behavior of $h_X(f^n(P))$ for $P \in X(\overline{\mathbb{Q}})$ where h_X is a height function on X. Kawaguchi and Silverman conjectured about a relation between δ_f and α_f in [20]. This conjecture is not completely proved, but there are many affirmative answers for several cases (for details, see [16], [18], [19], [27], [36], etc.).

The arithmetic degree is usually defined over $\overline{\mathbb{Q}}$. Hence this conjecture is mainly discussed on $\overline{\mathbb{Q}}$. But the dynamical degree can be defined over arbitrary field K. Thus, we extend the defining field of the arithmetic degree and Kawaguchi-Silverman's conjecture. In this paper, we use the notion of adelic curves introduced by Chen and Moriwaki in [8]. An adelic curve S is a field K equipped with the set of absolute values M_K on K which is indexed by a measure space $(\Omega, \mathcal{A}, \nu)$ and a map $\phi : \Omega \ni \omega \mapsto |\cdot|_\omega \in M_K$, and verifies the following relation which is called the product formula:

$$\forall a \in K^\times, \quad \int_\Omega \log |a|_\omega \, \nu(\mathrm{d}\omega) = 0.$$

As examples of adelic curves, we can construct adelic structures for number fields, function fields and finitely generated fields over $\mathbb{Q}$ or $\mathbb{F}_q$. In particular, it is important that a finitely generated field over $\mathbb{Q}$ or $\mathbb{F}_q$ forms an adelic curve because for any variety X over any field, we can reduce the base field of X to a finitely generated field over $\mathbb{Q}$ or $\mathbb{F}_q$. Hence by using this framework, we can consider Kawaguchi-Silverman's conjecture on any dynamical system over arbitrary field. Of course, it depends on the choice of a finitely generated field as a defining field and its adelic structure.

A height theory works on any adelic curve, but the notion of adelic curves is too general to consider Kawaguchi-Silverman's conjecture. For example, height functions on a trivially valued field (which is an adelic curve that consists of only the trivial absolute value) are very simple. In fact, the arithmetic degree calculated by a height function on a trivially valued field is always equal to one. Hence we need to choose a class of adelic curves which is convenient for this conjecture. In this paper, we use adelic curves which have the Northcott property. It is a class which holds a condition such like the Northcott theorem on $\mathbb{P}^n(\overline{\mathbb{Q}})$ (for details, see [34]). In this way, once we fix a good adelic curve, we can expect a sufficient background for Kawaguchi-Silverman's conjecture. This is our motivation of this paper.

Height functions on a trivially valued field as above are simple and they do not have the Northcott property. However Arakerov theory over a trivially valued field has many interesting results. In this paper, we will see two topics on it. The first one is the bigness of adelic Cartier divisors. Let X be a normal projective variety over a trivially valued field. On classical algebraic geometry, one of tools to study big divisors is a volume function. Hence we define a volume function of adelic Cartier divisor $\overline{D}$ on X as follows:

$$\widehat{\mathrm{vol}}(\overline{D}) := \limsup_{n\to+\infty} \frac{\widehat{\deg}_+(n\overline{D})}{n^{d+1}/(d+1)!},$$

where $d = \dim X$. The invariant $\widehat{\deg}_+(n\overline{D})$ is one which plays a similar role to $h^0(X, nD)$ (for details, see [4], [5] and [8]). An adelic Cartier divisor $\overline{D}$ is said to be *big* if $\widehat{\mathrm{vol}}(\overline{D}) > 0$. We will see the simple criterion of big adelic Cartier divisors and prove the properties of this volume function:

Theorem A. *Let $\overline{D}, \overline{E}$ be adelic Cartier divisors. The arithmetic volume function has the following properties:*

(1) (integral formula).

$$\widehat{\mathrm{vol}}(\overline{D}) = (d+1)\int_0^\infty F_{\overline{D}}(t)\ \mathrm{d}t,$$

where $F_{\overline{D}}$ is a function given by $\overline{D}$.

(2) (limit existence).

$$\widehat{\mathrm{vol}}(\overline{D}) = \lim_{n\to\infty} \frac{\widehat{\deg}_+(n\overline{D})}{n^{d+1}/(d+1)!}.$$

(3) (continuity). If D is big, then we have

$$\lim_{\epsilon\to 0}\widehat{\mathrm{vol}}(\overline{D}+\epsilon\overline{E})=\widehat{\mathrm{vol}}(\overline{D}).$$

(4) (homogeneity). For $a\in\mathbb{R}_{>0}$,

$$\widehat{\mathrm{vol}}(a\overline{D})=a^{d+1}\widehat{\mathrm{vol}}(\overline{D}).$$

(5) (log concavity). If $\overline{D},\overline{E}$ are big, then we have

$$\widehat{\mathrm{vol}}(\overline{D}+\overline{E})^{\frac{1}{d+1}}\geq\widehat{\mathrm{vol}}(\overline{D})^{\frac{1}{d+1}}+\widehat{\mathrm{vol}}(\overline{E})^{\frac{1}{d+1}}.$$

The second topic is the ampleness of adelic Cartier divisors. In this paper, we discuss several results about ample adelic Cartier divisors (which does not mean that only the underlying Cartier divisor is ample). We will see that one of them is related to height functions, which is the simple criterion of ampleness:

Theorem B. *An adelic Cartier divisor D is ample if and only if the height function $h^{\mathrm{an}}_{\overline{D}}(x)>0$ for all $x\in X^{\mathrm{an}}$.*

The above height function $h^{\mathrm{an}}_{\overline{D}}$ is an extension of the height function $h_{\overline{D}}$ on X. Hence it follows that height functions are deeply related to ampleness of adelic Cartier divisors.

Let $S=(K,(\Omega,\mathcal{A},\nu),\phi)$ be an adelic curve. For any adelic divisor $\overline{D}$ on X, we can define a height function $h_{\overline{D}}:X(\overline{K})\to\mathbb{R}$. We fix an adelic divisor $\overline{D}$ whose underlying Cartier divisor is ample, and take the height function $h_X:=h_{\overline{D}}$. We set $h_X^+:=\max\{h_X,1\}$. Let I_f be the indeterminacy locus of f and

$$X_f(\overline{K}):=\{P\in X(\overline{K})\mid f^n(P)\notin I_f\text{ for all }n>0\}.$$

Let P be an element of $X_f(\overline{K})$. We define the *upper and lower arithmetic degrees* of P with respect to f as

$$\overline{\alpha}_f(P):=\limsup_{n\to\infty}h_X^+(f^n(P))^{\frac{1}{n}},$$
$$\underline{\alpha}_f(P):=\liminf_{n\to\infty}h_X^+(f^n(P))^{\frac{1}{n}}.$$

Note that the above definitions are independent of the choice of $\overline{D}$. By definition, we have

$$1\leq\underline{\alpha}_f(P)\leq\overline{\alpha}_f(P).$$

If $\underline{\alpha}_f(P)=\overline{\alpha}_f(P)$, the *arithmetic degree* $\alpha_f(P)$ of P with respect to f is defined as $\overline{\alpha}_f(P)$.

By using the arithmetic degree over an adelic curve, we can consider the following conjecture:

Conjecture C (Kawaguchi-Silverman's conjecture over adelic curves). *Let $S=(K,\Omega,\nu)$ be an proper adelic curve, and $P\in X_f(\overline{K})$.*

(1) $\underline{\alpha}_f(P)=\overline{\alpha}_f(P)$. In particular, the arithmetic degree $\alpha_f(P)$ exists.

(2) We assume that S has the Northcott property. If the orbit $\mathcal{O}_f(P)=\{f^n(P)\,|\,n=0,1,\dots\}$ of P is Zariski dense in X, then we have $\alpha_f(P)=\delta_f$.

In the original paper, Kawaguchi and Silverman [20] conjectured that the set $\{\alpha_f(Q)\,|\,Q\in X_f(\overline{K})\}$ is finite. This set is finite for many cases, but unfortunately, Lesieutre and Satriano found an example of a birational map f on $\mathbb{P}^4$ such that the set $\{\alpha_f(Q)\,|\,Q\in(\mathbb{P}^4)_f(\overline{\mathbb{Q}})\}$ is infinite. For details, see [23, Theorem 2]. Moreover, Kawaguchi and Silverman [20] conjectured that the arithmetic degree is an algebraic integer. By [1], there exists a birational map f on $\mathbb{P}^2$ such that the dynamical degree δ_f is transcendental. Hence, while this remains an open problem, there might exist a transcendental arithmetic degree.

Another results consist of mainly two parts. The first one is to prove the fundamental inequality

$$\forall P\in X_f(\overline{K}),\quad \alpha_f(P)\le\delta_f.$$

This inequality is proved by using Matsuzawa's method in [26].

Theorem D (c.f. [26, Theorem 1.4]). *Let $S=(K,\Omega,\nu)$ be a proper adelic curve. Let X be a smooth projective variety over an algebraic closure $\overline{K}$ of K and $f:X\dashrightarrow X$ be a dominant rational map over $\overline{K}$. For any $\epsilon>0$, there is a constant $C>0$ such that*

$$\forall n\ge 0,\forall P\in X_f(\overline{K}),\quad h_X(f^n(P))\le C(\delta_f+\epsilon)^n h_X(P).$$

In the second part, we prove extended Kawaguchi-Silverman's conjecture for some cases:

Theorem E (c.f. [19, Theorem 3]). *Let $S=(K,\Omega,\nu)$ be a proper adelic curve. Let X be a normal projective variety over an algebraic closure $\overline{K}$ of K and $f:X\to X$ be a morphism. For any $\overline{K}$-rational point P of X,*

(1) $\overline{\alpha}_f(P)=\underline{\alpha}_f(P)$. In particular, the limit $\alpha_f(P)$ exists.

(2) The arithmetic degree $\alpha_f(P)$ is an algebraic integer.

(3) The set $\{\alpha_f(Q)\,|\,Q\in X(\overline{K})\}$ is finite.

Theorem F (c.f. [18, Theorem 2(a)]). *Let $S=(K,(\Omega,\mathcal{A},\nu),\phi)$ be a proper adelic curve. Let X be a normal projective variety over an algebraic closure $\overline{K}$ of K such that $\dim N^1(X)_{\mathbb{R}}=1$, and $f:X\to X$ be a morphism. Then for any $P\in X(\overline{K})$, the arithmetic degree $\alpha_f(P)$ exists and is equal to 1 or δ_f. Moreover, if S has the Northcott property and the orbit $\mathcal{O}_f(P)$ is infinite, then we have $\alpha_f(P)=\delta_f$.*

Theorem G (c.f. [18, Theorem 2(b)]). *Let $S = (K, (\Omega, \mathcal{A}, \nu), \phi)$ be a proper adelic curve with the Northcott property. Let $f : \mathbb{A}^n \to \mathbb{A}^n$ be a regular affine automorphism of degree $d \geq 2$ defined over an algebraic closure $\overline{K}$ of K. We denote by f' the restriction of f onto $\mathbb{P}^n \setminus \mathbb{A}^n$. Then for $P \in (\mathbb{P}^n)_f(\overline{K})$, we have*

$$\alpha_f(P) = \begin{cases} 1 & (\mathcal{O}_f(P) \text{ is finite}), \\ \delta_f & (\mathcal{O}_f(P) \text{ is infinite and } P \in \mathbb{A}^n(\overline{K})), \\ \delta_{f'} & (\mathcal{O}_f(P) \text{ is infinite and } P \in (\mathbb{P}^n \setminus \mathbb{A}^n)_f(\overline{K})). \end{cases}$$

Theorem H (c.f. [18, Theorem 2(c)]). *Let $S = (K, (\Omega, \mathcal{A}, \nu), \phi)$ be a proper adelic curve with the Northcott property. Let X be a smooth projective surface over an algebraic closure $\overline{K}$ of K and $f : X \to X$ be an automorphism. Then for $P \in X(\overline{K})_f$, we have*

$$\alpha_f(P) = \begin{cases} 1 & (\mathcal{O}_f(P) \text{ is finite or } P \in E_f(\overline{K})), \\ \delta_f & (\mathcal{O}_f(P) \text{ is infinite and } P \notin E_f(\overline{K})), \end{cases}$$

where E_f is the union of the f-periodic irreducible curves in X.

We will show these theorems by using slight modified methods in [18], [19] and [20].

In Chapter 1, we recall the basic results of algebraic geometry, normed vector spaces and Berkovich spaces. Next, we introduce the notion of adelic curves and see some examples in Chapter 2. In addition, we define a base change of an adelic curve and a height function on it. In Chapter 3, we study height theory on arithmetic varieties over an adelic curve. It contains the usual height theories over $\overline{\mathbb{Q}}$, for example, Weil height theory and Néron-Tate theory. In Chapter 4, we see some results on Arakelov geometry over a trivially valued field. We study properties of big adelic Cartier divisors, ample ones and volume functions. Finally in Chapter 5, we restate extended Kawaguchi-Silverman's conjecture and we prove our main theorems.

Acknowledgements

I'd like to thank Professor Shu Kawaguchi and Professor Huayi Chen for many helpful advice. I'm grateful to Professor Atsushi Moriwaki for valuable comments on my drafts of this paper.

Chapter 1

Preliminary

1.1 $\mathbb{Q}$- and $\mathbb{R}$-divisors

Let X be a variety over a field K and $K(X)$ be a function field of X. By abuse of notation, we also denote the (constant) sheaf of rational functions on X by $K(X)$. Firstly, we recall the definitions of Cartier divisors and Weil divisors (for details, see [13] and [24]).

Definition 1.1.1. Let $\mathrm{Div}(X) := H^0(X, K(X)^\times/\mathcal{O}_X^\times)$, whose element is called a *Cartier divisor*. By definition, for $D \in \mathrm{Div}(X)$, there is an open covering $\{U_i\}$ of X such that D is given by some non-zero rational function $f_i \in K(X)^\times$ on U_i and $f_i/f_j \in \mathcal{O}_X(U_i \cap U_j)^\times$ for $i \neq j$. A Cartier divisor $D \in \mathrm{Div}(X)$ is said to be *effective* if f_i is regular on U_i, that is, $f_i \in \mathcal{O}_X(U_i)$ for all i in the above setting. A non-zero rational function $f \in K(X)^\times$ naturally gives rise to a Cartier divisor, which is called a *principal Cartier divisor* (or simply a *principal divisor*) and denoted by (f). We denote the group law on $\mathrm{Div}(X)$ in additive way. We say that two Cartier divisors $D_1, D_2 \in \mathrm{Div}(X)$ are *linearly equivalent* if $D_1 - D_2$ is principal, which is denoted by $D_1 \sim D_2$. We set $\mathrm{Pic}(X) := \mathrm{Div}(X)/\!\sim$, which is called the *Picard group of* X.

For two Cartier divisors D_1, D_2, we write $D_1 \geq D_2$ if $D_1 - D_2$ is effective. In particular, we write $D \geq 0$ if D is effective. For an open subset U of X, let $D|_U$ be the image of D by the canonical restriction $H^0(X, K(X)^\times/\mathcal{O}_X) \to H^0(U, K(X)^\times/\mathcal{O}_X^\times)$, which gives a Cartier divisor on U.

We can associate any Cartier divisor $D = \{(U_i, f_i)\} \in \mathrm{Div}(X)$ with a subsheaf $\mathcal{O}_X(D) \subset K(X)$, which is given by $\mathcal{O}_X(D)|_{U_i} := f_i^{-1}\mathcal{O}_X|_{U_i}$. It is well-known that this construction is independent of the choice of a representation $\{(U_i, f_i)\}$ of D, and $\mathcal{O}_X(D)$ is an invertible $\mathcal{O}_X$-module on X.

Proposition 1.1.2 (c.f. [13, Proposition 6.13] and [24, Proposition 1.18]). *Let D_1, D_2 be Cartier divisors.*

(1) $\mathcal{O}_X(D_1) \simeq \mathcal{O}_X(D_2)$ *if* $D_1 \sim D_2$.

(2) $\mathcal{O}_X(D_1 + D_2) \simeq \mathcal{O}_X(D_1) \otimes_{\mathcal{O}_X} \mathcal{O}_X(D_2)$.

We denote $\Gamma(U, \mathcal{O}_X(D))$ by $\Gamma(U, D)$ for an open subset U of X. For any open subset U of X, we have

$$\Gamma(U, D) = \{f \in K(X)^\times \mid (D + (f))|_U \geq 0\} \cup \{0\} \tag{1.1.1}$$

by definition.

Conversely, we can associate any invertible $\mathcal{O}_X$-module $\mathcal{L}$ with a Cartier divisor D such that $\mathcal{L} \simeq \mathcal{O}_X(D)$. Let s be a non-zero rational section of $\mathcal{L}$, that is, $s \in \mathcal{L}_\eta \setminus \{0\}$ where η is the generic point of X. Let $\{U_i\}$ be an open covering of X in which $\mathcal{L}$ is trivialized, and $\omega_i \in \mathcal{L}(U_i)$ be a local basis of $\mathcal{L}$ for each i. Then s is denoted by $f_i\omega_i$ on U_i for some $f_i \in K(X)$. The data $\{(U_i, f_i)\}$ gives the required Cartier divisor, which is denoted by $\mathrm{div}(s)$. For example, if we choose 1 as a rational section of $\mathcal{O}_X(D)$, then we have $\mathrm{div}(1) = D$ by its construction.

Next, we assume that X is normal. Let $X^{(1)} = \{x \in X \mid \mathrm{codim}_X \overline{\{x\}} = 1\}$. For $x \in X^{(1)}$, let $[x] := \overline{\{x\}}$, which is an irreducible closed subset of X and called a *prime divisor*.

Definition 1.1.3. Let $\mathrm{WDiv}(X) := \bigoplus_{x \in X^{(1)}} \mathbb{Z}[x]$, whose element is called a *Weil divisor*. For a Weil divisor

$$D = \sum_{x \in X^{(1)}} n_x[x],$$

n_x is denoted by $\mathrm{ord}_x(D)$. We say that a Weil divisor $D \in \mathrm{WDiv}(X)$ is *effective* if $\mathrm{ord}_x(D) \geq 0$ for all $x \in X^{(1)}$. For two Weil divisors D_1, D_2, we write $D_1 \geq D_2$ if $D_1 - D_2$ is effective. In particular, we write $D \geq 0$ if D is effective. For a non-empty open subset U of X, let

$$D|_U := \sum_{x \in X^{(1)} \cap U} \mathrm{ord}_x(D)[x],$$

which is called the restriction of a Weil divisor D on U.

If $x \in X^{(1)}$, $\mathcal{O}_{X,x}$ is a discrete valuation ring since X is normal. Hence we have the normalized discrete valuation ord_x on $K(X)$ associated with $\mathcal{O}_{X,x}$. For a non-zero rational function $f \in K(X)^\times$, let

$$(f) := \sum_{x \in X^{(1)}} \mathrm{ord}_x(f)[x].$$

This is a Weil divisor and such a divisor is called a *principal Weil divisor* (or simply a *principal divisor*). We say that two Weil divisors $D_1, D_2 \in \mathrm{WDiv}(X)$ are *linearly equivalent* if $D_1 - D_2$ is principal. Then we write $D_1 \sim D_2$.

We can associate any Cartier divisor $D \in \mathrm{Div}(X)$ with a Weil divisor as follows: For any $x \in X^{(1)}$, let $f \in K(X)$ be a local equation of D around x. Then we set $\mathrm{ord}_x(D) := \mathrm{ord}_x(f)$. It is independent of the choice of a local equation. Hence we can define that

$$D := \sum_{x \in X^{(1)}} \mathrm{ord}_x(D)[x].$$

This construction gives a homomorphism $\varphi : \mathrm{Div}(X) \to \mathrm{WDiv}(X)$.

Proposition 1.1.4 (c.f. [24, Proposition 2.14]). *(1) The homomorphism φ is injective. Moreover, φ is an isomorphism if X is locally factorial (which means that $\mathcal{O}_{X,x}$ is UFD for all $x \in X$). Hence we often identify a Cartier divisor with a Weil divisor.*

(2) For any $D_1, D_2 \in \mathrm{Div}(X)$, $D_1 \sim D_2$ as Cartier divisors if and only if $D_1 \sim D_2$ as Weil divisors.

(3) For any $D \in \mathrm{Div}(X)$, $D \geq 0$ as Cartier divisors if and only if $D \geq 0$ as Weil divisors.

We can associate any Weil divisor D with a subsheaf $\mathcal{O}_X(D) \subset K(X)$, which is defined by

$$\mathcal{O}_X(D)|_U := \{f \in K(X)^\times \mid (D + (f))|_U \geq 0\} \cup \{0\}$$

for any open subset U of X. By (1.1.1), if D is Cartier, the above construction gives the same invertible $\mathcal{O}_X$-module $\mathcal{O}_X(D)$. However $\mathcal{O}_X(D)$ is not invertible if D is not Cartier. Note that in this case, $\mathcal{O}_X(D)$ is reflexive (which means that the canonical morphism $\mathcal{O}_X(D) \to \mathcal{H}\mathrm{om}(\mathcal{H}\mathrm{om}(\mathcal{O}_X(D), \mathcal{O}_X), \mathcal{O}_X)$ is an isomorphism).

Let $\mathbb{K} = \mathbb{Q}$ or $\mathbb{R}$. Let us introduce the definition of $\mathbb{K}$-divisors.

Definition 1.1.5. Let $\mathrm{Div}(X)_\mathbb{K} := \mathrm{Div}(X) \otimes_\mathbb{Z} \mathbb{K}$, $\mathrm{WDiv}(X)_\mathbb{K} := \mathrm{WDiv}(X) \otimes_\mathbb{Z} \mathbb{K}$ and $K(X)^\times_\mathbb{K} := K(X)^\times \otimes_\mathbb{Z} \mathbb{K}$. An element of $\mathrm{Div}(X)_\mathbb{K}$ (resp. $\mathrm{WDiv}(X)_\mathbb{K}$, $K(X)^\times_\mathbb{K}$) is called a $\mathbb{K}$-*Cartier divisor* (resp. a $\mathbb{K}$-*Weil divisor*, a $\mathbb{K}$-*rational function*) on X. Clearly, Cartier divisors and $\mathbb{Q}$-Cartier divisors (resp. Weil divisors and $\mathbb{Q}$-Weil divisors) are $\mathbb{R}$-Cartier divisors (resp. $\mathbb{R}$-Weil divisors). A non-zero $\mathbb{K}$-rational function $f \in K(X)^\times_\mathbb{K}$ naturally gives rise to a $\mathbb{K}$-Cartier divisor (or equivalently a $\mathbb{K}$-Weil divisor), which is called a $\mathbb{K}$-*principal divisor* and denoted by (f). We say that two $\mathbb{R}$-Cartier divisors (resp. $\mathbb{R}$-Weil divisors) D_1, D_2 are $\mathbb{K}$-*linearly equivalent* if $D_1 - D_2$ is $\mathbb{K}$-principal, which is denoted by $D_1 \sim_\mathbb{K} D_2$. A $\mathbb{K}$-Cartier divisor (resp. a $\mathbb{K}$-Weil divisor) D is said to be *effective* if D is a linear combination of effective divisors with positive coefficients in $\mathbb{K}$. We write $D_1 \geq D_2$ if $D_1 - D_2$ is effective. In particular, we write $D \geq 0$ if D is effective.

Similarly to Cartier divisors, for $D \in \mathrm{Div}(X)_\mathbb{K}$, there is an open covering $\{U_i\}$ of X such that D is given by some non-zero $\mathbb{K}$-rational functions $f_i \in K(X)^\times_\mathbb{K}$ on U_i and $f_i/f_j \in (\mathcal{O}_X(U_i \cap U_j) \otimes_\mathbb{Z} \mathbb{K})^\times$ for $i \neq j$.

Let $D \in \mathrm{WDiv}(X)_\mathbb{K}$. By definition, we can write $D = \sum_{x \in X^{(1)}} k_x[x]$, where $k_x \in \mathbb{K}$ and $k_x = 0$ for all but finitely many $x \in X^{(1)}$. Then we define the round down of D as follows:

$$\lfloor D \rfloor := \sum_{x \in X^{(1)}} \lfloor k_x \rfloor [x].$$

This is a Weil divisor and $\lfloor D \rfloor = D$ if and only if $D \in \mathrm{WDiv}(X)$.

For $D \in \mathrm{WDiv}(X)_\mathbb{K}$, the *associated $\mathcal{O}_X$-module* $\mathcal{O}_X(D)$ is defined by $\mathcal{O}_X(\lfloor D \rfloor)$. Then we have $H^0(X, D) = \{f \in K(X)^\times \mid D + (f) \geq 0\} \cup \{0\}$. We remark that $D + (f) \geq 0$ if and only if $\lfloor D \rfloor + (f) \geq 0$ for any $f \in K(X)^\times$, and $\mathcal{O}_X(2D)$ is not isomorphic to $\mathcal{O}_X(D) \otimes_{\mathcal{O}_X} \mathcal{O}_X(D)$ in general.

Proposition 1.1.6 (c.f. [24, Theorem 3.2]). *Let $D \in \mathrm{WDiv}(X)_\mathbb{K}$. Then $H^0(X, D)$ is a finite-dimensional vector space over K.*

1.2 Semiample, ample and big divisors

We recall the definitions of the semiampleness, ampleness and bigness of Cartier divisors. Let X be a projective variety over a field K.

Definition 1.2.1. We say that a Cartier divisor D is *semiample* if $\mathcal{O}_X(nD)$ is generated by global sections for some $n \in \mathbb{Z}_{>0}$, that is, the canonical morphism $H^0(X, nD) \otimes \mathcal{O}_X \to \mathcal{O}_X(nD)$ is surjective.

Definition 1.2.2. We say that D is *ample* if for a sufficiently large $n > 0$, there is a closed immersion $j : X \hookrightarrow \mathbb{P}^n_K$ such that $\mathcal{O}_X(nD) \simeq j^*\mathcal{O}_{\mathbb{P}^n_K}(1)$.

Definition 1.2.3. Let D be a Cartier divisor on X. Let $h^0(D) := \dim_K H^0(X, D)$ and $d = \dim X$. We define the *volume* $\mathrm{vol}(D)$ of D as follows:

$$\mathrm{vol}(D) := \limsup_{n\to+\infty} \frac{h^0(nD)}{n^d/d!}.$$

We say that D is *big* if $\mathrm{vol}(D) > 0$.

Later we will consider the volume of an $\mathbb{R}$-Weil divisor. Hence we extends the above definition.

Definition 1.2.4. Let D be an $\mathbb{R}$-Weil divisor on a normal variety X. We define a function $\mathfrak{h}_D : \mathbb{R}_+ \to \mathbb{Z}$ by $\mathfrak{h}_D(t) := \dim_K H^0(tD) = \dim_K H^0(\lfloor tD \rfloor)$. The *volume* of D is defined by

$$\mathrm{vol}(D) := \limsup_{t\to+\infty} \frac{\mathfrak{h}_D(t)}{t^d/d!},$$

where $d = \dim X$. We say that D is *big* if $\mathrm{vol}(D) > 0$.

By Fulger, Kollár and Lehmann [11], the above definition agrees with one in Definition 1.2.3 if D is Cartier.

Finally we recall the well-known properties of the volume function $\mathrm{vol}(\cdot)$ without a proof (for details, see [22]).

Proposition 1.2.5. *Let X be a proper normal variety and $d = \dim X$. Let D, E be $\mathbb{R}$-Weil divisors on X.*

(1) $\mathrm{vol}(D) = \lim_{t\to+\infty} \frac{h_D(t)}{t^d/d!} = \lim_{t\to+\infty} \frac{\mathfrak{h}_D(t)}{t^d/d!}$.

(2) For $a \in \mathbb{R}_{>0}$, $\mathrm{vol}(aD) = a^d \mathrm{vol}(D)$.

(3) The volume function $\mathrm{vol}(\cdot)$ *is continuous, that is,* $\mathrm{vol}(E) \to \mathrm{vol}(D)$ *as $E \to D$ (which means that each coefficients of E converge coefficients of D as an $\mathbb{R}$-Weil divisor).*

(4) The volume function $\mathrm{vol}(\cdot)$ *is d-concave on big divisors, that is, if D and E are big, then*

$$\mathrm{vol}(D + E)^{1/d} \geq \mathrm{vol}(D)^{1/d} + \mathrm{vol}(E)^{1/d}.$$

1.3 Normed vector space

In this section, we study fundamental properties of a normed vector space over a field equipped with an absolute value. However we mainly consider a trivially valued field.

Throughout this section, let K be a field.

Definition 1.3.1. We say that a map $|\cdot| : K \to \mathbb{R}_+$ is an *absolute value on* K if it satisfies the following conditions:

(1) $\forall a \in K, \quad |a| = 0 \Leftrightarrow a = 0.$

(2) $\forall a, b \in K, \quad |a|\cdot|b| = |ab|.$

(3) (*triangle inequality*) $\forall a, b \in K, \quad |a + b| \leq |a| + |b|.$

If an absolute value $|\cdot|$ also satisfies the following stronger inequality

$$\forall a, b \in K, \quad |a + b| \leq \max\{|a|, |b|\},$$

we say that $|\cdot|$ is *non-Archimedean*. Otherwise, $|\cdot|$ is said to be *Archimedean*.

Definition 1.3.2. We say that an absolute value $|\cdot|$ on K is *trivial* if it satisfies that $|a| = 1$ for any $a \in K \setminus \{0\}$. A field K equipped with the trivial absolute value $|\cdot|$ is called a *trivially valued field*. Clearly, the trivial absolute value is non-Archimedean and a trivially valued field is complete as a metric space.

Let V be a vector space over K.

Definition 1.3.3. A map $\|\cdot\| : V \to \mathbb{R}_+$ is said to be a *(multiplicative) norm over* $(K, |\cdot|)$ if it satisfies the following conditions:

(1) $\forall v \in V, \quad \|v\| = 0 \Leftrightarrow v = 0.$

(2) $\forall a \in K$ and $v \in V, \quad \|av\| = |a|\cdot\|v\|.$

(3) (*triangle inequality*) $\forall v, w \in V, \quad \|v + w\| \leq \|v\| + \|w\|.$

If a norm $\|\cdot\|$ also satisfies the following stronger inequality

$$\forall v, w \in V, \quad \|v + w\| \leq \max\{\|v\|, \|w\|\},$$

we say that $\|\cdot\|$ is *ultrametric*. A pair $(V, \|\cdot\|)$ is called a *normed vector space*.

Let $V_\bullet = \bigoplus_{n=0}^{\infty} V_n$ be a graded ring over K such that V_n is a vector space over K for all n and $V_0 = K$. Let $|\cdot|$ be an absolute value on K and $\|\cdot\|_n$ be a norm of V_n over $(K, |\cdot|)$ for $n \in \mathbb{Z}_{\geq 0}$ such that $\|\cdot\|_0 = |\cdot|$ on $V_0 = K$.

Definition 1.3.4. We say that

$$(V_\bullet, \|\cdot\|_\bullet) := \bigoplus_{n=0}^{\infty} (V_n, \|\cdot\|_n)$$

is a *normed graded ring over* $(K, |\cdot|)$ if $\|v_m \cdot v_n\|_{m+n} \leq \|v_m\|_m \cdot \|v_n\|_n$ for all $v_m \in V_m$ and $v_n \in V_n$.

Let $W_\bullet = \bigoplus_{n=0}^{\infty} W_n$ be a $V_\bullet$-module such that W_n is a vector space over K for all n. Let $h \in \mathbb{Z}_{>0}$. We say that $W_\bullet$ is a *h-graded $V_\bullet$-module* if $v_m \cdot w_n \in W_{hm+n}$ for all $v_m \in V_m$ and $w_n \in W_n$. If $h = 1$, $W_\bullet$ is simply called a *graded $V_\bullet$-module*.

Let $\|\cdot\|_{W_n}$ be a norm on W_n over $(K, |\cdot|)$ for $n \in \mathbb{Z}_{\geq 0}$.

Definition 1.3.5.

$$(W_\bullet, \|\cdot\|_{W_\bullet}) := \bigoplus_{n=0}^{\infty} (W_n, \|\cdot\|_{W_n})$$

is called a *normed h-graded $(V_\bullet, \|\cdot\|_\bullet)$-module* if $\|v_m \cdot w_n\|_{W_{hm+n}} \leq \|v_m\|_m \cdot \|w_n\|_{W_n}$ for all $v_m \in V_m$ and $w_n \in W_n$. If $h = 1$, $(W_\bullet, \|\cdot\|_{W_\bullet})$ is simply called a *normed graded $(V_\bullet, \|\cdot\|_\bullet)$-module.*

Next, we consider a norm induced by another norms. Let $(V, \|\cdot\|_V)$ be a normed vector space over $(K, |\cdot|)$.

Definition 1.3.6. Let W be a vector space over K. Let $f : V \to W$ be a surjective K-linear map. Then we define the *quotient norm* $\|\cdot\|_W$ on W induced by $\|\cdot\|_V$ and f as follows:

$$\|w\|_W := \inf\{\|v\|_V \mid f(v) = w,\ v \in V\} \quad \text{for } \forall w \in W.$$

Note that if $\|\cdot\|_V$ is ultrametric, then $\|\cdot\|_W$ is also ultrametric.

Definition 1.3.7. Let $(W, \|\cdot\|_W)$ be a normed vector space over $(K, |\cdot|)$. Then we define the *operator norm* $\|\cdot\|_{\mathrm{Hom}_K(V,W)}$ on $\mathrm{Hom}_K(V, W)$ as follows:

$$\|\phi\|_{\mathrm{Hom}_K(V,W)} := \sup\left\{ \frac{\|\phi(v)\|_W}{\|v\|_V} \,\middle|\, v \in V \setminus \{0\} \right\} \quad \text{for } \forall \phi \in \mathrm{Hom}_K(V, W).$$

If $(W, \|\cdot\|_W) = (K, |\cdot|)$, then we denote $\mathrm{Hom}_K(V, K)$ by $V^\vee$ and $\|\cdot\|_{\mathrm{Hom}_K(V,K)}$ by $\|\cdot\|_V^\vee$, which is called the *dual norm of* $\|\cdot\|_V$.

Definition 1.3.8. Let K' be an extension field of K and $|\cdot|'$ be an absolute value on K' which is an extension of $|\cdot|$. Let $V_{K'} = V \otimes_K K'$. Then $V_{K'}$ is identified with $\mathrm{Hom}_K(V^\vee, K')$. Hence we equip $V_{K'}$ with the operator norm $\|\cdot\|_{\mathrm{Hom}_K(V^\vee, K')}$, which is denoted by $\|\cdot\|_{V,K'}$. This norm $\|\cdot\|_{V,K'}$ is called the *scalar extension of* $\|\cdot\|_V$.

Remark 1.3.9. This scalar extension is called ϵ-extension of scalars in [8]. As another definition, Chen and Moriwaki [8] define the notion of π-extension of scalars. For the relation between two definitions, see [8, Section 1.3].

In the following, let $(V, \|\cdot\|)$ be an ultrametrically normed vector space over a trivially valued field $(K, |\cdot|)$ and $\dim_K(V) < +\infty$.

Lemma 1.3.10. *(1) Let $v_1, \ldots, v_n \in V$. If $\|v_1\|, \ldots, \|v_n\|$ are all distinct, then we have $\|v_1 + \cdots + v_n\| = \max\{\|v_1\|, \ldots, \|v_n\|\}$.*

(2) $\#\{\|v\| \mid v \in V\} \leq \dim_K(V) + 1$.

Proof. See [8, Proposition 1.1.5] for the proofs. □

We set

$$\mathcal{F}^t(V, \|\cdot\|) := \{v \in V \mid \|v\| \leq e^{-t}\} \quad \text{for } t \in \mathbb{R}.$$

Remark that $\mathcal{F}^t(V, \|\cdot\|)$ is a vector space over K for any $t \in \mathbb{R}$ because $|\cdot|$ is trivial. Then $\{\mathcal{F}^t(V, \|\cdot\|)\}_{t\in\mathbb{R}}$ satisfies the following conditions:

Proposition 1.3.11. *(1) For sufficiently positive $t \in \mathbb{R}$, $\mathcal{F}^t(V, \|\cdot\|) = \{0\}$.*

(2) For sufficiently negative $t \in \mathbb{R}$, $\mathcal{F}^t(V, \|\cdot\|) = V$.

(3) For any $t \geq s$, $\mathcal{F}^t(V, \|\cdot\|) \subseteq \mathcal{F}^s(V, \|\cdot\|)$.

(4) The function $\mathbb{R} \ni t \mapsto \dim_K \mathcal{F}^t(V, \|\cdot\|)$ is left-continuous.

Proof. (1) and (2) follow from Lemma 1.3.10, and (3) and (4) are trivial by definition. □

We set

$$\lambda_{\max}(V, \|\cdot\|) := \sup\{t \in \mathbb{R} \mid \mathcal{F}^t(V, \|\cdot\|) \neq \{0\}\},$$
$$\lambda_{\min}(V, \|\cdot\|) := \sup\{t \in \mathbb{R} \mid \mathcal{F}^t(V, \|\cdot\|) = V\}.$$

By convention, $\lambda_{\max}(V, \|\cdot\|) = -\infty, \lambda_{\min}(V, \|\cdot\|) = +\infty$ if $V = \{0\}$. By Proposition 1.3.11, we have $\lambda_{\max}(V, \|\cdot\|) < +\infty$, and by Lemma 1.3.10, we can replace "sup" by "max" in the above definition.

1.4 Analytification in the sense of Berkovich

Let K be a field equipped with an absolute value $|\cdot|$. We assume that K is complete with respect to $|\cdot|$. Let X be a scheme over $\operatorname{Spec} K$. We recall the analytification of X in the sense of Berkovich (for details, see [2]).

Definition 1.4.1. The *analytification of X in the sense of Berkovich*, or *Berkovich space associated to X* is the set of all pairs $x = (p, |\cdot|_x)$ where $p \in X$ and $|\cdot|_x$ is an absolute value on the residue field $\kappa(x) := \kappa(p)$ which is an extension of $|\cdot|$. We denote it by X^{an}. The map $j : X^{\mathrm{an}} \to X, (p, |\cdot|_x) \mapsto p$ is called the *specification map*.

Let U be a non-empty Zariski open subset of X. The subset $U^{\mathrm{an}} := j^{-1}(U)$ of X^{an} is called a *Zariski open subset of* X^{an}. A regular function $f \in \mathcal{O}_X(U)$ on U define a function $|f|$ on U^{an} as follows:

$$|f|(x) := |f(j(x))|_x \quad \text{for } x \in U^{\mathrm{an}}.$$

We also denote $|f|(x)$ by $|f|_x$.

We define a topology on X^{an} as the most coarse topology which makes j and $|f|$ continuous for any Zariski open subset U of X and any $f \in \mathcal{O}_X(U)$. This is called the *Berkovich topology*. Note that X^{an} is Hausdorff (resp. compact) if X is separated (resp. proper) over $\operatorname{Spec} K$.

Let $f : X \to Y$ be a morphism of schemes over $\operatorname{Spec} K$. There is a continuous map $f^{\mathrm{an}} : X^{\mathrm{an}} \to Y^{\mathrm{an}}$ such that the following diagram is commutative:

$$\begin{array}{ccc} X & \xrightarrow{\ f\ } & Y \\ \big\downarrow{\scriptstyle j} & & \big\downarrow{\scriptstyle j} \\ X^{\mathrm{an}} & \xrightarrow{\ f^{\mathrm{an}}\ } & Y^{\mathrm{an}}. \end{array}$$

Concretely, f^{an} is constructed as follows: Let $x = (p, |\cdot|_x) \in X^{\mathrm{an}}$ and $q = f(p) \in Y$. We remark that $\kappa(y) = \kappa(q)$ is a subfield of $\kappa(x) = \kappa(p)$. Then $y = f^{\mathrm{an}}(x)$ is given by $q = f(p)$ and the absolute value $|\cdot|_y$ on $\kappa(q)$ which is the restriction of $|\cdot|_x$.

In the following, $(K, |\cdot|)$ is a trivially valued field. For $x \in X$, let $x^{\mathrm{an}} = (x, |\cdot|_0) \in X^{\mathrm{an}}$ where $|\cdot|_0$ is the trivial absolute value on the residue field $\kappa(x)$. This correspondence gives a section of j, which is denoted by $\sigma : X \to X^{\mathrm{an}}$.

Now we introduce an important subset of X^{an}. We assume that X is normal projective variety over $\operatorname{Spec} K$. Let $\eta \in X$ be the generic point of X and

$$X^{(1)} = \{x \in X \mid \operatorname{codim}_X \overline{\{x\}} = 1\}.$$

Let $K(X)$ be the function field of X. Firstly, for $x \in X^{(1)}$, we set

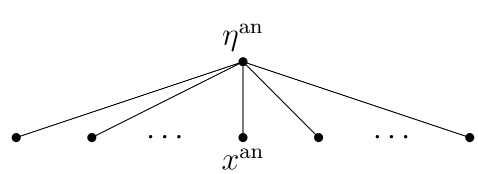
$$(\eta^{\mathrm{an}}, x^{\mathrm{an}}) := \left\{\xi \in X^{\mathrm{an}} \,\middle|\, \text{there is } t \in (0, +\infty) \text{ such that } j(\xi) = \eta, |\cdot|_\xi = e^{-t\operatorname{ord}_x(\cdot)} \text{ on } K(X)\right\}$$

and

$$[\eta^{\mathrm{an}}, x^{\mathrm{an}}] := \{\eta^{\mathrm{an}}\} \cup (\eta^{\mathrm{an}}, x^{\mathrm{an}}) \cup \{x^{\mathrm{an}}\}.$$

We often denote the above t by $t(\xi)$. Then the correspondence $\xi \mapsto t(\xi)$, $\eta^{\mathrm{an}} \mapsto 0$ and $x^{\mathrm{an}} \mapsto +\infty$ gives a homeomorphism from $(\eta^{\mathrm{an}}, x^{\mathrm{an}})$ (resp. $[\eta^{\mathrm{an}}, x^{\mathrm{an}}]$) to $(0, +\infty)$ (resp. $[0, +\infty]$). Hence we often identify $(\eta^{\mathrm{an}}, x^{\mathrm{an}})$ (resp. $[\eta^{\mathrm{an}}, x^{\mathrm{an}}]$) with $(0, +\infty)$ (resp. $[0, +\infty]$).

We set $X^{\mathrm{an}}_{\mathrm{div}} := \bigcup_{x \in X^{(1)}} [\eta^{\mathrm{an}}, x^{\mathrm{an}}]$. Then we can illustrate $X^{\mathrm{an}}_{\mathrm{div}}$ by an infinite tree as follows:



We remark that $X^{\mathrm{an}}_{\mathrm{div}} = X^{\mathrm{an}}$ if $\dim X = 1$.

Lemma 1.4.2. *$X^{\mathrm{an}}_{\mathrm{div}}$ is dense in X^{an}.*

Proof. For the proof, it is sufficient to show that, for any regular function f on a Zariski open set U in X and any $x \in U^{\mathrm{an}}$, the value $|f|(x)$ belongs to the closure W of $\{|f|(z) \mid z \in X^{\mathrm{an}}_{\mathrm{div}} \cap U^{\mathrm{an}}\} \subset \mathbb{R}_+$. If f has no pole on X, then f is regular on the whole X, so f is a constant function and algebraic over K because X is normal and projective. Therefore $|f|(z) = 1$ on X^{an}, so it is clear that $|f|(x) \in W$.

We next assume that f has poles on $X \setminus U$. In this case, there are $y, y' \in X^{(1)}$ such that $f(y) = 0$ and f has a pole at y' because X is normal. Then, $|f|(t) = e^{-at}$ for $t \in (\eta^{\mathrm{an}}, y^{\mathrm{an}})$, $|f|(t') = e^{a't'}$ for $t' \in (\eta^{\mathrm{an}}, y'^{\mathrm{an}})$ for some $a, a' > 0$ and $|f|(\eta^{\mathrm{an}}) = 1$, which implies that $W = \mathbb{R}_+$ and we complete the proof. □

Let $\mathbb{R}_{>0}$ be the multiplicative group of positive real numbers. There is an action of $\mathbb{R}_{>0}$ to X^{an}. For $r \in \mathbb{R}_{>0}$ and $x = (p, |\cdot|_x) \in X^{\mathrm{an}}$, we define

$$r^* x := (p, |\cdot|_x^r).$$

We also denote $r^* x$ by x^r. This action is called the *scaling action*. The scaling action is free faithful and preserves the subset $[\eta^{\mathrm{an}}, x^{\mathrm{an}}]$ for all $x \in X^{(1)}$.

Finally, we introduce the reduction map $\mathrm{red} : X^{\mathrm{an}} \to X$. For $x \in X^{\mathrm{an}}$, let $\widehat{\kappa}(x)$ be the completion of $\kappa(x)$ with respect to $|\cdot|_x$ and we denote the absolute value on $\widehat{\kappa}(x)$ by $|\cdot|_x$ by abuse of notation. We set $o_x := \{f \in \widehat{\kappa}(x) \mid |f|_x \leq 1\}$ and $m_x := \{f \in \widehat{\kappa}(x) \mid |f|_x < 1\}$. Then o_x is a local ring and m_x is the maximal ideal of o_x. If $|\cdot|_x$ is trivial on $\kappa(x)$, then $o_x = \kappa(x)$ and $m_x = \{0\}$. Let $p_x : \operatorname{Spec} \widehat{\kappa}(x) \to X$ be a K-morphism of schemes defined by $j(x)$, and $\iota_x : \operatorname{Spec} \widehat{\kappa}(x) \to \operatorname{Spec} o_x$ be a K-morphism defined by the inclusion $o_x \hookrightarrow \widehat{\kappa}(x)$. By the valuation criterion of properness (for instance, see [13]), there is a unique K-morphism $\phi_x : \operatorname{Spec} o_x \to X$ such that $p_x = \phi_x \circ \iota_x$:

$$\begin{array}{ccc} \operatorname{Spec} \widehat{\kappa}(x) & \xrightarrow{p_x} & X \\ \downarrow{\scriptstyle \iota_x} & \overset{\exists! \phi_x}{\nearrow} & \downarrow \\ \operatorname{Spec} o_x & \longrightarrow & \operatorname{Spec} K. \end{array}$$

Then we define $\mathrm{red}(x) \in X$ to be the image of m_x by ϕ_x. The map $\mathrm{red} : X^{\mathrm{an}} \to X$ defined by the above correspondence is called the *reduction map*. The morphism ϕ_x induces a homomorphism $\mathcal{O}_{X,\mathrm{red}(x)} \to o_x$. Hence we have

$$\forall f \in \mathcal{O}_{X,\mathrm{red}(x)}, \quad |f|_x \leq 1. \tag{1.4.1}$$

We remark that $j \neq \mathrm{red}$. For example, for any $x \in X$, $\mathrm{red}(x^{\mathrm{an}}) = x$ and for any $\xi \in (\eta^{\mathrm{an}}, x^{\mathrm{an}})$, $\mathrm{red}(\xi) = x$. It is known that $\mathrm{red} : X^{\mathrm{an}} \to X$ is anti-continuous, that is, for any open set U of X, $\mathrm{red}^{-1}(U)$ is closed in X^{an}.

Chapter 2

Adelic curve

2.1 Definition of adelic curves

Let K be a field and M_K be the set of all absolute values on K. Let $(\Omega, \mathcal{A}, \nu)$ be a measure space, where Ω is a set, $\mathcal{A}$ is a σ-algebra on Ω and ν is a measure on $(\Omega, \mathcal{A})$. Let $\phi : \Omega \to M_K$ be a map and we denote the image of $\omega \in \Omega$ by $|\cdot|_\omega \in M_K$. We assume that the function $\omega \mapsto \log |a|_\omega$ is $\mathcal{A}$-measurable and ν-integrable for all $a \in K^\times$.

Definition 2.1.1. The data $S = (K, (\Omega, \mathcal{A}, \nu), \phi)$ is called an *adelic curve*. Moreover, we say that S is *proper* if it satisfies a *product formula*:

$$\forall a \in K^\times, \quad \int_\Omega \log |a|_\omega \ \nu(\mathrm{d}\omega) = 0.$$

We set

$$\begin{aligned} \Omega_\infty &:= \{\omega \in \Omega \mid |\cdot|_\omega \text{ is Archimedean}\}, \\ \Omega_{\mathrm{fin}} &:= \{\omega \in \Omega \mid |\cdot|_\omega \text{ is non-Archimedean}\}, \\ \Omega_0 &:= \{\omega \in \Omega \mid |\cdot|_\omega \text{ is trivial}\}. \end{aligned}$$

Clearly, we have $\Omega = \Omega_\infty \cup \Omega_{\mathrm{fin}}$ and $\Omega_0 \subset \Omega_{\mathrm{fin}}$. We denote the completion of K with respect to $|\cdot|_\omega$ by K_ω. For any $\omega \in \Omega_\infty$, the field K_ω is isomorphic to $\mathbb{R}$ or $\mathbb{C}$. By Ostrowski's theorem (for example, see [33, Chapter II, Theorem 4.2]), there is a real number $\kappa(\omega) \in (0, 1]$ such that $|\cdot|_\omega = |\cdot|_\infty^{\kappa(\omega)}$ on $\mathbb{Q}$ where $|\cdot|_\infty$ is the usual absolute value on $\mathbb{Q}$. Then we can define a function $\kappa : \Omega \to (0, 1]$ by setting $\kappa(\omega) = 0$ for all $\omega \in \Omega_{\mathrm{fin}}$.

Proposition 2.1.2. *The function κ is $\mathcal{A}$-measurable and integrable with respect to ν. In particular, if $\inf_{\omega \in \Omega_\infty} \kappa(\omega) > 0$, then we have $\nu(\Omega_\infty) < \infty$.*

Proof. See [8, Proposition 3.1.2] for its proof. □

From now on, we always assume that $\kappa(\omega) = 1$ for all $\omega \in \Omega_\infty$, so $\nu(\Omega_\infty) < \infty$.
We see some examples of adelic curves.

Example 2.1.3 (Number fields)**.** Let K be a number field. We set

$$\Omega_{\mathbb{Q}} = \{|\cdot|_\infty\} \cup \{|\cdot|_p \,|\, p \text{ is a prime number}\},$$

where $|\cdot|_\infty$ is the usual Archimedean absolute value on $\mathbb{Q}$ and $|\cdot|_p$ is the p-adic absolute value on $\mathbb{Q}$ such that $|p|_p = 1/p$. Let Ω be the set of absolute values of K such that the restriction onto $\mathbb{Q}$ belongs to $\Omega_{\mathbb{Q}}$ and $\mathcal{A}$ be the discrete σ-algebra on Ω. Then there is a natural map $\phi : \Omega \to M_K$. Let ν be the measure on $(\Omega, \mathcal{A})$ such that $\nu(\{\omega\}) = [K_\omega : \mathbb{Q}_\omega]$. Then by the usual product formula on K, the adelic curve $(K, (\Omega, \mathcal{A}, \nu), \phi)$ is proper.

Example 2.1.4 (Function fields)**.** Let C be a regular projective curve over a field k and K be the rational function field of C. We define Ω as the set of all closed points of C equipped with the discrete σ-algebra $\mathcal{A}$. For $x \in \Omega$, let $\mathrm{ord}_x(\cdot) : \mathcal{O}_{C,x} \to \mathbb{Z} \cup \{\infty\}$ be the discrete valuation. We can uniquely extend this valuation onto K and let $|\cdot|_x$ be the absolute value defined by

$$\forall a \in K^\times, \quad |a|_x := e^{-\mathrm{ord}_x(a)}.$$

This gives a map $\phi : \Omega \to M_K$. Let ν be the measure on $(\Omega, \mathcal{A})$ such that $\nu(\{x\}) = [k(x) : k]$. By the residue formula, we have

$$\forall a \in K^\times, \quad \sum_{x \in \Omega} [k(x) : k] \mathrm{ord}_x(a) = 0,$$

which says that $(K, (\Omega, \mathcal{A}, \nu), \phi)$ is a proper adelic curve.

Example 2.1.5 (Copies of the trivial absolute value)**.** Let K be any field and $(\Omega, \mathcal{A}, \nu)$ be any measure space. We define the map ϕ such that $\Omega = \Omega_0$. Then we can easily see that $(K, (\Omega, \mathcal{A}, \nu), \phi)$ is a proper adelic curve. In particular, we say that $(K, (\Omega, \mathcal{A}, \nu), \phi)$ is a *trivially valued field* if Ω is a single set.

Example 2.1.6 (Finitely generated field over $\mathbb{Q}$)**.** For simplicity, we consider the case of a function field $\mathbb{Q}(T)$. For a general case, see [8, Section 3.2.6], [9, Chapter 2] and [32, Section 2].

Let $K = \mathbb{Q}(T)$ and we consider it as the field of all rational functions on $\mathbb{P}^1_{\mathbb{Q}}$. For any closed point x, we can define a discrete valuation $\mathrm{ord}_x(\cdot)$ on K. Let ∞ be the rational point of $\mathbb{P}^1_{\mathbb{Q}}$ such that

$$\mathrm{ord}_\infty(f/g) = \deg(f) - \deg(g)$$

for all polynomials $f, g \in \mathbb{Q}[T]$ and $g \neq 0$. Since $\mathbb{P}^1_{\mathbb{Q}} \setminus \{\infty\}$ is isomorphic to $\mathbb{A}^1_{\mathbb{Q}}$, we can associate a closed point $x \in \mathbb{P}^1_{\mathbb{Q}} \setminus \{\infty\}$ with an irreducible polynomial $F_x \in \mathbb{Z}[T]$ such that the coefficients of F_x are coprime. Let $H(x)$ be the Mahler measure of F_x which is defined by

$$H(x) := \exp\left(\int_0^1 \log |F_x(e^{2\pi t\sqrt{-1}})| \,\mathrm{d}t \right).$$

We fix a non-negative real number $\lambda \geq 0$. Then for $\varphi \in K$, we define an absolute value $|\cdot|_x$ on K as

$$|\varphi|_x := (\exp(\lambda \deg(F_x)) H(x))^{-\mathrm{ord}_x(\varphi)},$$

and

$$|\varphi|_\infty := \exp(\lambda)^{\deg(\varphi)}.$$

Let p is a prime number. For any polynomial

$$f = a_d T^d + \cdots + a_1 T + a_0 \in \mathbb{Q}[T],$$

we define an absolute value of f as $\max |a_i|_p$ where $|\cdot|_p$ is the p-adic absolute value on $\mathbb{Q}$ such that $|p|_p = 1/p$. By abuse of notation, we denote it by $|f|_p$. It is uniquely extended to an absolute value on K.

Let $[0,1]_* := \{t \in [0,1] \,|\, e^{2\pi t\sqrt{-1}} \text{ is transcendental}\}$. For any $t \in [0,1]_*$, we define an absolute value $|\cdot|_t$ on K as

$$|\varphi|_t := |\varphi(e^{2\pi t\sqrt{-1}})|$$

for all $\varphi \in K = \mathbb{Q}(T)$, where $|\cdot|$ is the usual absolute value on $\mathbb{C}$.

We set $\Omega_\lambda = \Omega_{\lambda,h} \coprod \mathcal{P} \coprod [0,1]_*$ where $\Omega_{\lambda,h}$ is the disjoint union of the set of all closed points of $\mathbb{P}_{\mathbb{Q}} \setminus \{\infty\}$ and $\{\infty\}$, and $\mathcal{P}$ is the set of all prime numbers. Let $\phi_\lambda : \Omega_\lambda \to M_K$ be a map such that $\phi(\omega) = |\cdot|_\omega$ for any $\omega \in \Omega_\lambda$. We equip $\Omega_{\lambda,h}$ and $\mathcal{P}$ with the discrete σ-algebras, $[0,1]_*$ with the restriction of the Borel σ-algebra on $[0,1]$, and Ω_λ with the σ-algebra $\mathcal{A}_\lambda$ generated by these σ-algebras. Let ν_λ be the measure on Ω_λ such that $\nu(\{x\}) = 1$ for $x \in \Omega_{\lambda,h}$ and $x \in \mathcal{P}$, and the restriction on $[0,1]_*$ coincides with the Lebesgue measure. Then the date $(K, (\Omega_\lambda, \mathcal{A}_\lambda, \nu_\lambda), \phi_\lambda)$ gives a proper adelic curve (for details, see [8, Section 3.2.5] and [32, Section 2]).

2.2 Base change

Here we briefly recall the notion of base change of adelic curves. for details, see [8, Section 3.3, 3.4].

Let $S = (K, (\Omega_K, \mathcal{A}_K, \nu_K), \phi_K)$ be an adelic curve and L be an extension field of K. For simplicity, we assume that L is a finite separable extension of K. For any $\omega \in \Omega_K$, let $M_{L,\omega}$ be the set of all absolute values on L which extends the absolute values $|\cdot|_\omega$ on K. Let Ω_L be the disjoint union

$$\coprod_{\omega \in \Omega_K} M_{L,\omega}.$$

Then we have a natural projection $\pi_{L/K} : \Omega_L \to \Omega_K$ which sends an element of $M_{L,\omega}$ to ω. Since $M_{L,\omega}$ is a subset of M_L, we naturally get the map $\phi_L : \Omega_L \to M_L$. We equip Ω_L with the σ-algebra $\mathcal{A}_L$ generated by $\pi_{L/K}$ and real-valued functions $\Omega_L \to \mathbb{R}, \omega \mapsto |a|_\omega$ for all $a \in L$.

We next construct a measure ν_L on $(\Omega_L, \mathcal{A}_L)$. We define the measure ν_L on $(\Omega_L, \mathcal{A}_L)$ as follows:

$$\forall A \in \mathcal{A}_L, \quad \nu_L(A) := \int_{\Omega_K} \sum_{x \in M_{L,\omega}} \frac{[L_x : K_\omega]}{[L : K]} 1_A(x)\ \nu_K(\mathrm{d}x),$$

where 1_A is the characteristic function of A. Then, $S_L = (L, (\Omega_L, \mathcal{A}_L, \nu_L), \phi_L)$ is an adelic curve and we also denote it by $S \otimes_K L$. Moreover, if S is proper, then so is S_L.

For a general algebraic extension L over K, let $\mathcal{E}_{L/K}$ be the set of finite extension fields of K in L. This set is ordered by the relation of inclusion and filtered. For any $K' \in \mathcal{E}_{L/K}$, we can construct the adelic curve $S \otimes_K K' = (K', (\Omega_{K'}, \mathcal{A}_{K'}, \nu_{K'}), \phi_{K'})$. By the same way in the case of finite extensions, we define the sets $M_{L,\omega}$ for each $\omega \in \Omega_K$,

$$\Omega_L := \coprod_{\omega \in \Omega_K} M_{L,\omega}$$

and the canonical inclusion $\phi_L : \Omega_L \to M_L$. For any $K' \in \mathcal{E}_{L/K}$, the restriction map gives the projection map $\pi_{L/K'} : \Omega_L \to \Omega_{K'}$. We can easily show that

$$\pi_{L/K'} = \pi_{K''/K'} \circ \pi_{L/K''}$$

where $K' \subset K'' \in \mathcal{E}_{L/K}$. Then Ω_L is identified with the projective limit of $\{\Omega_{K'}\}_{K' \in \mathcal{E}_{L/K}}$ in the category of sets. We equip Ω_L with the smallest σ-algebra $\mathcal{A}_L$ such that the maps $\pi_{L/K'}$ are measurable for all $K' \in \mathcal{E}_{L/K}$. Then $(\Omega_L, \mathcal{A}_L)$ is the projective limit of $\{(\Omega_{K'}, \mathcal{A}_{K'})\}_{K' \in \mathcal{E}_{L/K}}$ in the category of measurable spaces. By equipping $(\Omega_L, \mathcal{A}_L)$ with a suitable measure ν_L, we can define the adelic curve $S_L = (L, (\Omega_L, \mathcal{A}_L, \nu_L), \phi_L)$. But this process is very technical, we omit it (for details, see [8, Section 3.4] and [9, Chapter 2]).

2.3 Height functions

Let $S = (K, (\Omega, \mathcal{A}, \nu), \phi)$ be a proper adelic curve. Let $\overline{K}$ be an algebraic closure of K and $S \otimes_K \overline{K} = (\overline{K}, (\Omega_{\overline{K}}, \mathcal{A}_{\overline{K}}, \nu_{\overline{K}}), \phi_{\overline{K}})$.

Definition 2.3.1. We define the map $h_S : (\overline{K})^{n+1} \setminus \{(0, \ldots, 0)\} \to \mathbb{R}$ as follows:

$$\forall (a_0, \ldots, a_n) \in (\overline{K})^{n+1} \setminus \{(0, \ldots, 0)\}, \quad h_S(a_0, \ldots, a_n) := \int_{\Omega_{\overline{K}}} \log \max\{|a_0|_\chi, \ldots, |a_n|_\chi\} \, \nu_{\overline{K}}(\mathrm{d}\chi).$$

By the product formula on $S \otimes_K \overline{K}$, we have

$$\forall \lambda \in \overline{K}^\times, \quad h_S(\lambda a_0, \ldots, \lambda a_n) = h_S(a_0, \ldots, a_n).$$

Hence we can get the well-defined map $h_S : \mathbb{P}^n(\overline{K}) \to \mathbb{R}$. For $x \in \mathbb{P}^n(\overline{K})$, the value $h_S(x)$ is called the *height of x with respect to S*.

Example 2.3.2. Let $S = (\mathbb{Q}, \Omega, \nu)$ be the adelic curve in Example 2.1.3. For $(a_0, \ldots, a_n) \in$

$\mathbb{P}^n(\overline{\mathbb{Q}})$, we take a number field K such that it contains all a_i's. Then we have

$$\begin{aligned} h_S(a_0, \ldots, a_n) &= \int_{\Omega_{\overline{\mathbb{Q}}}} \log\max\{|a_0|_{\pi_{\overline{\mathbb{Q}}/K}(\chi)}, \ldots, |a_n|_{\pi_{\overline{\mathbb{Q}}/K}(\chi)}\}\ \nu_{\overline{\mathbb{Q}}}(\mathrm{d}\chi) \\ &= \int_{\Omega_K} \log\max\{|a_0|_x, \ldots, |a_n|_x\}\ \nu_K(\mathrm{d}x) \\ &= \int_{\Omega} \sum_{x\in M_{K,\omega}} \frac{[K_x : \mathbb{Q}_\omega]}{[K : \mathbb{Q}]} \log\max\{|a_0|_x, \ldots, |a_n|_x\}\ \nu(\mathrm{d}\omega) \\ &= \frac{1}{[K : \mathbb{Q}]} \sum_{x\in\Omega_K} [K_x : \mathbb{Q}_x] \log\max\{|a_0|_x, \ldots, |a_n|_x\}. \end{aligned}$$

Here we denote $[K_x : \mathbb{Q}_\omega]$ for all $\omega \in \Omega$ and $x \in M_{K.\omega}$ by $[K_x : \mathbb{Q}_x]$. Hence in this case, h_S coincides with the absolute logarithmic height on $\mathbb{P}^n(\overline{\mathbb{Q}})$.

By using this height function, we define an important class of adelic curves:

Definition 2.3.3. We say that S has the *Northcott property* if the set

$$\{a \in K \mid h_S(1 : a) \le C\}$$

is finite for any $C \ge 0$.

For instance, the adelic curves of Example 2.1.3 and the case of $\lambda > 0$ in Example 2.1.6 have the Northcott property (for details, see [30, 31, 32]).

2.4 Adelic vector bundles

Let $S = (K, (\Omega, \mathcal{A}, \nu), \phi)$ be an adelic curve.

Let V be a finite-dimensional vector space over K. We define a *norm family* ξ as a family $\{\|\cdot\|_\omega\}_{\omega\in\Omega}$ where $\|\cdot\|_\omega$ is norm on $V_{K_\omega} = V \otimes_K K_\omega$. A norm family ξ is said to be *ultrametric* if the norm $\|\cdot\|_\omega$ is ultrametric for any $\omega \in \Omega_{\mathrm{fin}}$. We define the *dual* of ξ as $\xi^\vee = \{\|\cdot\|_{\omega,*}\}_{\omega\in\Omega}$ where $\|\cdot\|_{\omega,*}$ is the dual norm of $\|\cdot\|_\omega$ on $(V_{K_\omega})^\vee = V^\vee \otimes_K K_\omega$ for any $\omega \in \Omega$.

Before defining an adelic vector bundle, we need some notions of metric families, due to Chen and Moriwaki (for details, see [8, Chapter 4]).

Definition 2.4.1. (1) A real-valued function f on Ω is said to be *upper dominated* if there exists a ν-integrable function $A(\cdot)$ on Ω such that $f(\omega) \le A(\omega)$ ν-almost everywhere. Similarly, we say that f is *lower dominated* if $-f$ is upper dominated. Finally, we say that f is *ν-dominated* if $|f|$ is upper dominated.

(2) A norm family ξ is called *upper dominated* (resp. *lower dominated*) if for any non-zero element $v \in V$, $\log\|v\|_\omega$ is upper dominated (resp. lower dominated).

(3) We say that ξ is *dominated* if ξ and $\xi^\vee$ are upper dominated.

Remark 2.4.2. If ξ is dominated, then ξ is upper and lower dominated. However, an upper and lower dominated metric family is *not* always dominated. For example, see [8, Remark 4.1.4].

Definition 2.4.3. We say that a norm family ξ is $\mathcal{A}$-*measurable* if for any non-zero element $v \in V$, $\log \|v\|_\omega$ is $\mathcal{A}$-measurable.

Now we define an adelic vector bundle.

Definition 2.4.4. Let V be a finite-dimensional vector space over K and ξ be a metric family. We say that a couple (V, ξ) is an *adelic vector bundle* on S if ξ and $\xi^\vee$ are $\mathcal{A}$-measurable and if ξ is dominated. Moreover if $\dim(V) = 1$, it is called an *adelic line bundle* on S.

Chapter 3

Arithmetic variety

In this section, we fix a proper adelic curve $S = (K, (\Omega, \mathcal{A}, \nu), \phi)$. For simplicity, we assume that S is a trivially valued field or $\Omega_0 = \emptyset$. Let X be a geometrically integral projective scheme over $\operatorname{Spec} K$. For each $\omega \in \Omega$, we set $X_\omega := X \times_{\operatorname{Spec} K} \operatorname{Spec} K_\omega$.

3.1 Adelic Cartier divisors

For a Cartier divisor D on X, we denote by D_ω the pull-back of D by the canonical morphism $X_\omega \to X$ for all $\omega \in \Omega$.

Definition 3.1.1. Let ω be an element of Ω.

(1) Let g_ω be a function on a dense open subset of X_ω^{an}. We say that g_ω is a *Green function* of D_ω if for any non-empty Zariski open subset U_ω of X_ω and any local equation f_ω of D_ω on U_ω, the function $g_\omega + \log |f_\omega|$ extends to a continuous function on U_ω^{an}.

(2) A *Green function family* g of D is a family $\{g_\omega\}_{\omega \in \Omega}$ where g_ω is a Green function of D_ω.

Example 3.1.2. (1) Let s be a non-zero rational function on X. Then we can consider s as a non-zero rational function on X_ω for any $\omega \in \Omega$. The family $\{-\log |s|_\omega\}_{\omega \in \Omega}$ gives a Green function family of the principal Cartier divisor (s), which is denoted by $-\log |s|$.

(2) Let 0 be the zero divisor on X. Then each element g_ω of a Green function family $\{g_\omega\}_{\omega \in \Omega}$ of 0 is a continuous function on X_ω^{an}.

For two Green function families $g = \{g_\omega\}_{\omega \in \Omega}$ and $g' = \{g'_\omega\}_{\omega \in \Omega}$ of D, we define the *local distance* of g and g' at $\omega \in \Omega$ as

$$d_\omega(g, g') := \sup_{x \in X_\omega^{\mathrm{an}}} |g_\omega - g'_\omega|(x).$$

Since X is projective, X_ω^{an} is compact for all $\omega \in \Omega$. Hence this value is well-defined.

Let V be a finite-dimensional vector space over K and $\xi = \{\|\cdot\|_\omega\}_{\omega\in\Omega}$ be a norm family of V. We assume that there is a surjective morphism $f : V \otimes \mathcal{O}_X \to \mathcal{O}_X(D)$. Then for each $\omega \in \Omega$, we obtain the norm $|\cdot|_\omega$ on $\mathcal{O}_X(D)_x$ induced by $(V_{K_\omega}, \|\cdot\|_\omega)$. Let s be the rational section of $\mathcal{O}_X(D)$ such that $D = \mathrm{div}(s)$. Then $g_{\overline{V},\omega} = -\log|s|_\omega$ gives a Green function of D_ω. The Green function family $g_{\overline{V}} = \{g_{\overline{V},\omega}\}_{\omega\in\Omega}$ is called the *quotient Green function family* of D induced by $\overline{V} = (V, \xi)$ and f.

Definition 3.1.3. (1) Let D be a very ample divisor on X. A Green function family g of D is said to be *dominated* if there exists a quotient Green function family $g_{\overline{V}}$ induced by a finite-dimensional vector space V with a dominated norm family, and a closed immersion $X \to \mathbb{P}(V)$ associated with D such that the local distance function

$$\Omega \ni \omega \mapsto d_\omega(g, g_{\overline{V}})$$

is ν-dominated.

(2) Let D be a Cartier divisor on X and g be a Green function family of D. We say that g is *dominated* if there exist two very ample divisors D_1 and D_2 together with dominated Green function families g_1 and g_2 of D_1 and D_2 , respectively, such that $D = D_1 - D_2$ and $g = g_1 - g_2$.

Remark 3.1.4. (1) If D is a very ample divisor on X, Definition 3.1.3 (1) and (2) are equivalent (for details, see [8, Remark 6.1.10]).

(2) Let D be a Cartier divisor on X and $g_{\overline{V}}$ be a quotient Green function family of D induced by a normed vector space $\overline{V} = (V, \xi)$. We assume that ξ is dominated. Then $g_{\overline{V}}$ is dominated in the sense of Definition 3.1.3 (for details, see [8, Proposition 6.1.11]).

Example 3.1.5. For any $s \in K(X)^\times$, the Green function family $-\log|s|$ of (s) is dominated. In fact, let V be a vector space of dimension one over K. Then the norm family $\{|\cdot|_\omega\}_{\omega\in\Omega}$ of V is dominated, and $-\log|s|$ is given by $(V, \{|\cdot|_\omega\}_{\omega\in\Omega})$ and $V \otimes \mathcal{O}_X \cong \mathcal{O}_X \to \mathcal{O}_X$.

Next, we define the notion of the measurability of Green function families. Here we use the notation of Section 2.2. Let g be a Green function family of D. Let $P \in X(\overline{K})$ be a closed point of X outside of $\mathrm{Supp}(D)$. Then we can represent P as a K-morphism $P : \mathrm{Spec}\, L \to X$ for some finite extension L of K. Let $S \otimes_K L = (L, (\Omega_L, \mathcal{A}_L, \nu_L), \phi_L)$. For each $\omega \in \Omega$, the Berkovich analytification of $\mathrm{Spec}\, L$ with respect to ω is identified with $M_{L,\omega}$ as a set. Since $\Omega_L = \coprod_{\omega\in\Omega} M_{L,\omega}$, we can consider the pull-back of g by P as a function on Ω_L. We denote this function by $g_L(P)(\cdot)$, or simply $g(P)(\cdot)$. If $\mathrm{Spec}\, L' \to X$ is another representation of P where L' is a finite extension of L, we have

$$g_{L'}(P) = g_L(P) \circ \pi_{L'/L}. \tag{3.1.1}$$

Definition 3.1.6. We say that a Green function family $g = \{g_\omega\}_{\omega\in\Omega}$ of D is *measurable* if the function $g(P) - \log|s|(P)$ is measurable with respect to $\mathcal{A}$ for all $s \in K(X)^\times$ and $P \in X(\overline{K})$ outside of $\mathrm{Supp}(D + (s))$.

By the equation (3.1.1), the above definition does not depend on representations of P. This definition is a little strange because it is defined by using the notion of Green function families. Usually, the measurability is defined by using the metric family of the invertible sheaf associated with the Green function family (for details, see [8, Section 6.1.4 and 6.2.3]).

Example 3.1.7. Let $s \in K(X)^\times$ be a non-zero rational function on X. For $P \in X(\overline{K})$ outside of $\mathrm{Supp}(s)$, the function $-\log|s|(P)$ is measurable. In fact, let $\mathrm{Spec}\, L \to X$ be a representation of P. Then we can consider $s(P)$ is a non-zero element of L. Hence by the definition of adelic curves, the function $\Omega_L \ni \omega \mapsto -\log|s|(P)(\omega) = -\log|s(P)|_\omega$ is measurable. In particular, $-\log|s|$ is a measurable Green function family of (s).

Remark 3.1.8. (1) Let D be a Cartier divisor on X and $g_{\overline{V}}$ be a quotient Green function family of D induced by a normed vector space $\overline{V} = (V, \xi)$. We assume that the σ-algebra $\mathcal{A}$ is discrete or K contains a countable subfield K_0 such that K_0 is dense in K_ω for all $\omega \in \Omega$. Then if ξ is $\mathcal{A}$-measurable, $g_{\overline{V}}$ is measurable in the sense of Definition 3.1.6 (for details, see [8, Proposition 6.1.30]).

(2) If $\Omega_0 \neq \emptyset$, we need a further condition to define the measurability of Green function families (for details, see [8, Section 6.1.3 and 6.1.4]).

Now we can define adelic Cartier divisors:

Definition 3.1.9. Let D be a Cartier divisor on X and g be a Green function family of D. We say that a pair $\overline{D} = (D, g)$ is an *adelic Cartier divisor* on X if g is dominated and measurable. The set of adelic Cartier divisors forms an abelian group, which is denoted by $\widehat{\mathrm{Div}}(X)$.

Example 3.1.10. Let $s \in K(X)^\times$ be a non-zero rational function on X. By Example 3.1.5 and Example 3.1.7, the pair $\widehat{(s)} = ((s), -\log|s|)$ is an adelic Cartier divisor on X, which is called a *principal adelic Cartier divisor* on X. Two adelic Cartier divisors $\overline{D}_1$ and $\overline{D}_2$ are said to be *linearly equivalent* if $\overline{D}_1 - \overline{D}_2 = \widehat{(s)}$ for some $s \in K(X)^\times$.

Next we consider the extension of scalars. Let $C^0_G(X)$ be the set of all Green function families of the trivial Cartier divisor on X. Then $C^0_G(X)$ naturally has an $\mathbb{R}$-vector space structure. Let $\mathbb{K}$ denote either $\mathbb{Q}$ or $\mathbb{R}$.

Definition 3.1.11. Let $\widehat{\mathrm{Div}}(X)_{\mathbb{K}}$ be the set $\widehat{\mathrm{Div}}(X) \otimes_{\mathbb{Z}} \mathbb{K}$ modulo the vector space generated by the following elements:

$$(0, g_1) \otimes \lambda_1 + \cdots + (0, g_n) \otimes \lambda_n - (0, \lambda_1 g_1 + \cdots + \lambda_n g_n),$$

for all $n \geq 1$ and $g_i \in C^0_G(X)$. Then $\widehat{\mathrm{Div}}(X)_{\mathbb{K}}$ forms a $\mathbb{K}$-vector space. An element of $\widehat{\mathrm{Div}}_{\mathbb{K}}(X)$ is called an *adelic* $\mathbb{K}$-*Cartier divisor*.

3.2 Height functions

Let $\overline{D} = (D, g = \{g_\omega\}_{\omega\in\Omega})$ be an adelic Cartier divisor. Let $P \in X(\overline{K})$ and $\operatorname{Spec} L \to X$ be a representation of P. We assume that $P \notin \operatorname{Supp}(D)$. Since g is dominated and measurable, $g_L(P)(\cdot)$ gives an ν_L-integrable function on Ω_L. Hence we define the *height* of P with respect to $\overline{D}$ as

$$h_{\overline{D}}(P) := \int_{\Omega_L} g_L(P)(x)\ \nu_L(\mathrm{d}x).$$

By the equation (3.1.1), the height of P does not depend on the choice of L.

Lemma 3.2.1. *Let $s \in K(X)^\times$ and $P \in X(\overline{K}) \setminus \operatorname{Supp}(s)$. Then we have $h_{\widehat{(s)}}(P) = 0$.*

Proof. It follows from the properness of S. □

Definition 3.2.2. For $P \in X(\overline{K})$, we define the *height* of P with respect to $\overline{D}$ as

$$h_{\overline{D}}(P) := h_{\overline{D}+\widehat{(s)}}(P)$$

for some $s \in K(X)^\times$ with $P \notin \operatorname{Supp}(D + (s))$. By Lemma 3.2.1, this definition does not depend on the choice of s.

This height function has the following properties:

Proposition 3.2.3. *Let $\overline{D} = (D, g)$ and $\overline{D}' = (D', g')$ be adelic Cartier divisors on X.*

(1) For any $P \in X(\overline{K})$, we have $h_{\overline{D}+\overline{D}'}(P) = h_{\overline{D}}(P) + h_{\overline{D}'}(P)$.

(2) If $D = D'$, then we have $h_{(D,g')} = h_{(D,g)} + O(1)$, where $O(1)$ is a bounded function on $X(\overline{K})$.

Proof. See [8, Proposition 6.2.2] for their proofs. □

Example 3.2.4. Let $\mathbb{P}^n = \operatorname{Proj} K[T_0, \dots, T_n]$ be the n-dimensional projective space. We set $z_i = T_i/T_0$ for $i = 0, \dots, n$ and $H = \{T_0 = 0\}$. For each $\omega \in \Omega$, let

$$g_{0,\omega} = \log\max\{1, |z_1|_\omega, \dots, |z_n|_\omega\}.$$

Then $(H, g_0 = \{g_{0,\omega}\}_{\omega\in\Omega})$ is an adelic Cartier divisor on $\mathbb{P}^n$ and $h_S = h_{(H,g_0)}$.

Let D be a very ample divisor on X and $\pi : X \to \mathbb{P}^n$ be a closed immersion associated with $\mathcal{O}_X(D)$. We have $D = \pi^*H + (s)$ for some $s \in K(X)^\times$ and define the Green function family g of D as $\pi^*g_0 - \log|s|$. Then we obtain that

$$h_{(D,g)} = h_{\pi^*H+(s),\pi^*g_0-\log|s|} = h_S \circ \pi,$$

which gives the usual height function on X.

For $\mathbb{K} = \mathbb{Q}$ or $\mathbb{R}$, we define the height function $h_{\overline{D}}$ of an adelic $\mathbb{K}$-Cartier divisor $\overline{D}$ as follows: We write $\overline{D} = a_1\overline{D}_1 + \cdots + a_n\overline{D}_n$ for some $a_i \in \mathbb{K}$ and $\overline{D}_i \in \widehat{\mathrm{Div}}(X)$ for $i = 1, \ldots, n$. Then we set

$$h_{\overline{D}} := a_1 h_{\overline{D}_1} + \cdots + a_n h_{\overline{D}_n}.$$

By Proposition 3.2.3, it is well-defined.

For Chapter 5, we need several propositions:

Proposition 3.2.5. *Let $\overline{D}$ be an adelic Cartier divisor on X. Let $s \in H^0(X, D) \setminus \{0\}$. Then there is a real number C such that $h_{\overline{D}}(P) > C$ for all $P \in X(\overline{K}) \setminus \mathrm{Supp}(D + (s))$.*

Proof. See [8, Proposition 6.2.6] for its proof. □

Corollary 3.2.6. *Let $\overline{D}$ be an adelic Cartier divisor on X whose underlying Cartier divisor D is ample. Then there exists a constant C such that $h_{\overline{D}}(P) > C$ for all $P \in X(\overline{K})$.*

Proof. For any positive integer $n > 0$, we have $h_{n\overline{D}} = nh_{\overline{D}}$. Hence by replacing D by nD, we can assume that the linear system $|D|$ is base point free. Let $\{s_1, \ldots, s_m\}$ be a basis of $H^0(X, D)$. For each $i = 1, \ldots, m$, let C_i be a real number in Proposition 3.2.5 for s_i. For any $P \in X(\overline{K})$, we have $P \notin \mathrm{Supp}(s_i)$ for some i and $h_{\overline{D}}(P) > C_i$. Hence the constant C is given by $C = \min\{C_1, \ldots, C_m\}$ for example. □

Corollary 3.2.7. *Let $\overline{D}$ and $\overline{E}$ be adelic Cartier divisors on X such that D is ample and $h_{\overline{D}} \geq 1$. Then there is a constant $C > 0$ such that*

$$\forall P \in X(\overline{K}), \quad h_{\overline{E}}(P) \leq Ch_{\overline{D}}(P).$$

Proof. Since D is ample, for sufficiently large $n > 0$, the Cartier divisor $nD - E$ is ample. By Corollary 3.2.6, there is a Green function family g of $nD - E$ such that $h_{(nD-E,g)} > 0$. For any $P \in X(\overline{K})$, we have

$$\begin{aligned} h_{\overline{E}}(P) &= h_{n\overline{D}}(P) - h_{(nD-E,g)}(P) + O(1) \\ &< nh_{\overline{D}}(P) + O(1). \end{aligned}$$

Hence for sufficiently large $C > 0$, we get the conclusion. □

Proposition 3.2.8. *Let $\overline{D}$ be an adelic Cartier divisor on X such that D is ample. We assume that S has the Northcott property. Then for all positive real numbers δ and C, the set*

$$\{P \in X(\overline{K}) \,|\, h_{\overline{D}}(P) \leq C, [K(P) : K] \leq \delta\}$$

is finite.

Proof. See [8, Proposition 6.2.3] for its proof. □

3.3 Canonical compactification of adelic Cartier divisors

In this subsection, we assume that either the σ-algebra $\mathcal{A}$ is discrete or there is a countable subfield K_0 of K which is dense in K_ω for all $\omega \in \Omega$.

Let $\pi : X \to X$ be a surjective morphism. We assume that there is a Cartier divisor on X, a positive integer $d > 1$ and $s \in K(X)^\times$ such that $\pi^* D = dD + (s)$. By [8, Proposition 2.5.11], there exists a Green function g_ω of D_ω such that $\pi^* g = dg - \log |s|_\omega$ for all $\omega \in \Omega$. Moreover, the Green family $g = \{g_\omega\}_{\omega\in\Omega}$ of D is measurable and dominated by [8, Proposition 6.2.19].

Definition 3.3.1. We say that an adelic Cartier divisor $\overline{D} = (D, g)$ is the *canonical compactification* of D with respect to π if it satisfies $\pi^*(\overline{D}) = d\overline{D} + \widehat{(s)}$.

Remark 3.3.2. By considering the construction carefully, the notion of canonical compactification can be extended to an adelic $\mathbb{R}$-Cartier divisor and a positive real number $d > 1$.

As an example of canonical compactifications, we consider Néron-Tate theory, that is, the case of abelian varieties. Let A be an abelian variety over $\operatorname{Spec} K$. For an integer n, we denote by $[n]$ the multiplication morphism by n. Let a be a positive integer such that $a > 1$ and a is not divisible by the characteristic of K. We remark that $[a] : A \to A$ is surjective. Let D be a symmetric divisor on A (which means that $[-1]^* D \sim D$). Since $[a]^* D \sim a^2 D$, there is $s \in K(A)^\times$ such that $[a]^* D = a^2 D + (s)$. Hence there exists a Green function family g of D such that $\overline{D} = (D, g)$ is the canonical compactification of D with respect to $[a]$. We set $\hat{h}_D = h_{\overline{D}}$. It is the Néron-Tate height on the setting over an adelic curve and it has the following property:

Proposition 3.3.3. *(1) For any integer n, we have*

$$\forall P \in A(\overline{K}), \quad \hat{h}_D([n]P) = n^2 \hat{h}_D(P).$$

(2) For any $P, Q \in A(\overline{K})$, we have

$$\hat{h}_D(P + Q) + \hat{h}_D(P - Q) = 2\hat{h}_D(P) + 2\hat{h}_D(Q).$$

The height $\hat{h}_D : A(\overline{K}) \to \mathbb{R}$ is a quadratic form. The associated pairing $\langle \cdot, \cdot \rangle_D : A(\overline{K}) \times A(\overline{K}) \to \mathbb{R}$ is defined by

$$\forall P, Q \in A(\overline{K}), \quad \langle P, Q \rangle_D := \frac{\hat{h}_D(P + Q) - \hat{h}_D(P) - \hat{h}_D(Q)}{2},$$

and it is bilinear and $\langle P, P \rangle_D = \hat{h}_D(P)$.

Proof. We can prove the assertions in a similar way as in [14, Theorem B.5.1]. □

Corollary 3.3.4. *Let X be a smooth projective variety over* $\operatorname{Spec} K$*. Let $\overline{D}$ and $\overline{E}$ be adelic Cartier divisors on X such that D is ample, $h_{\overline{D}} \geq 1$ and E is numerically equivalent to 0. Then there is a constant $C > 0$ such that*

$$\forall P \in X(\overline{K}), \quad h_{\overline{E}}(P) \leq C\sqrt{h_{\overline{D}}(P)}.$$

Proof. By [25, Theorem 4], nE is algebraically equivalent to 0 for some positive integer n. Then we can prove the assertion in a similar way as in [14, Theorem B.5.9]. □

Remark 3.3.5. In Corollary 3.3.4, we can drop the assumption of smoothness of X (for details, see [26, Appendix B]).

Chapter 4

Arakelov geometry over a trivially valued field

In this section, we study some topics in Arakelov geometry over a trivially valued field. Here, we introduce the arithmetic volume function and the bigness of adelic $\mathbb{R}$-Cartier divisors, and study their properties. Throughout this section, let K be a trivially valued field and X be a normal projective variety over $\operatorname{Spec} K$.

4.1 Adelic $\mathbb{R}$-Cartier divisors

Let $\mathbb{K} = \mathbb{Q}, \mathbb{R}$ or $\mathbb{Z}$. Let D be a $\mathbb{K}$-Cartier divisor on X. In the trivially valued field case, a Green function family of D consists of only one function, so we call it Green function simply. Let $\widehat{\operatorname{Pic}}(X)$ be $\widehat{\operatorname{Div}}(X)$ modulo linearly equivalence and it is called the *arithmetic Picard group*. An adelic $\mathbb{K}$-Cartier divisor (D, g) is *effective* if D is effective and g is a non-negative. We denote it by $(D, g) \geq 0$.

Proposition 4.1.1 (c.f. [6, Proposition 2.6])**.** *Let (D, g) be an effective adelic $\mathbb{R}$-Cartier divisor on X. Then the function e^{-g} extends to a non-negative continuous function on X^{an}.*

Proof. Let U be a non-empty Zariski open subset of X and f be a local equation of D on U. Since $g + \log|f|$ extends a continuous function on U^{an}, $e^{-g} = |f| \cdot e^{-(g+\log|f|)}$ extends a non-negative continuous function on U^{an}. We remark that $|f|$ is a continuous function on U^{an} because D is effective. By gluing continuous functions, e^{-g} extends a non-negative continuous function on X^{an}. □

By Proposition 4.1.1, we often consider a Green function of an effective $\mathbb{R}$-Cartier divisor as a map $X^{\mathrm{an}} \to \mathbb{R} \cup \{+\infty\}$.

Let $\overline{D} = (D, g)$ be an adelic $\mathbb{R}$-Cartier divisor on X. Then the set of "global sections" $H^0(D)$ is given by

$$H^0(D) = \{f \in K(X)^\times \mid D + (f) \geq 0\} \cup \{0\}.$$

Let s be a non-zero element of $H^0(D)$. By Proposition 4.1.1, the function $|s|e^{-g} = e^{-g+\log|s|}$ extends to a non-negative function on X^{an}. We denote this function by $|s|_g : X^{\mathrm{an}} \to \mathbb{R}_+$. Then we define

$$\|s\|_g := \sup_{x \in X^{\mathrm{an}}} |s|_g(x).$$

Note that $\|s\|_g$ exists since X^{an} is compact. The map $\|\cdot\|_g : H^0(D) \to \mathbb{R}_+$ gives an ultrametric norm on $H^0(D)$ over K and it coincides with the supremum norm induced by the continuous metric on $\mathcal{O}_X(D)$ corresponding to g. Moreover, by definition, it is easy to see that $\bigoplus_{n=0}^{\infty}(H^0(nD), \|\cdot\|_{ng})$ is a normed graded ring over K.

Let us recall the notations in Section 1.3: for a normed vector space $(V, \|\cdot\|)$,

$$\begin{aligned} \mathcal{F}^t(V, \|\cdot\|) &= \{v \in V \mid \|v\| \le e^{-t}\} \quad \text{for } t \in \mathbb{R}, \\ \lambda_{\max}(V, \|\cdot\|) &= \sup\{t \in \mathbb{R} \mid \mathcal{F}^t(V, \|\cdot\|) \ne \{0\}\}, \\ \lambda_{\min}(V, \|\cdot\|) &= \sup\{t \in \mathbb{R} \mid \mathcal{F}^t(V, \|\cdot\|) = V\}. \end{aligned}$$

We set

$$\begin{aligned} \lambda_{\max}(D, g) &:= \lambda_{\max}(H^0(D), \|\cdot\|_g), \\ \lambda_{\min}(D, g) &:= \lambda_{\min}(H^0(D), \|\cdot\|_g), \end{aligned}$$

and

$$\lambda_{\max}^{\mathrm{asy}}(D, g) := \limsup_{n \to +\infty} \frac{1}{n} \lambda_{\max}(nD, ng).$$

Since $\bigoplus_{n=0}^{\infty}(H^0(nD), \|\cdot\|_{ng})$ is a normed graded ring, the sequence $\{\lambda_{\max}(nD, ng)\}_n$ is super-additive, that is,

$$\lambda_{\max}((m+n)D, (m+n)g) \ge \lambda_{\max}(mD, mg) + \lambda_{\max}(nD, ng) \quad \text{for } \forall m, n \in \mathbb{Z}_+.$$

Hence by Fekete's lemma, we have

$$\lambda_{\max}^{\mathrm{asy}}(D, g) = \lim_{n \to +\infty} \frac{1}{n} \lambda_{\max}(nD, ng) = \sup_{n \ge 1} \frac{1}{n} \lambda_{\max}(nD, ng).$$

Later, we will show that $\lambda_{\max}^{\mathrm{asy}}(D, g) < +\infty$.

Definition 4.1.2. Let (D, g) be an adelic $\mathbb{R}$-Cartier divisor. We say that a non-zero global section $s \in H^0(D) \setminus \{0\}$ is a *small section* if $\|s\|_g \le 1$ or equivalently $s \in \mathcal{F}^0(H^0(D), \|\cdot\|_g)$. Moreover, if $\|s\| < 1$, it is called a *strictly small section*.

Proposition 4.1.3. *Let $\overline{D} = (D, g)$ be an adelic $\mathbb{R}$-Cartier divisor on X. Then we have*

$$\mathcal{F}^0(H^0(D), \|\cdot\|_g) = \left\{ s \in K(X)^\times \,\middle|\, \overline{D} + \widehat{(s)} \ge 0 \right\} \cup \{0\}.$$

Proof. Let $s \in H^0(D) \setminus \{0\}$. By definition,

$$\begin{aligned} \|s\|_g \le 1 &\Leftrightarrow e^{-g+\log|s|} \le 1 \text{ on } X^{\mathrm{an}} \\ &\Leftrightarrow g - \log|s| \ge 0 \text{ on } X^{\mathrm{an}}, \end{aligned}$$

as required. □

Small sections play a similar role as global sections in algebraic geometry. Therefore we are interested in the asymptotic behavior of $\mathcal{F}^0(H^0(D), \|\cdot\|_{ng})$ as $n \to +\infty$.

4.2 Associated ℝ-Weil divisors

In this section, we use the notations in Section1.4.

Definition 4.2.1. Let (D, g) be an adelic $\mathbb{R}$-Cartier divisor on X. For any $x \in X^{(1)}$,

$$\mu_x(g) := \inf_{\xi \in (\eta^{\mathrm{an}}, x^{\mathrm{an}})} \frac{g(\xi)}{t(\xi)} \in \mathbb{R} \cup \{-\infty\}.$$

Clearly $\mu_x(g) \geq 0$ if and only if $g \geq 0$ on $(\eta^{\mathrm{an}}, x^{\mathrm{an}})$. Moreover $\mu_x(g) = -\infty$ if and only if $g(\eta^{\mathrm{an}}) < 0$, which implies that if $\mu_x(g) = -\infty$ for some $x \in X^{(1)}$, then $\mu_x(g) = -\infty$ for every $x \in X^{(1)}$.

The above invariant $\mu_x(g)$ has following properties:

Proposition 4.2.2 (c.f. [6, Proposition 5.7])**.** *Let (D, g) be an adelic $\mathbb{R}$-Cartier divisor on X. For all but finitely many $x \in X^{(1)}$, we have $\mu_x(g) \leq 0$.*

Proof. Let U be a non-empty Zariski open subset of X such that g is a continuous function on U^{an}. Then g is continuous on $[\eta^{\mathrm{an}}, x^{\mathrm{an}}]$ for all $x \in U \cap X^{(1)}$. Since $[\eta^{\mathrm{an}}, x^{\mathrm{an}}]$ is compact, $g|_{[\eta^{\mathrm{an}}, x^{\mathrm{an}}]}$ is bounded above. Hence we have $\mu_x(g) \leq 0$ for for all $x \in U \cap X^{(1)}$, which implies the assertion because $X^{(1)} \setminus U$ is a finite set. □

Proposition 4.2.3 (c.f. [6, Lemma 5.8])**.** *Let (D, g) be an adelic $\mathbb{R}$-Cartier divisor on X and $x \in X^{(1)}$.*

(1) For any $s \in K(X)^{\times}_{\mathbb{R}}$, we have

$$\mu_x(g - \log|s|) = \mu_x(g) + \mathrm{ord}_x(s).$$

(2) We have $\mu_x(g) \leq \mathrm{ord}_x(D)$.

Proof. (1) It follows from the definition of $X^{\mathrm{an}}_{\mathrm{div}}$ that

$$-\log|s|(\xi) = t(\xi)\mathrm{ord}_x(s), \quad \xi \in (\eta^{\mathrm{an}}, x^{\mathrm{an}})$$

for all $s \in K(X)^{\times}_{\mathbb{R}}$. Hence we obtain that

$$\mu_x(g - \log|s|) = \inf_{\xi \in (\eta^{\mathrm{an}}, x^{\mathrm{an}})} \frac{g(\xi) - \log|s|(\xi)}{t(\xi)} = \mu_x(g) + \mathrm{ord}_x(s).$$

(2) Let $f \in K(X)^{\times}_{\mathbb{R}}$ be a local equation of D around x. Then $g + \log|f|$ extends to a continuous function on $[\eta^{\mathrm{an}}, x^{\mathrm{an}}]$. Since $(g + \log|f|)|_{[\eta^{\mathrm{an}}, x^{\mathrm{an}}]}$ is bounded above, we have $\mu_x(g + \log|f|) \leq 0$. Hence by (1), we get $\mu_x(g) \leq \mathrm{ord}_x(D)$. □

Now we introduce an important divisor.

Definition 4.2.4. Let (D, g) be an adelic $\mathbb{R}$-Cartier divisor on X. We say that (D, g) is *μ-finite* if $\mu_x(g) = 0$ for all but finitely many $x \in X^{(1)}$, which is equivalent to $\mu_x(g) \geq 0$ for all but finitely many $x \in X^{(1)}$ by Proposition 4.2.2. If (D, g) is μ-finite, it follows from definition that $\mu_x(g) \in \mathbb{R}$ for all $x \in X^{(1)}$. Hence we can define an $\mathbb{R}$-Weil divisor on X as follows:

$$D_{\mu(g)} := \sum_{x \in X^{(1)}} \mu_x(g)[x].$$

It is called an $\mathbb{R}$-*Weil divisor associated with* (D, g). Note that $D_{\mu(g)}$ may not be an $\mathbb{R}$-Cartier divisor.

For example, if (D, g) has a Dirichlet property (which means that (D, g) is $\mathbb{R}$-linearly equivalent to an effective adelic $\mathbb{R}$-Cartier divisor), then (D, g) is μ-finite.

By Proposition 4.2.3, we have $D_{\mu(g)} \leq D$ and

$$(D + (s))_{\mu(g - \log|s|)} = D_{\mu(g)} + (s). \tag{4.2.1}$$

Proposition 4.2.5. *Let (D, g) be a μ-finite adelic $\mathbb{R}$-Cartier divisor on X. Then (D, g) is effective if and only if $D_{\mu(g)}$ is effective.*

Proof. We first assume that (D, g) is effective. Then g is non-negative on X^{an}, so $\mu_x(g) \geq 0$ for any $x \in X^{(1)}$, which implies $D_{\mu(g)}$ is effective.

Conversely we suppose that $D_{\mu(g)}$ is effective. Then g is non-negative on $X^{\mathrm{an}}_{\mathrm{div}}$, but $X^{\mathrm{an}}_{\mathrm{div}}$ is dense in X^{an} by Lemma 1.4.2, so it follows that g is non-negative on the whole X^{an}. Moreover by Proposition 4.2.3, we have

$$\mathrm{ord}_x(D) \geq \mu_x(g) \geq 0$$

for any $x \in X^{(1)}$, which completes the proof. $\square$

By the above proposition and the equation (4.2.1), we have the following corollary:

Corollary 4.2.6. $H^0(D_{\mu(g)}) = \mathcal{F}^0(H^0(D), \|\cdot\|_g) = \{s \in H^0(D) \mid \|s\|_g \leq 1\}$.

4.3 Canonical Green functions

For any $\mathbb{R}$-Cartier divisor D, we can naturally give a Green function of D as follows: For any $x \in X^{\mathrm{an}}$, let $f \in K(X)^{\times}_{\mathbb{R}}$ be a local equation of D around $\mathrm{red}(x) \in X$. Then we define

$$g^{\mathrm{c}}_D(x) := -\log|f|_x.$$

This definition is independent of the choice of a local equation. In fact, let $f' \in K(X)^{\times}_{\mathbb{R}}$ be another local equation. Then there is an element $a \in (\mathcal{O}_{X,\mathrm{red}(x)})^{\times}_{\mathbb{R}}$ such that $f' = af$. Since $|a|_x = 1$ by (1.4.1), we have $-\log|f'|_x = -\log|f|_x$.

Proposition 4.3.1. *The function g^{c}_D is a Green function of D.*

Proof. It is enough to show that for any non-empty Zariski open subset U of X and local equation f of D on U, $g^{\mathrm{c}}_D + \log|f|$ extends to a continuous function on U^{an}. Let $x \in U^{\mathrm{an}}$. If $\mathrm{red}(x) \in U$, then $g^{\mathrm{c}}_D(x) = -\log|f|_x$. Hence we have $g^{\mathrm{c}}_D(x) + \log|f|_x = 0$. Next, we assume that $\mathrm{red}(x) \notin U$. Let U' be a non-empty Zariski open neighborhood of $\mathrm{red}(x)$ and f' be a local equation of D on U'. Then we have $g^{\mathrm{c}}_D(x) = -\log|f'|_x$ We remark that $j(x) \in U'$, hence $U \cap U' \neq \emptyset$. There is a non-zero regular function $u \in (\mathcal{O}_X(U \cap U'))^{\times}_{\mathbb{R}}$ such that $f' = uf$ on $U \cap U'$. Therefore we obtain that $g^{\mathrm{c}}_D(x) + \log|f|_x = -\log|u|_x$, which is continuous on $U^{\mathrm{an}} \cap U'^{\mathrm{an}}$. Finally, let $y \in U^{\mathrm{an}} \cap U'^{\mathrm{an}}$ such that $\mathrm{red}(y) \in U$. Since $u \in (\mathcal{O}_{X,\mathrm{red}(y)})^{\times}_{\mathbb{R}}$, we have $|u|_y = 1$ by (1.4.1). Hence $g^{\mathrm{c}}_D(y) + \log|f|_y = -\log|u|_y = 0$, which completes the proof. □

Remark 4.3.2. Proposition 4.3.1 also prove the existence of Green functions of Cartier divisors in the case of a trivially valued field.

Definition 4.3.3. The function g^{c}_D is called the *canonical Green function of D.*

Proposition 4.3.4. *(1) For any $s \in K(X)^{\times}_{\mathbb{R}}$, $g^{\mathrm{c}}_{(s)} = -\log|s|$.*

(2) For any $D, D' \in \mathrm{Div}(X)_{\mathbb{R}}$ and $a, a' \in \mathbb{R}$, $g^{\mathrm{c}}_{aD+a'D'} = ag^{\mathrm{c}}_D + a'g^{\mathrm{c}}_{D'}$.

Proof. (1) Since (s) is globally defined by s, it follows from the definition of the canonical Green function.

(2) Let $x \in X^{\mathrm{an}}$ and f, f' be local equations of D, D' around $\mathrm{red}(x)$ respectively. Then $f^a f'^{a'}$ is a local equation of $aD + a'D'$ around $\mathrm{red}(x)$. Hence we have

$$g^{\mathrm{c}}_{aD+a'D'}(x) = -\log|f^a f'^{a'}|_x = -a\log|f|_x - a'\log|f'|_x = ag^{\mathrm{c}}_D(x) + a'g^{\mathrm{c}}_{D'}(x).$$

□

Using the canonical Green function, we can define the following injective homomorphism:

$$\varphi : \mathrm{Div}(X) \to \widehat{\mathrm{Div}}(X), \quad D \mapsto (D, g^{\mathrm{c}}_D).$$

By Proposition 4.3.4, it induces an injective homomorphism $\overline{\phi} : \mathrm{Pic}(X) \to \widehat{\mathrm{Pic}}(X)$ such that the following diagram is commutative:

$$\begin{array}{ccc} \mathrm{Div}(X) & \xrightarrow{\varphi} & \widehat{\mathrm{Div}}(X) \\ \downarrow & & \downarrow \\ \mathrm{Pic}(X) & \xrightarrow{\overline{\varphi}} & \widehat{\mathrm{Pic}}(X). \end{array}$$

4.4 Height functions

Here we see the height function on X^{an} associated with an adelic $\mathbb{R}$-Cartier divisor, which is introduced by Chen and Moriwaki [6].

Definition 4.4.1. Let (D, g) be an adelic $\mathbb{R}$-Cartier divisor on X. We set $h^{\mathrm{an}}_{(D,g)} := g - g^{\mathrm{c}}_D$, which is called a *hight function on* X^{an} *associated with* (D, g).

Proposition 4.4.2 (c.f. [6, Proposition 4.3])**.** *Let* $\overline{D}$ *and* $\overline{D}'$ *be adelic* $\mathbb{R}$*-Cartier divisors on* X*.*

(1) For any $s \in K(X)^{\times}_{\mathbb{R}}$, $h^{\mathrm{an}}_{\widehat{(s)}} = 0$ *on* X^{an}*.*

(2) For any $a, a' \in \mathbb{R}$, $h^{\mathrm{an}}_{a\overline{D}+a'\overline{D}'} = ah^{\mathrm{an}}_{\overline{D}} + a'h^{\mathrm{an}}_{\overline{D}'}$ *on* X^{an}*.*

Proof. It immediately follows from Proposition 4.3.4. $\square$

By using the map $\sigma : X \to X^{\mathrm{an}}$ in Section 1.4, we can consider X is a subset of X^{an}. Hence we can compare this height function $h^{\mathrm{an}}_{\overline{D}}$ and $h_{\overline{D}}$ in Section 3.2.

Proposition 4.4.3. *Let* $\overline{D}$ *be an adelic Cartier divisor on* X*.*

$$h_{\overline{D}} = h^{\mathrm{an}}_{\overline{D}} \circ \sigma.$$

Proof. By definition, it is sufficient to show that $g^c_D(x) = 0$ for all $x \in X$. Let $x \in X$ and $f \in K(X)^{\times}_{\mathbb{R}}$ be a local equation of D around $\mathrm{red}(\sigma(x))$. Then we have

$$g^c_D(x) = -\log|f|_{\sigma(x)}.$$

Since $\mathrm{red}(\sigma(x)) = x$ and $|\cdot|_{\sigma(x)}$ is trivial, we get the conclusion. $\square$

For any adelic $\mathbb{R}$-Cartier divisor (D, g) on X, $h^{\mathrm{an}}_{(D,g)}$ is a continuous function on X^{an}. Hence we have the following homomorphism:

$$\psi : \widehat{\mathrm{Div}}(X) \to C^0(X^{\mathrm{an}}), \quad (D, g) \mapsto h^{\mathrm{an}}_{(D,g)},$$

where $C^0(X^{\mathrm{an}})$ is the set of all continuous functions on X^{an}. This homomorphism is surjective. In fact, let $\rho : C^0(X^{\mathrm{an}}) \to \widehat{\mathrm{Div}}(X)$ be a homomorphism such that $u \mapsto (0, u)$. Then we have $\psi \circ \rho(u) = u$ for all $u \in C^0(X^{\mathrm{an}})$. By Proposition 4.4.2, it induces a surjective homomorphism $\overline{\psi} : \widehat{\mathrm{Pic}}(X) \to C^0(X^{\mathrm{an}})$ such that the following diagram is commutative:

$$\begin{array}{ccc} \widehat{\mathrm{Div}}(X) & \xrightarrow{\ \psi\ } & C^0(X^{\mathrm{an}}) \\ \downarrow & \nearrow_{\overline{\psi}} & \\ \widehat{\mathrm{Pic}}(X) & & \end{array}.$$

Theorem 4.4.4. *The following sequence is exact:*

$$0 \longrightarrow \mathrm{Pic}(X) \xrightarrow{\ \overline{\varphi}\ } \widehat{\mathrm{Pic}}(X) \xrightarrow{\ \overline{\psi}\ } C^0(X^{\mathrm{an}}) \longrightarrow 0.$$

In particular, $\widehat{\mathrm{Pic}}(X) \simeq \mathrm{Pic}(X) \oplus C^0(X^{\mathrm{an}})$*.*

Proof. Since $\psi \circ \varphi = 0$ by definition, we have $\overline{\psi} \circ \overline{\varphi} = 0$. Let $(D, g) \in \widehat{\mathrm{Div}}(X)$ such that $\overline{\psi}(D, g) = 0$. Then there are $H \in \mathrm{Div}(X)$ and $s \in K(X)$ such that $(D, g) = (H, g_H^{\mathrm{c}}) + (s, -\log|s|)$. By Proposition 4.3.4, we get $g = g_H^{\mathrm{c}} - \log|s| = g_D^{\mathrm{c}}$, which implies that $(D, g) = \phi(D)$. Hence we obtain that $\mathrm{Im}\,\overline{\phi} = \mathrm{Ker}\,\overline{\psi}$.

Finally, we denote by $\overline{\rho}$ the composition of $\rho : C^0(X^{\mathrm{an}}) \to \widehat{\mathrm{Div}}(X)$ and the natural homomorphism $\widehat{\mathrm{Div}}(X) \to \widehat{\mathrm{Pic}}(X)$. Then it follows from $\overline{\psi} \circ \overline{\rho} = \mathrm{id}$ that this exact sequence is splitting. □

For $X = \mathrm{Spec}\, K$, the Berkovich space X^{an} associated with X is a single point. Hence we have $C^0(X^{\mathrm{an}}) = \mathbb{R}$, which implies that

$$\widehat{\mathrm{Pic}}(X) \simeq \mathbb{R}.$$

This result corresponds to the fact that the Picard group of $\mathbb{P}^1$ is isomorphic to $\mathbb{Z}$ and the arithmetic Picard group of $\mathrm{Spec}\,\mathbb{Z}$ is isomorphic to $\mathbb{R}$.

4.5 Scaling action for Green functions

We saw that the multiplicative group $\mathbb{R}_{>0}$ acts X^{an} (see Section 1.4). Here we see that it also acts the set of Green functions of D. Let (D, g) be an adelic $\mathbb{R}$-Cartier divisor on X. For $r \in \mathbb{R}_{>0}$, we define

$$r^*g(x) := rg(x^{1/r}).$$

Note that $x^{1/r} = (1/r)^*x = (p, |\cdot|_x^{1/r})$ for $x = (p, |\cdot|_x) \in X^{\mathrm{an}}$. The function r^*g is also a Green function of D. In fact, let U be a non-empty Zariski open subset of X and f be a local equation of D on U. Then we have

$$r^*g(x) + \log|f|_x = rg(x^{1/r}) + r\log|f|_x^{1/r} = r(g(x^{1/r}) + \log|f|_{x^{1/r}})$$

on U^{an}, which is continuous. This action is also called the *sacling action*.

Proposition 4.5.1. *Let D be an $\mathbb{R}$-Cartier divisor on X, and g and g' be Green functions of D.*

(1) The scaling action is linear, that is, for $r \in \mathbb{R}_{>0}$,

$$r^*(g + g') = r^*g + r^*g'.$$

(2) The scaling action preserves the canonical Green function g_D^{c}.

Proof. (1) It is clear by definition.

(2) Let $x \in X^{\mathrm{an}}$ and f be a local equation of D around $\mathrm{red}(x)$. Then $g_D^{\mathrm{c}}(x) = -\log|f|_x$. Hence we have

$$r^*g_D^{\mathrm{c}}(x) = -r\log|f|_{x^{1/r}} = -r\log|f|_x^{1/r} = -\log|f|_x = g_D^{\mathrm{c}}(x)$$

for any $r \in \mathbb{R}_{>0}$. □

4.6 Big adelic $\mathbb{R}$-Cartier divisors

4.6.1 Definition

We introduce the counterparts of $h^0(D)$ and $\mathrm{vol}(\cdot)$ in Arakelov geometry, which is given by Chen and Moriwaki [6]. We set

$$\widehat{\deg}_+(D,g) := \int_0^{+\infty} \dim_K \mathcal{F}^t(H^0(D), \|\cdot\|_g)\, \mathrm{d}t,$$

and

$$\widehat{\mathrm{vol}}(D,g) := \limsup_{n\to+\infty} \frac{\widehat{\deg}_+(nD,ng)}{n^{d+1}/(d+1)!},$$

where $d = \dim X$.

Definition 4.6.1. An adelic $\mathbb{R}$-Cartier divisor (D,g) is said to be *big* if $\widehat{\mathrm{vol}}(D,g) > 0$.

Proposition 4.6.2. *Let (D,g) be an adelic $\mathbb{R}$-Cartier divisor on X. If (D,g) is big, then (D,g) is μ-finite, $D_{\mu(g)}$ is big and $\lambda^{\mathrm{asy}}_{\max}(D,g) > 0$. In particular, D is big.*

Proof. If (D,g) is big, (D,g) is $\mathbb{Q}$-linearly equivalent to an effective adelic $\mathbb{R}$-Cartier divisor, which implies that (D,g) is μ-finite. By definition, for any integer $n > 0$, we have

$$\widehat{\deg}_+(nD,ng) \le \dim_k \mathcal{F}^0(H^0(nD), \|\cdot\|_{ng}) \max\{\lambda^{\mathrm{asy}}_{\max}(nD,ng), 0\}.$$

Therefore we obtain that

$$\widehat{\mathrm{vol}}(D,g) \le (d+1)\mathrm{vol}(D_{\mu(g)}) \max\{\lambda^{\mathrm{asy}}_{\max}(D,g), 0\},$$

by Corollary 5.3.5. Since $\widehat{\mathrm{vol}}(D,g) > 0$, it follows that $\mathrm{vol}(D_{\mu(g)}) > 0$ and $\lambda^{\mathrm{asy}}_{\max}(D,g) > 0$. □

4.6.2 Existence of limit of the arithmetic volume

Firstly, we define

$$\nu_{\max}(D,g) := \sup\{t \in \mathbb{R} \mid (D, g-t) \text{ is } \mu\text{-finite}\}.$$

Lemma 4.6.3. *Let (D,g) be an adelic $\mathbb{R}$-Cartier divisor on X. We have*

$$\lambda^{\mathrm{asy}}_{\max}(D,g) \le \nu_{\max}(D,g) \le g(\eta^{\mathrm{an}}).$$

Proof. Clearly we can assume $\lambda^{\mathrm{asy}}_{\max}(D,g) \in \mathbb{R}$. For a sufficiently large integer $n > 0$, there is a non-zero element $s \in H^0(nD) \setminus \{0\}$ such that $\|s\|_{ng} \le e^{-\lambda_{\max}(nD,ng)}$, which is equivalent to $\|s\|_{ng-\lambda_{\max}(nD,ng)} \le 1$. Therefore $(D, g - \lambda_{\max}(nD,ng)/n)$ is effective, which implies

$$\frac{1}{n}\lambda_{\max}(nD,ng) \le \nu_{\max}(D,g).$$

Taking a supremum, we get $\lambda^{\mathrm{asy}}_{\max}(D,g) \le \nu_{\max}(D,g)$.

Next we show $\nu_{\max}(D,g) \le g(\eta^{\mathrm{an}})$. For any $\epsilon > 0$, $g(\zeta) - (g(\eta^{\mathrm{an}}) + \epsilon)$ is negative around $\zeta = \eta^{\mathrm{an}}$. So we have $\mu_x(g - (g(\eta^{\mathrm{an}}) + \epsilon)) = -\infty$ for any $x \in X^{(1)}$, which implies that $(D, g - (g(\eta^{\mathrm{an}}) + \epsilon))$ is not μ-finite and $\nu_{\max}(D,g) \le g(\eta^{\mathrm{an}}) + \epsilon$. Since ϵ is arbitrary, we conclude that $\nu_{\max}(D,g) \le g(\eta^{\mathrm{an}})$. □

Remark 4.6.4. The above inequality is sometimes strict. For example, let $X = \mathbb{P}^1_K = \mathrm{Proj} K[T_0, T_1]$, $z = T_1/T_0$, $D = \{T_0 = 0\}$ and $x_\infty = (0:1)$. Let $g_1 = 2\log\max\{2, |z|\} - \log\max\{1,|z|\}$. Then $g_1(\xi) = 2\log 2$ for $\xi \in [\eta^{\mathrm{an}}, x^{\mathrm{an}}]$ for $x \ne x_\infty$ and

$$g_1(\xi) = \begin{cases} 2\log 2 - \xi & (0 \le \xi \le \log 2), \\ \xi & (\log 2 \le \xi), \end{cases}$$

on $[\eta^{\mathrm{an}}, x^{\mathrm{an}}_\infty]$. Hence we have

$$\mu_x(g_1 - t) = \begin{cases} 0 & (x \ne x_\infty), \\ \log 2 - t & (x = x_\infty), \end{cases}$$

for $t \le 2\log 2$ and $\mu_x(g_1 - t) = -\infty$ for $t > 2\log 2$ and all closed point x of X. Therefore we obtain that $\lambda^{\mathrm{asy}}_{\max}(D,g) = \log 2$ and $\nu_{\max}(D,g) = 2\log 2$.

Next, we set

$$h(\xi) = \begin{cases} -\xi & (0 \le \xi \le 1), \\ -1 & (1 \le \xi), \end{cases}$$

for $\xi \in [\eta^{\mathrm{an}}, x^{\mathrm{an}}]$ and all closed point x of X, which is a continuous function on X^{an}. We define a Green function g_2 of D as $\log\max\{1,|z|\} + h$. Then we have

$$\mu_x(g_2 - t) = \begin{cases} 0 & (x \ne x_\infty), \\ 1 & (x = x_\infty), \end{cases}$$

for $t \le -1$,

$$\mu_x(g_2 - t) = \begin{cases} -1 - t & (x \ne x_\infty), \\ -t & (x = x_\infty), \end{cases}$$

for $-1 \le t \le 0$ and $\mu_x(g_2 - t) = -\infty$ for $t > 0$ and all closed point x of X. Hence we obtain that $\nu_{\max}(D, g_2) = -1$ and $g(\eta^{\mathrm{an}}) = 0$.

For any integer $n > 0$, let

$$P_n^{(D,g)}(t) := \frac{\dim_K \mathcal{F}^{nt}(H^0(nD), \|\cdot\|_{ng})}{n^d/d!},$$

where $d = \dim X$. If there is no confusion, we write it simply $P_n(t)$. By definition, if (D,g) is μ-finite,

$$P_n(0) = \frac{\dim_K \mathcal{F}^0(H^0(nD), \|\cdot\|_{ng})}{n^d/d!} = \frac{\dim_K H^0(nD_{\mu(g)})}{n^d/d!},$$

so we have

$$\lim_{n\to+\infty} P_n(0) = \mathrm{vol}(D_{\mu(g)}). \tag{4.6.1}$$

Lemma 4.6.5. *Let (D, g) be an adelic $\mathbb{R}$-Cartier divisor on X. For any $\epsilon \in \mathbb{R}$, we have*

$$P_n^{(D,g-\epsilon)}(t) = P_n^{(D,g)}(t+\epsilon).$$

Proof. For any $s \in H^0(nD)$, it follows that

$$\|s\|_{n(g-\epsilon)} \le e^{-nt} \Leftrightarrow \|s\|_{ng} \le e^{-n(t+\epsilon)}.$$

Hence we get

$$\dim_K \mathcal{F}^{nt}(H^0(nD), \|\cdot\|_{n(g-\epsilon)}) = \dim_K \mathcal{F}^{n(t+\epsilon)}(H^0(nD), \|\cdot\|_{ng}),$$

which implies that $P_n^{(D,g-\epsilon)}(t) = P_n^{(D,g)}(t+\epsilon)$. □

In particular,

$$P_n^{(D,g)}(t) = P_n^{(D,g-t)}(0),$$

and hence by the equation (4.6.1), we obtain that

$$\lim_{n\to+\infty} P_n(t) = \mathrm{vol}(D_{\mu(g-t)}) \tag{4.6.2}$$

for any $t < \nu_{\max}(D, g)$.

If we define

$$F_{(D,g)}(t) := \begin{cases} \mathrm{vol}(D_{\mu(g-t)}) & (t < \lambda_{\max}^{\mathrm{asy}}(D,g)), \\ 0 & (t > \lambda_{\max}^{\mathrm{asy}}(D,g)), \end{cases}$$

then we have the following theorem by the equation (4.6.2):

Theorem 4.6.6. *Let (D, g) be an adelic $\mathbb{R}$-Cartier divisor on X. The sequence $\{P_n(t)\}_{n\ge1}$ converges pointwise to $F_{(D,g)}(t)$ on $\mathbb{R} \setminus \{\lambda_{\max}^{\mathrm{asy}}(D,g)\}$.*

The sequence $\{P_n(t)\}_{n\ge1}$ is uniformly bounded on $(0, \lambda_{\max}^{\mathrm{asy}}(D,g))$. In fact, $P_n(t)$ is monotonically decreasing function with respect to t and $P_n(0)$ is bounded with respect to n. Hence we get the main theorem in this section by using bounded convergence theorem:

Theorem 4.6.7. *Let (D, g) be an adelic $\mathbb{R}$-Cartier divisor on X. We have*

$$\begin{aligned}\widehat{\mathrm{vol}}(D,g) &= \lim_{n\to+\infty} \frac{\widehat{\deg}_+(nD, ng)}{n^{d+1}/(d+1)!} \\ &= (d+1)\int_0^{\lambda_{\max}^{\mathrm{asy}}(D,g)} F_{(D,g)}(t)\ \mathrm{d}t.\end{aligned}$$

Proof. By definition,

$$\widehat{\deg}_+(nD, ng) = \int_0^{\lambda_{\max}(nD,ng)} \dim_K \mathcal{F}^t(H^0(nD), \|\cdot\|_{ng})\ \mathrm{d}t.$$

Substituting t for nt, we have

$$\widehat{\deg}_+(nD, ng) = n\int_0^{\frac{1}{n}\lambda_{\max}(nD,ng)} \dim_K \mathcal{F}^{nt}(H^0(nD), \|\cdot\|_{ng})\ \mathrm{d}t.$$

Therefore we get

$$\frac{\widehat{\deg}_+(nD, ng)}{n^{d+1}/(d+1)!} = (d+1)\int_0^{\lambda^{\mathrm{asy}}_{\max}(D,g)} P_n(t)\ \mathrm{d}t.$$

We remark that $\lambda_{\max}(nD, ng)/n \leq \lambda^{\mathrm{asy}}_{\max}(D, g)$ and $P_n(t) = 0$ if $t > \lambda_{\max}(nD, ng)/n$. Hence by using bounded convergence theorem, we get the conclusion. □

Corollary 4.6.8. *The arithmetic volume* $\widehat{\mathrm{vol}}(\cdot)$ *is* $(d+1)$*-homogeneous. Namely, for any adelic* ℝ*-Cartier divisor* (D, g) *and* $a \in \mathbb{R}_{>0}$*, we have*

$$\widehat{\mathrm{vol}}(aD, ag) = a^{d+1}\widehat{\mathrm{vol}}(D, g).$$

Proof. We have $\lambda^{\mathrm{asy}}_{\max}(aD, ag) = a\lambda^{\mathrm{asy}}_{\max}(D, g)$ and $F_{(aD,ag)}(at) = a^d F_{(D,g)}(t)$ because the algebraic volume is d-homogeneous. Therefore by Theorem 4.6.7, we have

$$\begin{aligned}\widehat{\mathrm{vol}}(aD, ag) &= (d+1)\int_0^{\lambda^{\mathrm{asy}}_{\max}(aD,ag)} F_{(aD,ag)}(t)\ \mathrm{d}t\\ &= a(d+1)\int_0^{\lambda^{\mathrm{asy}}_{\max}(D,g)} a^d F_{(D,g)}(t)\ \mathrm{d}t\\ &= a^{d+1}\widehat{\mathrm{vol}}(D, g).\end{aligned}$$

□

Finally, we prove a simple criterion of the bigness of an adelic ℝ-Cartier divisor.

Theorem 4.6.9 (c.f. [3, Lemma 1.6] and [6, Proposition 4.10])**.** *Let* (D, g) *be an adelic* ℝ*-Cartier divisor on* X*. We assume that* D *is big. Then the following conditions are equivalent:*

(1) (D, g) *is big.*

(2) $\lambda^{\mathrm{asy}}_{\max}(D, g) > 0$.

(3) For $\forall n \gg 0$*, there is a strictly small section of* $H^0(nD)$*.*

Proof. (1) $\Rightarrow$ (2) It follows from Proposition 4.6.2.

(2) $\Rightarrow$ (1) It is sufficient to show that $D_{\mu(g)}$ is big. In fact, if $D_{\mu(g)}$ is big, $D_{\mu(g-t)}$ is also big for $t < \lambda_{\max}^{\mathrm{asy}}(D,g)$ because $\lambda_{\max}^{\mathrm{asy}}(D,g-t) = \lambda_{\max}^{\mathrm{asy}}(D,g) - t$ for $t \in \mathbb{R}$. Then we have

$$\widehat{\mathrm{vol}}(D,g) = (d+1)\int_0^\infty \mathrm{vol}(D_{\mu(g-t)})\ \mathrm{d}t > 0$$

by Theorem 4.6.7. Now we prove that $D_{\mu(g)}$ is big. Since D is big, there is an ample divisor A such that $mD - A$ is effective for some $m \in \mathbb{Z}_{>0}$. Let $s \in H^0(mD-A)\setminus\{0\}$ be a non-zero section such that the map $H^0(kA) \to H^0(kmD)$ is given by multiplication by $s^{\otimes k}$ for all $k > 0$. We denote the image of the map $H^0(kA) \to H^0(kmD)$ by V_k and $V_0 = K$. Since the graded ring $\bigoplus_{k=0}^\infty V_k$ is finitely generated, there is $a \in \mathbb{R}$ such that $\|v\|_{kmg} \le e^{-akm}$ for all $v \in V_k$ and a sufficiently large $k > 0$. Let ϵ be a real number such that $0 < \epsilon < \lambda_{\max}^{\mathrm{asy}}(D,g)$. Then we can find $p \in \mathbb{Z}_{>0}$ such that there is a non-zero element $s_p \in H^0(pD)\setminus\{0\}$ with $\|v_p\|_{pg} \le e^{-p\epsilon}$ and $p > -am/\epsilon$ because $\epsilon < \lambda_{\max}(pD,pg)/p$ for a sufficiently large $p > 0$. The image W_k of the composition of the map $H^0(kA) \to H^0(kmD) \to H^0(k(m+p)D)$ is given by multiplication by $(ss_p)^{\otimes k}$ for all $k > 0$. Hence for any $w \in W_k$, we can write $w = v \otimes (s_p)^{\otimes k}$ with $v \in V_k$ and we have

$$\|w\|_{k(m+p)g} \le \|v\|_{kmg} \cdot \|s_p\|_{pg}^k \le e^{-akm}e^{-kp\epsilon} = e^{-k(am+p\epsilon)} \le 1,$$

which implies that $W_k \subset H^0(k(m+p)D_{\mu(g)})$ for a sufficiently large $k > 0$. Therefore we obtain that $\mathrm{vol}((m+p)D_{\mu(g)}) \ge \mathrm{vol}(A) > 0$, which is required.

(2) $\Rightarrow$ (3) Since $\lambda_{\max}^{\mathrm{asy}}(D,g) > 0$, we have $\lambda_{\max}(nD,ng) > 0$ for a sufficiently large $n > 0$. Hence there is a non-zero section $s \in H^0(nD)\setminus\{0\}$ such that $\|s\|_{ng} \le e^{-\lambda_{\max}(nD,ng)} < 1$.

(3) $\Rightarrow$ (2) Let s be a strictly small section of $H^0(nD)$. Then we have $\lambda_{\max}(nD,ng) \ge -\log\|s\|_{ng} > 0$. Therefore we obtain that $\lambda_{\max}^{\mathrm{asy}}(D,g) \ge \lambda_{\max}(nD,ng)/n > 0$. □

4.6.3 Continuity of $F_{(D,g)}(t)$

Firstly, we will prove a very useful lemma:

Lemma 4.6.10. *Let C be a convex cone and let $f : C \to \mathbb{R}$ be a concave function. Namely, for any $v, v' \in C$ and $a, a' \ge 0$,*

$$f(av + a'v') \ge af(v) + a'f(v').$$

If $g(t) := v + tv'$ is a map from some open interval $(a,b) \subset \mathbb{R}$ to C for fixed elements $v, v' \in C$, then $f \circ g$ is a concave function on (a,b). In particular, $f \circ g$ is continuous on (a,b).

Proof. For any $t, t' \in (a,b)$ and $0 \le \epsilon \le 1$, we have

$$\begin{aligned} f\circ g(\epsilon t + (1-\epsilon)t') &= f(v + (\epsilon t + (1-\epsilon)t')v') \\ &= f(\epsilon(v+tv') + (1-\epsilon)(v+t'v')) \\ &\ge \epsilon f(v+tv') + (1-\epsilon)f(v+t'v') \\ &= \epsilon f\circ g(t) + (1-\epsilon) f\circ g(t'), \end{aligned}$$

as required. □

For an adelic $\mathbb{R}$-Cartier divisor (D,g), it follows immediately from the above lemma that $\mu_x(g-t)$ is a continuous concave function on $(-\infty, \lambda^{\mathrm{asy}}_{\max}(D,g))$ for every $x \in X^{(1)}$. Hence we get the following proposition:

Proposition 4.6.11. *Let (D,g) be an adelic $\mathbb{R}$-Cartier divisor on X. For any $t, t' < \lambda^{\mathrm{asy}}_{\max}(D,g)$ and $0 \le \epsilon \le 1$, we have*

$$D_{\mu(g-(\epsilon t+(1-\epsilon)t'))} \ge \epsilon D_{\mu(g-t)} + (1-\epsilon) D_{\mu(g-t')}.$$

Theorem 4.6.12. *Let (D,g) be an adelic $\mathbb{R}$-Cartier divisor on X and $d = \dim X$. Then $F_{(D,g)}(t)$ is a d-concave function on $(-\infty, \lambda^{\mathrm{asy}}_{\max}(D,g))$, that is, $F_{(D,g)}(t)^{1/d}$ is concave on $(-\infty, \lambda^{\mathrm{asy}}_{\max}(D,g))$. In particular, $F_{(D,g)}(t)$ is continuous on $\mathbb{R} \setminus \{\lambda^{\mathrm{asy}}_{\max}(D,g)\}$.*

Proof. By definition,

$$F_{(D,g)}(t) = \mathrm{vol}(D_{\mu(g-t)})$$

for $t < \lambda^{\mathrm{asy}}_{\max}(D,g)$. Since the algebraic volume is d-concave on a big cone, for any $t, t' < \lambda^{\mathrm{asy}}_{\max}(D,g)$ and $0 \le \epsilon \le 1$, we have

$$\begin{aligned} F_{(D,g)}(\epsilon t + (1-\epsilon)t')^{\frac{1}{d}} &= \mathrm{vol}(D_{\mu(g-(\epsilon t+(1-\epsilon)t'))})^{\frac{1}{d}} \\ &\ge \mathrm{vol}(\epsilon D_{\mu(g-t)} + (1-\epsilon) D_{\mu(g-t')})^{\frac{1}{d}} \quad \text{(by Proposition 4.6.11)} \\ &\ge \epsilon\ \mathrm{vol}(D_{\mu(g-t)})^{\frac{1}{d}} + (1-\epsilon)\mathrm{vol}(D_{\mu(g-t')})^{\frac{1}{d}} \\ &= \epsilon F_{(D,g)}(t)^{\frac{1}{d}} + (1-\epsilon) F_{(D,g)}(t')^{\frac{1}{d}}, \end{aligned}$$

as desired. □

Remark 4.6.13. In general, we cannot extend $F_{(D,g)}$ to a continuous function on whole $\mathbb{R}$. For example, let $X = \mathbb{P}^1_K = \mathrm{Proj} K[T_0, T_1]$, $z = T_1/T_0$, $D = \{T_0 = 0\}$ and $x_\infty = (0:1)$. Let $g = \log \max\{1, |z|\}$. Then we have

$$\mu_x(g-t) = \begin{cases} 0 & (x \ne x_\infty), \\ 1 & (x = x_\infty), \end{cases}$$

for $t < \lambda^{\mathrm{asy}}_{\max}(D,g) = 0$. Hence we obtain that

$$F_{(D,g)}(t) = \begin{cases} 1 & (t < 0), \\ 0 & (t > 0). \end{cases}$$

4.6.4 Continuity of the arithmetic volume

First of all, we will prove the continuity of $\lambda^{\mathrm{asy}}_{\max}(D,g)$ for an adelic $\mathbb{R}$-Cartier divisor (D,g).

Lemma 4.6.14. *Let $(D,g), (D',g')$ be adelic $\mathbb{R}$-Cartier divisors on X. We have*

$$\lambda^{\mathrm{asy}}_{\max}(D+D', g+g') \ge \lambda^{\mathrm{asy}}_{\max}(D,g) + \lambda^{\mathrm{asy}}_{\max}(D',g').$$

Proof. For any integers $n, n' > 0$, there are non-zero elements $s \in H^0(nD) \setminus \{0\}$ and $s' \in H^0(n'D') \setminus \{0\}$ such that

$$\|s\|_{ng} \leq e^{-\lambda_{\max}(nD,ng)} \text{ and } \|s'\|_{n'g'} \leq e^{-\lambda_{\max}(n'D',n'g')}.$$

Since $s^{\otimes n'} \otimes s'^{\otimes n} \in H^0(nn'(D + D')) \setminus \{0\}$, we have

$$\|s^{\otimes n'} \otimes s'^{\otimes n}\|_{nn'(g+g')} \leq (\|s\|_{ng})^{n'}(\|s'\|_{n'g'})^n \leq e^{-n'\lambda_{\max}(nD,ng)-n\lambda_{\max}(n'D',n'g')},$$

which implies

$$\frac{1}{nn'}\lambda_{\max}(nn'(D + D'), nn'(g + g')) \geq \frac{1}{n}\lambda_{\max}(nD, ng) + \frac{1}{n'}\lambda_{\max}(n'D', n'g').$$

Since $\lambda^{\text{asy}}_{\max}(D, g) \geq \lambda_{\max}(nD, ng)/n$, we get

$$\lambda^{\text{asy}}_{\max}(D + D', g + g') \geq \frac{1}{n}\lambda_{\max}(nD, ng) + \frac{1}{n'}\lambda_{\max}(n'D', n'g').$$

Taking a supremum with respect to n and n', we complete the proof. □

Proposition 4.6.15. *Let $\overline{D} = (D, g), \overline{D}' = (D', g')$ be adelic $\mathbb{R}$-Cartier divisors on X. We assume D is big. Then $\lambda(t) := \lambda^{\text{asy}}_{\max}(\overline{D} + t\overline{D}')$ is a real-valued function on some open interval $(a, b) \subset \mathbb{R}$ containing 0, and concave on (a, b). In particular $\lambda(t)$ is continuous on (a, b).*

Proof. Since D is big, $D + tD'$ is big for $|t| \ll 1$, which implies that $\lambda(t)$ is definable on a sufficiently small open neighborhood of 0. Moreover, using Lemma 4.6.14, we can prove the concavity of $\lambda(t)$ by Lemma 4.6.10. □

Next, we prove the continuity of the arithmetic volume $\widehat{\text{vol}}(\cdot)$. Let $(D, g), (D', g')$ be adelic $\mathbb{R}$-Cartier divisors on X and we assume D is big. We set

$$(D_\epsilon, g_\epsilon) := (D, g) + \epsilon(D', g'),$$

and

$$F_\epsilon(t) := \begin{cases} \text{vol}((D_\epsilon)_{\mu(g_\epsilon - t)}) & (t < \lambda^{\text{asy}}_{\max}(D_\epsilon, g_\epsilon)), \\ 0 & (t > \lambda^{\text{asy}}_{\max}(D_\epsilon, g_\epsilon)). \end{cases}$$

We remark that this function is well-defined if $|\epsilon| \ll 1$ by Proposition 4.6.15.

Proposition 4.6.16. *The function $F_\epsilon(t)$ converges pointwise to $F_{(D,g)}(t)$ on $\mathbb{R} \setminus \{\lambda^{\text{asy}}_{\max}(D, g)\}$ as $|\epsilon| \to 0$. More precisely, for any $t \in \mathbb{R} \setminus \{\lambda^{\text{asy}}_{\max}(D, g)\}$, $F_\epsilon(t)$ is continuous with respect to ϵ on a sufficiently small open neighborhood of $\epsilon = 0$.*

Proof. We first assume $t > \lambda^{\text{asy}}_{\max}(D, g)$. By Proposition 4.6.15, there is $\delta > 0$ such that $\lambda^{\text{asy}}_{\max}(D_\epsilon, g_\epsilon) < t$ if $|\epsilon| < \delta$. Then $F_\epsilon(t) = F_{(D,g)}(t) = 0$, which is required.

Next we assume $t < \lambda^{\rm asy}_{\max}(D, g)$. Similarly, there is $\delta > 0$ such that $\lambda^{\rm asy}_{\max}(D_\epsilon, g_\epsilon) > t$ if $|\epsilon| < \delta$. Then $F_\epsilon(t)$ is d-concave with respect to ϵ on $(-\delta, \delta)$, where $d = \dim X$. In fact, by Lemma 4.6.10, for any $\epsilon, \epsilon' \in (-\delta, \delta)$ and $0 \leq \zeta \leq 1$, we have

$$(D_{\zeta\epsilon+(1-\zeta)\epsilon'})_{\mu(g_{\zeta\epsilon+(1-\zeta)\epsilon'}-t)} \geq \zeta(D_\epsilon)_{\mu(g_\epsilon-t)} + (1-\zeta)(D_{\epsilon'})_{\mu(g_{\epsilon'}-t)}.$$

Therefore $F_\epsilon(t)$ is d-concave with respect to ϵ on $(-\delta, \delta)$ because $F_\epsilon(t) = \mathrm{vol}((D_\epsilon)_{\mu(g_\epsilon-t)})$ and the algebraic volume is d-concave. In particular, $F_\epsilon(t)$ is continuous with respect to ϵ on $(-\delta, \delta)$. □

Since $F_\epsilon(t)$ is monotonically decreasing with respect to t and $F_\epsilon(0)$ is bounded, $F_\epsilon(t)$ is uniformly bounded with respect to ϵ, and

$$\widehat{\mathrm{vol}}(D_\epsilon, g_\epsilon) = (d+1)\int_0^{+\infty} F_\epsilon(t)\ \mathrm{d}t$$

by Theorem 4.6.7, we get the continuity of the arithmetic volume by bounded convergence theorem:

Theorem 4.6.17. *Let $\overline{D} = (D, g), \overline{D}' = (D', g')$ be adelic ℝ-Cartier divisors on X. We assume D is big. Then $\widehat{\mathrm{vol}}(\overline{D} + \epsilon\overline{D}')$ converges to $\widehat{\mathrm{vol}}(\overline{D})$ as $|\epsilon| \to 0$.*

4.6.5 Log concavity of the arithmetic volume

Firstly, we will prove some inequalities:

Lemma 4.6.18. *Let a, b, p and ϵ be real numbers such that $a, b \geq 0$, $p > 0$ and $0 < \epsilon < 1$. Then we have the following inequality:*

$$(\epsilon a^p + (1-\epsilon)b^p)^{\frac{1}{p}} \geq a^\epsilon b^{1-\epsilon} \geq \min\{a, b\}.$$

Proof. If $ab = 0$, the assertion is clear, so we assume that $a, b > 0$. Moreover, the inequality $a^\epsilon b^{1-\epsilon} \geq \min\{a, b\}$ is also clear. Now, we will show the first inequality. Since $\log x$ is concave on $(0, +\infty)$, we have

$$\log(\epsilon x + (1-\epsilon)y) \geq \epsilon \log x + (1-\epsilon)\log y$$

for any $x, y > 0$. Substituting x for a^p and y for b^p,

$$\begin{aligned}\log(\epsilon a^p + (1-\epsilon)b^p) \geq \epsilon \log a^p + (1-\epsilon)\log b^p &\Longleftrightarrow \log(\epsilon a^p + (1-\epsilon)b^p)^{\frac{1}{p}} \geq \log a^\epsilon b^{1-\epsilon} \\ &\Longleftrightarrow (\epsilon a^p + (1-\epsilon)b^p)^{\frac{1}{p}} \geq a^\epsilon b^{1-\epsilon},\end{aligned}$$

as required. □

Lemma 4.6.19. *Let C be a convex cone. Let $f : C \to (0, +\infty)$ be a non-negative d-homogeneous function for some $d > 0$, that is,*

$$f(av) = a^d f(v)$$

for any $a > 0$ and $v \in C$. Then the following conditions are equivalent:

(1) f is d-concave, that is,

$$f(\epsilon v + (1-\epsilon)v')^{\frac{1}{d}} \geq \epsilon f(v)^{\frac{1}{d}} + (1-\epsilon) f(v')^{\frac{1}{d}}$$

for every $v, v' \in C$ and $0 \leq \epsilon \leq 1$.

(2) $f(\epsilon v + (1-\epsilon)v') \geq \min\{f(v), f(v')\}$ for every $v, v' \in C$ and $0 \leq \epsilon \leq 1$.

Proof. Firstly, we assume (1) and we can assume $\min\{f(v), f(v')\} = f(v)$. Then we have

$$f(\epsilon v + (1-\epsilon)v')^{\frac{1}{d}} \geq \epsilon f(v)^{\frac{1}{d}} + (1-\epsilon) f(v')^{\frac{1}{d}} \geq f(v)^{\frac{1}{d}}.$$

Raising both sides to d-th power, we have

$$f(\epsilon v + (1-\epsilon)v') \geq f(v) = \min\{f(v), f(v')\}.$$

Next we suppose (2). If we set

$$w = f(v)^{-\frac{1}{d}} v,\ w' = f(v')^{-\frac{1}{d}} v',\ \epsilon = \frac{f(v)^{\frac{1}{d}}}{f(v)^{\frac{1}{d}} + f(v')^{\frac{1}{d}}},$$

we have

$$\begin{aligned} \epsilon w + (1-\epsilon)w' &= \frac{1}{f(v)^{\frac{1}{d}} + f(v')^{\frac{1}{d}}}(v+v'), \\ \min\{f(w), f(w')\} &= 1. \end{aligned}$$

By the inequality (2) for w, w' and ϵ, we have

$$\begin{aligned} (f(v)^{\frac{1}{d}} + f(v')^{\frac{1}{d}})^{-d} f(v+v') \geq 1 &\iff f(v+v') \geq (f(v)^{\frac{1}{d}} + f(v')^{\frac{1}{d}})^d \\ &\iff f(v+v')^{\frac{1}{d}} \geq f(v)^{\frac{1}{d}} + f(v')^{\frac{1}{d}}, \end{aligned}$$

which implies the inequality (1) because f is d-homogeneous. □

Moreover, we will use the following inequality so called "Prékopa-Leindler inequality" (for details, see [12]).

Theorem 4.6.20 (Prékopa-Leindler inequality)**.** *Let $0 < \epsilon < 1$ and $f, g, h : \mathbb{R}^n \to [0, +\infty)$ be measurable functions. We assume*

$$h(\epsilon x + (1-\epsilon)y) \geq f(x)^{\epsilon} g(y)^{1-\epsilon}$$

for any $x, y \in \mathbb{R}^n$. Then we have $||h||_1 \geq ||f||_1^{\epsilon} ||g||_1^{1-\epsilon}$, that is,

$$\int_{\mathbb{R}^n} h \,\mathrm{d}\nu \geq \left(\int_{\mathbb{R}^n} f \,\mathrm{d}\nu\right)^{\epsilon} \left(\int_{\mathbb{R}^n} g \,\mathrm{d}\nu\right)^{1-\epsilon}$$

where ν is the Lebesgue measure on $\mathbb{R}^n$.

Now, we start to prove the log concavity of $\widehat{\mathrm{vol}}(\cdot)$.

Theorem 4.6.21. *The arithmetic volume $\widehat{\mathrm{vol}}(\cdot)$ is $(d+1)$-concave for $d = \dim X$. More precisely, for any big adelic $\mathbb{R}$-Cartier divisors $(D, g), (D', g')$, we have*

$$\widehat{\mathrm{vol}}(D + D', g + g')^{\frac{1}{d+1}} \geq \widehat{\mathrm{vol}}(D, g)^{\frac{1}{d+1}} + \widehat{\mathrm{vol}}(D', g')^{\frac{1}{d+1}}.$$

Proof. For $0 < \epsilon < 1$, we set

$$(D_\epsilon, g_\epsilon) := \epsilon(D, g) + (1 - \epsilon)(D', g'),$$

and

$$\Theta_{(D,g)}(t) := \begin{cases} (d+1)\mathrm{vol}(D_{\mu(g-t)}) & (0 \leq t < \lambda^{\mathrm{asy}}_{\max}(D, g)), \\ 0 & (\text{otherwise}). \end{cases}$$

Then, we have

$$\widehat{\mathrm{vol}}(D, g) = ||\Theta_{(D,g)}||_1,\ \widehat{\mathrm{vol}}(D', g') = ||\Theta_{(D',g')}||_1,\ \widehat{\mathrm{vol}}(D_\epsilon, g_\epsilon) = ||\Theta_{(D_\epsilon,g_\epsilon)}||_1 \tag{4.6.3}$$

by Theorem 4.6.7. We claim that

$$\Theta_{(D_\epsilon,g_\epsilon)}(\epsilon x + (1 - \epsilon)y) \geq \Theta_{(D,g)}(x)^\epsilon \Theta_{(D',g')}(y)^{1-\epsilon} \quad \text{for any } x, y \in \mathbb{R}. \tag{4.6.4}$$

In fact, if $x < 0, \lambda^{\mathrm{asy}}_{\max}(D, g) \leq x, y < 0$ or $\lambda^{\mathrm{asy}}_{\max}(D', g') \leq y$, we obtain that $\Theta_{(D,g)}(x) = 0$ or $\Theta_{(D',g')}(y) = 0$, so the inequality (4.6.4) is clear in this case. And if $0 \leq x < \lambda^{\mathrm{asy}}_{\max}(D, g)$ and $0 \leq y < \lambda^{\mathrm{asy}}_{\max}(D', g')$, it follows that

$$\mu_z(g_\epsilon - (\epsilon x + (1 - \epsilon)y)) \geq \epsilon\mu_z(g - x) + (1 - \epsilon)\mu_z(g' - y)$$

for any $z \in X^{(1)}$, which implies that

$$(D_\epsilon)_{\mu(g_\epsilon-(\epsilon x+(1-\epsilon)y))} \geq \epsilon D_{\mu(g-x)} + (1 - \epsilon)D'_{\mu(g'-y)}.$$

Since the algebraic volume is d-concave, we obtain

$$\mathrm{vol}((D_\epsilon)_{\mu(g_\epsilon-(\epsilon x+(1-\epsilon)y))})^{\frac{1}{d}} \geq \epsilon\ \mathrm{vol}(D_{\mu(g-x)})^{\frac{1}{d}} + (1 - \epsilon)\mathrm{vol}(D'_{\mu(g'-y)})^{\frac{1}{d}}.$$

By Lemma 4.6.18, we get

$$\mathrm{vol}((D_\epsilon)_{\mu(g_\epsilon-(\epsilon x+(1-\epsilon)y))}) \geq \mathrm{vol}(D_{\mu(g-x)})^\epsilon \mathrm{vol}(D'_{\mu(g'-y)})^{1-\epsilon},$$

which is equivalent to the inequality (4.6.4). Therefore by Prékopa-Leindler inequality, we have $||\Theta_{(D_\epsilon,g_\epsilon)}||_1 \geq ||\Theta_{(D,g)}||_1^\epsilon ||\Theta_{(D',g')}||_1^{1-\epsilon}$. By Lemma 4.6.18 again, it follows that $||\Theta_{(D_\epsilon,g_\epsilon)}||_1 \geq \min\{||\Theta_{(D,g)}||_1, ||\Theta_{(D',g')}||_1\}$, which is the inequality

$$\widehat{\mathrm{vol}}(D_\epsilon, g_\epsilon) \geq \min\{\widehat{\mathrm{vol}}(D, g), \widehat{\mathrm{vol}}(D', g')\}$$

by (4.6.3). Since the arithmetic volume is $(d+1)$-homogeneous by Corollary 4.6.8, we obtain that

$$\widehat{\mathrm{vol}}(D_\epsilon, g_\epsilon)^{\frac{1}{d+1}} \geq \epsilon\ \widehat{\mathrm{vol}}(D, g)^{\frac{1}{d+1}} + (1 - \epsilon)\widehat{\mathrm{vol}}(D', g')^{\frac{1}{d+1}},$$

by Lemma 4.6.19, which completes the proof. □

4.7 Ample adelic Cartier divisors

4.7.1 Plurisubharmonic Green functions

Let (D, g) be an adelic Cartier divisor on X. Let $\{g_n\}_{n\in\mathbb{Z}_{>0}}$ be a sequence of Green functions of D. We say that $\{g_n\}_{n\in\mathbb{Z}_{>0}}$ *converges to g uniformly on X^{an}* if the sequence

$$\|g - g_n\|_{\sup} := \sup_{x\in X^{\mathrm{an}}} |g(x) - g_n(x)|$$

converges to 0.

Definition 4.7.1. Let (D, g) be an adelic Cartier divisor on X. We assume that D is semiample. We say that g is *plurisubharmonic* if there is a sequence $\{e_n\}_{n\in\mathbb{Z}_{>0}}$ of positive integers and a sequence $\{\overline{V}_n\}_{n\in\mathbb{Z}_{>0}}$ of a finite-dimensional ultrametrically normed vector space over a tirivially valued field K such that there is a surjective morphism $f_n : V_n \otimes \mathcal{O}_X \to \mathcal{O}_X(e_n D)$ and the sequence

$$\left\{\frac{1}{e_n}\log|1|^{\mathrm{quot}}_{\overline{V}_n}\right\}_{n\in\mathbb{Z}_{>0}}$$

converges to g uniformly on X^{an}, where 1 is a rational section of $\mathcal{O}_X(D)$.

Several properties can be observed for plurisubharmonic Green functions. But we only recall them without proofs (for details, see [7] and [8]).

Proposition 4.7.2. *Let (D, g) and (D', g') be adelic Cartier divisors on X. We assume that D and D' are semiample.*

(1) If g and g' are plurisubharmonic, then $g + g'$ is also plurisubharmonic.

(2) Let $\{g_n\}_{n\in\mathbb{Z}_{>0}}$ be a sequence of plurisubharmonic Green functions of D. If $\{g_n\}_{n\in\mathbb{Z}}$ converges to g uniformly on X^{an}, then g is also plurisubharmonic.

(3) The following conditions are equivalent:

(a) g is plurisubharmonic.

(b) ng is plurisubharmonic for all $n \in \mathbb{Z}_{>0}$.

(c) ng is plurisubharmonic for some $n \in \mathbb{Z}_{>0}$.

Proof. See [6, Proposition 2.11] for their proofs. □

We assume that $\mathcal{O}_X(D)$ is generated by global sections. Then there is a surjective morphism $f : H^0(X, nD) \otimes \mathcal{O}_X \to \mathcal{O}_X(nD)$ for all $n \in \mathbb{Z}_{>0}$. Hence we have a quotient Green function g_n of nD induced by $(H^0(X, nD), \|\cdot\|_{ng})$ and f.

Proposition 4.7.3 (c.f. [8, Proposition 2.2.22])**.**

(1) $ng \geq g_n$ on X^{an}.

(2) $g_{m+n} \geq g_m + g_n$ *on* X^{an}.

(3) $\|\cdot\|_{ng} = \|\cdot\|_{g_n}$ *on* $H^0(X, nD)$.

Proof. See [7, Lemma 3.5] for their proofs. □

Finally we see that the canonical Green function of a semiample Cartier divisor is plurisubharmonic. Let (D, g) be an adelic Cartier divisor on X and we assume that D is semiample.

Proposition 4.7.4. *If g is plurisubharmonic, then r^*g is also plurisubharmonic for all $r \in \mathbb{R}_{>0}$.*

Proof. Since g is plurisubharmonic, g is the uniform limit of Green functions induced by ultrametrically normed vector spaces $(V_n, \|\cdot\|_n)$. Then r^*g is the uniform limit of Green functions induced by $(V_n, \|\cdot\|_n^r)$, so it is plurisubharmonic by definition. □

Proposition 4.7.5. *The sequence $\{r^*g\}_{r\in\mathbb{R}_{>0}}$ converges to g_D^{c} uniformly on X^{an} as $r \to 0$.*

Proof. We set $u = g - g_D^{\mathrm{c}}$, which is a continuous function on X^{an}. Since $r^*g = r^*(g_D^{\mathrm{c}} + u) = g_D^{\mathrm{c}} + r^*u$ by Proposition 4.5.1, we have

$$\|r^*g - g_D^{\mathrm{c}}\|_{\sup} = \|r^*u\|_{\sup} = r\|u\|_{\sup},$$

which completes the proof. □

Proposition 4.7.6. *The canonical Green function g_D^{c} is plurisubharmonic.*

Proof. By replacing g by g_n/n for some $n \in \mathbb{Z}_{>0}$ if necessary, we can assume that g is plurisubharmonic. By Proposition 4.7.4 and 4.7.5, g_D^{c} is the uniform limit of the sequence $\{(1/n)^*g\}_{n\in\mathbb{Z}_{>0}}$ of plurisubharmonic Green functions. Hence g_D^{c} is plurisubharmonic by Proposition 4.7.2. □

4.7.2 Ample adelic Cartier divisors

In this section, we assume that K is perfect.

Definition 4.7.7. We say that an adelic Cartier divisor (D, g) is *vertically ample* if D is ample and g is plurisubharmonic. Moreover, if $\lambda_{\min}(nD, ng) > 0$ for any sufficiently large integer $n > 0$, (D, g) is said to be *ample*.

Remark 4.7.8. This definition is equivalent to the one in [8]. More precisely, we have "Nakai-Moishezon's criterion" for ample adelic Cartier divisors as follows: An adelic Cartier divisor (D, g) on X is ample if and only if for any closed subvariety Y of X, the restriction $(D|_Y, g|_{Y^{\mathrm{an}}})$ of (D, g) on Y is big, and g is plurisubharmonic.

For the criterion of ampleness, we introduce the following invariant:

Definition 4.7.9. Let (D, g) be an adelic $\mathbb{R}$-Cartier divisor on X. We define

$$\nu_{\min}(D, g) := \sup\{t \in \mathbb{R} \mid D_{\mu(g-t)} = D\}.$$

Proposition 4.7.10. *Let (D, g) be an adelic Cartier divisor on X. We assume D is ample. Then the following conditions are equivalent:*

(1) For $\forall n \gg 0$, $\lambda_{\min}(nD, ng) > 0$.

(2) For $\forall n > 0$, $\lambda_{\min}(nD, ng) > 0$.

(3) $\nu_{\min}(D, g) > 0$.

Proof. (1) $\Rightarrow$ (3) Since D is ample, there is $n_0 \in \mathbb{Z}_{>0}$ such that $\lambda_{\min}(n_0 D, n_0 g) > 0$ and

$$\mathrm{Sym}^n(H^0(X, n_0 D)) \to H^0(X, nn_0 D)$$

is surjective for all $n \in \mathbb{Z}_{>0}$. Let ϵ be a real number such that $0 < n_0\epsilon < \lambda_{\min}(n_0 D, n_0 g)$ and $g' = g - \epsilon$. Then we have $\lambda_{\min}(n_0 D, n_0 g') = \lambda_{\min}(n_0 D, n_0 g) - n_0\epsilon > 0$. Let $s_1, \ldots, s_m$ be a basis of $H^0(X, n_0 D)$ and we set $C = \max\{\|s_1\|_{n_0 g'}, \ldots, \|s_m\|_{n_0 g'}\} < 1$. Since $H^0(X, nn_0 D)$ is generated by $\{s_1^{a_1}, \ldots, s_m^{a_m}\}_{a_1+\cdots+a_m=n}$, we have

$$\lambda_{\min}(nn_0 D, nn_0 g') \geq -n \log C \tag{4.7.1}$$

for all $n \in \mathbb{Z}_{>0}$.

Claim 1. *There is a real number $A \in \mathbb{R}$ such that $\lambda_{\min}(nD, ng') \geq A - (n/n_0) \log C$ for all $n \in \mathbb{Z}_{\geq 0}$.*

Proof. Since $R = \bigoplus_{n=0}^{\infty} H^0(X, nD)$ is a finitely generated graded $S = \bigoplus_{n=0}^{\infty} H^0(X, nn_0 D)$-module, there is a generator $m_1, \ldots, m_l$ of R over S. Let d_i be the degree of m_i. By the equation (4.7.1), for each $n \in \mathbb{Z}_{>0}$, we have a basis $x_{n,1}, \ldots, x_{n,i_n}$ of $H^0(X, nn_0 D)$ such that $\|x_{n,j}\|_{nn_0 g'} \leq C^n$ for $j = 1, \ldots, i_n$. Then $H^0(X, nD)$ is generated by elements of the form $x_{i,j} m_k$ with $in_0 + d_k = n$. We set

$$B = \max_{k=1,\ldots,l}\{\|m_k\|_{d_k g'} C^{-d_k/n_0}\}.$$

Then we have

$$\begin{aligned}\|x_{i,j} m_k\|_{ng'} &\leq \|x_{i,j}\|_{in_0 g'} \|m_k\|_{d_k g'} \leq C^i \|m_k\|_{d_k g'} \\ &= C^{(n-d_k)/n_0} \|m_k\|_{d_k g'} \leq BC^{n/n_0}\end{aligned}$$

for $x_{i,j} m_k \in H^0(X, nD)$. Since $H^0(X, nD)$ is generated by such an element, we obtain that $\lambda_{\min}(nD, ng') \geq -\log B - n/n_0 \log C$. □

Since $\log C < 0$, there is $N \in \mathbb{Z}_{>0}$ such that $\lambda_{\min}(nD, n(g-\epsilon)) > 0$ for all $n \geq N$, which implies that

$$H^0(X, nD) = \mathcal{F}^0(H^0(X, nD), \|\cdot\|_{n(g-\epsilon)}) = H^0(X, nD_{\mu(g-\epsilon)})$$

for $n \geq N$. Hence we have $\mathrm{vol}(D) = \mathrm{vol}(D_{\mu(g-\epsilon)})$. Since $D \geq D_{\mu(g-\epsilon)}$ and K is perfect, we have $D = D_{\mu(g-\epsilon)}$ by [11]. Therefore, we obtain that $\nu_{\min}(D, g) \geq \epsilon > 0$.

(3) $\Rightarrow$ (2) Let ϵ be a real number such that $0 < \epsilon < \nu_{\min}(D, g)$. Since $D = D_{\mu(g-\epsilon)}$, we have $\|s\|_{n(g-\epsilon)} \leq 1$ for all $s \in H^0(X, nD)$ and $n \in \mathbb{Z}_{>0}$, which is equivalent to $\|s\|_{ng} \leq e^{-n\epsilon} < 1$. Thus (2) follows.

(2) $\Rightarrow$ (1) It is clear. $\square$

Corollary 4.7.11. *Let (D, g) be an adelic Cartier divisor on X. We assume that D is ample and g is plurisubharmonic. Then (D, g) is ample if and only if $\nu_{\min}(D, g) > 0$.*

Finally, we see the ampleness of (D, g) in terms of the height function defined by (D, g).

Proposition 4.7.12. *Let (D, g) be an adelic $\mathbb{R}$-Cartier divisor on X. Then we have*

$$\min_{x \in X^{\mathrm{an}}} h^{\mathrm{an}}_{(D,g)}(x) = \nu_{\min}(D, g).$$

Proof. Firstly, we show that

$$\mu_x(g^{\mathrm{c}}_D) = \mathrm{ord}_x(D) \tag{4.7.2}$$

for all $x \in X^{(1)}$. Let f be a local equation of D around $\mathrm{red}(x)$. Then we have

$$\mu_x(g^{\mathrm{c}}_D) = \inf_{\xi \in (\eta^{\mathrm{an}}, x^{\mathrm{an}})} \frac{-\log|f|_\xi}{t(\xi)} = \inf_{\xi \in (\eta^{\mathrm{an}}, x^{\mathrm{an}})} \frac{\mathrm{ord}_x(f) t(\xi)}{t(\xi)} = \mathrm{ord}_x(D).$$

Let a be a real number. For $x \in X^{(1)}$, we obtain that

$$\begin{aligned}\mu_x(g - a) &= \inf_{\xi \in (\eta^{\mathrm{an}}, x^{\mathrm{an}})} \frac{g(\xi) - g^{\mathrm{c}}_D(\xi) - a + g^{\mathrm{c}}_D(\xi)}{t(\xi)} \\ &= \inf_{\xi \in (\eta^{\mathrm{an}}, x^{\mathrm{an}})} \frac{h^{\mathrm{an}}_{(D,g)}(\xi) - a}{t(\xi)} + \mathrm{ord}_x(D) = \mu_x(h^{\mathrm{an}}_{(D,g)} - a) + \mathrm{ord}_x(D)\end{aligned}$$

by the equation (4.7.2). Hence it follows that

$$D_{\mu(g-a)} = 0_{\mu(h^{\mathrm{an}}_{(D,g)} - a)} + D.$$

Since $0_{\mu(h^{\mathrm{an}}_{(D,g)} - a)} \leq 0$, and $0_{\mu(h^{\mathrm{an}}_{(D,g)} - a)} = 0$ if and only if $h^{\mathrm{an}}_{(D,g)} - a \geq 0$ on X^{an}, we have $\min_{x \in X^{\mathrm{an}}} h^{\mathrm{an}}_{(D,g)}(x) = \nu_{\min}(D, g)$. $\square$

By the above proposition, we have the following corollary:

Corollary 4.7.13 (Theorem B)**.** *Let (D, g) be an adelic $\mathbb{R}$-Cartier divisor on X. Then $h^{\mathrm{an}}_{(D,g)} > 0$ on X^{an} if (D, g) is ample. In particular, a vertically ample adelic Cartier divisor (D, g) on X is ample if and only if $h^{\mathrm{an}}_{(D,g)} > 0$ on X^{an}.*

4.7.3 Plurisubharmonic approximation

In algebraic geometry, as a theorem related to the ampleness and bigness, there is the Fujita's approximation theorem. It states that the volume of a big divisor D can be approximated by the one of a ample divisor A such that $D - A$ is effective. We expect that a similar theorem holds for a trivial valued case. We only have a partial answer, that is, the volume of an adelic Cartier divisor $\overline{D} = (D, g)$ can be approximated by replacing g by a plurisubharmonic Green function g' of D such that $g \geq g'$. We believe that it might be useful for complete proof of Fujita's approximation theorem.

Theorem 4.7.14. *Let (D, g) be an adelic Cartier divisor. We assume that $\mathcal{O}_X(D)$ is generated by global sections. For each $n \geq 1$, let g_n be the quotient Green function of nD induced by $(H^0(X, nD), \|\cdot\|_{ng})$. Then we have the followings:*

(1) $\lim_{n\to+\infty} \lambda^{\text{asy}}_{\max}(D, g_n/n) = \lambda^{\text{asy}}_{\max}(D, g)$.

(2) $\lim_{n\to+\infty} \widehat{\text{vol}}(D, g_n/n) = \widehat{\text{vol}}(D, g)$.

(3) If D is ample and $\nu_{\min}(D, g) > 0$, we have $\nu_{\min}(D, g_n/n) > 0$ for $\forall n \gg 0$.

Proof. (1) By Proposition 4.7.3 (1), we have $\lambda^{\text{asy}}_{\max}(D, g_n/n) \leq \lambda^{\text{asy}}_{\max}(D, g)$. By the definition of $\lambda^{\text{asy}}_{\max}(D, g)$, for any $\epsilon > 0$, there is a positive integer $N > 0$ such that $\lambda^{\text{asy}}_{\max}(D, g) - \epsilon \leq \lambda_{\max}(nD, ng)/n$ for any integer $n \geq N$. Since $\lambda_{\max}(nD, g_n) = \lambda_{\max}(nD, ng)$ by Proposition 4.7.3 (3), we have

$$\lambda^{\text{asy}}_{\max}(D, g) - \epsilon \leq \lambda_{\max}(nD, g_n)/n \leq \lambda^{\text{asy}}_{\max}(D, g_n/n), \quad \text{for } \forall n \geq N,$$

which completes the proof.

(2) If (D, g) is not big, then we have $\widehat{\text{vol}}(D, g_n/n) = \widehat{\text{vol}}(D, g) = 0$ for every positive integer $n > 0$ because $\widehat{\text{vol}}(D, g_n/n) \leq \widehat{\text{vol}}(D, g)$ by Proposition 4.7.3 (1). So we assume (D, g) is big. By Theorem 4.6.7, we obtain that

$$\widehat{\text{vol}}(D, g) = (d + 1) \int_0^{\lambda^{\text{asy}}_{\max}(D,g)} \text{vol}(D_{\mu(g-t)})\ \mathrm{d}t.$$

In addition, the Green function of nD induced by the surjective morphism $H^0(nD) \otimes \mathcal{O}_X \to \mathcal{O}_X(nD)$ and $ng - t$ for all $t \in \mathbb{R}$ is $g_n - t$. Hence, once we show that $\text{vol}(D_{\mu(g_n/n)})$ converges to $\text{vol}(D_{\mu(g)})$ as $n \to \infty$, we get the required result.

Firstly, we can assume that (D, g_1) is big by replacing D by mD for a sufficiently large $m > 0$ by (1). For any $x \in X^{(1)}$, we have $\mu_x(g_{n+m}) \geq \mu_x(g_n) + \mu_x(g_m)$ for any $n, m > 0$ by Proposition 4.7.3 (2). Hence $\{\mu_x(g_n)\}_{n\geq 1}$ is superadditive, which implies that there exists

$$\mu_x(g_\infty) := \lim_{n\to+\infty} \mu_x(g_n/n) = \sup_{n\geq 1} \mu_x(g_n/n). \tag{4.7.3}$$

Since $\mu_x(g_1) \leq \mu_x(g_n/n) \leq \mu_x(g)$ for all $n > 0$ by Proposition 4.7.3, we have $\mu_x(g_n/n) = 0$ for all but finitely many $x \in X^{(1)}$. Therefore the $\mathbb{R}$-Weil divisor

$$D_\infty := \sum_{x \in X^{(1)}} \mu_x(g_\infty)[\overline{\{x\}}]$$

is well-defined by (4.7.3). Moreover, it satisfies that $D_{\mu(g_n/n)} \leq D_\infty \leq D_{\mu(g)}$ for all $n > 0$ by (4.7.3) and Proposition 4.7.3 (1), and $\mathrm{vol}(D_{\mu(g_n/n)}) \to \mathrm{vol}(D_\infty)$ as $n \to +\infty$ because of the continuity of the algebraic volume. On the other hand, for each $n > 0$ we have $H^0(D_{\mu(g_n)}) \subseteq H^0(nD_\infty) \subseteq H^0(nD_{\mu(g)})$. By Proposition 4.7.3 (3), we have $H^0(D_{\mu(g_n)}) = H^0(nD_{\mu(g)})$ for all $n > 0$, which implies $\bigoplus_{n \geq 1} H^0(nD_\infty) = \bigoplus_{n \geq 1} H^0(nD_{\mu(g)})$. Hence we obtain that $\mathrm{vol}(D_\infty) = \mathrm{vol}(D_{\mu(g)})$. So we get $\mathrm{vol}(D_{\mu(g_n/n)}) \to \mathrm{vol}(D_{\mu(g)})$ as $n \to +\infty$.

(3) Since $\lambda_{\min}(nD, ng) = \lambda_{\min}(nD, g_n)$ for all $n \in \mathbb{Z}_{>0}$ by Proposition 4.7.3 (3), it follows from Proposition 4.7.10. □

Immediately we have the following corollary:

Corollary 4.7.15 (Plurisubharmonic approximation)**.** *Let (D, g) be a big adelic Cartier divisor with an ample divisor D. For any $\epsilon > 0$, there is a plurisubharmonic Green function g' of D such that $g' \leq g$ and*

$$\widehat{\mathrm{vol}}(D, g') \geq \widehat{\mathrm{vol}}(D, g) - \epsilon.$$

Moreover, if $\nu_{\min}(D, g) > 0$, we can choose g' such that (D, g') is ample.

Proof. The first assertion easily follows from Theorem 4.7.14 (2). The last one is given by Corollary 4.7.11 and Theorem 4.7.14 (3). □

Chapter 5

Kawaguchi-Silverman's conjecture over adelic curves

Throughout this chapter, we fix a proper adelic curve $S = (K, (\Omega, \mathcal{A}, \nu), \phi)$. For simplicity, we assume that $\Omega_0 = \emptyset$.

5.1 Conjecture

Let X be a smooth projective variety over an algebraic closure $\overline{K}$ of K and $f : X \dashrightarrow X$ be a dominant rational map. We denote by I_f the indeterminacy locus of f and set

$$X_f(\overline{K}) := \{P \in X(\overline{K}) \mid f^n(P) \notin I_f \text{ for all } n > 0\}.$$

We take a height function h_X associated with some adelic Cartier divisor $\overline{D}$ on X whose underlying Cartier divisor D is ample. Let $h_X^+ := \max\{h_X, 1\}$. First, we recall the arithmetic degree.

Definition 5.1.1. For $P \in X_f(\overline{K})$, we define the *upper and lower arithmetic degrees* of P with respect to f as

$$\overline{\alpha}_f(P) := \limsup_{n\to\infty} h_X^+(f^n(P))^{\frac{1}{n}},$$
$$\underline{\alpha}_f(P) := \liminf_{n\to\infty} h_X^+(f^n(P))^{\frac{1}{n}}.$$

Note that the above definitions are independent of the choice of $\overline{D}$. In fact, let $\overline{D}'$ be another adelic Cartier divisor on X whose underlying divisor D' be ample. We denote the height function associated with $\overline{D}'$ by h'_X, we set $h'^+_X = \max\{h'_X, 1\}$. By Corollary 3.2.7, there are positive constants c_1 and c_2 such that

$$c_1 h'^+_X \le h_X^+ \le c_2 h'^+_X.$$

Hence by taking a limit, we obtain that

$$\overline{\alpha}_f(P) = \limsup_{n\to\infty} h'^{+}_X(f^n(P))^{\frac{1}{n}},$$
$$\underline{\alpha}_f(P) = \liminf_{n\to\infty} h'^{+}_X(f^n(P))^{\frac{1}{n}}.$$

By definition, we have

$$1 \le \underline{\alpha}_f(P) \le \overline{\alpha}_f(P).$$

If $\underline{\alpha}_f(P) = \overline{\alpha}_f(P)$, the *arithmetic degree* $\alpha_f(P)$ of P with respect to f is defined as $\overline{\alpha}_f(P)$.

Here, we restate the conjecture over adelic curves:

Conjecture 5.1.2 (Conjecture C)**.** *Let $S = (K, (\Omega, \mathcal{A}, \nu), \phi)$ be a proper adelic curve. Let X be a smooth projective variety over an algebraic closure $\overline{K}$ of K and $f : X \dashrightarrow X$ be a dominant rational map. For any $P \in X_f(\overline{K})$,*

(1) The arithmetic degree $\alpha_f(P) = \lim_{n\to\infty} h^+_X(f^n(P))^{1/n}$ exists.

(2) We assume that S has the Northcott property. If the orbit $\mathcal{O}_f(P) = \{f^n(P) \mid n = 0, 1, \dots\}$ of P is Zariski dense, then we have $\alpha_f(P) = \delta_f$.

In Conjecture 5.1.2 (2), we cannot drop the assumption that S has the Northcott property even if f is a morphism. Here is an easy example:

Example 5.1.3. Let S be an adelic curve in Example 2.1.4. We assume that $C = \mathbb{P}^1$, that is, $K = k(t)$. Then S does not have the Northcott property. In fact, $h_S(1 : a) = 0$ for all $a \in \overline{k}$.

Let $f : \mathbb{P}^1 \to \mathbb{P}^1$ be the morphism defined by $x \mapsto x^2$ for $x \in \mathbb{P}^1$. Then we have $\delta_f = 2$. On the other hand, let $n \in \mathbb{P}^1(\overline{k}) \subset \mathbb{P}^1(\overline{K})$ be an integer which is not divisible by the characteristic of k. Then $\mathcal{O}_f(P) = \{n^{2m} \mid m = 1, 2, \dots\}$ is Zariski dense in $\mathbb{P}^1$ and $\alpha_f(n) = 1 < 2 = \delta_f$.

For another examples, see [28, Example 3.7].

5.2 Fundamental inequality

Firstly, we prove the fundamental inequality about the arithmetic degree and the dynamical degree.

Theorem 5.2.1 (Theorem D)**.** *Let X be a smooth projective variety over an algebraic closure $\overline{K}$ of K and $f : X \dashrightarrow X$ be a dominant rational map over $\overline{K}$. For any $\epsilon > 0$, there is a constant $C > 0$ such that*

$$\forall n \ge 0, \forall P \in X_f(\overline{K}), \quad h^+_X(f^n(P)) \le C(\delta_f + \epsilon)^n h^+_X(P).$$

In particular, we have

$$\overline{\alpha}_f(P) \le \delta_f.$$

Proof. Let $(D_1, g_{D_1}), \dots, (D_r, g_{D_r})$ be adelic Cartier divisors on X whose underlying divisors are very ample and make a basis of $N^1(X)_{\mathbb{R}}$ where $r = \dim N^1(X)_{\mathbb{R}}$. Let H be an ample divisor on X such that $H \pm D_i$ are ample for $i = 1, \dots, r$. We choose non-negative numbers $c_1, \dots, c_r$ which satisfy

$$H \equiv \sum_{k=1}^{r} c_k D_k.$$

Let $p : Y \to X$ be a resolution of f with a projective variety Y such that $f' = f \circ p$ is a morphism:

$$\begin{array}{ccc} Y & & \\ \downarrow{\scriptstyle p} & \searrow^{f'} & \\ X & \overset{f}{\dashrightarrow} & X \end{array}$$

We denote the exceptional locus of p by $\mathrm{Exc}(p)$. By the negativity lemma (for example, see [21, Lemma 3.35]),

$$Z_i := p^* p_* f'^* D_i - f'^* D_i$$

is effective and $\mathrm{Supp}(Z_i) \subset \mathrm{Exc}(p)$ for $i = 1, \dots, r$. We set $F_i := f'^* D_i$ for $i = 1, \dots, r$. Then we can write

$$Z_i = p^* p_* F_i - F_i \tag{5.2.1}$$

for $i = 1, \dots, r$.

As $F_1, \dots, F_r$ are linearly independent over $\mathbb{R}$, we can choose Cartier divisors $F_{r+1}, \dots, F_s$ on Y such that $F_1, \dots, F_s$ form a basis of $N^1(Y)_{\mathbb{R}}$ where $s = \dim N^1(Y)_{\mathbb{R}}$.

We can take an ample $\mathbb{Q}$-divisor H' on Y such that $p^*H - H'$ is effective and its support is contained in $\mathrm{Exc}(p)$. In fact, let G be an effective p-exceptional divisor such that $-G$ is p-ample. Then, $H' = -G/n + p^*H$ is required one for sufficiently large n. We set

$$p_* F_j \equiv \sum_{k=1}^{r} b_{kj} D_k$$

for $j = 1, \dots, s$. Let A and B be $s \times r$ matrix and $r \times s$ matrix respectively such that

$$A = \left.\begin{pmatrix} \overbrace{\begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix}}^{r} \\ \begin{matrix} 0 & & 0 \\ & \ddots & \\ 0 & & 0 \end{matrix} \end{pmatrix}\right\} s$$

and $B = (b_{ij})$. We set $\boldsymbol{D} = \begin{pmatrix} D_1 \\ \vdots \\ D_r \end{pmatrix}$, $\boldsymbol{F} = \begin{pmatrix} F_1 \\ \vdots \\ F_s \end{pmatrix}$, $\boldsymbol{c} = \begin{pmatrix} c_1 \\ \vdots \\ c_r \end{pmatrix}$ and $\boldsymbol{Z} = \begin{pmatrix} Z_1 \\ \vdots \\ Z_r \end{pmatrix}$. Then we have

$$f'^* \boldsymbol{D} \equiv {}^t A \boldsymbol{F}, \quad p_* \boldsymbol{F} \equiv {}^t B \boldsymbol{D}, \quad f'^* H \equiv \langle A\boldsymbol{c}, \boldsymbol{F} \rangle,$$

where $\langle \cdot, \cdot \rangle$ is the usual inner product. Let

$$\begin{cases} E := f'^* H - \langle A\boldsymbol{c}, \boldsymbol{F} \rangle, & (5.2.2) \\ E_j := p_* F_j - \sum_{k=1}^{r} b_{kj} D_k, & (5.2.3) \end{cases}$$

for $j = 1, \ldots, s$. Note that they are numerically equivalent to zero.

By Corollary 3.2.6, we can find a Green function g_H of H such that

$$\begin{cases} h_{(H, g_H)} \geq 1, \\ h_{(H, g_H)} \geq |h_{(D_i, g_{D_i})}|, \end{cases} \tag{5.2.4}$$

for $i = 1, \ldots, r$. We define a Green function family g'_{F_i} of F_i by $g'_{F_i} = g_{D_i} \circ f'$ for $i = 1, \ldots, r$, and take a Green function family g'_{F_j} of F_j arbitrary for $j = r+1, \ldots, s$. Moreover, we can fix a Green function family $g_{p_* F_j}$ of $p_* F_j$ for $j = 1, \ldots, s$ such that

$$h_{(Z_i, g_{p_* F_i} \circ p - g'_{F_i})} \geq 0 \text{ on } Y \setminus Z_i$$

for $i = 1, \ldots, r$ by Proposition 3.2.5. Let g'_E and g_{E_j} be Green function families of E and E_j which are defined by equations (5.2.2) and (5.2.3) for $j = 1, \ldots, s$, respectively, that is,

$$\begin{cases} g'_E = g_H \circ f' - \sum_{k=1}^{r} c_k g'_{F_k}, \\ g_{E_j} = g_{p_* F_j} - \sum_{k=1}^{r} b_{kj} g_{D_k}. \end{cases}$$

Let $g'_{H'}$ be a Green function of H' such that $h_{(H', g'_{H'})} \geq 1$ and $g'_{p^*H - H'}$ be a Green function of $p^*H - H'$ such that $h_{(p^*H - H', g'_{p^*H-H'})} \geq 0$ on $Y \setminus \mathrm{Exc}(p)$. Since

$$h_{(p^*H, g_H \circ p)} = h_{(p^*H - H', g_{p^*H-H'})} + h_{(H', g'_{H'})} + O(1),$$

there is a constant $\gamma \geq 0$ such that

$$h_{(p^*H, g_H \circ p)} \geq h_{(p^*H - H', g_{p^*H-H'})} + h_{(H', g'_{H'})} - \gamma. \tag{5.2.5}$$

By Corollary 3.3.4, there exists a constant $C > 0$ such that

$$\begin{cases} |h_{(E, g'_E)}| \leq C \sqrt{h_{(H', g'_{H'})}}, \\ |h_{(E_j, g_{E_j})}| \leq C \sqrt{h_{(H, g_H)}}, \end{cases} \tag{5.2.6}$$

for $j = 1, \ldots, s$.

Let P be an element of $X_f(\overline{K})$. We remark that $p^{-1}(f^i(P))$ is well-defined and it is not contained in $\mathrm{Exc}(p)$ for $i \geq 0$. For a positive integer n,

$$
\begin{aligned}
h_{(H,g)}(f^n(P)) &= h_{(f'^*H,g\circ f')}(p^{-1}(f^{n-1})(P)) \\
&= h_{(f'^*H,g\circ f')}(p^{-1}(f^{n-1})(P)) - \langle A\boldsymbol{c}, \boldsymbol{h}_{p^*p_*\boldsymbol{F}}\rangle(p^{-1}(f^{n-1}(P))) \\
&\qquad + \langle A\boldsymbol{c}, \boldsymbol{h}_{p_*\boldsymbol{F}}\rangle(f^{n-1}(P)) \\
&= \langle A\boldsymbol{c}, \boldsymbol{h}_{\boldsymbol{F}} - \boldsymbol{h}_{p^*p_*\boldsymbol{F}}\rangle(p^{-1}(f^{n-1}(P))) + h_{(E,g'_E)}(p^{-1}(f^{n-1}(P))) \\
&\qquad + \langle BA\boldsymbol{c}, \boldsymbol{h}_{\boldsymbol{D}}\rangle(f^{n-1}(P)) + \langle A\boldsymbol{c}, \boldsymbol{h}_{\boldsymbol{E}'}\rangle(f^{n-1}(P)) \quad \text{(by (5.2.2), (5.2.3))} \\
&= \langle \boldsymbol{c}, -\boldsymbol{h}_{\boldsymbol{Z}}\rangle(p^{-1}(f^{n-1}(P))) + h_{(E,g'_E)}(p^{-1}(f^{n-1}(P))) \\
&\qquad + \langle BA\boldsymbol{c}, \boldsymbol{h}_{\boldsymbol{D}}\rangle(f^{n-1}(P)) + \langle \boldsymbol{c}, {}^tA\boldsymbol{h}_{\boldsymbol{E}'}\rangle(f^{n-1}(P)), \quad \text{(by (5.2.1))} \qquad (5.2.7)
\end{aligned}
$$

where $\boldsymbol{h}_{\boldsymbol{D}'} = (h_{\overline{D}'})$ for $D' = p^*p_*F, p_*F, F, D, E'$ and Z. Since $h_{Z_i} \geq 0$ on $Y \setminus \mathrm{Exc}(p)$ for $i = 1, \ldots, r$, we have

$$
\begin{aligned}
&h_{(H,g_H)}(f^n(P)) \\
&\leq h_{(E,g'_E)}(p^{-1}(f^{n-1}(P))) + \langle BA\boldsymbol{c}, \boldsymbol{h}_{\boldsymbol{D}}\rangle(f^{n-1}(P)) + \langle \boldsymbol{c}, {}^tA\boldsymbol{h}_{\boldsymbol{E}'}\rangle(f^{n-1}(P)) \\
&\leq C\sqrt{h_{(H',g'_{H'})}(p^{-1}(f^{n-1}(P)))} + r^2\|\boldsymbol{c}\|\|BA\|h_{(H,g_H)}(f^{n-1}(P)) \\
&\qquad + r\|\boldsymbol{c}C\|\sqrt{h_{(H,g_H)}(f^{n-1}(P))}, \quad \text{(by (5.2.4), (5.2.6))}
\end{aligned}
$$

where $\|\cdot\|$ means vector norm or matrix norm on real vector spaces. Moreover, since $h_{(p^*H-H',h'')} \geq 0$ on $Y \setminus \mathrm{Exc}(p)$ and (5.2.5), it follows that

$$
\begin{aligned}
h_{(H,g_H)}(f^n(P)) \leq r^2\|\boldsymbol{c}\|\|BA\|h_{(H,g_H)}(f^{n-1}(P)) + r\|\boldsymbol{c}\|C\sqrt{h_{(H,g_H)}(f^{n-1}(P))} \\
+ C\sqrt{h_{(H,g_H)}(f^{n-1}(P)) + \gamma}. \qquad (5.2.8)
\end{aligned}
$$

For a linear map $F : N^1(X)_{\mathbb{R}} \to N^1(X)_{\mathbb{R}}$, we denote by $M(F)$ the representation matrix of F with respect to D_i's. Then we have $BA = M(f^*)$. Let $R(f) := \max\{1, r^2\|\boldsymbol{c}\|\|M(f^*)\|\}$. By dividing the inequality (5.2.8) by $R(f)^n$, we obtain that

$$
\begin{aligned}
\frac{h_{(H,g_H)}(f^n(P))}{R(f)^n} &\leq \frac{r^2\|\boldsymbol{c}\|\|BA\|}{R(f)^n}h_{(H,g_H)}(f^{n-1}(P)) + \frac{r\|\boldsymbol{c}\|C}{R(f)^n}\sqrt{h_{(H,g_H)}(f^{n-1}(P))} \\
&\qquad + \frac{C}{R(f)^n}\sqrt{h_{(H,g_H)}(f^{n-1}(P)) + \gamma} \\
&\leq \frac{h_{(H,g_H)}(f^{n-1}(P))}{R(f)^{n-1}} + r\|\boldsymbol{c}\|C\sqrt{\frac{h_{(H,g_H)}(f^{n-1}(P))}{R(f)^{n-1}}} \\
&\qquad + C\sqrt{\frac{h_{(H,g_H)}(f^{n-1}(P))}{R(f)^{n-1}} + \gamma}.
\end{aligned}
$$

By [26, Appendix Lemma A.1], there is a constant C_1 which is independent of n and P such that

$$\frac{h_{(H,g_H)}(f^n(P))}{R(f)^n} \leq C_1 n^2 h_{(H,g_H)}(P),$$

that is,

$$h_{(H,g_H)}(f^n(P)) \leq C_1 n^2 R(f)^n h_{(H,g_H)}(P). \tag{5.2.9}$$

Let $\epsilon > 0$ be an arbitrary positive number. Since $\lim_{k\to\infty} \|M((f^k)^*)\|^{1/k} = \delta_f$, there is a positive integer $k > 0$ such that

$$\frac{\|M((f^k)^*)\|}{(\delta_f + \epsilon)^k} r^2 \|\boldsymbol{c}\| < 1. \tag{5.2.10}$$

By the inequality (5.2.9) for f^k, we have

$$h_{(H,g_H)}(f^{kn}(P)) \leq C_1 n^2 \left(\frac{R(f^k)}{(\delta_f + \epsilon)^k}\right)^n (\delta_f + \epsilon)^{kn} h_{(H,g_H)}(P).$$

Since $R(f^k) = \max\{1, r^2\|\boldsymbol{c}\|\|M((f^k)^*)\|\}$ and by the inequality (5.2.10), it follows that

$$\frac{R(f^k)}{(\delta_f + \epsilon)^k} < 1.$$

Hence there is a constant $C_2 > 0$ such that

$$C_1 n^2 \left(\frac{R(f^k)}{(\delta_f + \epsilon)^k}\right)^n \leq C_2$$

for all n. Then we obtain that

$$h_{(H,g_H)}(f^{kn}(P)) \leq C_2 (\delta_f + \epsilon)^{kn} h_{(H,g_H)}(P).$$

By Corollary 3.2.7, there is a constant $C_3 > 0$ such that

$$h_X^+ \leq C_3 h_{(H,g_H)}, \quad h_{(H,g_H)} \leq C_3 h_X^+.$$

Hence we have

$$h_X^+(f^{kn}(P)) \leq C_3 h_{(H,g_H)}(f^{kn}(P)) \leq C_2 C_3 (\delta_f + \epsilon)^{kn} h_{(H,g_H)}(P) \leq C_2 C_3^2 (\delta_f + \epsilon)^{kn} h_X^+(P).$$

By the same argument as the proof after [26, Lemma 3.3], we get the conclusion. □

Remark 5.2.2. If f is a morphism, we have a little stronger inequality and we can drop the assumption of smoothness of X (for details, see [26]).

5.3 Morphism case

We prove (1) in Conjecture 5.1.2 when f is a morphism by using methods in [19].

Theorem 5.3.1 (Theorem E). *Let $S = (K, (\Omega, \mathcal{A}, \nu), \phi)$ be a proper adelic curve. Let X be a normal projective variety over an algebraic closure $\overline{K}$ of K and $f : X \to X$ be a morphism. For $P \in X(\overline{K})$, we have*

(1) $\overline{\alpha}_f(P) = \underline{\alpha}_f(P)$. In particular, the limit $\alpha_f(P) = \lim_{n\to\infty} h_X^+(f^n(P))^{1/n}$ exists.

(2) The arithmetic degree $\alpha_f(P)$ is an algebraic integer.

(3) The set $\{\alpha_f(Q) \mid Q \in X(\overline{K})\}$ is finite.

Before starting proof, we recall the canonical height theory associated with Jordan blocks in $\mathrm{Pic}(X)_{\mathbb{C}}$ introduced by [19].

Proposition 5.3.2 (c.f. [19, Theorem 13]). *Let λ be a complex number. Let $\overline{D}_0, \overline{D}_1, \dots, \overline{D}_l \in \widehat{\mathrm{Div}}(X)_{\mathbb{C}}$ be adelic divisors which satisfy the Jordan block condition in $\mathrm{Pic}(X)_{\mathbb{C}}$:*

$$f^*D_0 \sim \lambda D_0, f^*D_1 \sim \lambda D_1 + D_0, \dots, f^*D_l \sim \lambda D_l + D_{l-1},$$

where the symbol "$\sim$" means $\mathbb{C}$-linearly equivalence.

(1) There exists a constant $C > 0$ such that

$$\forall n \geq 0, \forall P \in X(\overline{K}), \quad \|\boldsymbol{h_D}(f^n(P))\| \leq Cn^l \max\{|\lambda|, 1\}^n (\|\boldsymbol{h_D}(P)\| + 1),$$

where $\boldsymbol{h_D}(Q) = (h_{\overline{D}_0}(Q), \dots, h_{\overline{D}_l}(Q))$ for $Q \in X(\overline{K})$.

(2) Suppose that $|\lambda| > 1$. Then there exists a function $\hat{\boldsymbol{h}}_{\boldsymbol{D}} : X(\overline{K}) \to \mathbb{C}^{l+1}$ such that

$$\hat{\boldsymbol{h}}_{\boldsymbol{D}} \circ f = \Lambda \hat{\boldsymbol{h}}_{\boldsymbol{D}}$$

and

$$\hat{\boldsymbol{h}}_{\boldsymbol{D}} = \boldsymbol{h_D} + O(1),$$

where

$$\Lambda = \begin{pmatrix} \lambda & 0 & \cdots & \cdots & 0 \\ 1 & \lambda & & & \vdots \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \lambda & 0 \\ 0 & \cdots & \cdots & 1 & \lambda \end{pmatrix}.$$

Proof. (1) We set

$$\begin{cases} \overline{E}_0 := f^*\overline{D}_0 - \lambda\overline{D}_0, \\ \overline{E}_i := f^*\overline{D}_i - (\lambda\overline{D}_i + \overline{D}_{i-1}), \end{cases}$$

for $1 \le i \le l$ and $\boldsymbol{h_E} := (h_{\overline{E}_0}, \dots, h_{\overline{E}_l})$. Note that $E_0, \dots, E_l$ are linearly equivalent to zero. By definition, the function $\boldsymbol{h_E}$ satisfies that

$$\boldsymbol{h_E} = \boldsymbol{h_D} \circ f - \Lambda\boldsymbol{h_D}.$$

Thus for $n \ge 1$, we can see

$$\begin{aligned} \boldsymbol{h_D}(f^n(P)) &= \Lambda\boldsymbol{h_D}(f^{n-1}(P)) + \boldsymbol{h_E}(f^{n-1}(P)) \\ &= \Lambda^2\boldsymbol{h_D}(f^{n-2}(P)) + \Lambda\boldsymbol{h_E}(f^{n-2}(P)) + \boldsymbol{h_E}(f^{n-1}(P)) \\ &\vdots \\ &= \Lambda^n\boldsymbol{h_D}(P) + \sum_{k=0}^{n-1}\Lambda^{n-k-1}\boldsymbol{h_E}(f^k(P)). \end{aligned}$$

By [19, Lemma 12(a)], we have

$$\forall k \ge 0, \quad \|\Lambda^k\| \le k^l \max\{|\lambda|, 1\}^k. \tag{5.3.1}$$

Since $E_0, \dots, E_l$ are linearly equivalent to zero, the height functions $h_{E_0}, \dots, h_{E_l}$ are bounded. Hence there exists a constant $C > 0$ such that

$$\forall Q \in X(\overline{K}), \quad \|\boldsymbol{h_E}(Q)\| \le C. \tag{5.3.2}$$

Thus we obtain that

$$\begin{aligned} &\|\boldsymbol{h_D}(f^n(P))\| \\ &\le \|\Lambda^n\boldsymbol{h_D}(P)\| + \sum_{k=0}^{n-1}\|\Lambda^{n-k-1}\boldsymbol{h_E}(f^k(P))\| \\ &\le (l+1)\|\Lambda^n\|\|\boldsymbol{h_D}(P)\| + \sum_{k=0}^{n-1}(l+1)\|\Lambda^{n-k-1}\|\|\boldsymbol{h_E}(f^k(P))\| \quad \text{(by (5.3.1))} \\ &\le (l+1)n^l\max\{|\lambda|,1\}^n\|\boldsymbol{h_D}(P)\| + (l+1)\sum_{k=0}^{n-1}C(n-k-1)^l\max\{|\lambda|,1\}^{n-k-1} \quad \text{(by (5.3.2))} \\ &\le (l+1)n^l\max\{|\lambda|,1\}^n\|\boldsymbol{h_D}(P)\| + (l+1)Cn^l\max\{|\lambda|,1\}^n, \end{aligned}$$

which completes the proof.

(2) We set

$$\hat{\boldsymbol{h}}_{\boldsymbol{D}} := \boldsymbol{h_D} + \sum_{n=0}^{\infty}\Lambda^{-n-1}\boldsymbol{h_E} \circ f^n.$$

We claim that $\hat{\boldsymbol{h}}_{\boldsymbol{D}}$ is absolutely convergent. We write $\Lambda = \lambda I + N$, where I is the identity matrix. Note that $N^{l+1} = 0$. Then we have

$$\begin{aligned}
\|\Lambda^{-n}\| &= \|(\lambda I + N)^{-n}\| \\
&= \left\| \sum_{k=0}^{l} \binom{-n}{k} \lambda^{-n-k} N^k \right\| \\
&\leq |\lambda|^{-n}(l+1) \max_{0\leq k\leq l} \left| \binom{-n}{k} \right| \\
&\leq (l+1) n^l |\lambda|^{-n}.
\end{aligned}$$

Hence we obtain that

$$\begin{aligned}
\sum_{n=0}^{\infty} \|\Lambda^{-n-1} \boldsymbol{h}_{\boldsymbol{E}}(f^n(P))\| &\leq (l+1) \sum_{n=0}^{\infty} \|\Lambda^{-n-1}\| \|\boldsymbol{h}_{\boldsymbol{E}}(f^n(P))\| \\
&\leq (l+1)^2 C \sum_{n=0}^{\infty} n^l |\lambda|^{-n-1}.
\end{aligned}$$

Since $|\lambda| > 1$, $\sum_{n=0}^{\infty} n^l |\lambda|^{-n-1}$ is convergent, which implies that $\hat{\boldsymbol{h}}_{\boldsymbol{D}}$ is absolutely convergent and $\hat{\boldsymbol{h}}_{\boldsymbol{D}} - \boldsymbol{h}_{\boldsymbol{D}}$ is a bounded function. Finally, we have

$$\begin{aligned}
\hat{\boldsymbol{h}}_{\boldsymbol{D}} \circ f &= \boldsymbol{h}_{\boldsymbol{D}} \circ f + \sum_{n=0}^{\infty} \Lambda^{-n-1} \boldsymbol{h}_{\boldsymbol{E}} \circ f^{n+1} \\
&= \boldsymbol{h}_{\boldsymbol{D}} \circ f + \sum_{n=1}^{\infty} \Lambda^{-n} \boldsymbol{h}_{\boldsymbol{E}} \circ f^{n} \\
&= \boldsymbol{h}_{\boldsymbol{D}} \circ f - \boldsymbol{h}_{\boldsymbol{E}} + \sum_{n=0}^{\infty} \Lambda^{-n} \boldsymbol{h}_{\boldsymbol{E}} \circ f^{n} \\
&= \Lambda \boldsymbol{h}_{\boldsymbol{D}} + \Lambda \sum_{n=0}^{\infty} \Lambda^{-n-1} \boldsymbol{h}_{\boldsymbol{E}} \circ f^{n} = \Lambda \hat{\boldsymbol{h}}_{\boldsymbol{D}}.
\end{aligned}$$

This calculation works well because the series defining $\hat{\boldsymbol{h}}_{\boldsymbol{D}}$ is absolutely convergent. □

Proof of Theorem 5.3.1. Let $\overline{H}$ be an ample adelic Cartier divisor such that $h_{\overline{H}} \geq 1$. By [19, Lemma 19], there is a monic polynomial $P_f(t) \in \mathbb{Z}[t]$ such that

$$\forall D \in \mathrm{Pic}(X), \quad P_f(f^*)(D) \sim 0.$$

Let $d = \deg P_f(t)$ and V be a subspace of $\mathrm{Pic}(X)_{\mathbb{Q}}$ spanned by $H, f^*H, \ldots, (f^*)^{d-1}H$. Since $P_f(f^*)(H) \sim 0$, V is an f^*-invariant and finite-dimensional subspace of $\mathrm{Pic}(X)_{\mathbb{Q}}$.

Let $D_{10}, D_{11}, \ldots, D_{1\rho(1)}, D_{20} \ldots, D_{\kappa\rho(\kappa)} \in \mathrm{Div}(X)_{\mathbb{C}}$ such that the classes of these divisors give a basis of V and the associated matrix $f^*|_V$ is the following Jordan normal form:

$$\begin{pmatrix} \Lambda_1 & & & \\ & \Lambda_2 & & \\ & & \ddots & \\ & & & \Lambda_\kappa \end{pmatrix},$$

where Λ_i is a Jordan block of size $\rho(i)+1$ such that $(\Lambda_i - \lambda_i I_{\rho(i)+1})^{\rho(i)+1} = 0$ for the eigenvalue $\lambda_i \in \mathbb{C}$, the identity matrix $I_{\rho(i)+1}$ and $1 \leq i \leq \kappa$. Then the above divisors satisfy that

$$f^* D_{i0} \sim \lambda_i D_{i0}$$

and

$$f^* D_{ij} \sim \lambda_i D_{ij} + D_{i,j-1}$$

for all $1 \leq i \leq \kappa$ and $1 \leq j \leq \rho(i)$. We fix Green function families of these divisors.

By relabeling these divisors, we can assume that

$$|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_\sigma| > 1 \geq |\lambda_{\sigma+1}| \geq \cdots \geq |\lambda_\kappa|.$$

By Proposition 5.3.2(2), there are canonical height functions $\hat{h}_{D_{ij}}$ for $1 \leq i \leq \sigma$ such that

$$\hat{h}_{D_{ij}} = h_{\overline{D}_{ij}} + O(1)$$

and

$$\hat{h}_{D_{ij}}(f^n(P)) = \sum_{k=0}^{j} \binom{n}{k} \lambda_i^{n-k} \hat{h}_{D_{ik}}(P). \tag{5.3.3}$$

On the other hand, for $\sigma < i \leq \kappa$, there exists a constant $C > 0$ such that

$$|h_{\overline{D}_{ij}}(f^n(P))| \leq C n^l$$

by Proposition 5.3.2(1) and $|\lambda_i| \leq 1$. Hence we have

$$\limsup_{n \to \infty} |h_{\overline{D}_{ij}}(f^n(P))|^{\frac{1}{n}} \leq 1. \tag{5.3.4}$$

Firstly, we assume that there are non-negative integers $1 \leq i' \leq \sigma$ and $0 \leq j' \leq \rho(i')$ such that

$$\hat{h}_{D_{ij}}(P) = \hat{h}_{D_{i'k}}(P) = 0 \tag{5.3.5}$$

for $1 \leq i \leq i'-1$, $0 \leq j \leq \rho(i)$ and $0 \leq k < j'$, and $\hat{h}_{D_{i'j'}}(P) \neq 0$. Then by the equation (5.3.3), we have

$$\hat{h}_{D_{i'j'}}(f^n(P)) = \sum_{k=0}^{j'} \binom{n}{k} \lambda_{i'}^{n-k} \hat{h}_{D_{i'k}}(P) = \lambda_{i'}^n \hat{h}_{D_{i'j'}}(P).$$

By a similar proof of [19, Lemma 18], we obtain that

$$\begin{aligned}\underline{\alpha}_f(P) &\geq \liminf_{n\to\infty} |h_{\overline{D}_{i'j'}}(f^n(P))|^{\frac{1}{n}} \geq \liminf_{n\to\infty} |\hat{h}_{D_{i'j'}}(f^n(P)) - O(1)|^{\frac{1}{n}} \\ &= \liminf_{n\to\infty} |\lambda_{i'}^n \hat{h}_{D_{i'j'}}(P) - O(1)|^{\frac{1}{n}} = |\lambda_{i'}|.\end{aligned}$$

Here we use the assumption that $|\lambda_{i'}| > 1$ and $\hat{h}_{D_{i'j'}}(P) \neq 0$.

We write the ample divisor H defining V as

$$H \sim \sum_{i,j} c_{ij} D_{ij}$$

where $c_{ij} \in \mathbb{C}$. Since $|\lambda_{i'j'}| > 0$, there exists a positive number $\epsilon > 0$ such that $\epsilon < |\lambda_{i'j'}|$. By Proposition 5.3.2, we have

$$|h_{\overline{D}_{ij}}(f^n(P))| \leq O(n^{\rho(i)}|\lambda_i|^n) \tag{5.3.6}$$

for $1 \leq i \leq i'$. Moreover, it follows from the inequality (5.3.4) that

$$|h_{\overline{D}_{ij}}(f^n(P))| = O((1+\epsilon)^n) \tag{5.3.7}$$

for $\sigma < i \leq \kappa$. Then we obtain that

$$\begin{aligned}h_{\overline{H}}(f^n(P)) &= \sum_{i,j} c_{ij} h_{\overline{D}_{ij}}(f^n(P)) \\ &= \sum_{i=1}^{\sigma}\sum_{j=0}^{\rho(i)} c_{ij} h_{\overline{D}_{ij}}(f^n(P)) + \sum_{i=\sigma+1}^{\kappa}\sum_{j=0}^{\rho(i)} c_{ij} h_{\overline{D}_{ij}}(f^n(P)) \\ &= \sum_{i=1}^{\sigma}\sum_{j=0}^{\rho(i)} c_{ij} \hat{h}_{D_{ij}}(f^n(P)) + O(1) + O((1+\epsilon)^n) \quad \text{(by (5.3.7))} \\ &= \sum_{i=1}^{\sigma}\sum_{j=0}^{\rho(i)} c_{ij} \hat{h}_{D_{ij}}(f^n(P)) + O((1+\epsilon)^n) \qquad (5.3.8) \\ &= \sum_{i=i'}^{\sigma}\sum_{j=0}^{\rho(i)} c_{ij} \hat{h}_{D_{ij}}(f^n(P)) + O((1+\epsilon)^n) \quad \text{(by (5.3.5))} \\ &= \sum_{i=i'}^{\sigma}\sum_{j=0}^{\rho(i)} c_{ij} \hat{h}_{D_{ij}}(f^n(P)) + O((1+\epsilon)^n) \\ &\leq \sum_{i=i'}^{\sigma} O(n^{\rho(i)}\lambda_i^n) + O((1+\epsilon)^n) \quad \text{(by (5.3.6))}.\end{aligned}$$

Since $|\lambda_{i'}| \geq \cdots \geq |\lambda_\sigma|$ and $\epsilon < |\lambda_{i'}| - 1$, we have

$$h_{\overline{H}}(f^n(P)) \leq O(n^\rho \lambda_{i'}^n)$$

for some positive integer $\rho > 0$, which implies that

$$\overline{\alpha}_f(P) \leq |\lambda_{i'}|$$

because $|\lambda_{i'}| > 1$. Hence in the case of $\hat{h}_{D_{i'j'}}(P) \neq 0$ for some i', j', it follows that

$$\alpha_f(P) = \lim_{n\to\infty} h_{\overline{H}}(f^n(P))^{\frac{1}{n}} = |\lambda_{i'}|.$$

Finally, we assume that

$$\forall i, j, \quad \hat{h}_{D_{i,j}}(P) = 0,$$

which implies that

$$\forall n \geq 0, \forall i, j, \quad \hat{h}_{D_{i,j}}(f^n(P)) = 0$$

by the equation (5.3.3). Then by the inequality (5.3.8), we have

$$h_{\overline{H}}(f^n(P)) = O((1+\epsilon)^n).$$

Hence we obtain that

$$\overline{\alpha}_f(P) = \limsup_{n\to\infty} h_{\overline{H}}(f^n(P))^{\frac{1}{n}} \leq 1 + \epsilon.$$

Since ϵ is an arbitrary small positive number, we get $\overline{\alpha}_f(P) \leq 1$.

The above discussion says that the arithmetic degree α_f exists and is equal to 1 or the absolute value of some eigenvalue of $f^*|_V$. Since $P_f(f^*)$ annihilates $\mathrm{Pic}(X)_{\mathbb{Q}}$, the minimal polynomial of $f^*|_V$ divides $P_f(t) \in \mathbb{Z}[t]$. Hence the arithmetic degree is equal to 1 or the absolute value of some root of $P_f(t)$, which is an algebraic integer. □

5.4 Simple case

In this section, we prove Conjecture 5.1.2 for the simplest case: f is a surjective morphism and the Picard number of X is equal to one.

Theorem 5.4.1 (Theorem F)**.** *Let $S = (K, (\Omega, \mathcal{A}, \nu), \phi)$ be a proper adelic curve. Let X be a normal projective variety over an algebraic closure $\overline{K}$ of K such that $\dim N^1(X)_{\mathbb{R}} = 1$ and $f : X \to X$ be a morphism. Then for any $P \in X(\overline{K})$, the arithmetic degree $\alpha_f(P)$ exists and is equal to 1 or δ_f. Moreover, if S has the Northcott property and the orbit $\mathcal{O}_f(P)$ is infinite, we have $\alpha_f(P) = \delta_f$.*

Proof. If $\mathcal{O}_f(P)$ is finite or $\delta_f = 1$, then it follows from the definition and Theorem 5.2.1 that $\alpha_f(P) = 1$. Hence we assume that $\mathcal{O}_f(P)$ is infinite and $\delta_f > 1$. Let $\overline{H}$ be an adelic Cartier divisor such that H is ample and $h_{\overline{H}} \geq 1$. By Theorem 5.2.1 again, we have $\overline{\alpha}_f(P) \leq \delta_f$. Thus it is sufficient to prove the opposite inequality $\underline{\alpha}_f(P) \geq \delta_f$. Since $\dim N^1(X)_{\mathbb{R}} = 1$ and f is a morphism, we obtain that

$$(f^n)^*H \equiv (f^*)^n H \equiv \delta_f^n H$$

for all $n \geq 0$. We set $E := f^*H - \delta_f H$, which is numerically equivalent to zero. We define a Green function family g_E of E by the definition of E. By Corollary 3.3.4, there exists a constant $C > 0$ such that

$$\forall Q \in X(\overline{K}), \quad |h_{\overline{E}}(Q)| \leq C\sqrt{h_{\overline{H}}(Q)}, \tag{5.4.1}$$

where $\overline{E} = (E, g_E)$. Since $\delta_f > 1$, we can fix $\epsilon > 0$ such that

$$\sqrt{\delta_f + \epsilon} < \delta_f.$$

By Theorem 5.2.1, there exists a constant $C' > 0$ such that

$$\forall k \geq 0, \quad h_{\overline{H}}(f^k(P)) \leq C'(\delta_f + \epsilon)^k h_{\overline{H}}(P). \tag{5.4.2}$$

Firstly, we prove the following claim:

Claim 2. *For any $Q \in X(\overline{K})$, the limit*

$$\hat{h}_X(Q) = \lim_{n \to \infty} \frac{h_{\overline{H}}(f^n(Q))}{\delta_f^n}$$

exists. Moreover, $\hat{h}_X$ satisfies that

$$\forall Q \in X(\overline{K}), \quad \hat{h}_X(f(Q)) = \delta_f \hat{h}_X(Q)$$

and there exists a constant $C'' > 0$ such that

$$\forall Q \in X(\overline{K}), \quad |\hat{h}_X(Q) - h_{\overline{H}}(Q)| \leq C''\sqrt{h_{\overline{H}}(Q)}.$$

Proof. Let m, n be non-negative integers such that $m > n$. Then we have

$$\begin{aligned}
\left| \frac{h_{\overline{H}}(f^m(Q))}{\delta_f^m} - \frac{h_{\overline{H}}(f^n(Q))}{\delta_f^n} \right| &= \left| \sum_{k=n+1}^{m} \frac{h_{\overline{H}}(f^k(Q))}{\delta_f^k} - \frac{h_{\overline{H}}(f^{k-1}(Q))}{\delta_f^{k-1}} \right| \\
&\leq \sum_{k=n+1}^{m} \frac{1}{\delta_f^k} \left| h_{\overline{H}}(f^k(Q)) - \delta_f h_{\overline{H}}(f^{k-1}(Q)) \right| \\
&= \sum_{k=n+1}^{m} \frac{1}{\delta_f^k} \left| h_{\overline{E}}(f^{k-1}(Q)) \right| \\
&\leq \sum_{k=n+1}^{m} \frac{C}{\delta_f^k} \sqrt{h_{\overline{H}}(f^{k-1}(Q))} \quad \text{(by (5.4.1))} \\
&\leq \sum_{k=n+1}^{m} \frac{C\sqrt{C'}}{\delta_f^k} \left(\sqrt{\delta_f + \epsilon}\right)^{k-1} \quad \text{(by (5.4.2))}.
\end{aligned}$$

Hence we obtain that

$$\left| \frac{h_{\overline{H}}(f^m(Q))}{\delta_f^m} - \frac{h_{\overline{H}}(f^n(Q))}{\delta_f^n} \right| \le \frac{C\sqrt{C'}}{\delta_f} \sum_{n=k+1}^{m} \left(\frac{\sqrt{\delta_f + \epsilon}}{\delta_f} \right)^{k-1}. \tag{5.4.3}$$

Because of the choice of ϵ, the right side of the inequality (5.4.3) converges to zero as $m, n \to \infty$, which shows that $\hat{h}_X(Q)$ exists. The second equation $\hat{h}_X(f(Q)) = \delta_f \hat{h}_X(Q)$ immediately follows from the definition. By taking $m \to \infty$ and $n = 0$ in (5.4.3), we complete the proof. $\square$

By Claim 2, we have

$$\begin{aligned} h_{\overline{H}}(f^n(P)) &\ge \hat{h}_X(f^n(P)) - C''\sqrt{h_{\overline{H}}(f^n(P))} \\ &= \delta_f^n \hat{h}_X(P) - C''\sqrt{h_{\overline{H}}(f^n(P))}. \end{aligned}$$

It follows from the inequality (5.4.2) that

$$h_{\overline{H}}(f^n(P)) \ge \delta_f^n \hat{h}_X(P) - C'\sqrt{C''(\delta_f + \epsilon)^n h_{\overline{H}}(P)}. \tag{5.4.4}$$

By definition, we obtain that $\hat{h}_X(P) \ge 0$. If $\hat{h}_X(P) = 0$, by the inequality (5.4.4), we have

$$0 = \hat{h}_X(f^n(P)) \ge h_{\overline{H}}(f^n(P)) - C''\sqrt{h_{\overline{H}}(f^n(P))}.$$

Hence we get $h_{\overline{H}}(f^n(P)) \le C''^2$ for all n, which implies that $\alpha_f(P) = 1$. We assume that $\hat{h}_X(P) > 0$. Since $\sqrt{\delta_f + \epsilon} < \delta_f$, by taking n-th roots of the inequality (5.4.4) and letting $n \to \infty$, we have

$$\underline{\alpha}_f(P) = \liminf_{n \to \infty} h_{\overline{H}}(f^n(P))^{\frac{1}{n}} \ge \delta_f.$$

Finally, if S has the Northcott property, the condition that $\mathcal{O}_f(P)$ is infinite implies that the height $h_{\overline{H}}(f^n(P))$ is not bounded above by [8, Proposition 6.2.3]. $\square$

5.5 Regular affine automorphism case

Let us recall the definition of a regular affine automorphism, due to Sibony (for details, see [35]).

Definition 5.5.1. Let $f : \mathbb{A}^n \to \mathbb{A}^n$ be an automorphism. By abuse of notation, we also denote by f and f^{-1} rational maps $\mathbb{P}^n \dashrightarrow \mathbb{P}^n$ which is the extensions of f and f^{-1}, respectively.

(1) The *degree* of f is the maximal degree of defining polynomials of f.

(2) Let I_f and $I_{f^{-1}}$ be the indeterminacy loci in $\mathbb{P}^n$ of f and f^{-1}. We say that f is a *regular affine automorphism* if $I_f \cap I_{f^{-1}} = \emptyset$.

Theorem 5.5.2 (Theorem G). *Let $S = (K, (\Omega, \mathcal{A}, \nu), \phi)$ be a proper adelic curve with the Northcott property. Let $f : \mathbb{A}^n \to \mathbb{A}^n$ be a regular affine automorphism of degree $d \geq 2$ defined over an algebraic closure $\overline{K}$ of K. We denote by f' the restriction of f onto $\mathbb{P}^n \setminus \mathbb{A}^n$. Then for $P \in \mathbb{P}^n(\overline{K})_f$, we have*

$$\alpha_f(P) = \begin{cases} 1 & (\mathcal{O}_f(P) \text{ is finite}), \\ \delta_f & (\mathcal{O}_f(P) \text{ is infinite and } P \in \mathbb{A}^n(\overline{K})), \\ \delta_{f'} & (\mathcal{O}_f(P) \text{ is infinite and } P \in (\mathbb{P}^n \setminus \mathbb{A})^n(\overline{K})_f). \end{cases}$$

Before proving Theorem 5.5.2, we extend the canonical height theory of regular affine automorphisms in [17].

Proposition 5.5.3 (c.f. [17, Theorem 6.3]). *Let $S = (K, (\Omega, \mathcal{A}, \nu), \phi)$ be a proper adelic curve with the Northcott property. Let $f : \mathbb{A}^n \to \mathbb{A}^n$ be a regular affine automorphism of degree $d \geq 2$ defined over $\overline{K}$. Then for any $P \in \mathbb{A}^n(\overline{K})$, the limit*

$$\hat{h}_f(P) = \lim_{k \to \infty} \frac{h_{\overline{H}}(f^k(P))}{d^k}$$

exists. Moreover, we have

$$\hat{h}_f(P) = 0 \Longleftrightarrow \mathcal{O}_f(P) \text{ is finite.}$$

Proof. We set $\mathbb{A}^n = \operatorname{Spec} \overline{K}[T_1, \ldots, T_n]$ and $\mathbb{P}^n = \operatorname{Proj} \overline{K}[T_0, T_1, \ldots, T_n]$. Let $H = \{T_0 = 0\}$ be an ample divisor on $\mathbb{P}^n$ and $g = \{g_\chi\}_{\chi \in \Omega_{\overline{K}}}$ be a Green function family of H defined by

$$g_\chi(T) = \log \max\{|T_0|_\chi, |T_1|_\chi, \ldots, |T_n|_\chi\}$$

for $\chi \in \Omega_{\overline{K}}$. It follows from the definition that

$$h_{\overline{H}}(f^k(P)) = \int_{\Omega_{\overline{K}}} \log \max\{1, |f_{1,k}(P)|_\chi, \ldots, |f_{n,k}(P)|_\chi\} \, \nu_{\overline{K}}(\mathrm{d}\chi)$$

for $P \in \mathbb{A}^n(\overline{K})$, where $f^k = (f_{1,k}, \ldots, f_{n,k})$ for $k \geq 1$. Hence we have

$$\frac{h_{\overline{H}}(f^k(P))}{d^k} = \int_{\Omega_{\overline{K}}} \frac{1}{d^k} \log \max\{1, |f_{1,k}(P_1)|_\chi, \ldots, |f_{n,k}(P)|_\chi\} \, \nu_{\overline{K}}(\mathrm{d}\chi).$$

For each $\chi \in \Omega_{\overline{K}}$, the limit

$$G_\chi(P) = \lim_{k \to \infty} \frac{1}{d^k} \log \max\{1, |f_{1,k}(P_1)|_\chi, \ldots, |f_{n,k}(P)|_\chi\}$$

exists by [17] if χ is non-Archimedean and [35] if χ is Archimedean. To complete the proof, we need to estimate G_χ more precisely. We write $f = (f_{1,1}, \ldots, f_{n,1}) : \mathbb{A}^n \to \mathbb{A}^n$ as

$$f_{i,1} := \sum_{\alpha \in I, |\alpha| \leq d} a_{i,\alpha} T^\alpha \quad (i = 1, \ldots, n),$$

where $I = \mathbb{Z}_{\geq 0}^n$, and for $\alpha = (\alpha_1, \ldots, \alpha_n)$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and $T^\alpha = T_1^{\alpha_1} \cdots T_n^{\alpha_n}$. Note that we have

$$G_\chi(P) = \lim_{k\to\infty} \frac{1}{d^k} g_\chi(f^k(P))$$

and

$$h_{\overline{H}}(P) = \int_{\Omega_{\overline{K}}} g_\chi(P)\ \nu_{\overline{K}}(\mathrm{d}\chi).$$

Now we start to estimate G_χ.

Firstly, we assume that χ is non-Archimedean. By the stronger triangle inequality, we obtain that

$$|f_{i,1}(P)|_\chi \leq \max_{|\alpha|\leq d}\{|a_{i,\alpha}|_\chi |P^\alpha|_\chi\} \leq \max_{|\alpha|\leq d}\{|P^\alpha|_\chi\} \max_{|\alpha|\leq d}\{|a_{i,\alpha}|_\chi\}$$

for each i, where $|P^\alpha|_\chi = |P_1|_\chi^{\alpha_1} \cdots |P_n|_\chi^{\alpha_n}$. Hence it follows that

$$\begin{aligned} g_\chi(f(P)) &\leq \log \max_{|\alpha|\leq d}\{1, |P^\alpha|_\chi\} + \log \max_{i,\alpha}\{1, |a_{i,\alpha}|_\chi\} \\ &\leq \log \max_i\{1, |P_i|_\chi^d\} + \log \max_{i,\alpha}\{1, |a_{i,\alpha}|_\chi\} \\ &= d g_\chi(P) + \log \max_{i,\alpha}\{1, |a_{i,\alpha}|_\chi\}, \end{aligned}$$

which implies that

$$G_\chi(P) \leq g_\chi(P) + \frac{1}{d-1} \log \max_{i,\alpha}\{1, |a_{i,\alpha}|_\chi\}.$$

Next, we suppose that χ is Archimedean. This case is slightly complicated. By the triangle inequality, we have

$$\begin{aligned} |f_{i,1}(P)|_\chi &\leq \sum_{|\alpha|\leq d} |a_{i,\alpha}|_\chi |P^\alpha|_\chi \leq \max_{|\alpha|\leq d}\{|P^\alpha|_\chi^d\} \sum_{|\alpha|\leq d} |a_{i,\alpha}|_\chi \\ &\leq \max_j\{1, |P_j|_\chi^d\} \cdot \sum_{j=0}^d \binom{j+n-1}{n-1} \max_\alpha\{1, |a_{i,\alpha}|_\chi\}. \end{aligned}$$

Hence it follows that

$$g_\chi(f(P)) \leq d g_\chi(P) + \log \max_{i,\alpha}\{1, |a_{i,\alpha}|_\chi\} + \log \sum_{j=0}^d \binom{j+n-1}{n-1},$$

which implies that

$$G_\chi(P) \leq g_\chi(P) + \frac{1}{d-1} \log \max_{i,\alpha}\{1, |a_{i,\alpha}|_\chi\} + \frac{1}{d-1} \log \sum_{j=0}^d \binom{j+n-1}{n-1}.$$

By the above discussions, we conclude that

$$G_\chi(P) \leq g_\chi(P) + \frac{1}{d-1} \log \max_{i,\alpha}\{1, |a_{i,\alpha}|_\chi\} + C 1_{\Omega_{\overline{K},\infty}},$$

where $C = \frac{1}{d-1}\log\sum_{j=0}^{d}\binom{j+n-1}{n-1}$. Since $\nu(\Omega_\infty) < \infty$, the right side functions are $\nu_{\overline{K}}$-integrable on $\Omega_{\overline{K}}$. Hence we obtain that

$$\begin{aligned}\lim_{k\to\infty}\frac{h_{\overline{H}}(f^k(P))}{d^k} &= \lim_{k\to\infty}\int_{\Omega_{\overline{K}}}\frac{1}{d^k}\log\max\{1, |f_{1,k}(P)|_\chi, \ldots, |f_{n,k}(P)|_\chi\}\ \nu_{\overline{K}}(\mathrm{d}\chi)\\ &= \int_{\Omega_{\overline{K}}}\lim_{k\to\infty}\frac{1}{d^k}\log\max\{1, |f_{1,k}(P)|_\chi, \ldots, |f_{n,k}(P)|_\chi\}\ \nu_{\overline{K}}(\mathrm{d}\chi)\\ &= \int_{\Omega_{\overline{K}}} G_\chi(P)\ \nu_{\overline{K}}(\mathrm{d}\chi)\end{aligned}$$

by the Lebesgue's dominated convergence theorem.

The last assertion is given by a similar proof of [15, Theorem 4.2]. □

Proof of Theorem 5.5.2. If $\mathcal{O}_f(P)$ is finite, it follows from the definition that $\alpha_f(P) = 1$. Hence we assume that $\mathcal{O}_f(P)$ is infinite. Firstly, let P be a $\overline{K}$-rational point of $\mathbb{A}^n$. By Proposition 5.5.3, we have

$$\hat{h}_f(P) = \lim_{k\to\infty}\frac{h_{\overline{H}}(f^k(P))}{d^k} > 0.$$

Hence there exists an integer N such that

$$\forall k \geq N, \quad \frac{h_{\overline{H}}(f^k(P))}{d^k} \geq \frac{\hat{h}_f(P)}{2}.$$

Then we get

$$\underline{\alpha}_f(P) = \liminf_{k\to\infty} h_{\overline{H}}(f^k(P))^{\frac{1}{k}} \geq \liminf_{k\to\infty}\left(\frac{\hat{h}_f(P)}{2}d^k\right)^{\frac{1}{k}} = d,$$

which implies that $\underline{\alpha}_f(P) \geq \delta_f$. Note that $d = \delta_f$. By Theorem 5.2.1, we obtain that $\alpha_f(P) = \delta_f$.

Next we assume that $P \in (\mathbb{P}^n \setminus \mathbb{A}^n)(\overline{K})$. We write $X = I_{f^{-1}}$. By abuse of notation, we also denote by f' the restriction of f onto X, which is a morphism since $I_f \cap X = \emptyset$. By [18, Proposition 9], f' is surjective and $f(P) \in X(\overline{K})$. Let $p : \tilde{X} \to X$ be the normalization of X and $\tilde{f}' : \tilde{X} \to \tilde{X}$ be the induced morphism by f'. Let $\overline{D}$ be an adelic Cartier divisor on X whose underlying Cartier divisor is ample. Since p is a finite morphism, p^*D is also ample. Let $Q \in p^{-1}(f(P)) \subset \tilde{X}(\overline{K})$. Then we have

$$h_{p^*\overline{D}}(\tilde{f}'^k(Q)) = h_{\overline{D}}(p \circ \tilde{f}'^k(Q)) = h_{\overline{D}}(f'^k(f(P))),$$

which implies that $\alpha_{\tilde{f}'}(Q)$ exists if and only if $\alpha_{f'}(f(P))$ exists, and that $\alpha_{\tilde{f}'}(Q) = \alpha_{f'}(f(P))$ if they exist. Moreover, since the dynamical degree is a birational invariant, we have $\delta_{f'} = \delta_{\tilde{f}'}$. By the above discussion, we can assume that X is normal. By [18, Proposition 9] again, we have a surjective morphism $\mathbb{P}^l \to X$ where $l = \dim X$. Hence we obtain that $\dim N^1(X)_\mathbb{R} = \dim N^1(\mathbb{P}^l)_\mathbb{R} = 1$. So it follows from Theorem 5.4.1 that $\alpha_{f'}(f(P)) = \delta_{f'}$. By choosing a very ample divisor H on $\mathbb{P}^n$ which is also very ample on X for computing the arithmetic degree, we have

$$\alpha_f(f(P)) = \alpha_{f'}(f(P)) = \delta_{f'}.$$

Finally, we obtain that

$$\begin{aligned}\alpha_f(f(P)) &= \lim_{k\to\infty} h_X^+(f^{k+1}(P))^{\frac{1}{k}} \\ &= \lim_{k\to\infty} \left(h_X^+(f^{k+1}(P))^{\frac{1}{k+1}}\right)^{1+\frac{1}{k}} \\ &= \lim_{k\to\infty} h_H^+(f^k(P))^{\frac{1}{k}} = \alpha_f(P),\end{aligned}$$

which completes the proof. □

5.6 Surface automorphism case

In this section, we consider the case of surfaces.

Definition 5.6.1. Let X be a smooth projective surface over a field and $f : X \to X$ be an automorphism. Let C be an irreducible curve in X. We say that C is f-*periodic* if $f^n(C) = C$ for some $n > 0$. We denote the union of all f-periodic curves in X by E_f.

Theorem 5.6.2 (Theorem H)**.** *Let $S = (K, (\Omega, \mathcal{A}, \nu), \phi)$ be a proper adelic curve with the Northcott property. Let X be a smooth projective surface over an algebraic closure $\overline{K}$ of K and $f : X \to X$ be an automorphism. Then for $P \in X(\overline{K})_f$, we have*

$$\alpha_f(P) = \begin{cases} 1 & (\mathcal{O}_f(P) \text{ is finite or } P \in E_f(\overline{K})), \\ \delta_f & (\mathcal{O}_f(P) \text{ is infinite and } P \notin E_f(\overline{K})). \end{cases}$$

To prove this theorem, we construct the canonical height function on X in several steps. These proofs are the extension of ones in [16].

Proposition 5.6.3. *Let X be a smooth projective surface over a field and $f : X \to X$ be an automorphism with $\delta_f > 1$.*

(1) There are non-zero nef classes $\nu_+, \nu_- \in N^1(X)_{\mathbb{R}}$ such that

$$f^*(\nu_+) = \delta_f \nu_+, \quad f^*(\nu_-) = \delta_f^{-1} \nu_-.$$

(2) We have $(\nu_+)^2 = (\nu_-)^2 = 0$.

(3) Let $\nu := \nu_+ + \nu_-$. Then we have ν is nef and big. Moreover,

$$f^*(\nu) + (f^{-1})^*(\nu) = (\delta_f + \delta_f^{-1})\nu.$$

Proof. (1) Firstly, we consider the eigenvalues of f^*. By the Hodge index theorem, the signature of $N^1(X)_{\mathbb{R}}$ is $(1, \rho - 1)$. Hence by [29, Lemma 3.1], f^* has at most one eigenvalue λ such that $|\lambda| > 1$. Since δ_f is the maximum of absolute values of eigenvalues of f^*, we have $\lambda = \delta_f$. By the same way for $(f^{-1})^*$, the set of eigenvalues of f^* is

$$\{\delta_f, \delta_f^{-1}, \alpha_1, \ldots, \alpha_l\},$$

where $|\alpha_i| = 1$ for all i.

Let $\beta \in N^1(X)_{\mathbb{R}}$ be an ample class. Let ν'_+ be an eigenvector of f^* with eigenvalue δ_f. Since δ_f is the only one eigenvalue of f^* whose absolute value is greater than 1, the sequence $\{(f^*)^n(\beta)/\delta_f^n\}$ converges to $c\nu'_+$ for some $c \neq 0$ as $n \to \infty$. We set $\nu_+ = c\nu'_+$, which is nef because $(f^*)^n(\beta)/\delta_f^n$ is ample for all n. By similar way, we can take nef class ν_- as the eigenvector of f^* with eigenvalue δ_f^{-1}.

(2) We have

$$\delta_f^2(\nu_+)^2 = (f^*(\nu_+))^2 = (\nu_+)^2,$$

which implies that $(\nu_+)^2 = 0$. Similarly, we have $(\nu_-)^2 = 0$.

(3) Let $x, y_1, \ldots, y_m$ be the basis of $N^1(X)_{\mathbb{R}}$ such that

$$(x, x) = 1, \quad (y_i, y_i) = -1, \quad (x, y_i) = 0 \text{ (for all } i), \quad (y_i, y_j) = 0 \text{ (for } i \neq j).$$

We set

$$\nu_+ = ax + b_1y_1 + \cdots + b_my_m,$$
$$\nu_- = a'x + b'_1y_1 + \cdots + b'_my_m.$$

and

$$\boldsymbol{y} = b_1y_1 + \cdots + b_my_m, \quad \boldsymbol{y}' = b'_1y_1 + \cdots + b'_my_m.$$

Since ν_+ and ν_- are nef, we have $a \geq 0$ and $a' \geq 0$. By (2), we obtain that

$$(\nu_+^2) = ((ax + \boldsymbol{y})^2) = a^2 + (\boldsymbol{y}^2) = 0,$$

which implies that $(\boldsymbol{y}^2) = -a^2$. Similarly, $(\boldsymbol{y}'^2) = -a'^2$. Again by (2), we get

$$(\nu^2) = (\nu_+^2) + 2(\nu_+, \nu_-) + (\nu_-^2) = 2(\nu_+, \nu_-) = aa' - \tilde{\boldsymbol{y}} \cdot \tilde{\boldsymbol{y}}',$$

where $\tilde{\boldsymbol{y}} = (b_1, \ldots, b_m)$, $\tilde{\boldsymbol{y}}' = (b'_1, \ldots, b'_m)$ and $\tilde{\boldsymbol{y}} \cdot \tilde{\boldsymbol{y}}' = b_1b'_1 + \cdots + b_mb'_m$, which is the usual inner product of a real vector space. If $\tilde{\boldsymbol{y}} \cdot \tilde{\boldsymbol{y}}' < 0$, there is nothing to prove. We assume that $\tilde{\boldsymbol{y}} \cdot \tilde{\boldsymbol{y}}' \geq 0$. By the Cauthy-Schwartz inequality, we have

$$\tilde{\boldsymbol{y}} \cdot \tilde{\boldsymbol{y}}' \leq \sqrt{(\tilde{\boldsymbol{y}} \cdot \tilde{\boldsymbol{y}})(\tilde{\boldsymbol{y}}' \cdot \tilde{\boldsymbol{y}}')} = \sqrt{(-(\boldsymbol{y}^2))(-(\boldsymbol{y}'^2))} = \sqrt{a^2a'^2} = aa'.$$

Hence it follows that

$$(\nu^2) = aa' - \tilde{\boldsymbol{y}} \cdot \tilde{\boldsymbol{y}}' \geq aa' - aa' = 0.$$

Since ν_+ and ν_- are linearly independent, this inequality is strict. Thus ν is big. Finally, we have

$$\begin{aligned} f^*(\nu) + (f^{-1})^*(\nu) &= f^*(\nu_+) + f^*(\nu_-) + (f^{-1})^*(\nu_+) + (f^{-1})^*(\nu_-) \\ &= \delta_f \nu_+ + \delta_f^{-1}\nu_- + \delta_f^{-1}\nu_+ + \delta_f \nu_- \\ &= (\delta_f + \delta_f^{-1})(\nu_+ + \nu_-) = (\delta_f + \delta_f^{-1})\nu, \end{aligned}$$

as required. □

Proposition 5.6.4. *Let X be a smooth projective surface over a field and $f : X \to X$ be an automorphism with $\delta_f > 1$.*

(1) Let $\nu \in N^1(X)_{\mathbb{R}}$ be a nef and big class in Proposition 5.6.3. Let C be an irreducible curve on X. Then C is f-periodic if and only if $([C], \nu) = 0$.

(2) There are only finitely many f-periodic curves on X.

Proof. See [16, Proposition 3.1] for their proofs. □

Proposition 5.6.5. *Let X be a smooth projective surface over a field and $f : X \to X$ be an automorphism with $\delta_f > 1$.*

(1) There are $\mathbb{R}$-Cartier divisors D_+ and D_- on X such that

$$[D_\pm] = \nu_\pm, \quad f^*(D_\pm) \sim_{\mathbb{R}} \delta_f^{\pm} D_\pm.$$

(2) We set $D = D_+ + D_-$. Then we have

$$[D] = \nu, \quad f^*(D) + (f^{-1})^*(D) \sim_{\mathbb{R}} (\delta_f + \delta_f^{-1})D.$$

Proof. See [16, Lemma 3.8] for their proofs. □

Proposition 5.6.6 (c.f. [16, Theorem 5.2])**.** *Let $S = (K, (\Omega, \mathcal{A}, \nu), \phi)$ be a proper adelic curve with the Northcott property, X be a smooth projective surface over an algebraic closure $\overline{K}$ of K and $f : X \to X$ be an automorphism with $\delta_f > 1$. Let D be a nef and big $\mathbb{R}$-Cartier divisor in Proposition 5.6.5 and E_f be the union of all f-periodic curves on X. Then there is a Green function family g of D which satisfies the following properties:*

(1) $\forall P \in X(\overline{K}), \quad h_{(D,g)}(f(P)) + h_{(D,g)}(f^{-1}(P)) = (\delta_f + \delta_f^{-1})h_{(D,g)}(P)$.

(2) $\forall P \in E_f(\overline{K}), \quad h_{(D,g)}(P) = 0$.

(3) $\forall P \in X(\overline{K}), \quad h_{(D,g)}(P) \geq 0$.

(4) For all positive real numbers δ and C, the set

$$\{P \in (X \setminus E_f)(\overline{K}) \,|\, h_{(D,g)}(P) \leq C, [K(P) : K] \leq \delta\}$$

is finite.

(5) *Let* $P \in (X \setminus E_f)(\overline{K})$*. Then* $h_{(D,g)}(P) = 0$ *if and only if* $\mathcal{O}_f(P)$ *is finite.*

Proof. (1) Let D_+ and D_- be nef $\mathbb{R}$-Cartier divisors in Proposition 5.6.5. Then we have

$$f^*(D_+) \sim_{\mathbb{R}} \delta_f D_+, \quad (f^{-1})^*(D_-) \sim_{\mathbb{R}} \delta_f D_-. \tag{5.6.1}$$

Let $\overline{D}_+ = (D_+, g_+)$ and $\overline{D}_- = (D_-, g_-)$ be the canonical compactifications of D_+ and D_- with respect to f and f^{-1}, respectively (see Section 3.3 for notations). By the equations (5.6.1), we have

$$h_{\overline{D}_+}(f(P)) = \delta_f h_{\overline{D}_+}(P), \quad h_{\overline{D}_-}(f(P)) = \delta_f^{-1} h_{\overline{D}_+}(P).$$

We set $\overline{D} := \overline{D}_+ + \overline{D}_-$. Then the height function $h_{\overline{D}}$ clearly satisfies (1).

(2) Let P be a $\overline{K}$-rational point of E_f. Let C be an irreducible component of E such that $P \in C(\overline{K})$. Since C is f-periodic, we have $f^n(C) = C$ for some n. Let D' be an $\mathbb{R}$-Cartier divisor on X which is $\mathbb{R}$-linearly equivalent to D and $\operatorname{Supp}(C) \not\subset \operatorname{Supp}(D')$. We set $\overline{L} := \overline{D}'|_C$. Let $\varphi : \widetilde{C} \to C$ be the normalization and $\widetilde{L} := \varphi^*(L)$. Then a morphism f^n induces an automorphism $\widetilde{f}^n : \widetilde{C} \to \widetilde{C}$. It follows from the equations (5.6.1) that

$$\widetilde{f}^n(\widetilde{L}) + \widetilde{f}^{-n}(\widetilde{L}) \sim_{\mathbb{R}} (\delta_f^n + \delta_f^{-n})\widetilde{L},$$

which implies that $\widetilde{L} \sim_{\mathbb{R}} 0$ by [16, Lemma 5.3]. Hence the height function $h_{\overline{L}}$ on $\widetilde{C}$ is a bounded function. Moreover, this height function also satisfies the equation in (1), we have

$$h_{\overline{D}}(P) = h_{\overline{L}}(P) = 0.$$

(3) By (2), it is sufficient to show that

$$\forall P \in (X \setminus E_f)(\overline{K}), \quad h_{\overline{D}}(P) \geq 0.$$

By [16, Proposition 1.3 (2)], there is an effective divisor Z and sufficiently small $\epsilon > 0$ such that $\operatorname{Supp}(Z) \subset \operatorname{Supp}(E_f)$ and $D - \epsilon Z$ is ample. Let g_Z be a Green function family of Z. By Proposition 3.2.5, there is a constant c_1 such that

$$\forall P \in (X \setminus E_f)(\overline{K}), \quad h_{(Z,g_Z)} > c_1. \tag{5.6.2}$$

Moreover by Corollary 3.2.6, there exists a constant c_2 such that

$$\forall P \in X(\overline{K}), \quad h_{\overline{D}} - \epsilon h_{(Z,g_Z)} > c_2.$$

Then by (1), we have

$$\begin{aligned} h_{\overline{D}}(P) &= \frac{1}{\delta_f^n + \delta_f^{-n}} (h_{\overline{D}}(f^n(P)) + h_{\overline{D}}(f^{-n}(P))) \\ &= \frac{1}{\delta_f^n + \delta_f^{-n}} \{(h_{\overline{D}}(f^n(P)) - \epsilon h_{(Z,g_Z)}(f^n(P))) + \epsilon h_{(Z,g_Z)}(f^n(P)) \\ &\qquad + (h_{\overline{D}}(f^{-n}(P)) - \epsilon h_{(Z,g_Z)}(f^{-n}(P))) + \epsilon h_{(Z,g_Z)}(f^{-n}(P))\} \\ &> \frac{2(\epsilon c_1 + c_2)}{\delta_f^n + \delta_f^{-n}}. \end{aligned}$$

By letting $n \to \infty$, we get the conclusion.

(4) Let $P \in (X \setminus E_f)(\overline{K})$ such that $h_{\overline{D}} \leq C$ and $[K(P) : K] \leq \delta$. Then by the inequality (5.6.2), we have

$$h_{\overline{D}}(P) - \epsilon h_{(Z,g_Z)} \leq C - \epsilon c_1.$$

Hence we obtain that

$$\begin{aligned} &\{P \in (X \setminus E_f)(\overline{K}) \mid h_{(D,g)}(P) \leq C, [K(P) : K] \leq \delta\} \\ &\qquad \subset \{P \in X(\overline{K}) \mid h_{(D,g)}(P) - \epsilon h_{(Z,g_Z)} \leq C - \epsilon c_1, [K(P) : K] \leq \delta\}. \end{aligned}$$

Since $D - \epsilon Z$ is ample, the latter set is finite by Proposition 3.2.8.

(5) Let $P \in (X \setminus E_f)(\overline{K})$. Firstly we assume that $h_{\overline{D}}(P) = 0$. Then by (1) and (3), we have $h_{\overline{D}}(f^n(P)) = 0$ for all $n \in \mathbb{Z}$, which implies that

$$\mathcal{O}_f(P) \subset \{P \in (X \setminus E_f)(\overline{K}) \mid h_{(D,g)}(P) \leq 0, [K(P) : K] \leq \delta\}$$

for some δ. Hence by (4), the set $\mathcal{O}_f(P)$ is finite. Conversely, we suppose that $\mathcal{O}_f(P)$ is finite. Then we can find some integer n such that $f^n(P) = P$. By (1), we conclude that $h_{\overline{D}}(P) = 0$. □

Proposition 5.6.7 (c.f. [16, Proposition 5.3])**.** *We use the notation in Proposition 5.6.6.*

(1) $\forall P \in X(\overline{K}), \quad h_{\overline{D}_+}(P) \geq 0$ *and* $h_{\overline{D}_-}(P) \geq 0$.

(2) For $P \in (X \setminus E_f)(\overline{K})$*, we have*

$$h_{\overline{D}_+}(P) = 0 \Leftrightarrow h_{\overline{D}_-}(P) = 0 \Leftrightarrow h_{\overline{D}}(P) = 0.$$

Proof. (1) By Proposition 5.6.6 (3), we have

$$\begin{aligned} h_{\overline{D}_+}(P) &= \delta_f^{-n} h_{\overline{D}_+}(f^n(P)) \\ &= \delta_f^{-n}(h_{\overline{D}}(f^n(P)) - h_{\overline{D}_-}(f^n(P))) \\ &\geq -\delta_f^{-n} h_{\overline{D}_-}(f^n(P)) = -\delta_f^{-2n} h_{\overline{D}_-}(P). \end{aligned}$$

By letting $n \to \infty$, we obtain that $h_{\overline{D}_+}(P) \geq 0$. Similarly, we get $h_{\overline{D}_-}(P) \geq 0$.

(2) Firstly we assume that $h_{\overline{D}_+}(P) = 0$. Then we obtain that

$$h_{\overline{D}}(f^n(P)) = h_{\overline{D}_-}(f^n(P)) = \delta_f^{-n} h_{\overline{D}_-}(P).$$

Hence it follows that

$$\mathcal{O}_f(P) \subset \{Q \in (X \setminus E_f)(\overline{K}) \mid h_{\overline{D}}(Q) \leq h_{\overline{D}_-}(P), [K(Q) : K] \leq [K(P) : K]\}.$$

By Proposition 5.6.6(4), the set $\mathcal{O}_f(P)$ is finite, which implies that $h_{\overline{D}}(P) = 0$ by Proposition 5.6.6 (5). Moreover we have $h_{\overline{D}_-}(P) = 0$. Similarly, $h_{\overline{D}_-}(P) = 0$ implies that $h_{\overline{D}}(P) = h_{\overline{D}_+}(P) = 0$. Finally, we assume that $h_{\overline{D}}(P) = 0$. Since $h_{\overline{D}}(P) = h_{\overline{D}_+}(P) + h_{\overline{D}_-}(P)$, we clearly get $h_{\overline{D}_+}(P) = h_{\overline{D}_-}(P) = 0$ by (1). □

Now, we start to prove Theorem 5.6.2.

Proof of Theorem 5.6.2. If $\delta_f = 1$ or $\mathcal{O}_f(P)$ is finite, we clearly get $\alpha_f(P) = 1$. Hence we assume that $\delta_f > 1$ and $\mathcal{O}_f(P)$ is infinite.

We suppose that E_f is nonempty. Let $f_E : E_f \to E_f$ be the restriction of f on E_f. We write $E_f = \bigcup C_i$ where C_i is an f-periodic curve. By Proposition 5.6.4 (2), this union is finite. Hence we can find some integer m such that $f_E^m \in \mathrm{Aut}(C_i)$ for all i. Since any automorphism of a curve has dynamical degree 1, we have $\alpha_{f^m}(P) = \alpha_{f_E^m}(P) = 1$ for all $P \in E_f(\overline{K})$.

We assume that $P \notin E_f(\overline{K})$. Let $\overline{D}, \overline{D}_+$ and $\overline{D}_-$ be adelic $\mathbb{R}$-Cartier divisors in Proposition 5.6.6, and $\overline{H}$ be an adelic Cartier divisor on X whose underlying Cartier divisor is ample and $h_{\overline{H}} \geq 1$. By Corollary 3.2.7, we have

$$\begin{aligned}
\underline{\alpha}_f(P) &= \liminf_{n\to\infty} h_{\overline{H}}(f^n(P))^{1/n} \\
&\geq \liminf_{n\to\infty} h_{\overline{D}}(f^n(P))^{1/n} \\
&= \liminf_{n\to\infty} (h_{\overline{D}_+}(f^n(P)) + h_{\overline{D}_-}(f^n(P)))^{1/n} \\
&= \liminf_{n\to\infty} (\delta_f^n h_{\overline{D}_+}(P) + \delta_f^{-n} h_{\overline{D}_-}(P))^{1/n} \\
&= \delta_f.
\end{aligned}$$

Hence we obtain that $\alpha_f(P) = \delta_f$. Note that $h_{\overline{D}_+}(P) > 0$ by Proposition 5.6.7. □

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