Demazure slices of type $A_{2l}^{(2)}$

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Abstract

We consider a Demazure slice of type $A_{2l}^{(2)}$, that is an associated graded piece of an infinite-dimensional version of a Demazure module. We show that a global Weyl module of a hyperspecial current algebra of type $A_{2l}^{(2)}$ is filtered by Demazure slices. We calculate extensions between a Demazure slice and a usual Demazure module and prove that a graded character of a Demazure slice is equal to a suitably specialized nonsymmetric Macdonald-Koornwinder polynomial divided by its square norm. In the last section, we prove that a global Weyl module of the special current algebra of type $A_{2l}^{(2)}$ is a free module over the polynomial ring arising as the endomorphism ring of itself.

Contents

Chapter 1

Introduction

In this chapter, we review the previous works related to orthogonal polynomials and representations of Lie algebras as a motivation and explain main results and the organization of the thesis. Throughout this thesis, the base field is the field of complex numbers $\mathbb{C}$.

1.1 Motivations

1.1.1 Finite simple Lie algebras and orthogonal polynomials

For a natural number $n \in \mathbb{N}$, we denote the Lie algebra of $(n+1) \times (n+1)$ complex matrices with trace 0 by $\mathfrak{sl}_{n+1}$ i.e. simple Lie algebra of type A_n. Let $\mathring{P}_{\mathfrak{sl}_{n+1}+}$ be the set of dominant integral weights of $\mathfrak{sl}_{n+1}$. It is known that simple finite dimensional modules $L(\lambda)$ of $\mathfrak{sl}_{n+1}$ are in one-to-one correspondence with $\lambda \in \mathring{P}_{\mathfrak{sl}_{n+1}+}$. Let $\mathring{\mathfrak{h}}$ be the set of diagonal matrices in $\mathfrak{sl}_{n+1}$ and $\mathring{P}_{\mathfrak{sl}_{n+1}}$ be the set of integral weights of $\mathfrak{sl}_{n+1}$. Then the module

$L(\lambda)$ has a decomposition

$$L(\lambda) = \bigoplus_{\mu \in \mathring{P}_{\mathfrak{sl}_{n+1}}} L(\lambda)_\mu,$$

$$L(\lambda)_\mu = \{v \in L(\lambda) \mid hv = \mu(h)v \text{ for } h \in \mathring{\mathfrak{h}}\}$$

Using this decomposition, the character $\operatorname{ch} L(\lambda)$ of $L(\lambda)$ is defined by

$$\operatorname{ch} L(\lambda) = \sum_{\mu \in \mathring{P}_{\mathfrak{sl}_{n+1}}} e^\mu \dim_{\mathbb{C}} L(\lambda)_\mu \in \mathbb{C}[\mathring{P}_{\mathfrak{sl}_{n+1}}].$$

Here, e^μ is the element of $\mathbb{C}[\mathring{P}_{\mathfrak{sl}_{n+1}}]$ corresponding to $\mu \in \mathring{P}_{\mathfrak{sl}_{n+1}}$.

Let $\mathfrak{S}_n$ be the n-th symmetric group. Let ϖ_i $(i = 1, ..., n)$ be the fundamental weight of $\mathfrak{sl}_{n+1}$. The group $\mathfrak{S}_n$ acts on $\mathring{P}_{\mathfrak{sl}_{n+1}}$ by $\sigma(\varpi_i) = \varpi_{\sigma(i)}$ for $\sigma \in \mathfrak{S}_n$. Schur polynomials $s_\lambda(x)$ are the family of symmetric polynomials in $\mathbb{C}[\mathring{P}_{\mathfrak{sl}_{n+1}}]^{\mathfrak{S}_n}$ indexed by $\lambda \in \mathring{P}_{\mathfrak{sl}_{n+1}+}$. Schur polynomials satisfy the following orthogonality relations:

$$\langle s_\lambda, s_\mu \rangle = \begin{cases} 1 & \text{if } \lambda = \mu \\ 0 & \text{if } \lambda \neq \mu. \end{cases}$$

Here $\langle -, - \rangle$ is the inner product on $\mathbb{C}[\mathring{P}_{\mathfrak{sl}_{n+1}}]^{\mathfrak{S}_n}$ called the Hall inner product. It is known that the character of a simple module is equal to a Schur polynomial:

$$\operatorname{ch} L(\lambda) = s_\lambda(x) \ \ \forall \lambda \in \mathring{P}_{\mathfrak{sl}_{n+1}+}.$$

Let $\mathring{\mathfrak{g}}$ be a finite-dimensional simple Lie algebra. For $\mathring{\mathfrak{g}}$-modules M and N, let $\operatorname{Ext}^\bullet_{\mathcal{O}}(M, N)$ be the extensiton between them in the category $\mathcal{O}$ ([Hum]). For finite-dimensional $\mathring{\mathfrak{g}}$-modules M and N, the Euler-Poincaré pairing

$$\langle M, N \rangle_{\mathrm{Ext}} = \sum_{i=0}^{\infty} (-1)^i \dim_{\mathbb{C}} \operatorname{Ext}^i_{\mathcal{O}}(M, N)$$

depends only on $\operatorname{ch} M$ and $\operatorname{ch} N$. Hence this induces a pairing on $\mathbb{C}[\mathring{P}]$, where $\mathring{P}$ is the integral weight lattice of $\mathring{\mathfrak{g}}$. Moreover, this pairing coincides with the Hall inner product ([Mac95]). The equality $\operatorname{ch} L(\lambda) = s_\lambda(x)$ can be interpreted in terms of the following equalities of extenstions:

$$\dim_{\mathbb{C}} \operatorname{Ext}^i_{\mathcal{O}}(L(\lambda), L(\mu)) = \delta_{i,0}\delta_{\lambda,\mu},$$

where $\delta_{a,b}$ is Kronecker's delta.

This connection between representation theory of $\mathfrak{sl}_{n+1}$ and Schur polynomials enables us to interpret combinatorial properties such as Littlewood-Richardson coefficients of product expansions and its positivity from the view point of representation theory.

1.1.2 Demazure modules and nonsymmetric Macdonald polynomials

The connection between representation theory and orthogonal polynomials is generalized to affine Kac-Moody algebras and Macdonald polynomials.

Let $\mathfrak{g}$ be the affine Kac-Moody Lie algebra of type $X_n^{(r)}$. For a dominant integral weight Λ, we denote the highest weight simple module of $\mathfrak{g}$ with its highest weight Λ by $L(\Lambda)$. For an element w of Weyl group W of $\mathfrak{g}$, the thin Demazure module $D_{w\Lambda}$ and the thick Demazure module $D^{w\Lambda}$ are representations of the lower Borel subalgebra $\mathfrak{b}_-$ of $\mathfrak{g}$ defined by

$$D_{w\Lambda} = U(\mathfrak{b}_-)\left(L(\Lambda)_{-w\Lambda}\right)^* \subset L(\Lambda)^\vee, \quad D^{w\Lambda} = U(\mathfrak{b}_-)L(\Lambda)_{w\Lambda} \subset L(\Lambda),$$

where $L(\Lambda)^\vee$ is the restricted dual of $L(\Lambda)$ and $L(\Lambda)_{w\Lambda}$ is the weight space with its weight $w\Lambda$.

Let $\mathfrak{h}$ be the Cartan subalgera of $\mathfrak{g}$. We denote the simple imaginary root of $\mathfrak{g}$ by δ and the integral weight lattice of simple Lie algebra of type X_n by

$\mathring{P}$. When a $\mathfrak{h}$-module M has a weight decomposition

$$M = \bigoplus_{\lambda+n\delta \in \mathring{P} \oplus \frac{\mathbb{Z}}{2}\delta} M_{\lambda+n\delta}, \quad M_{\lambda+n\delta} = \{v \in M \mid hm = (\lambda+n\delta)(h)m, \ h \in \mathfrak{h}\}$$

such that every weight space $M_{\lambda+n\delta}$ is finite-dimensional, its graded character $\operatorname{gch} M$ is the formal sum

$$\operatorname{gch} M = \sum_{\lambda+n\delta \in \mathring{P} \oplus \frac{\mathbb{Z}}{2}\delta} e^{\lambda} q^{-n} \dim_{\mathbb{C}} M_{\lambda+n\delta}.$$

The graded characters of $D_{w\Lambda}$ and $D^{w\Lambda}$ are well-defined.

Let $\mathbb{F}$ be the field of rational functions in $t^{\pm 1}$ and $q^{\pm\frac{1}{2}}$. The nonsymmetric Macdonald polynomials $\{E_{\lambda}(x,q,t)\}_{\lambda \in \mathring{P}}$ are the family of elements in $\mathbb{F}[\mathring{P}]$ orthogonal with respect to the Macdonald-Cherednik pairing $\langle -, - \rangle$ ([Che, Mac, Sahi00]). In [Ion], it is proved that the specialization of Macdonald polynomials $E_{\lambda}(x,q,t)|_{t=0}$ coincide with the graded characters of level one thin Demazure modules of type $X_n^{(r)}$ for $X = A, D, E$. The specialization of Macdonald polynomials $E_{\lambda}(x,q,t)|_{t=\infty}$ at $t = \infty$ are dual to $\{E_{\lambda}(x,q,t)|_{t=0}\}_{\lambda \in \mathring{P}}$ with respect to the specialized Macdonald-Cherednik pairing $\langle -, - \rangle|_{t=0}$ at $t = 0$. In [CK], Cherednik-Kato proved that the graded characters of the level one Demazure slices $\mathbb{D}^{\lambda}$ which are quotient $\mathfrak{b}_{-}$-modules of thick Demazure modules are equal to $E_{\lambda}(x,q,t)|_{t=\infty}$ divided by their norm with respect to $\langle -, - \rangle|_{t=0}$ for type $X_n^{(r)}$ with $X = A, D, E$ except for $A_{2l}^{(2)}$. They identify the Macodonald-Cherednik pairing with the Euler-Poincaré pairing in some category of $\mathfrak{b}_{-}$-modules.

1.1.3 Weyl modules for current algebras and symmetric Macdonald polynomials

The local Weyl modules and the global Weyl modules for $\mathfrak{g}$ are introduced in [CP] to study finite-dimensional representations of $\mathfrak{g}$. These modules are defined in terms of generator and relations and characterized by a certain universal property.

In this thesis, we refer to a maximal parabolic subalgebra of $\mathfrak{g}$ corresponding to the affine node of the affine Dynkin diagram of $\mathfrak{g}$ as the current algebra $\mathfrak{Cg}$ and let $\mathfrak{Cg}' = [\mathfrak{Cg}, \mathfrak{Cg}]$. The Lie algebra $\mathfrak{Cg}'$ is referred to as current algebra in the literature. In the literature, two types of the current algebra are considered. They are called hyperspecial current algebra and special current algebra. The analogue of local Weyl modules $W(\lambda)_{loc}$ and the global Weyl modules $W(\lambda)$ for current algebras are defined in the similar way as the Weyl modules for $\mathfrak{g}$ ([CFK, CIK, CL]). Let $X = A, D, E$. In this case, local Weyl modules $W(\lambda)_{loc}$ are isomorphic to Demazure modules ([CIK, FK, FL]). Therefore, the graded characters of local Weyl modules are specialized symmetric Macdonald polynomials.

Another important result on Weyl modules are freeness of $W(\lambda)$ over their endomorphism rings $\mathrm{End}_{\mathfrak{Cg}'}(W(\lambda))$ which is isomorphic to a polynomial ring $\mathbb{A}_\lambda$. This result is proved for all hyperspecial current algebras [CIK, CL, FL, Na] and allows us to obtain the graded character formula for $W(\lambda)$

1.1.4 Demazure slices and Rogers-Ramanujan type identities

The relation between characters of $L(\Lambda)$ and Rogers-Ramanujan type identity also motivates the study of Demazure slices. The Rogers-Ramanujan identities are the following identities between infinite sums and infinite products:

$$\sum_{n\geq 0}\frac{q^{n^2}}{(1-q)\cdots(1-q^n)}=\prod_{n\geq 0}\frac{1}{(1-q^{5n+1})(1-q^{5n+4})}$$

$$\sum_{n\geq 0}\frac{q^{n^2+n}}{(1-q)\cdots(1-q^n)}=\prod_{n\geq 0}\frac{1}{(1-q^{5n+2})(1-q^{5n+3})}.$$

Rogers-Ramanujan identities can be obtained from characters of highest weight integrable simple modules $L(\Lambda)$ of affine Kac-Moody Lie algebras of type $A_1^{(1)}$[LW78, LW82, LW84, LW85]. It is interesting to obtain Rogers-Ramanujan type identities from affine Kac-Moody Lie algebras other than type $A_1^{(1)}$.

Recently, Charednik and Feigin [ChFe] showed that the expansions of the products of the characters of level one integrable modules for affine Kac-Moody algebras in terms Macdonald polynomials specialized at $t=\infty$ give infinite sums with non-negative coefficients similar to the sum side of Rogers-Ramanujan identities. Charednik and Kato [CK] showed that this can be intepreted in terms of filtrations on integrable modules by Demazure slices except for type $A_{2l}^{(2)}$.

1.2 Main Results

This thesis consists of the study of Demazure slices of type $A_{2l}^{(2)}$ and the Weyl module of the special current algebras of type $A_{2l}^{(2)}$. Let $\mathfrak{g}$ be an affine Kac-Moody Lie algebra of type $A_{2l}^{(2)}$ and $\mathring{\mathfrak{g}}$ be a simple Lie algebra of type C_l contained in $\mathfrak{g}$. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. Let $\mathring{P}$ be the integral weight lattice of $\mathring{\mathfrak{g}}$ and $\mathring{P}_+$ be the set of dominant integral weights of $\mathring{\mathfrak{g}}$. For each $\lambda\in\mathring{P}_+$, we have a $\mathfrak{C}\mathfrak{g}$-module $W(\lambda)$, that is called a global Weyl module.

Level one Demazure slices and thin Demazure modules are parametrized by $\lambda \in \mathring{P}$ as $\mathbb{D}^\lambda$ and D_λ, respectively. Let Λ_0 be the unique level one dominant integral weight of $\mathfrak{g}$ up to $\mathbb{Z}\delta$ and let δ be the simple imaginary root of $\mathfrak{g}$. Let $\mathring{W}$ be the Weyl group of $\mathring{\mathfrak{g}}$. Let $\mathfrak{b}_-$ be a lower-triangular Borel subalgebra of $\mathfrak{g}$.

Theorem A (=Theorem 5.2.3)**.** *For each $\lambda \in \mathring{P}_+$, the global Weyl module $W(\lambda) \otimes_{\mathbb{C}} \mathbb{C}_{\Lambda_0}$ has a filtration by Demazure slices as $\mathfrak{b}_-$-module and each $\mathbb{D}^\mu$ $(\mu \in \mathring{W}\lambda)$ appears exactly once.*

Let $\mathfrak{B}$ be a full subcategory of the category of $U(\mathfrak{b}_-)$-modules and $\langle -, - \rangle_{\mathrm{Ext}}$ be the Euler-Poincaré-pairing associated to $\mathrm{Ext}_{\mathfrak{B}}$ (see Section 1 for their precise definitions).

Theorem B (=Thorem 5.3.4)**.** *For each λ, $\mu \in \mathring{P}$, $m \in \frac{1}{2}\mathbb{Z}$ and $k \in \mathbb{Z}$, we have*

$$\dim_{\mathbb{C}} \mathrm{Ext}^n_{\mathfrak{B}}(\mathbb{D}^\lambda \otimes_{\mathbb{C}} \mathbb{C}_{m\delta + k\Lambda_0}, D_\mu^\vee) = \delta_{n,0}\delta_{m,0}\delta_{k,0}\delta_{\lambda,\mu} \quad n \in \mathbb{Z}_{\geq 0},$$

where $\vee$ means the restricted dual.

For each $\lambda \in \mathring{P}$, let $\bar{E}_\lambda(x_1, .., x_l, q)$ and $E^\dagger_\lambda(x_1, ..., x_l, q)$ be nonsymmetric Macdonald-Koornwinder polynomials specialized at $t = 0, \infty$ respectively. Let $(-, -)$ be the Weyl group invariant inner product on the dual of a Cartan subalgebra $\mathfrak{h}^*$ normalized so that the square length of the shortest roots of $\mathfrak{g}$ with respect to $(-, -)$ is 1. Let $\mathrm{gch}\, M$ be a graded character of M (see §4.1 for the definition). As a corollary of Theorem B, we have

Theorem C (=Corollary 5.3.6)**.** *For each $\lambda \in \mathring{P}$, we have*

$$\mathrm{gch}\, \mathbb{D}^\lambda = q^{\frac{(b|b)}{2}} E^\dagger_\lambda(x_1^{-1}, ..., x_l^{-1}, q^{-1}) / \langle \bar{E}_\lambda, E^\dagger_\lambda \rangle_{\mathrm{Ext}}.$$

For an affine Lie algebra of type $A_{2l}^{(2)}$, two kind of current algebras are studied in the literature. They contain simple Lie algebras of type C_l and B_l, respectively. The former is called a hyperspecial current algebra. A dimension formula of a local Weyl module of a hyperspecial current algebra and freeness of a global Weyl module over its endomorphism ring are proved in [CIK]. The latter is called a special current algebra and a dimension formula of a local Weyl module of a special current algebra is proved in [FK] and [FM]. Let $\mathfrak{Cg}^\dagger$ be a special current algebra of $\mathfrak{g}$. Then $\mathfrak{Cg}^\dagger$ contains a simple Lie algebra $\mathring{\mathfrak{g}}^\dagger$ of type B_l. Let $W(\lambda)^\dagger$ be a global Weyl module of $\mathfrak{Cg}^\dagger$. Let $\mathfrak{Cg}^{\dagger\prime} = [\mathfrak{Cg}^\dagger, \mathfrak{Cg}^\dagger]$. In the last section, we prove the following theorem, which is proved for hyperspecial current algebras in ([CIK, CL, FL, Na]) .

Theorem D (=Theorem 6.5.1+Theorem 6.5.2)**.** *Let λ be a dominant integral weight of $\mathring{\mathfrak{g}}^\dagger$. The endomorphism ring* $\mathrm{End}_{\mathfrak{Cg}^{\dagger\prime}}(W(\lambda)^\dagger)$ *is a polynomial ring and $W(\lambda)^\dagger$ is free over* $\mathrm{End}_{\mathfrak{Cg}^{\dagger\prime}}(W(\lambda)^\dagger)$.

The organization of the thesis is as follows: In chapter two, we prepare basic notation and definitions. Chapter three is on the representation theory of $\mathfrak{b}_-$. We define the category $\mathfrak{B}$ of $\mathfrak{b}_-$-modules and Demazure modules in chapter three. Chapter four is on the representation theory of $\mathfrak{Cg}$. We define local and global Weyl module for $\mathfrak{Cg}$ and compute extensions between them in $\mathfrak{B}$ in chapter four. In Chapter five, we prove the relation between a global Weyl module and a Demazure slice (Theorem A) and calculate the extensions between a Demazure slice and a thin Demazure module (Theorem B). As a corollary, we prove a character formula of a Demazure slice (Theorem C). In chapter six, we study a global Weyl module of a special current algebra of type $A_{2l}^{(2)}$. We prove the endomorphism ring of a global Weyl module is isomorphic to a polynomial ring and a global Weyl module is free over its endomorphism ring (Theorem D).

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Chapter 2

Preliminaries

This chapter is a review of the basic materials on affine Kac-Moody algebras of type $A_{2l}^{(2)}$ and Macdonald-Koorinwinder polynomials. We refer to [Sahi00], [Kac, Chapter 6] and [CI] for general terminologies throughout this chapter. Mainly we refer to [Kac] for §2.2 and §2.4 and refer to [CI] for the §2.3.

2.1 Notations

We denote the field of complex numbers by $\mathbb{C}$, the ring of integers by $\mathbb{Z}$, the set of nonnegative integers by $\mathbb{Z}_{\geq 0}$, the field of rational numbers by $\mathbb{Q}$, and the set of natural numbers by $\mathbb{N}$. We work over the field of complex numbers. In particular, a vector space is a $\mathbb{C}$-vector space. For each $x \in \mathbb{Q}$, we set $\lfloor x \rfloor := \max\{z \in \mathbb{Z} | \ x \geq z\}$. We set $x^{(r)} := x^r/r!$ for an element x of a $\mathbb{C}$ algebra. We denote Kronecker's delta by $\delta_{i,j}$.

2.2 Affine Kac-Moody algebra of type $A_{2l}^{(2)}$

Let $\mathfrak{g}$ be an affine Kac-Moody algebra of type $A_{2l}^{(2)}$ with the scaling element d and let $\mathfrak{h}$ be its Cartan subalgebra. We denote the set of roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$ by Δ and fix a set of simple roots $\{\alpha_0, \alpha_1, ..., \alpha_l\}$, where α_0 is the shortest simple root of $\mathfrak{g}$. Let Δ_+ and Δ_- be the set of positive and negative roots, respectively. We set the simple imaginary root as $\delta := 2\alpha_0 + \alpha_1 + \cdots + \alpha_l$, the set of imaginary roots as $\Delta_{im} := \mathbb{Z}\delta$, and the set of real roots $\Delta_{re} := \Delta \backslash \Delta_{im}$. We set $Q := \bigoplus_{i=0}^{l} \mathbb{Z}\alpha_i$, $\mathring{Q} := \bigoplus_{i=1}^{l} \mathbb{Z}\alpha_i$, and $\mathring{Q}^{\dagger} := \bigoplus_{i=0}^{l-1} \mathbb{Z}\alpha_i$. We set $Q_+ := \bigoplus_{i=0}^{l} \mathbb{Z}_{\geq 0}\alpha_i$, $\mathring{Q}_+ := \bigoplus_{i=1}^{l} \mathbb{Z}_{\geq 0}\alpha_i$, and $\mathring{Q}_+^{\dagger} := \bigoplus_{i=0}^{l-1} \mathbb{Z}_{\geq 0}\alpha_i$. Let $\mathring{\Delta} = \Delta \cap \mathring{Q}$. The set $\mathring{\Delta}$ is a root system of type C_l. We denote the set of short roots of $\mathring{\Delta}$ by $\mathring{\Delta}_s$ and the set of long roots of $\mathring{\Delta}$ by $\mathring{\Delta}_l$. Using the standard basis $\varepsilon_1, ..., \varepsilon_l$ of $\mathbb{R}^l$, we have:

$$\mathring{\Delta} = \{\pm(\varepsilon_i \pm \varepsilon_j),\ \pm 2\varepsilon_i |\ i, j = 1, ..., l\} \backslash \{0\}$$

$$\mathring{\Delta}_s = \{\pm(\varepsilon_i \pm \varepsilon_j) |\ i, j = 1, ..., l\} \backslash \{0\}, \quad \mathring{\Delta}_l = \{\pm 2\varepsilon_i |\ i, j = 1, ..., l\} \backslash \{0\}$$

We have

$$\Delta_{re} = (\mathring{\Delta}_s + \mathbb{Z}\delta) \cup (\mathring{\Delta}_l + 2\mathbb{Z}\delta) \cup \frac{1}{2}(\mathring{\Delta}_l + (2\mathbb{Z} + 1)\delta)$$

and

$$\alpha_0 = \frac{\delta}{2} + \varepsilon_1, \ \ \alpha_1 = -\varepsilon_1 + \varepsilon_2, \ \cdots, \ \alpha_{l-1} = -\varepsilon_{l-1} + \varepsilon_l, \ \ \alpha_l = -2\varepsilon_l.$$

We set $\mathring{\Delta}_{l\pm} := \Delta_\pm \cap \mathring{\Delta}_l$, $\mathring{\Delta}_{s\pm} := \Delta_\pm \cap \mathring{\Delta}_s$ and $\mathring{\Delta}_\pm := \Delta_\pm \cap \mathring{\Delta}$. For each $\alpha \in \Delta_{re}$, let $\check{\alpha} \in \mathfrak{h}$ be the corresponding coroot of $\mathfrak{g}$. Let θ be the highest root of $\mathring{\Delta}$. Let $d \in \mathfrak{h}$ be the scaling element that satisfies $\alpha_i(d) = \delta_{i,0}$. We denote a central element of $\mathfrak{g}$ by $K = \check{\alpha}_0 + 2\check{\alpha}_1 + \cdots + 2\check{\alpha}_l$. For each $\alpha \in \Delta$, we denote the root space corresponding to α by $\mathfrak{g}_\alpha$. For each $\alpha \in \Delta_{re}$, the root space $\mathfrak{g}_\alpha$ is one dimensional and we denote a fixed nonzero vector in $\mathfrak{g}_\alpha$ by e_α. A Borel subalgebra $\mathfrak{b}_\pm$ and a maximal nilpotent subalgebra $\mathfrak{n}_\pm$ of $\mathfrak{g}$ are

$$\mathfrak{b}_+ = \mathfrak{h} \oplus \mathfrak{n}_+,\ \mathfrak{n}_+ = \bigoplus_{\alpha \in \Delta_+} \mathfrak{g}_\alpha,\ \mathfrak{b}_- = \mathfrak{h} \oplus \mathfrak{n}_-,\ \text{and } \mathfrak{n}_- = \bigoplus_{\alpha \in \Delta_-} \mathfrak{g}_\alpha.$$

For each $i \in \{0, 1, ..., l\}$, we define $\Lambda_i \in \mathfrak{h}^*$ by

$$\Lambda_i(\check{\alpha}_j) = \delta_{i,j},\ \Lambda_i(d) = 0.$$

We set

$$P := \mathbb{Z}\Lambda_0 \oplus \cdots \oplus \mathbb{Z}\Lambda_l \oplus \mathbb{Z}\frac{\delta}{2}, \ \text{ and } \ P_+ := \mathbb{Z}_{\geq 0}\Lambda_0 \oplus \cdots \oplus \mathbb{Z}_{\geq 0}\Lambda_l \oplus \mathbb{Z}\frac{\delta}{2}.$$

We set $\varpi_i := \Lambda_i - 2\Lambda_0$ $(i \in \{1, ..., l\})$,

$$\mathring{P} = \mathbb{Z}\varpi_1 \oplus \cdots \oplus \mathbb{Z}\varpi_l, \ \text{ and } \ \mathring{P}_+ = \mathbb{Z}_{\geq 0}\varpi_1 \oplus \cdots \oplus \mathbb{Z}_{\geq 0}\varpi_l.$$

We set

$$\mathring{Q}' := \mathring{Q} + \frac{1}{2}\mathbb{Z}\mathring{\Delta}_l, \ \text{ and } \ \mathring{Q}'_+ := \mathring{Q}_+ + \frac{1}{2}\mathbb{Z}_{\geq 0}\mathring{\Delta}_{l+}.$$

2.3 Hyperspecial current algebra of $A_{2l}^{(2)}$

We set

$$\mathring{\mathfrak{h}} := \bigoplus_{i=1}^{l} \mathbb{C}\alpha_i, \ \ \mathring{\mathfrak{g}} := \bigoplus_{\alpha \in \mathring{\Delta}} \mathfrak{g}_\alpha \oplus \mathring{\mathfrak{h}}, \ \text{ and } \ \mathring{\mathfrak{b}}_+ := \bigoplus_{\alpha \in \mathring{\Delta}_+} \mathfrak{g}_\alpha.$$

Then $\mathring{\mathfrak{g}}$ is a finite dimensional simple Lie algebra of type C_l, the Lie subalgebra $\mathring{\mathfrak{h}}$ is a Cartan subalgebra of $\mathring{\mathfrak{g}}$, the Lie subalgebra $\mathring{\mathfrak{b}}_+$ is a Borel subalgebra of $\mathring{\mathfrak{g}}$, and $\mathring{\Delta}$ is the set of roots of $\mathring{\mathfrak{g}}$ with respect to $\mathring{\mathfrak{h}}$. The lattice $\mathring{P}$ is the integral weight lattice of $\mathring{\mathfrak{g}}$, and $\mathring{P}_+$ is the set of dominant integral weight of

$\mathring{\mathfrak{g}}$. A hyperspecial current algebra $\mathfrak{C}\mathfrak{g}$ is a maximal parabolic subalgebra of $\mathfrak{g}$ that contains $\mathring{\mathfrak{g}}$. I.e.

$$\mathfrak{C}\mathfrak{g} := \mathring{\mathfrak{g}} + \mathfrak{b}_-.$$

We set $\mathfrak{C}\mathfrak{g}' := [\mathfrak{C}\mathfrak{g}, \mathfrak{C}\mathfrak{g}]$.

Remark 2.3.1. *Usually* $\mathfrak{C}\mathfrak{g}'$ *is called current algebra in the literature. We have* $\mathfrak{C}\mathfrak{g} = \mathfrak{C}\mathfrak{g}' \oplus \mathbb{C}d \oplus \mathbb{C}K$.

We define a subalgebra $\mathfrak{C}\mathfrak{g}_{im}$ of $\mathfrak{C}\mathfrak{g}$ by

$$\mathfrak{C}\mathfrak{g}_{im} := \bigoplus_{n \in -\mathbb{N}} \mathfrak{g}_{n\delta},$$

and define a subalgebra $\mathfrak{C}\mathfrak{n}_+$ of $\mathfrak{C}\mathfrak{g}$ by

$$\mathfrak{C}\mathfrak{n}_+ := \bigoplus_{\alpha \in (\mathring{\Delta}_{s+} - \mathbb{Z}_{\geq 0}\delta) \cup (\mathring{\Delta}_{l+} - 2\mathbb{Z}_{\geq 0}\delta) \cap \frac{1}{2}(\mathring{\Delta}_{l+} - (2\mathbb{Z}_{\geq 0}+1)\delta)} \mathfrak{g}_\alpha.$$

2.4 Weyl group

Let $s_\alpha \in \operatorname{Aut}(\mathfrak{h}^*)$ be the simple reflection corresponding to $\alpha \in \Delta_{re}$. We have

$$s_\alpha(\lambda) = \lambda - \langle \lambda, \check{\alpha} \rangle \alpha, \text{ for } \lambda \in \mathfrak{h}^*.$$

We set W as the subgroup of $\operatorname{Aut}(\mathfrak{h}^*)$ generated by s_α ($\alpha \in \Delta_{re}$), and $\mathring{W}$ as the subgroup generated by s_α ($\alpha \in \mathring{\Delta}$). For each $i = 0, ..., l$, let $s_i := s_{\alpha_i}$. Then W is generated by s_i ($i = 0, ..., l$), and $\mathring{W}$ is generated by s_i ($i = 1, ..., l$).

Definition 2.4.1 (Reduced expression)**.** *Each* $w \in W$ *can be written as a product* $w = s_{i_1} s_{i_2} \cdots s_{i_n}$ $(i_j \in \{0, ..., l\})$. *If* n *is minimal number among such expressions, then* $s_{i_1} s_{i_2} \cdots s_{i_n}$ *is called a reduced expression of* w *and* n *is called the length of* w *(written as* $l(w)$*).*

Let $(-|-)$ be a W-invariant bilinear form on $\mathfrak{h}^*$ normalized so that $(\alpha_0|\alpha_0) = 1$. For each $\mu \in \mathring{P}$, we define $t_\mu \in \mathrm{Aut}(\mathfrak{h}^*)$ by

$$t_\mu(\lambda) = \lambda + \langle \lambda, K \rangle \mu - ((\lambda|\mu) + \frac{1}{2}(\mu|\mu)\langle \lambda, K \rangle)\delta.$$

We have $t_\mu \in W$ and

$$W = \mathring{W} \ltimes \mathring{P}. \tag{2.4.1}$$

For each $\lambda \in \mathring{P}$, we denote the unique element of $\mathring{W}\lambda \cap \pm P_+$ by $\lambda_\pm$, respectively. We set $\rho := \frac{1}{2}\sum_{\alpha \in \mathring{\Delta}_+} \alpha$. For each $w \in \mathring{W}$ and $\lambda \in \mathring{P}$, we define $w \circ \lambda := w(\lambda + \rho) - \rho$. For each $\Lambda \in P$, we set $W^\Lambda := \{w \in W |\ w\Lambda = \Lambda\}$. We denote the set of minimal length coset representatives of $\mathring{W}\backslash W$ by W^0.

Definition 2.4.2 (Left weak Bruhat order)**.** *Let $w \in W$ and $i = 0, .., l$. We write $s_i w > w$ if $l(s_i w) > l(w)$ holds. Left weak Bruhat order is the partial order on W generated by $>$.*

Definition 2.4.3 (Macdonald order)**.** *We write $\mu \succeq \lambda$ if and only if one of the following two conditions holds:*

(1) *$\mu - \lambda \in \mathring{Q}_+$ if $\mu \in \mathring{W}\lambda$;*

(2) *$\lambda_+ - \mu_+ \in \mathring{Q}'_+$ if $\mu_+ \neq \lambda_+$.*

For $w \in W$ and $\mu \in \mathring{P}$, let $w(\!(\mu)\!) \in \mathring{P}$ be the restriction of $w(\mu + \Lambda_0)$ to $\mathring{\mathfrak{h}}$. For each $\lambda \in \mathring{P}$, let $\pi_\lambda \in W$ be a minimal length element such that $\pi_\lambda(\!(0)\!) = \lambda$. For each $\mu \in \mathring{P}$, we denote the convex hull of $\mathring{W}\mu$ by $C(\mu)$.

Lemma 2.4.4 ([Mac] Proposition 2.6.2)**.** *If $\mu \in \mathring{P}$, then $C(\mu) \cap (\mu + \mathring{Q}') \subseteq \bigcap_{w \in \mathring{W}} w(\mu_+ - \mathring{Q}'_+)$.*

Proof. The set $w(\mu_+ - \mathring{Q}'_+)$ is the intersection of $\mu_+ + \mathring{Q}'$ with the convex hull of $w(\mu_+ - \mathring{Q}'_+)$. The set $\mathring{W}\mu$ is contained in $\bigcap_{w \in \mathring{W}} w(\mu_+ - \mathring{Q}'_+)$. Hence we have $C(\mu) \cap (\mu + \mathring{Q}') \subseteq \bigcap_{w \in \mathring{W}} w(\mu_+ - \mathring{Q}'_+)$. $\square$

Lemma 2.4.5.

(1) *If* $w > v \in W$*, then* $v(\!(0)\!) \succeq w(\!(0)\!)$*;*

(2) *Let* b*,* $c \in \mathring{P}$ *satisfy* $b = s_i(\!(c)\!)$ *for some* $i = 0, ..., l$*. Then*

$$c \succ b \iff \pi_b = s_i\pi_c > \pi_c.$$

Proof. First, we prove (1). It is enough to prove the assertion for $w = s_i v$. Since $w > v$, we have $\langle v\Lambda_0, \check{\alpha}_i\rangle \geq 0$. This implies $v\Lambda_0 - w\Lambda_0 \in Q_+$. Hence we have $v(\!(0)\!) \succeq w(\!(0)\!)$ if $i \neq 0$. If $i = 0$, then we have $w(\!(0)\!) - v(\!(0)\!) = \langle v\Lambda_0, \check{\alpha}_0\rangle\theta/2$. We set $N = \langle v\Lambda_0, \check{\alpha}_0\rangle$. We have $\langle w(\!(0)\!), \check{\theta}\rangle = (N+1)/2$. Hence $s_\theta(w(\!(0)\!)) = w(\!(0)\!) - \frac{N+1}{2}\theta$ and $v(\!(0)\!) = \frac{N}{N+1}s_\theta(w(\!(0)\!)) + \frac{1}{N+1}w(\!(0)\!)$. Therefore, $v(\!(0)\!) \in C(w(\!(0)\!)) \cap (w(\!(0)\!) + \mathring{Q}')$. By Lemma 2.4.4, $w(\!(0)\!)_+ - v(\!(0)\!)_+ \in \mathring{Q}'_+$. Hence $v(\!(0)\!) \succeq w(\!(0)\!)$.

Next, we prove (2). We already proved ($\Leftarrow$). So we prove ($\Rightarrow$). By Definition 2.4.2, we have either $s_i\pi_c > \pi_c$ or $s_i\pi_c < \pi_c$. From $c \succ b$ and (1), we have $s_i\pi_c > \pi_c$ and $\pi_b > s_i\pi_b$. We have $(s_i\pi_c)(\!(0)\!) = b$ thanks to $b = s_i(\!(c)\!)$. We show that $\pi_b = s_i\pi_c$. If $\pi_b \neq s_i\pi_c$, then we have $l(s_i\pi_c) > l(\pi_b)$ by the minimality of $l(\pi_b)$. Since $l(\pi_b) = l(s_i\pi_b) + 1$, $l(s_i\pi_c) = l(\pi_c) + 1$ and $l(s_i\pi_c) > l(\pi_b)$, we get $l(\pi_c) > l(s_i\pi_b)$. This contradicts the minimality of $l(\pi_c)$. Hence the assertion follows. □

2.5 Macdonald-Koornwinder polynomials

In this section, we recall materials presented in [Sahi00, §3] and [Ion], and we specialize parameters t, t_0, u_0, t_l, u_l in [Sahi00] as $t_0 = t_l = u_0 = t$ and $u_l = 1$ ([Ion]).

2.5.1 Nonsymmetric case

Let $\mathbb{F}$ be the field of rational functions in $t^{\pm 1}$ and $q^{\pm\frac{1}{2}}$. Let $\mathbb{F}[\mathring{P}]$ be a group ring of $\mathring{P}$ over $\mathbb{F}$ and X^λ be an element of $\mathbb{F}[\mathring{P}]$ corresponding to $\lambda \in \mathring{P}$. We identify $\mathbb{F}[x_1^{\pm 1}, ..., x_l^{\pm 1}]$ with $\mathbb{F}[\mathring{P}]$ by $x_i = X^{\varepsilon_i}$ for each $i \in \{1, ..., l\}$. We define

$$\Delta(x) := \Delta(x)_+ \Delta(x^{-1})_+ \prod_{n \in \mathbb{N}} (1 - q^n)^l \in \mathbb{F}[\![x_1^{\pm 1}, ..., x_l^{\pm 1}]\!]$$

by using

$$\Delta(x)_+ := \prod_{i=1,\ldots,l} \frac{(x_i)_\infty (-x_i)_\infty (q^{1/2} x_i)_\infty}{(tx_i)_\infty (-tx_i)_\infty (q^{1/2} t^2 x_i)_\infty} \prod_{1 \leq i < j \leq l} \frac{(x_i x_j)_\infty (x_i x_j^{-1})_\infty}{(t x_i x_j)_\infty (t x_i x_j^{-1})_\infty},$$

where $(u)_\infty = \prod_{n \in \mathbb{Z}_{\geq 0}} (1 - q^n u)$. We define

$$\varphi(x) := \prod_{i=1,\ldots,l} \frac{(x_i - t)(x_i + t)}{x_i^2 - 1} \prod_{1 \leq i < j \leq l} \frac{(x_i x_j - t)(x_i x_j^{-1} - t)}{(x_i x_j - 1)(x_i x_j^{-1} - 1)}$$

and $\mathcal{C}(x) := \Delta(x)\varphi(x)$. We have

$$\Delta(x)_+|_{t=0} = \prod_{i=1,\ldots,l} (x_i)_\infty (-x_i)_\infty (q^{1/2} x_i)_\infty \prod_{1 \leq i < j \leq l} (x_i x_j)_\infty (x_i x_j^{-1})_\infty$$

and

$$\varphi(x)|_{t=0} = \prod_{i=1,\ldots,l} \frac{1}{1 - x_i^{-2}} \prod_{1 \leq i < j \leq l} \frac{1}{(1 - x_i^{-1} x_j^{-1})(1 - x_i^{-1} x_j)}.$$

Under the identification $x_i = X^{\varepsilon_i}$ and $q = e^\delta$, we have

$$\Delta(x)|_{t=0} = \prod_{\alpha \in \Delta \text{ and } \alpha(d) \leq 0} (1 - X^\alpha)^{\dim \mathfrak{g}_\alpha} \quad \text{and} \quad \varphi(x)|_{t=0} = \prod_{\alpha \in \mathring{\Delta}_+} \frac{1}{1 - X^\alpha}.$$

Hence we have

$$\mathcal{C}|_{t=0} = \prod_{\alpha \in \Delta_-} (1 - X^\alpha)^{\dim \mathfrak{g}_\alpha}.$$

Definition 2.5.1. *We define an inner product on* $\mathbb{F}[x_1^{\pm 1}, ..., x_l^{\pm 1}]$ *by*

$$\langle f, g\rangle'_{nonsym} := \text{the constant term of } fg^{\star}\mathcal{C} \text{ in } x_1, ..., x_l \in \mathbb{F},$$

where $\star$ *is the involution on* $\mathbb{F}[x_1^{\pm 1}, ..., x_l^{\pm 1}]$ *such that* $q^{\star} = q^{-1}$, $x_i^{\star} = x_i^{-1}$ *and* $t^{\star} = t^{-1}$.

Definition 2.5.2. *The set of nonsymmetric Macdonald-Koornwinder polynomials* $\{E_\lambda(x, q, t)\}_{\lambda \in \mathring{P}}$ *is a collection of elements in* $\mathbb{F}[\mathring{P}]$ *indexed by* $\mathring{P}$ *with the following properties:*

(1) $\langle E_\lambda, E_\mu\rangle'_{nonsym} = 0$ *if* $\lambda \neq \mu$;

(2) $E_\lambda = X^\lambda + \sum\limits_{\mu \succ \lambda} c_\mu X^\mu$.

As in [Ion, §3.2], we set

$$\bar{E}_\lambda := \lim_{t \to 0} E_\lambda,\ \ E_\lambda^{\dagger} := \lim_{t \to 0} E_\lambda^{\star}.$$

Definition 2.5.3. *We define an inner product on* $\mathbb{C}(\!(q^{\frac{1}{2}})\!)[x_1^{\pm 1}, ..., x_l^{\pm 1}]$ *by*

$$\langle f, g\rangle_{nonsym} := \text{the constant term of } fg^{\star}(\mathcal{C}\mid_{t=0}) \text{ in } x_1, ..., x_l \in \mathbb{C}[\![q^{\pm\frac{1}{2}}]\!],$$

where $f,\ g \in \mathbb{C}(\!(q^{\frac{1}{2}})\!)[x_1^{\pm 1}, ..., x_l^{\pm 1}]$.

2.5.2 Symmetric case

The Weyl group $\mathring{W}$ acts linearly on $\mathbb{F}[\mathring{P}]$ by $w(e^\lambda) = e^{w(\lambda)}$ for each $w \in \mathring{W}$ and $\lambda \in \mathring{P}$.

Definition 2.5.4. *We define an inner product on* $\mathbb{F}[x_1^{\pm 1}, ..., x_l^{\pm 1}]$ *by*

$$\langle f, g\rangle'_{sym} := \text{the constant term of } fg\Delta(x) \text{ in } x_1, ..., x_l \in \mathbb{F}.$$

Definition 2.5.5. *The set of symmetric Macdonald-Koornwinder polynomials* $\{P_\lambda(x,q,t)\}_{\lambda\in\mathring{P}}$ *is a collection of elements in* $\mathbb{F}[\mathring{P}]^{\mathring{W}}$ *indexed by* $\mathring{P}_+$ *with the following properties:*

(1) $\langle P_\lambda, P_\mu\rangle'_{sym} = 0$ *if* $\lambda \neq \mu$*;*

(2) $P_\lambda = X^\lambda + \sum\limits_{\mu\succ\lambda} c_\mu X^\mu$.

We set

$$\bar{P}_\lambda := \lim_{t\to 0} P_\lambda.$$

Definition 2.5.6. *We define an inner product on* $\mathbb{C}(\!(q^{\frac{1}{2}})\!)[x_1^{\pm 1},...,x_l^{\pm 1}]^{\mathring{W}}$ *by*

$$\langle f,g\rangle_{sym} := \textit{the constant term of } fg(\Delta(x)\mid_{t=0}) \textit{ in } x_1,...,x_l \in \mathbb{C}[\![q^{\pm\frac{1}{2}}]\!],$$

where $f,\ g \in \mathbb{C}(\!(q^{\frac{1}{2}})\!)[x_1^{\pm 1},...,x_l^{\pm 1}]^{\mathring{W}}$.

We abbreviate $\bar{E}_\lambda(x_1,...,x_l,q)$, $E^\dagger_\lambda(x_1,...,x_l,q)$and $\bar{P}_\lambda(x_1,...,x_l,q)$ as $\bar{E}_\lambda(X,q)$, $E^\dagger_\lambda(X,q)$ and $\bar{P}_\lambda(X,q)$, respectively.

Proposition 2.5.7 ([Ion] Theorem 4.2)**.** *For each* $\lambda \in \mathring{P}_+$*, we have*

$$\bar{P}_\lambda(X^{-1},q^{-1}) = \bar{E}_\lambda(X^{-1},q^{-1}).$$

□

Chapter 3

Representation theory of $\mathfrak{b}_-$

In this chapter, we collect materials on representations of $\mathfrak{b}_-$ such as categories of $\mathfrak{b}_-$-modules, Demazure modules and their graded characters. Basic references in this chapter are [CK, FKM, Ion, Kac]. We continue to work in the setting of the previous chapter.

3.1 Categories of representations of $\mathfrak{b}_-$

For each $\mathfrak{b}_-$-module M and $\lambda \in P$, we set

$$M_\lambda := \{m \in M| \ hm = \lambda(h)m \text{ for } h \in \mathfrak{h}\}$$

and

$$\text{wt } M := \{\lambda \in P \mid M_\lambda \neq \{0\}\}.$$

For $v \in M_\lambda$, we set $\text{wt}(v) = \lambda$.

Definition 3.1.1. *The category $\mathfrak{B}$ is the full subcategory of the category of $U(\mathfrak{b}_-)$-modules such that a $\mathfrak{b}_-$-module M is an object of $\mathfrak{B}$ if and only if M*

has a weight decomposition

$$M = \bigoplus_{\lambda \in P} M_\lambda$$

such that M_λ has at most countable dimension for all $\lambda \in P$.

Definition 3.1.2. *The category $\mathfrak{B}'$ is the full subcategory of $\mathfrak{B}$ such that $M \in \mathfrak{B}$ is an object of $\mathfrak{B}'$ if and only if the following are satisfied:*

(1) *The module M is a $\mathfrak{b}_-$-module such that the set of weights* wt M *is contained in $\bigcup_{i=1,...,k} (\mu_i - Q_+)$ for some $\{\mu_1, ..., \mu_k\} \subset P$;*

(2) *Every weight space M_λ is finite dimensional.*

Let $\mathfrak{B}_0$ be the full subcategory of $\mathfrak{B}'$ consisting of finite-dimensional $\mathfrak{b}_-$-modules.

Definition 3.1.3. *For each $M \in \mathfrak{B}'$, we define a graded character of M by the following formal sum*

$$\mathrm{gch}\, M := \sum_{\lambda - m\delta \in \mathring{P} \oplus \frac{1}{2}\mathbb{Z}\delta} q^m X^\lambda \dim_{\mathbb{C}} \mathrm{Hom}_{\mathring{\mathfrak{h}} \oplus \mathbb{C}d} (\mathbb{C}_{\lambda - m\delta}, M),$$

where $\mathbb{C}_{\lambda - m\delta}$ is a 1-dimensional $\mathring{\mathfrak{h}} \oplus \mathbb{C}d$-module with its weight $\lambda - m\delta$.

For each $\Lambda \in P$, let $\mathbb{C}'_\Lambda$ be the 1-dimensional $\mathfrak{h}$-module with its weight Λ, and $\mathbb{C}_\Lambda$ be the 1-dimensional simple module of $\mathfrak{b}_-$ with its $\mathfrak{h}$-weight Λ. For each $\Lambda \in P$, we set $P(\Lambda) := U(\mathfrak{b}_-) \underset{U(\mathfrak{h})}{\otimes} \mathbb{C}'_\Lambda$ and $N(\Lambda) := \sum_{\mu \in P \setminus \{\Lambda\}} P(\Lambda)_\mu$. Then $N(\Lambda)$ is a $\mathfrak{b}_-$-submodule of $P(\Lambda)$ and $\mathbb{C}_\Lambda \cong P(\Lambda)/N(\Lambda)$.

Proposition 3.1.4. *For each $\Lambda \in P$, the $\mathfrak{b}_-$-module $P(\Lambda) = U(\mathfrak{b}_-) \underset{U(\mathfrak{h})}{\otimes} \mathbb{C}'_\Lambda$ is a projective cover of $\mathbb{C}_\Lambda$ in $\mathfrak{B}$.*

Proof. For each $M \in \mathfrak{B}$, we have $\mathrm{Hom}_{\mathfrak{B}}(P(\Lambda), M) = \mathrm{Hom}_{\mathfrak{h}}(\mathbb{C}_\Lambda, M)$. Hence, $P(\Lambda)$ is a projective cover of $\mathbb{C}_\Lambda$ in $\mathfrak{B}$. $\square$

Proposition 3.1.5 ([FKM] Lemma 5.2)**.** *The category $\mathfrak{B}$ has enough projectives.* $\square$

Definition 3.1.6. *Let M be a $\mathfrak{b}_-$-module with $\mathfrak{h}$-weight decomposition $M = \bigoplus_{\mu \in \mathfrak{h}^*} M_\mu$. Then $M^\vee := \bigoplus_{\mu \in \mathfrak{h}^*} M^*$ is a $\mathfrak{b}_-$-module with a $\mathfrak{b}_-$-action defined by*

$$Xf(v) := -f(Xv) \text{ for } X \in \mathfrak{b}_-,\ f \in M^\vee \text{ and } v \in M.$$

Definition 3.1.7. *For each $M \in \mathfrak{B}'$ and $N \in \mathfrak{B}_0$, we define the Euler-Poincaré pairing $\langle M, N\rangle_{\mathrm{Ext}}$ as a formal sum by*

$$\langle M, N\rangle_{\mathrm{Ext}} := \sum_{p \in \mathbb{Z}_{\geq 0}, m \in \frac{1}{2}\mathbb{Z}} (-1)^p q^m \dim_{\mathbb{C}} \mathrm{Ext}^p_{\mathfrak{B}}(M \otimes_{\mathbb{C}} \mathbb{C}_{m\delta}, N^\vee).$$

Proposition 3.1.8. *For each $M \in \mathfrak{B}'$ and $N \in \mathfrak{B}_0$, the following hold:*

(1) *The pairing $\langle M, N\rangle_{\mathrm{Ext}}$ is an element of $\mathbb{C}(\!(q^{1/2})\!)$;*

(2) *This pairing depends only on the graded characters of M and N.*

Proof. First, we prove (1). Let S be the set of highest weight vectors of M. Since wt M is bounded from above, we have a surjection $\varphi^0 : P^0 := \bigoplus_{v \in S} P(\mathrm{wt}(v)) \to M$, where $\mathrm{wt}(v)$ is the $\mathfrak{h}$-weight of v. If $v \in S$ such that $(\mathrm{wt}(v) + Q_+ \backslash \{0\}) \cap \mathrm{wt}\ M = \emptyset$, then the vector v is not an element of $\mathrm{Ker}\,\varphi^0$. Hence the set $\mathrm{wt}\,\mathrm{Ker}\,\varphi^0$ is a proper subset of wt P^0. For $\mathrm{Ker}\,\varphi^0$, we define $\varphi^1 : P^1 \to \mathrm{Ker}\,\varphi^0$ in the same way. Repeating this procedure, we get a projective resolution $\cdots \to P^1 \to P^0 \to M \to 0$ such that wt P^{k+1} is a proper subset of wt P^k for all $k \in \mathbb{Z}_+$. The complex $P^\bullet \otimes_{\mathbb{C}} \mathbb{C}_{m\delta}$ is a projective resolution of $M \otimes_{\mathbb{C}} \mathbb{C}_{m\delta}$. For each $m \in \frac{1}{2}\mathbb{Z}$, we have wt $(P^k \otimes_{\mathbb{C}} \mathbb{C}_{m\delta}) \cap \mathrm{wt}\ N = \emptyset$ for all $k \gg 0$ since N and every weight space of M are finite-dimensional. This implies $\mathrm{Ext}^k_{\mathfrak{B}}(M \otimes_{\mathbb{C}} \mathbb{C}_{m\delta}, N^\vee) = \{0\}$ for all $k \gg 0$. Hence $\sum_{k \in \mathbb{Z}_+} (-1)^k q^m \dim_{\mathbb{C}} \mathrm{Ext}^k_{\mathfrak{B}}(M \otimes_{\mathbb{C}} \mathbb{C}_{m\delta}, N^\vee)$ is well-defined. Since $\mathfrak{b}_-$-action on P^0 does not increase d-eigenvalues, and the set of weights of an object of $\mathfrak{B}'$ is bounded from above, the intersection of the set of d-eigenvalues of $N^\vee$ and $P^0 \otimes_{\mathbb{C}} \mathbb{C}_{m\delta}$ is empty for all $m \ll 0$. This implies the assertion.

Next, we prove (2). Let N' be an object of $\mathfrak{B}_0$ such that $\mathrm{gch}\, N = \mathrm{gch}\, N'$. The sets of composition factors of N and N' are the same. We denote the set of composition factors by S. For each exact sequence $0 \to L_1 \to L_2 \to L_3 \to 0$, we have $\langle M, L_2 \rangle_{\mathrm{Ext}} = \langle M, L_1 \rangle_{\mathrm{Ext}} + \langle M, L_3 \rangle_{\mathrm{Ext}}$. This implies $\langle M, N \rangle_{\mathrm{Ext}} = \sum_{\mathbb{C}_\Lambda \in S} \langle M, \mathbb{C}_\Lambda \rangle_{\mathrm{Ext}} = \langle M, N' \rangle_{\mathrm{Ext}}$. Hence the assertion for the second argument follows. Let $K^0 := \bigoplus_{v \in S} N(\mathrm{wt}(v))$ be a $\mathfrak{b}_-$-submodule of P^0. We set $N^0 := M$ and $N^1 := \varphi^0(K^0)$. We define a $\mathfrak{b}_-$-submodule N^2 of N^1 in the same way for N^1 instead of M. Repeating this, we get a sequence of $\mathfrak{b}_-$-submodules $M = N^0 \supset N^1 \supset N^2 \supset \cdots$. Since every weight space of M is finite-dimensional, for each $\mu \in P$, we have $N^s_\mu = \{0\}$ for $s \gg 0$ by construction. We can take a composition series $M = M^0 \supset \cdots \supset M^s \supset M^{s+1} \supset \cdots$ of M as a refinement of the above sequence of $\mathfrak{b}_-$-modules. Since N is finite-dimensional, for $s \gg 0$, we have $\mathrm{wt}(v) - \mathrm{wt}(u) \notin Q_+$ for each $v \in M^s$ and $u \in N$. By taking a projective resolution of M^s as in the proof of (1), we have $\mathrm{Ext}^k_{\mathfrak{B}}(M^s \otimes_{\mathbb{C}} \mathbb{C}_{m\delta}, N^\vee) = \{0\}$ for $s \gg 0$. Using this composition series, we can prove the assertion for the first argument in the same way. $\square$

Thanks to Proposition 3.1.8, we get a bilinear map

$$\mathbb{C}(\!(q^{1/2})\!)[\mathring{P}] \times \mathbb{C}(\!(q^{1/2})\!)[\mathring{P}] \ni (f, g) \to \langle f, g \rangle_{\mathrm{Ext}} \in \mathbb{C}(\!(q^{1/2})\!)[\mathring{P}],$$

that we also denote by $\langle -, - \rangle_{\mathrm{Ext}}$

Proposition 3.1.9. *For each $M \in \mathfrak{B}'$ and $N \in \mathfrak{B}_0$, we have $\langle \mathrm{gch}\, M, \mathrm{gch}\, N \rangle_{\mathrm{Ext}} = \langle \mathrm{gch}\, M, \mathrm{gch}\, N \rangle_{nonsym}$.*

Proof. $\{\operatorname{gch}\mathbb{C}_\Lambda\}_{\Lambda\in P}$ and $\{\operatorname{gch}P(\Lambda)\}_{\Lambda\in P}$ are $\mathbb{C}(\!(q^{1/2})\!)$-basis of $\mathbb{C}(\!(q^{1/2})\!)[\mathring{P}]$. Therefore, it suffices to check the assertion for $M=\mathbb{C}_\Lambda$ and $N=P(\Lambda)$. By the PBW theorem, we have $\operatorname{gch}P(\Lambda)=X^\Lambda/\prod_{\alpha\in\Delta_-}(1-X^\alpha)^{\dim_\mathbb{C}\mathfrak{g}_\alpha}$. Hence we have $\operatorname{gch}P(\Lambda)=(X^\Lambda/\mathcal{C})|_{t=0}$. Hence we get

$$\langle\operatorname{gch}P(\Lambda),\ \operatorname{gch}\mathbb{C}_\Lambda\rangle_{\mathrm{Ext}}=1=\langle\operatorname{gch}P(\Lambda),\ \operatorname{gch}\mathbb{C}_\Lambda\rangle_{nonsym}.$$

The assertion follows. □

3.2 Demazure modules

3.2.1 Highest weight simple modules of $\mathfrak{g}$

Definition 3.2.1. *Let $\Lambda\in P$ and let $\mathbb{C}_\Lambda$ be the corresponding 1-dimensional module of $\mathfrak{b}_+$. The Verma module $M(\Lambda)$ of highest weight Λ is a $\mathfrak{g}$-module defined by*

$$M(\Lambda):=U(\mathfrak{g})\underset{U(\mathfrak{b}_+)}{\otimes}\mathbb{C}_\Lambda.$$

The Verma module $M(\Lambda)$ has a unique simple quotient (see [Kac] Proposition 9.2). We denote it by $L(\Lambda)$.

Theorem 3.2.2 (see [Kac] Proposition 3.7, Lemma 10.1 and §9.2)**.** *For each $\Lambda\in P$, the following hold:*

(1) *$L(\Lambda)$ is an integrable $\mathfrak{g}$-module if and only if $\Lambda\in P_+$;*

(2) *For each $\Lambda\in P_+$ and $w\in W$, we have $\dim_\mathbb{C}L(\Lambda)_{w\Lambda}=1$;*

(3) *$L(\Lambda)$ has a $\mathfrak{h}$-weight decomposition*

$$L(\Lambda)=\bigoplus_{\mu\in P}L(\Lambda)_\mu$$

and $L(\Lambda)_\mu$ is finite-dimensional for all $\mu\in P$. □

We remark that $\operatorname{gch}L(\Lambda)$ is well-defined thanks to Theorem 3.2.2 (3).

3.2.2 Realization of $L(\Lambda_0)$

Definition 3.2.3 (Heisenberg algebra)**.** *For each $l \in \mathbb{N}$, let S_l be a unital $\mathbb{C}$-algebra generated by $x_{i,n}$ $(i = 1, ..., l,\ 0 \neq n \in \mathbb{Z})$ and K which satisfy the following conditions:*

(1) $[x_{i,n}, x_{j,m}] = n\delta_{i,j}\delta_{n,-m}K$*;*

(2) $[K, S_l] = 0$*.*

We set $R = \mathbb{C}[y_{i,n} \mid i \in \{1, ..., l\},\ n \in \mathbb{N}]$. We define a representation $p : S_l \to \mathrm{End}_{\mathbb{C}}(R)$ by

$$p(x_{i,-n}) = y_{i,n}, \ \ p(x_{i,n}) = n\frac{\partial}{\partial y_{i,n}}, \ \ p(K) = \mathrm{id}_R \ \ (n > 0).$$

Let $\mathfrak{g}_{im} := \bigoplus_{n\in\mathbb{Z}\setminus\{0\}} \mathfrak{g}_{n\delta}$. The algebra S_l is a $\mathbb{Z}$-graded algebra by setting $\deg x_{i,n} = n$ and $\deg K = 0$, and $U(\mathfrak{g}_{im} \oplus \mathbb{C}K)$ is a $\mathbb{Z}$-graded algebra by the $\mathbb{Z}$-grading induced from the adjoint action of the scaling element d. For $\mathfrak{g}$ of type $A_{2l}^{(2)}$, we have $\dim_{\mathbb{C}} \mathfrak{g}_{n\delta} = l$ for $n \in \mathbb{Z}$, and we have the following.

Proposition 3.2.4 (see [Kac] Proposition 8.4)**.** *The algebras $U(\mathfrak{g}_{im} \oplus \mathbb{C}K)$ and S_l are isomorphic as $\mathbb{Z}$-graded algebras.* $\square$

By Proposition 3.2.4, we identify S_l with $U(\mathfrak{g}_{im} \oplus \mathbb{C}K)$. Since $\mathring{\mathfrak{h}}$ and $U(\mathfrak{g}_{im} \oplus \mathbb{C}K)$ are mutually commutative, the following $\mathbb{C}$-algebra homomorphism $p_\lambda : U(\mathfrak{g}_{im} \oplus \mathring{\mathfrak{h}} \oplus \mathbb{C}K) \to \mathrm{End}_{\mathbb{C}}(R)$ $(\lambda \in \mathring{P})$ is well-defined

$$p_\lambda|_{S_l} = p \ \text{ and } \ p_\lambda(h) = \lambda(h)\mathrm{id}_R \ \text{ for } \ h \in \mathring{\mathfrak{h}}.$$

We denote this $U(\mathfrak{g}_{im} \oplus \mathring{\mathfrak{h}} \oplus \mathbb{C}K)$-module by R_λ.

Theorem 3.2.5 ([LNX] Theorem 6.4)**.** *We put $\tilde{p} := \prod_{\lambda\in\mathring{P}} p_\lambda : U(\mathfrak{g}_{im} \oplus \mathring{\mathfrak{h}} \oplus \mathbb{C}K) \to \mathrm{End}_{\mathbb{C}}(\bigoplus_{\lambda\in\mathring{P}} R_\lambda)$. Then $\tilde{p}$ extends to an algebra homomorphism $U(\mathfrak{g}) \to \mathrm{End}_{\mathbb{C}}(\bigoplus_{\lambda\in\mathring{P}} R_\lambda)$ and $\bigoplus_{\lambda\in\mathring{P}} R_\lambda$ is isomorphic to $L(\Lambda_0)$ as a $\mathfrak{g}$-module.* $\square$

3.2.3 Thin and thick Demazure modules

Definition 3.2.6. *For each $w \in W$ and $\Lambda \in P_+$, we define $\mathfrak{b}_-$-modules*

$$D_{w\Lambda} := U(\mathfrak{b}_-)v^*_{w\Lambda} \subset L(\Lambda)^\vee \text{ and } D^{w\Lambda} := U(\mathfrak{b}_-)v_{w\Lambda} \subset L(\Lambda),$$

where $v_{w\Lambda} \in L(\lambda)_{w\Lambda}$ and $v^*_{w\Lambda} \in (L(\Lambda)_{-w\Lambda})^*$ are nonzero vectors. By Theorem 3.2.2 (3)*, these vectors are unique up to scalars. Hence $D_{w\Lambda}$ and $D^{w\Lambda}$ are well-defined. We call $D_{w\Lambda}$ a thin Demazure module and $D^{w\Lambda}$ a thick Demazure module.*

Remark 3.2.7. *Thin Demazure module is usually referred to as Demazure module in the literature.*

Lemma 3.2.8 ([Kac] Proposition 3.6)**.** *For each $w \in W$, $\Lambda \in P_+$ and $\alpha \in \Delta_+$, we have*

$$v_{s_\alpha w\Lambda} \in \begin{cases} \mathfrak{g}_{-\alpha}^{\langle w\Lambda, \check{\alpha}\rangle} v_{w\Lambda} & (\langle w\Lambda, \check{\alpha}\rangle > 0) \\ \mathfrak{g}_{\alpha}^{-\langle w\Lambda, \check{\alpha}\rangle} v_{w\Lambda} & (\langle w\Lambda, \check{\alpha}\rangle < 0) \\ \mathbb{C}v_{w\Lambda} & (\langle w\Lambda, \check{\alpha}\rangle = 0) \end{cases},$$

where $\mathfrak{g}^m_\alpha = \{X_1X_2 \cdots X_m \in U(\mathfrak{g}) \mid X_i \in \mathfrak{g}_\alpha\}$. □

Lemma 3.2.9 and Corollary 3.2.12 in the below are proved in [CK] for the dual of the untwisted affine Lie algebra. The proofs in [CK] are also valid for type $A^{(2)}_{2l}$.

Lemma 3.2.9 ([CK] Corollary 4.2)**.** *For each $w, v \in W$ and $\Lambda \in P_+$, we have the following:*

(1) *If $w \leq v$, then $D^{v\Lambda} \subseteq D^{w\Lambda}$;*

(2) *If w and v are minimal representatives of cosets in W/W^Λ and $D^{v\Lambda} \subseteq D^{w\Lambda}$, then $w \leq v$.* □

Lemma 3.2.9 allows us to define as follows:

Definition 3.2.10. *For each $w \in W$ and $\Lambda \in P_+$, we define a $U(\mathfrak{b}_-)$-module $\mathbb{D}^{w\Lambda}$ as*

$$\mathbb{D}^{w\Lambda} := D^{w\Lambda} / \sum_{w<v} D^{v\Lambda}.$$

We call this module Demazure slice.

Proposition 3.2.11 ([Kat] Corollary 2.22)**.** *For each $\Lambda \in P_+$ and $S \subset W$, there exists $S' \subset W$ such that*

$$\bigcap_{w \in S} D^{w\lambda} = \sum_{w \in S'} D^{w\Lambda}.$$

□

Corollary 3.2.12 ([CK] Corollary 4.4)**.** *For each $w,\ v \in W$ and $\Lambda \in P_+$, we have*

$$(D^{w\Lambda} \cap D^{v\Lambda}) / (D^{v\Lambda} \cap \sum_{u>w} D^{u\Lambda}) = \mathbb{D}^{w\Lambda} \text{ or } \{0\}.$$

□

3.2.4 Level one Demazure modules

In this subsection, we consider level one Demazure modules. The unique level one dominant integral weight of $A_{2l}^{(2)}$ is Λ_0. From (2.4.1),

$$\mathring{P} \ni \lambda \mapsto \lambda + \Lambda_0 + \frac{(\lambda|\lambda)}{2}\delta \in W\Lambda_0$$

is a bijection. For each $\lambda \in \mathring{P}$, we set

$$D_\lambda := D_{\pi_\lambda},\ D^\lambda := D^{\pi_\lambda},\ \mathbb{D}^\lambda := \mathbb{D}^{\pi_\lambda}.$$

Lemma 3.2.13. *For each $\lambda,\ \mu \in \mathring{P}$, we have $D^\lambda \subsetneq D^\mu$ if and only if $\mu \succ \lambda$.*

Proof. If $D^\lambda \subsetneq D^\mu$, then we have $\pi_\mu < \pi_\lambda$ by Lemma 3.2.9. Then, Lemma 2.4.5 (1) implies $\mu \succ \lambda$. Conversely, we assume that $\mu \succ \lambda$. There exists $w \in W$ such that $\mu \succ \lambda = w(\!(\mu)\!)$. Let $w = s_{i_1} \cdots s_{i_n}$ be a reduced expression of w such that $(s_{i_{k+1}} \cdots s_1)(\!(\mu)\!) \succ (s_{i_k} s_{i_{k+1}} \cdots s_1)(\!(\mu)\!)$ for all k. If $n = 1$, then Lemma 2.4.5 (2) implies $\pi_\mu < \pi_\lambda$. Hence, we have $D^\lambda \subsetneq D^\mu$. If $n > 1$, then we have $D^\lambda \subsetneq D^{(s_{i_2} \cdots s_{i_n})(\!(\mu)\!)} \subsetneq \cdots \subsetneq D^\mu$ inductively. □

Theorem 3.2.14 ([Ion] Theorem 1). *For each $\lambda \in \mathring{P}$, we have*

$$\mathrm{gch}\ D_\lambda = q^{\frac{(b|b)}{2}} \bar{E}_\lambda(X^{-1}, q^{-1}).$$

□

Chapter 4

Representation theory of $\mathfrak{Cg}$

In this chapter, we collect materials on representations of $\mathfrak{Cg}$ such as Weyl modules and categories of representations. Then we calculate extensions between global and local Weyl modules in the category $\mathfrak{B}$. We continue to work in the setting of the previous chapters.

4.1 Categories of representations of $\mathfrak{Cg}$

Definition 4.1.1. *The category* $\mathfrak{Cg}$-$\mathrm{mod}_{\mathrm{wt}}$ *is the full subcategory of the category of* $\mathfrak{Cg}$*-modules such that* M *is an object of* $\mathfrak{Cg}$-$\mathrm{mod}_{\mathrm{wt}}$ *if and only if* M *is a* $\mathfrak{Cg}$*-module which has a weight decomposition*

$$M = \bigoplus_{\Lambda \in P} M_\Lambda$$

such that every weight space has at most countable dimension.

Definition 4.1.2. *The* $\mathfrak{Cg}$-$\mathrm{mod}_{\mathrm{int}}$ *is the full subcategory of the category* $\mathfrak{Cg}$-$\mathrm{mod}_{\mathrm{wt}}$ *such that an object* M *of* $\mathfrak{Cg}$-$\mathrm{mod}_{\mathrm{wt}}$ *is an object of* $\mathfrak{Cg}$-$\mathrm{mod}_{\mathrm{int}}$ *if and only if every d-eigenspaces are finite-dimensional.*

Definition 4.1.3. *For each $\lambda \in \mathring{P}_+$, $\mu \in \mathring{P}$ and $n,\ 2m \in \mathbb{Z}$, we set*

$$P(\lambda + n\Lambda_0 + m\delta)_{\mathrm{int}} := U(\mathfrak{Cg}) \underset{U(\mathring{\mathfrak{g}}+\mathfrak{h})}{\otimes} V(\lambda + n\Lambda_0 + m\delta)$$

and

$$P(\mu + n\Lambda_0 + m\delta)_{\mathrm{wt}} := U(\mathfrak{Cg}) \underset{U(\mathfrak{h})}{\otimes} \mathbb{C}_{\mu+n\Lambda_0+m\delta},$$

where $V(\lambda + n\Lambda_0 + m\delta)$ is the highest weight simple module of $\mathring{\mathfrak{g}} + \mathfrak{h}$ with its highest weight $\lambda + n\Lambda_0 + m\delta$ and $\mathbb{C}_{\mu+n\Lambda_0+m\delta}$ is the 1-dimensional module of $\mathfrak{h}$ with its weight $\mu + n\Lambda_0 + m\delta$.

Let $\pi : \mathfrak{Cg} \to \mathring{\mathfrak{g}}$ be a homomorphism of Lie algebras defined by

$$\pi|_{\mathring{\mathfrak{g}}} = \mathrm{id}_{\mathring{\mathfrak{g}}}, \quad \pi(\mathfrak{Cg}_{\neq 0}) = \{0\},$$

where $\mathfrak{Cg}_{\neq 0} := \{X \in \mathfrak{Cg} \mid [d, X] \neq 0\}$. We can prove the following proposition in the same way as Proposition 3.1.4, and we omit its proof.

Proposition 4.1.4. *For each $\mu \in \mathring{P}$ and n, $2m \in \mathbb{Z}$, the $\mathfrak{Cg}$-module $P(\mu + n\Lambda_0 + m\delta)_{\mathrm{wt}}$ is a projective module.* □

Proposition 4.1.5 ([CI] Proposition 2.3)**.** *Let $\lambda \in \mathring{P}_+$ and $n,\ 2m \in \mathbb{Z}$.*

(1) *$\pi^*V(\lambda + n\Lambda_0 + m\delta)$ is a simple object in $\mathfrak{Cg}$-$\mathrm{mod}_{\mathrm{int}}$.*

(2) *$P(\lambda + n\Lambda_0 + m\delta)_{\mathrm{int}}$ is a projective cover of its unique simple quotient $\pi^*V(\lambda + n\Lambda_0 + m\delta)$ in $\mathfrak{Cg}$-$\mathrm{mod}_{\mathrm{int}}$.* □

Proposition 4.1.6. *The categories $\mathfrak{Cg}$-$\mathrm{mod}_{\mathrm{wt}}$ and $\mathfrak{Cg}$-$\mathrm{mod}_{\mathrm{int}}$ have enough projectives.*

Proof. We can prove that $\mathfrak{Cg}$-$\mathrm{mod}_{\mathrm{wt}}$ has enough projectives in the same way as Proposition 3.1.5. Let M be an object of $\mathfrak{Cg}$-$\mathrm{mod}_{\mathrm{int}}$. Since M is an integrable $\mathring{\mathfrak{g}}$-module, for each $\mathring{\mathfrak{g}}$-highest weight vector $v \in M$ with its weight

Λ, we have a morphism of $\mathfrak{C}\mathfrak{g}$-module $P(\Lambda)_{\mathrm{int}} \to M$. Collecting them for all $\mathring{\mathfrak{g}}$-highest weight vector, we obtain a surjection from a projective module to M. $\square$

Definition 4.1.7. *For each M, $N \in \mathfrak{C}\mathfrak{g}\text{-mod}_{\mathrm{int}}$ such that N is finite-dimensional, we define the Euler-Poincaré-pairing $\langle M, N \rangle_{int}$ as a formal sum by*

$$\langle M, N \rangle_{int} := \sum_{p \in \mathbb{Z}_{\geq 0}, m \in \frac{1}{2}\mathbb{Z}} (-1)^p q^m \dim_{\mathbb{C}} \mathrm{Ext}^p_{\mathfrak{C}\mathfrak{g}-\mathrm{mod}_{\mathrm{int}}}(M \otimes_{\mathbb{C}} \mathbb{C}_{m\delta}, N^{\vee}).$$

We can prove the following proposition in the same way as Proposition 3.1.8, and we omit its proof.

Proposition 4.1.8. *For each M, $N \in \mathfrak{C}\mathfrak{g}\text{-mod}_{\mathrm{int}}$ such that N is finite-dimensional, the following hold:*

(1) *The pairing $\langle M, N \rangle_{int}$ is an element of $\mathbb{C}(\!(q^{1/2})\!)$;*

(2) *This pairing depends only on the graded characters of M and N.* $\square$

4.2 Weyl modules

Definition 4.2.1 ([CIK] §3.3)**.** *For each $\lambda \in \mathring{P}_+$, the global Weyl module is a cyclic $\mathfrak{C}\mathfrak{g}$-module $W(\lambda)$ generated by a vector v_λ that satisfies following relations:*

(1) *$hv_\lambda = \lambda(h)v_\lambda$ for each $h \in \mathfrak{h}$;*

(2) *$e_{-\alpha}^{\langle \lambda, \check{\alpha} \rangle + 1} v_\lambda = 0$ for each $\alpha \in \mathring{\Delta}_+$;*

(3) *$\mathfrak{C}\mathfrak{n}_+ v_\lambda = 0$.*

Definition 4.2.2 ([CIK] §3.5 and §7.2)**.** *For each $\lambda \in \mathring{P}$, the local Weyl module is a cyclic $\mathfrak{C}\mathfrak{g}$-module $W(\lambda)_{loc}$ generated by a vector v_λ satisfies relations* (1), (2), (3) *of Definition 4.2.1 and*

(4) $Xv_\lambda = 0$ *for* $X \in \mathfrak{Cg}_{im}$.

Theorem 4.2.3 ([CIK] Theorem 2)**.** *Let* $\lambda \in \mathring{P}_+$. *Then* $D_\lambda \otimes_{\mathbb{C}} \mathbb{C}_{(\lambda|\lambda)\delta/2-\Lambda_0}$ *is isomorphic to* $W(\lambda)_{loc}$ *as* $\mathfrak{Cg}$*-module, where* $\mathbb{C}_{(\lambda|\lambda)\delta/2-\Lambda_0}$ *is the 1-dimensional module with its* $\mathfrak{h}$*-weight* $(\lambda|\lambda)\delta/2 - \Lambda_0$. □

Corollary 4.2.4. *For each* $\lambda \in \mathring{P}_+$, *we have*

$$\mathrm{gch}\, W(\lambda)_{loc} = q^{\frac{(b|b)}{2}} \bar{P}_\lambda(X^{-1}, q^{-1}).$$

Proof. By Theorem 4.2.3, we have

$$\mathrm{gch}\, W(\lambda)_{loc} = \mathrm{gch}\, D_\lambda.$$

By Proposition 2.5.7 and Theorem 3.2.14, the assertion follows. □

Theorem 4.2.5 ([CI] Theorem 2.5 (3), Theorem 4.7 and [Kle] Theorem 7.21)**.** *For each* λ, $\mu \in \mathring{P}_+$ *and* $m \in \frac{1}{2}\mathbb{Z}$, *we have*

$$\dim_{\mathbb{C}} \mathrm{Ext}^n_{\mathfrak{Cg}\text{-mod}_{\mathrm{int}}}(W(\lambda) \otimes_{\mathbb{C}} \mathbb{C}_{m\delta}, W(\mu)^\vee_{loc}) = \delta_{m,0}\delta_{0,n}\delta_{\lambda,\mu}.$$

□

Corollary 4.2.6. *For each* λ, $\mu \in \mathring{P}_+$, *we have* $\langle \mathrm{gch}\, W(\lambda), \mathrm{gch}\, W(\mu)_{loc}\rangle_{int} = \delta_{\lambda,\mu}$.

Proof. The assertion follows from Definition 4.1.7 and Theorem 4.2.5. □

4.2.1 Extensions between Weyl modules in $\mathfrak{B}$

In this subsection, we prove the following corollary of Theorem 4.2.5.

Theorem 4.2.7. *For each* λ, $\mu \in \mathring{P}_+$, $m \in \frac{1}{2}\mathbb{Z}$ *and* $n \in \mathbb{Z}_{\geq 0}$, *we have*

$$\dim_{\mathbb{C}} \mathrm{Ext}^n_{\mathfrak{B}}(W(\lambda) \otimes_{\mathbb{C}} \mathbb{C}_{m\delta}, W(\mu)^\vee_{loc}) = \delta_{m,0}\delta_{0,n}\delta_{\lambda,\mu}.$$

□

Definition 4.2.8 ([Gro] §2.1)**.** *Let $\mathfrak{C}$, $\mathfrak{D}$ be abelian categories. A contravariant δ-functor from $\mathfrak{C}$ to $\mathfrak{D}$ consists of the following data:*

(a) *A collection $T = \{T^i\}_{i \in \mathbb{Z}_{\geq 0}}$ of contravariant additive functors from $\mathfrak{C}$ to $\mathfrak{D}$;*

(b) *For each exact sequence $0 \to M'' \to M \to M' \to 0$, a collection of morphisms $\{\delta^n : T^n(M'') \to T^{n+1}(M')\}_{n \in \mathbb{Z}_{\geq 0}}$ with the following conditions:*

(1) *For each exact sequence $0 \to M'' \to M \to M' \to 0$, there is a long exact sequence*

$$\begin{aligned} 0 \ &\to \ T^0(M') \to T^0(M) \to T^0(M'') \stackrel{\delta^0}{\to} \\ &\to \ T^1(M') \to \ \cdots \ \to T^{n-1}(M'') \stackrel{\delta^{n-1}}{\to} \\ &\to \ T^n(M') \to T^n(M) \to T^n(M'') \stackrel{\delta^n}{\to} \cdots ; \end{aligned}$$

(2) *For each morphism of short exact sequence*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M'' & \longrightarrow & M & \longrightarrow & M' & \longrightarrow & 0 \\ & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\ 0 & \longrightarrow & N'' & \longrightarrow & N & \longrightarrow & N' & \longrightarrow & 0 \end{array} ,$$

we have the following commutative diagram

$$\begin{array}{ccc} T^{n-1}(N'') & \xrightarrow{\delta^{n-1}} & T^n(N') \\ \big\downarrow & & \big\downarrow \\ T^{n-1}(M'') & \xrightarrow{\delta^{n-1}} & T^n(M') \end{array} .$$

Definition 4.2.9 ([Gro] §2.1)**.** *For each contravariant δ-functors $T = \{T^i\}_{i \in \mathbb{Z}_{\geq 0}}$ and $S = \{S^i\}_{i \in \mathbb{Z}_{\geq 0}}$, a morphism of δ-functor from $T = \{T^i\}_{i \in \mathbb{Z}_{\geq 0}}$ to $S = \{S^i\}_{i \in \mathbb{Z}_{\geq 0}}$ is a collection of natural transformations $F = \{F^i : T^i \to S^i\}_{i \in \mathbb{Z}_{\geq 0}}$ with the following condition:*

(∗) *For each exact sequence* $0 \to M'' \to M \to M' \to 0$, *the following diagram is commutative*

$$\begin{array}{ccc} T^{n-1}(M'') & \xrightarrow{\delta^{n-1}} & T^n(M') \\ {\scriptstyle F^{n-1}(M'')}\Big\downarrow & & {\scriptstyle F^n(M')}\Big\downarrow \\ S^{n-1}(M'') & \xrightarrow{\delta^{n-1}} & S^n(M') \end{array}.$$

Definition 4.2.10 ([Gro] §2.2)**.** *A contravariant* δ*-functor* $T = \{T^i\}_{i\in\mathbb{Z}_{\geq 0}}$ *is called a universal* δ*-functor if for each* δ*-functor* $S = \{S^i\}_{i\in\mathbb{Z}_{\geq 0}}$ *and for each natural transformation* $F^0 : T^0 \to S^0$, *there exists a unique morphism of* δ*-functor* $\{F^i : T^i \to S^i\}_{i\in\mathbb{Z}_{\geq 0}}$.

Definition 4.2.11 ([Gro] §2.2)**.** *An additive functor* $F : \mathfrak{C} \to \mathfrak{D}$ *is called coeffaceable if for each object* M *of* $\mathfrak{C}$, *there is a epimorphism* $P \to M$ *such that* $F(P) = 0$.

Theorem 4.2.12 ([Gro] Proposition 2.2.1)**.** *For each* $\mathfrak{C}$, $\mathfrak{D}$ *be abelian categories and let* $T = \{T^i\}_{i\in\mathbb{Z}_{\geq 0}}$ *be a contravariant* δ*-functor from* $\mathfrak{C}$ *to* $\mathfrak{D}$*. If* T^i *is coeffaceable for* $i > 0$, *then* T *is universal.* □

Lemma 4.2.13 (Shapiro's lemma)**.** *For each* $M \in \mathfrak{B}$, $N \in \mathfrak{Cg}\text{-mod}_{\mathrm{wt}}$ *and* $n \in \mathbb{Z}_{\geq 0}$, *we have*

$$\mathrm{Ext}^n_{\mathfrak{B}}(M, N) = \mathrm{Ext}^n_{\mathfrak{Cg}\text{-mod}_{\mathrm{wt}}}(U(\mathfrak{Cg}) \underset{U(\mathfrak{b}_-)}{\otimes} M, N).$$

Proof. Let $P^\bullet \to M \to 0$ be a projective resolution of M in $\mathfrak{B}$. Since $U(\mathfrak{Cg})$ is free over $U(\mathfrak{b}_-)$, the complex $U(\mathfrak{Cg}) \underset{U(\mathfrak{b}_-)}{\otimes} P^\bullet$ is a projective resolution of $U(\mathfrak{Cg}) \underset{U(\mathfrak{b}_-)}{\otimes} M$ in $\mathfrak{Cg}\text{-mod}_{\mathrm{wt}}$. The assertion follows from the Frobenius reciprocity. □

Lemma 4.2.14. *We have the following:*

(1) *For each* $M,\ N \in \mathfrak{Cg}\text{-mod}_{\mathrm{int}}$, *we have*

$$\mathrm{Ext}^k_{\mathfrak{Cg}\text{-mod}_{\mathrm{wt}}}(M, N^\vee) = \mathrm{Ext}^k_{\mathfrak{Cg}\text{-mod}_{\mathrm{int}}}(M, N^\vee) \quad k \in \mathbb{Z}_{\geq 0};$$

(2) *For each* $N \in \mathfrak{Cg}\text{-mod}_{\mathrm{int}}$, *we have*

$$\mathrm{Ext}^k_{\mathfrak{Cg}\text{-mod}_{\mathrm{wt}}}(U(\mathfrak{Cg}) \underset{U(\mathfrak{b}_-)}{\otimes} \mathbb{C}_0, N^\vee) = \mathrm{Ext}^k_{\mathfrak{Cg}\text{-mod}_{\mathrm{wt}}}(\mathbb{C}_0, N^\vee) \quad k \in \mathbb{Z}_{\geq 0}.$$

Proof. First, we prove the first assertion. The sets of functors

$$\{\mathrm{Ext}^k_{\mathfrak{Cg}\text{-mod}_{\mathrm{wt}}}(-, N^\vee)\}_{k\in\mathbb{Z}_{\geq 0}}, \quad \{\mathrm{Ext}^k_{\mathfrak{Cg}\text{-mod}_{\mathrm{int}}}(-, N^\vee)\}_{k\in\mathbb{Z}_{\geq 0}}$$

are contravariant δ-functors from $\mathfrak{Cg}\text{-mod}_{\mathrm{int}}$ to the category of vector spaces. From Theorem 4.2.12, $\{\mathrm{Ext}^k_{\mathfrak{Cg}\text{-mod}_{\mathrm{int}}}(-, N^\vee)\}_{k\in\mathbb{Z}_{\geq 0}}$ is a universal δ-functor. We prove $\{\mathrm{Ext}^k_{\mathfrak{Cg}\text{-mod}_{\mathrm{wt}}}(-, N^\vee)\}_{k\in\mathbb{Z}_{\geq 0}}$ is also a universal δ-functor. From Theorem 4.2.12, it is sufficient to show $\mathrm{Ext}^l_{\mathfrak{Cg}\text{-mod}_{\mathrm{wt}}}(P(\lambda+n\Lambda_0+m\delta)_{\mathrm{int}}, N^\vee) = \{0\}$ for $\lambda \in \mathring{P}_+$, n, $2m \in \mathbb{Z}$ and $l > 0$. From the BGG-resolution, we have an exact sequence

$$\cdots \to \bigoplus_{w\in\mathring{W}, l(w)=n+1} \mathring{M}(w\circ\lambda+n\Lambda_0+m\delta) \to \bigoplus_{w\in\mathring{W}, l(w)=n} \mathring{M}(w\circ\lambda+n\Lambda_0+m\delta) \to \cdots$$

$$\cdots \to \bigoplus_{w\in\mathring{W}, l(w)=1} \mathring{M}(w\circ\lambda+n\Lambda_0+m\delta) \to \mathring{M}(\lambda+n\Lambda_0+m\delta) \to V(\lambda+n\Lambda_0+m\delta) \to 0,$$

where

$$\mathring{M}(\mu) := U(\mathring{\mathfrak{g}} + \mathfrak{h}) \underset{U(\mathring{\mathfrak{b}}_+ + \mathfrak{h})}{\otimes} \mathbb{C}_\mu.$$

Since $U(\mathfrak{Cg})$ is free over $U(\mathring{\mathfrak{g}}+\mathfrak{h})$, by tensoring $U(\mathfrak{Cg})$, we obtain a projective resolution $P^\bullet \to P(\lambda + n\Lambda_0 + m\delta)_{\mathrm{int}} \to 0$ of $P(\lambda + n\Lambda_0 + m\delta)_{\mathrm{int}}$ in $\mathfrak{Cg}$-mod$_{\mathrm{wt}}$ such that $P^n = \bigoplus_{w\in\mathring{W}, l(w)=n} U(\mathfrak{Cg}) \underset{U(\mathring{\mathfrak{g}}+\mathfrak{h})}{\otimes} \mathring{M}(w\circ\lambda+n\Lambda_0+m\delta)$. For each $l(w) > 0$, since $U(\mathfrak{Cg}) \underset{U(\mathring{\mathfrak{g}}+\mathfrak{h})}{\otimes} \mathring{M}(w\circ\lambda+n\Lambda_0+m\delta)$ does not have a $\mathring{\mathfrak{g}}$-integrable quotient, we have $\mathrm{Hom}_{\mathfrak{Cg}\text{-mod}_{\mathrm{wt}}}(U(\mathfrak{Cg}) \underset{U(\mathring{\mathfrak{g}}+\mathfrak{h})}{\otimes} \mathring{M}(w \circ \lambda + n\Lambda_0 + m\delta), N^\vee) = \{0\}$. This implies $\mathrm{Ext}^l_{\mathfrak{Cg}\text{-mod}_{\mathrm{wt}}}(P(\lambda + n\Lambda_0 + m\delta)_{\mathrm{int}}, N^\vee) = \{0\}$ for $l > 0$. Hence $\{\mathrm{Ext}^k_{\mathfrak{Cg}\text{-mod}_{\mathrm{wt}}}(-, N^\vee)\}_{k \in \mathbb{Z}_{\geq 0}}$ is a universal δ-functor by Theorem 4.2.12. Since $\mathrm{Ext}^0_{\mathfrak{Cg}\text{-mod}_{\mathrm{wt}}}(-, N^\vee) = \mathrm{Ext}^0_{\mathfrak{Cg}\text{-mod}_{\mathrm{int}}}(-, N^\vee)$, the assertion follows.

Next, we prove the second assertion. Two sets of functors

$$\{\mathrm{Ext}^k_{\mathfrak{Cg}\text{-mod}_{\mathrm{wt}}}(U(\mathfrak{Cg}) \underset{U(\mathfrak{b}_-)}{\otimes} \mathbb{C}_0, (-)^\vee)\}_{k \in \mathbb{Z}_{\geq 0}}, \quad \{\mathrm{Ext}^k_{\mathfrak{Cg}\text{-mod}_{\mathrm{wt}}}(\mathbb{C}_0, (-)^\vee)\}_{k \in \mathbb{Z}_{\geq 0}}$$

are contravariant δ-functors from $\mathfrak{Cg}$-mod$_{\mathrm{int}}$ to the category of vector spaces. Since $\mathbb{C}_0$ is an object of $\mathfrak{Cg}$-mod$_{\mathrm{int}}$, we can prove that the latter is a universal δ-functor by the same argument as in the proof of (1). We show that $\mathrm{Ext}^l_{\mathfrak{Cg}\text{-mod}_{\mathrm{wt}}}(U(\mathfrak{Cg}) \underset{U(\mathfrak{b}_-)}{\otimes} \mathbb{C}_0, P(\lambda + n\Lambda_0 + m\delta)^\vee_{\mathrm{int}}) = \{0\}$ for each $l > 0$. For each $w \in \mathring{W}$, by the PBW theorem and the Frobenius reciprocity, we have

$$\begin{aligned}
&\mathrm{Hom}_{\mathfrak{Cg}}(U(\mathfrak{Cg}) \underset{U(\mathfrak{b}_-)}{\otimes} \mathbb{C}_0, (U(\mathfrak{Cg}) \otimes_{U(\mathring{\mathfrak{g}}+\mathfrak{h})} \mathring{M}(w \circ \lambda + n\Lambda_0 + m\delta))^\vee) \\
= \;& \mathrm{Hom}_{\mathfrak{Cg}}(U(\mathfrak{Cg}) \otimes_{U(\mathring{\mathfrak{g}}+\mathfrak{h})} \mathring{M}(w \circ \lambda + n\Lambda_0 + m\delta), (U(\mathfrak{Cg}) \underset{U(\mathfrak{b}_-)}{\otimes} \mathbb{C}_0)^\vee) \\
= \;& \mathrm{Hom}_{\mathring{\mathfrak{b}}_+ + \mathfrak{h}}(\mathbb{C}_{w \circ \lambda + n\Lambda_0 + m\delta}, (U(\mathfrak{Cg}) \underset{U(\mathfrak{b}_-)}{\otimes} \mathbb{C}_0)^\vee) \\
= \;& \mathrm{Hom}_{\mathring{\mathfrak{b}}_+ + \mathfrak{h}}(\mathbb{C}_{w \circ \lambda + n\Lambda_0 + m\delta}, (U(\mathring{\mathfrak{b}}_+ + \mathfrak{h}) \underset{U(\mathfrak{h})}{\otimes} \mathbb{C}_0)^\vee) \\
= \;& \mathrm{Hom}_{\mathring{\mathfrak{b}}_+ + \mathfrak{h}}(U(\mathring{\mathfrak{b}}_+ + \mathfrak{h}) \underset{U(\mathfrak{h})}{\otimes} \mathbb{C}_0, \mathbb{C}_{-w \circ \lambda - n\Lambda_0 - m\delta}) \\
= \;& \mathrm{Hom}_{\mathfrak{h}}(\mathbb{C}_0, \mathbb{C}_{-w \circ \lambda - n\Lambda_0 - m\delta}).
\end{aligned}$$

If $l(w) > 0$, then $\mathrm{Hom}_{\mathring{\mathfrak{b}}_+ + \mathfrak{h}}(\mathbb{C}_{w \circ \lambda + n\Lambda_0 + m\delta}, (U(\mathring{\mathfrak{b}}_+ + \mathfrak{h}) \underset{\mathbb{C}}{\otimes} \mathbb{C}_0)^\vee) = \{0\}$. Using the projective resolution of $P(\lambda + n\Lambda_0 + m\delta)_{\mathrm{int}}$ considered in the proof of (1), this implies $\mathrm{Ext}^l_{\mathfrak{Cg}\text{-mod}_{\mathrm{wt}}}(U(\mathfrak{Cg}) \underset{U(\mathfrak{b}_-)}{\otimes} \mathbb{C}_0, P(\lambda + n\Lambda_0 + m\delta)^\vee_{\mathrm{int}}) = \{0\}$ for each $l > 0$. Hence $\mathrm{Ext}^k_{\mathfrak{Cg}\text{-mod}_{\mathrm{wt}}}(U(\mathfrak{Cg}) \underset{U(\mathfrak{b}_-)}{\otimes} \mathbb{C}_0, (-)^\vee) = \{0\}$ is a universal δ-functor. Since $\mathrm{Ext}^0_{\mathfrak{Cg}\text{-mod}_{\mathrm{wt}}}(U(\mathfrak{Cg}) \underset{U(\mathfrak{b}_-)}{\otimes} \mathbb{C}_0, N^\vee) = \mathrm{Ext}^0_{\mathfrak{Cg}\text{-mod}_{\mathrm{wt}}}(\mathbb{C}_0, N^\vee)$, the assertion follows. $\square$

Lemma 4.2.15. *For each M, $N \in \mathfrak{B}$ such that $M^\vee$, $N^\vee \in \mathfrak{B}$, we have*

$$\mathrm{Ext}^n_{\mathfrak{B}}(M, N^\vee) = \mathrm{Ext}^n_{\mathfrak{B}}(\mathbb{C}_0, M^\vee \otimes_{\mathbb{C}} N^\vee) \ \text{ for } \ n \in \mathbb{Z}_{\geq 0}.$$

Proof. We show that $\{\mathrm{Ext}^n_{\mathfrak{B}}(\mathbb{C}_0, (-)^\vee \otimes_{\mathbb{C}} N^\vee)\}_{n \in \mathbb{Z}_{\geq 0}}$ is a universal δ-functor. For each injective object $I \in \mathfrak{B}$, the object $I \otimes_{\mathbb{C}} N^\vee$ is an injective object in $\mathfrak{B}$. Hence we have $\mathrm{Ext}^k_{\mathfrak{B}}(\mathbb{C}_0, P^\vee \otimes_{\mathbb{C}} N^\vee) = \{0\}$ for each projective object $P \in \mathfrak{B}$ and $k \in \mathbb{N}$. From Theorem 4.2.12, this implies $\{\mathrm{Ext}^n_{\mathfrak{B}}(\mathbb{C}_0, (-)^\vee \otimes_{\mathbb{C}} N^\vee)\}_{n \in \mathbb{Z}_{\geq 0}}$ is a universal δ-functor. For each $R \in \mathfrak{B}$, we have $\mathrm{Hom}_{\mathfrak{B}}(R, N^\vee) = \mathrm{Hom}_{\mathfrak{B}}(\mathbb{C}_0, R^\vee \otimes_{\mathbb{C}} N^\vee)$. Since $\{\mathrm{Ext}^n_{\mathfrak{B}}(-, N^\vee)\}_{n \in \mathbb{Z}_{\geq 0}}$ is a universal δ-functor, this implies

$$\{\mathrm{Ext}^n_{\mathfrak{B}}(-, N^\vee)\}_{n \in \mathbb{Z}_{\geq 0}} \cong \{\mathrm{Ext}^n_{\mathfrak{B}}(\mathbb{C}_0, (-)^\vee \otimes_{\mathbb{C}} N^\vee)\}_{n \in \mathbb{Z}_{\geq 0}}.$$

Hence the assertion follows. □

Remark 4.2.16. *The conclusion of Lemma 4.2.15 remains valid if we replace* $\mathrm{Ext}^n_{\mathfrak{B}}$ *with* $\mathrm{Ext}^n_{\mathfrak{Cg}\text{-mod}_{\mathrm{int}}}$ *by the same argument.*

Theorem 4.2.17. *For M, $N \in \mathfrak{Cg}\text{-mod}_{\mathrm{int}}$ such that $M^\vee$, $N^\vee \in \mathfrak{Cg}\text{-mod}_{\mathrm{int}}$, we have*

$$\mathrm{Ext}^n_{\mathfrak{B}}(M, N^\vee) = \mathrm{Ext}^n_{\mathfrak{Cg}\text{-mod}_{\mathrm{int}}}(M, N^\vee).$$

Proof. We have

$$\begin{aligned}
\mathrm{Ext}^n_{\mathfrak{B}}(M, N^\vee) &= \mathrm{Ext}^n_{\mathfrak{B}}(\mathbb{C}_0, M^\vee \otimes_{\mathbb{C}} N^\vee) \quad \text{from Lemma 4.2.15} \\
&= \mathrm{Ext}^n_{\mathfrak{Cg}\text{-mod}_{\mathrm{wt}}}(U(\mathfrak{Cg}) \underset{U(\mathfrak{b}_-)}{\otimes} \mathbb{C}_0, M^\vee \otimes_{\mathbb{C}} N^\vee) \quad \text{from Lemma 4.2.13} \\
&= \mathrm{Ext}^n_{\mathfrak{Cg}\text{-mod}_{\mathrm{wt}}}(\mathbb{C}_0, M^\vee \otimes_{\mathbb{C}} N^\vee) \quad \text{from Lemma 4.2.14 (2)} \\
&= \mathrm{Ext}^n_{\mathfrak{Cg}\text{-mod}_{\mathrm{int}}}(\mathbb{C}_0, M^\vee \otimes_{\mathbb{C}} N^\vee) \quad \text{from Lemma 4.2.14 (1)} \\
&= \mathrm{Ext}^n_{\mathfrak{Cg}\text{-mod}_{\mathrm{int}}}(M, N^\vee) \quad \text{from Remark 4.2.16.}
\end{aligned}$$

□

Proof of Theorem 4.2.7. If we set $M = W(\lambda) \otimes_{\mathbb{C}} \mathbb{C}_{m\delta}$ and $N = W(\mu)_{loc}$ in Theorem 4.2.17, then we obtain Theorem 4.2.7. $\square$

Corollary 4.2.18. *For each* $f,\, g \in \mathbb{C}(\!(q^{1/2})\!)[\mathring{P}]^{\mathring{W}}$, *we have*

$$\langle f, g \rangle_{int} = \langle f, g \rangle_{\mathrm{Ext}}.$$

Proof. From Theorem 4.2.17, we have

$$\langle \mathrm{gch}\, W(\lambda), \mathrm{gch}\, W(\mu)_{loc} \rangle_{int} = \langle \mathrm{gch}\, W(\lambda), \mathrm{gch}\, W(\mu)_{loc} \rangle_{\mathrm{Ext}}$$

for each $\lambda,\, \mu \in \mathring{P}_{+}$. Since $\{\mathrm{gch}\, W(\lambda)\}_{\lambda \in \mathring{P}_{+}}$ and $\{\mathrm{gch}\, W(\lambda)_{loc}\}_{\lambda \in \mathring{P}_{+}}$ are $\mathbb{C}(\!(q^{1/2})\!)$-basis of $\mathbb{C}(\!(q^{1/2})\!)[\mathring{P}]^{\mathring{W}}$, we obtain the assertion. $\square$

Chapter 5

Extensions between $\mathbb{D}^\lambda$ and D_μ

In this chapter, we calculate the extensions between Demazure slices and thin Demazure modules. As a consequence, we obtain the orthogonality relations of their characters and the character formula of Demazure slices. We continue to work in the previous chapters.

5.1 Demazure-Joseph functor

For each $i = 0, ..., l$, let $\mathfrak{sl}(2, i)$ be a Lie subalgebra of $\mathfrak{g}$ isomorphic to $\mathfrak{sl}_2$ corresponding to α_i and $\mathfrak{p}_i := \mathfrak{b}_- + \mathfrak{sl}(2, i)$. For each $i = 0, ..., l$ and a $\mathfrak{b}_-$-module M with semisimple $\mathfrak{h}$-action, the $\mathfrak{b}_-$-module $\mathcal{D}_i(M)$ is the unique maximal $\mathfrak{sl}(2, i)$-integrable quotient of $U(\mathfrak{p}_i) \underset{U(\mathfrak{b}_-)}{\otimes} M$. Then $\mathcal{D}_i$ defines a functor called Demazure-Joseph functor ([Jos]).

Theorem 5.1.1 ([Jos]). *For each $i = 0, ..., l$ and a $\mathfrak{h}$-semisimple $\mathfrak{b}_-$-module M, the following hold:*

(1) *The functors $\{\mathcal{D}_i\}_{i=0,...,l}$ satisfy braid relations of W;*

(2) *There is a natural transformation* $\mathrm{Id} \to \mathcal{D}_i$*;*

(3) *If M is an $\mathfrak{sl}(2,i)$-integrable $\mathfrak{p}_i$-module, then $\mathcal{D}_i(M) \cong M$;*

(4) *If N is an $\mathfrak{sl}(2,i)$-integrable $\mathfrak{p}_i$-module, then $\mathcal{D}_i(M \otimes N) \cong \mathcal{D}_i(M) \otimes N$;*

(5) *The functor $\mathcal{D}_i$ is right exact.* $\square$

For a reduced expression $w = s_{i_1} s_{i_2} \cdots s_{i_n} \in W$, we define

$$\mathcal{D}_w := \mathcal{D}_{i_1} \circ \mathcal{D}_{i_2} \circ \cdots \circ \mathcal{D}_{i_n}.$$

This is well-defined by Theorem 5.1.1 (1).

Theorem 5.1.2. *For each $\Lambda \in P_+$, $w \in W$ and $i \in \{0, ..., l\}$, we have*

$$\mathcal{D}_i(D_{w\Lambda}) = \begin{cases} D_{w\Lambda} & (w \geq s_i w) \\ D_{s_i w\Lambda} & (w < s_i w). \end{cases}$$

Proof. By Lemma 3.2.8 and the PBW theorem, Demazure module $D_{w\Lambda}$ has an integrable $\mathfrak{sl}_{2(i)}$-action if $w \geq s_i w$. Hence Theorem 5.1.1 (3) implies $\mathcal{D}_i(D_{w\Lambda}) = D_{w\Lambda}$ if $w \geq s_i w$. If $w < s_i w$, then $D_{s_i w\Lambda}$ is a $\mathfrak{p}_i$-module with an integrable $\mathfrak{sl}(2,i)$ action by Lemma 3.2.8 and the PBW theorem, and we have an inclusion $D_{w\Lambda} \to D_{s_i w\Lambda}$. Hence we have a morphism of $\mathfrak{p}_i$-module $U(\mathfrak{p}_{\mathfrak{i}}) \underset{U(\mathfrak{b}_-)}{\otimes} D_{w\Lambda} \to D_{s_i w\Lambda}$. This morphism is surjective since $D_{s_i w\Lambda}$ is generated by a vector with its weight $w\Lambda$ as $\mathfrak{p}_i$-module. Therefore we obtain a surjection $\mathcal{D}_i(D_{w\Lambda}) \to D_{s_i w\Lambda}$ by taking a maximal $\mathfrak{sl}(2,i)$-integrable quotient. By [Kas, Proposition 3.3.4], we have gch $\mathcal{D}_i(D_{w\Lambda}) =$ gch $D_{s_i w\Lambda}$. Hence the above surjection is an isomorphism. $\square$

We set $\mathcal{D}_i^{\#} := \vee \circ \mathcal{D}_i \circ \vee$.

Proposition 5.1.3 ([FKM] Proposition 5.7)**.** *For each $i = 0, 1, ..., l$, $n \in \mathbb{Z}_{\geq 0}$, $M \in \mathfrak{B}'$, $N \in \mathfrak{B}_0$, we have*

$$\mathrm{Ext}^n_{\mathfrak{B}}(\mathcal{D}_i(M), N) \cong \mathrm{Ext}^n_{\mathfrak{B}}(M, \mathcal{D}_i^{\#}(N)).$$

$\square$

5.2 Realization of global Weyl modules

For each $\lambda \in \mathring{P}_+$, we define

$$\mathrm{Gr}^\lambda D := D^\lambda / \sum_{\lambda \succ \mu, \mu \notin \mathring{W}\lambda} D^\mu.$$

From the PBW theorem and Lemma 3.2.8, the modules D^λ and $\sum_{\lambda \succ \mu, \mu \notin \mathring{W}\lambda} D^\mu$ are stable under the action of $\mathfrak{C}\mathfrak{g}$. Hence $\mathrm{Gr}^\lambda D$ admits a $\mathfrak{C}\mathfrak{g}$-module structure.

Proposition 5.2.1. *Let $\lambda \in \mathring{P}_+$. Then $\mathrm{Gr}^\lambda D$ has a filtration of $\mathfrak{b}_-$-submodules*

$$\{0\} = F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_{N-1} \subset F_N = \mathrm{Gr}^\lambda D$$

such that

$$\{F_i/F_{i-1}\}_{i=1,\ldots,N} = \{\mathbb{D}^\mu\}_{\mu \in \mathring{W}\lambda}.$$

Proof. Let $\geq'$ be a total order on W such that if $w \geq v$ then $w \geq' v$. For each $w \geq \pi_\lambda$, define

$$F_w := (\sum_{v \geq' w} D^{w\Lambda_0} + \sum_{\lambda \succ \mu, \mu \notin \mathring{W}\lambda} D^\mu) / \sum_{\lambda \succ \mu, \mu \notin \mathring{W}\lambda} D^\mu.$$

This is a $\mathfrak{b}_-$-submodule of $\mathrm{Gr}^\lambda D$ and

$$F_w \subseteq F_v \quad \text{if} \quad w \geq' v.$$

By Corollary 3.2.12, $\{F_w\}_{w \in W}$ gives the assertion. $\square$

Lemma 5.2.2. *We have the following equality of graded characters.*

$$\mathrm{gch}\ L(\Lambda_0) = \sum_{\lambda \in \mathring{P}_+} q^{(\lambda|\lambda)/2} \mathrm{gch}\ W(\lambda).$$

Proof. Let $\lambda \in \mathring{P}_+$ and $k \in \mathbb{Z}_{\geq 0}$. By Theorem 4.2.3,

$$\mathrm{Ext}^k_{\mathfrak{Cg}\text{-mod}_{\mathrm{int}}}(L(\Lambda_0), (\mathbb{C}_{-(\lambda|\lambda)\delta/2} \otimes_{\mathbb{C}} W(\lambda)_{loc})^\vee) = \mathrm{Ext}^k_{\mathfrak{Cg}\text{-mod}_{\mathrm{int}}}(L(\Lambda_0), D^\vee_\lambda).$$

Applying Theorem 5.1.2 and Proposition 5.1.3 repeatedly, we have

$$\mathrm{Ext}^k_{\mathfrak{Cg}\text{-mod}_{\mathrm{int}}}(L(\Lambda_0), D^\vee_\lambda) = \mathrm{Ext}^k_{\mathfrak{Cg}\text{-mod}_{\mathrm{int}}}(L(\Lambda_0), D^\vee_0) \quad k \in \mathbb{Z}_{\geq 0},$$

where D_0 is isomorphic to the trivial $\mathfrak{Cg}$-module $\mathbb{C}_{\Lambda_0}$ with its weight Λ_0. By [HK, Theorem 3.6], We have a projective resolution of a $\mathfrak{Cg}$-module

$$\cdots \to P^1 \to P^0 \to \mathbb{C}_{\Lambda_0} \to 0,$$

where $P^n = \bigoplus_{w \in W^0,\, l(w)=n} P(w \circ 0 + \Lambda_0)_{\mathrm{int}}$. Since $\dim_{\mathbb{C}} \mathrm{Hom}_{\mathfrak{Cg}}(P^n, \mathbb{C}_{\Lambda_0}) = \delta_{0,n}$, we obtain

$$\dim_{\mathbb{C}}\mathrm{Ext}^k_{\mathfrak{Cg}\text{-mod}_{\mathrm{int}}}(L(\Lambda_0), (\mathbb{C}_{-(\lambda|\lambda)\delta/2} \otimes_{\mathbb{C}} W(\lambda)_{loc})^\vee) = \delta_{0,k} \quad k \in \mathbb{Z}_{\geq 0}.$$

Therefore, we have

$$\langle \mathrm{gch}\ L(\Lambda_0), \mathrm{gch}\ (\mathbb{C}_{-(\lambda|\lambda)\delta/2} \otimes_{\mathbb{C}} W(\lambda)_{loc})\rangle_{int} = 1.$$

By Corollary 4.2.4, the set of graded characters $\{\mathrm{gch}\ W(\lambda)_{loc}\}_{\lambda \in \mathring{P}_+}$ is an orthogonal $\mathbb{C}(\!(q^{1/2})\!)$-basis of $\mathbb{C}(\!(q^{1/2})\!)[\mathring{P}]$. Hence Corollary 4.2.6 implies the assertion. □

If a $\mathfrak{b}_-$-module M admits a finite sequence of $\mathfrak{b}_-$-submodules such that every successive quotient is isomorphic to some $\mathbb{D}^\mu$ ($\mu \in P$), then we say M is filtered by Demazure slices. Let $f,\, g \in \mathbb{C}(\!(q^{1/2})\!)[\mathring{P}]$. Here we make a convention that $f \geq g$ means all the coefficients of $f - g$ belong to $\mathbb{Z}_{\geq 0}$.

Theorem 5.2.3. *For each $\lambda \in \mathring{P}_+$, the global Weyl module $W(\lambda) \otimes_{\mathbb{C}} \mathbb{C}_{\Lambda_0}$ is isomorphic to $\mathrm{Gr}^\lambda D$ as $\mathfrak{Cg}$-module. In particular, $W(\lambda) \otimes_{\mathbb{C}} \mathbb{C}_{\Lambda_0}$ is filtered by Demazure slices and each $\mathbb{D}^\mu$ ($\mu \in \mathring{W}\lambda$) appears exactly once as a successive quotient.*

Proof. First, we show that there exists a surjection $W(\lambda) \otimes_{\mathbb{C}} \mathbb{C}_{\Lambda_0} \to \mathrm{Gr}^{\lambda} D$. Let $v_\lambda \in \mathrm{Gr}^{\lambda} D$ be the nonzero cyclic vector with its weight $\lambda + \Lambda_0 - \frac{(\lambda|\lambda)}{2}\delta$. We check v_λ satisfies Definition 4.2.1 (1), (2), (3). The condition (1) is trivial from the definition of v_λ. Since $L(\Lambda_0)$ is an integrable $\mathfrak{g}$-module, the vector v_λ is an extremal weight vector. This implies the condition (2). We check the condition (3) in the sequel. Since $\langle \lambda, \check{\alpha} \rangle \geq 0$ and v_λ is an extremal weight vector, we have $e_\alpha v_\lambda = 0$ for $\alpha \in \mathring{\Delta}_+$. For each $\mu \in \mathring{P}$, we set $|\mu\rangle := 1 \in R_\mu$. For each $\beta = \alpha + n\delta \in \Delta$ with $n \in -\mathbb{Z}_{\geq 0}/2$ and $\alpha \in \mathring{\Delta}_+ \cup \frac{1}{2}\mathring{\Delta}_{l+}$, we have $v := e_{\alpha+n\delta}|\lambda\rangle \in U(\mathfrak{C}\mathfrak{g}_{im})|\lambda + \alpha\rangle$ by Theorem 3.2.5. Since $U(\mathring{\mathfrak{g}})v$ is finite-dimensional, the $\mathring{\mathfrak{g}}$-module $U(\mathring{\mathfrak{g}})v$ has a highest weight vector whose weight is ν. Then, $v \in U(\mathfrak{C}\mathfrak{g}_{im})U(\mathring{\mathfrak{n}}_-)|\nu\rangle \subset D^\nu$. Hence $\nu - \lambda \in \mathring{Q}'_+$. Since λ and ν is dominant, we have $\lambda \succ \nu$. Therefore D^ν is 0 in $\mathrm{Gr}^\lambda D$ as $|\nu\rangle \in D^\nu$. This implies $v = 0$ and we have the desired surjection. In particular, we have an inequality

$$q^{(\lambda,\lambda)/2} \mathrm{gch}\, W(\lambda) \geq \mathrm{gch}\, \mathrm{Gr}^\lambda D.$$

On the other hand, we have,

$$\mathrm{gch}\, L(\Lambda_0) = \sum_{\lambda \in \mathring{P}_+} \mathrm{gch}\, \mathrm{Gr}^\lambda D$$

and

$$\mathrm{gch}\, L(\Lambda_0) = \sum_{\lambda \in \mathring{P}_+} q^{(\lambda|\lambda)/2} \mathrm{gch}\, W(\lambda)$$

by Lemma 5.2.2. Thus the above inequality is actually an equality and the assertion follows. $\square$

5.3 Calculation of $\mathrm{Ext}^n_{\mathfrak{B}}(\mathbb{D}^\lambda \otimes_{\mathbb{C}} \mathbb{C}_{m\delta+k\Lambda_0}, D_\mu)$

5.3.1 Demazure-Joseph functor and Demazure slices

Theorem 5.3.1. *For each $w \in W$ and $i \in \{0, ..., l\}$, we have the following:*

$$\mathcal{D}_i(D^w) = \begin{cases} D^{s_i w} & if \ s_i w < w \\ D^w & if \ s_i w \geq w. \end{cases}$$

Proof. The proof is the same as proof of Theorem 5.1.2 using the analog of [Kas, Proposition 3.3.4] for thick Demazure modules (cf. [Kas, §4]). □

For each $c \in \mathring{P}$, let $W(c)$ be the image of D^c in $\mathrm{Gr}^{c_+} D$, where c_+ is a unique dominant integrable weight in $\mathring{W}c$. From Theorem 5.2.3, the global Weyl module is isomorphic to the image of D^{c_+} as $\mathfrak{Cg}'$-module. Hence this notation is consistent with the previous notation and we use the same notation.

Proposition 5.3.2 ([CK] Proposition 4.13)**.** *For each $c \in \mathring{P}$ and $i \in \{1, ..., l\}$, we have*

$$\mathcal{D}_i(W(c)) = \begin{cases} W(s_i c) & (s_i c \succeq c) \\ W(c) & (s_i c \not\succeq c) \end{cases}.$$

Proof. We set $M_c := \sum_{c_- \succ a} D^a$. We have a short exact sequence

$$0 \to M_c \to D^c + M_c \to W(c) \to 0.$$

The module M_c is invariant under $\mathcal{D}_i$ by Theorem 5.3.1. Hence we obtain the following exact sequence

$$\mathbb{L}^{-1}\mathcal{D}_i(W(c)) \to M_c \to D^{c'} + M_c \to \mathcal{D}_i(W(c)) \to 0,$$

where

$$c' = \begin{cases} s_i c & (s_i c \succeq c) \\ c & (s_i c \not\succeq c) \end{cases}$$

and $\mathbb{L}^\bullet \mathcal{D}_i$ is the left derived functor of $\mathcal{D}_i$. By Theorem 5.1.1 (2), we have the following commutative diagram

$$\begin{array}{ccc} M_c & \longrightarrow & D^c + M_c \\ \Big\| & & \Big\downarrow \\ M_c & \longrightarrow & D^{c'} + M_c \end{array}.$$

Since $L(\Lambda_0)$ is completely reducible as a $\mathfrak{sl}(2,i)$-module and $D^c + M_c$ is a $\mathfrak{b}_-$-submodule of $L(\Lambda_0)$, the above morphism $D^c + M_c \to D^{c'} + M_c$ is injective by [Jos, Lemma 2.8 (1)]. Hence $M_c \to D^{c'} + M_c$ is injective. Therefore we obtain $\mathcal{D}_i(W(c)) \cong (D^{c'} + M_c)/M_c$ from the above exact sequence. $\Box$

Proposition 5.3.3 ([CK] Corollary 4.15). *Let $i \in \{0, 1, ..., l\}$ and $c \in \mathring{P}$. If $s_i c \succ c$, then we have an exact sequence*

$$0 \to \mathbb{D}^c \to \mathcal{D}_i(\mathbb{D}^c) \to \mathbb{D}^{s_i c} \to 0$$

and $\mathcal{D}_i(\mathbb{D}^{s_i c}) = \{0\}$.

Proof. We set $S := \{w \in W | w \not\leq \pi_c,\ s_i w \not\leq \pi_c\}$ and $M := \sum_{w \in S} D^w$. Then we have $D^c \cap M = \sum_{\pi_c < w} D^w$. Hence we have an exact sequence

$$0 \to M \to D^c + M \to \mathbb{D}^c \to 0.$$

As $s_i(S) \subset S$, we have $\mathcal{D}_i(M) \cong M$. By the same argument as in the proof of Proposition 5.3.2, applying $\mathcal{D}_i$, we obtain

$$0 \to M \to D^{s_i c} + M \to \mathcal{D}_i(\mathbb{D}^c) \to 0.$$

In particular, we have

$$\mathbb{D}^c \cong (D^c + M)/M \ \text{ and } \ \mathcal{D}_i(\mathbb{D}^c) \cong (D^{s_i c} + M)/M.$$

Hence we have

$$0 \to \mathbb{D}^c \to \mathcal{D}_i(\mathbb{D}^c) \to (D^{s_i c} + M)/(D^c + M) \to 0,$$

where $(D^{s_i c} + M)/(D^c + M) \cong D^{s_i c}/(D^{s_i c} \cap (D^c + M))$ is isomorphic to $\mathbb{D}^{s_i c}$ since $D^{s_i c} \cap (D^c + M) = \sum_{w > s_i \pi_c} D^w$. Hence the first assertion follows. Applying $\mathcal{D}_i$ to the last exact sequence, from right exactness of $\mathcal{D}_i$, we have an exact sequence

$$\mathcal{D}_i(\mathbb{D}^c) \to \mathcal{D}_i^2(\mathbb{D}^c) \to \mathcal{D}_i(\mathbb{D}^{s_i c}) \to 0.$$

From Theorem 5.1.1 (3), the above homomorphism $\mathcal{D}_i(\mathbb{D}^c) \to \mathcal{D}_i^2(\mathbb{D}^c)$ is an isomorphism. This implies the second assertion. $\square$

5.3.2 Calculation of $\mathrm{Ext}^n_{\mathfrak{B}}(\mathbb{D}^\lambda \otimes_{\mathbb{C}} \mathbb{C}_{m\delta+k\Lambda_0}, D_\mu^\vee)$

The following theorem is an $A_{2l}^{(2)}$ version of [CK, Theorem 4.18].

Theorem 5.3.4. *For each $\lambda, \mu \in \mathring{P}$, $m \in \frac{1}{2}\mathbb{Z}$ and $k \in \mathbb{Z}$, we have*

$$\dim_{\mathbb{C}} \mathrm{Ext}^n_{\mathfrak{B}}(\mathbb{D}^\lambda \otimes_{\mathbb{C}} \mathbb{C}_{m\delta+k\Lambda_0}, D_\mu^\vee) = \delta_{n,0}\delta_{m,0}\delta_{k,0}\delta_{\lambda,\mu} \quad n \in \mathbb{Z}_{\geq 0}.$$

Proof. By comparing the level, the extension vanishes if $k \neq 0$. We prove the assertion by induction on μ with respect to $\succ$. By Theorem 5.1.1 (3), we have $\mathcal{D}_w(D_0) = D_0$ for all $w \in \mathring{W}$. If λ is not anti-dominant, then there exists $i \in \{1, ..., l\}$ such that $s_i\lambda > \lambda$. Hence

$$\begin{aligned}
\mathrm{Ext}^n_{\mathfrak{B}}(\mathbb{D}^\lambda \otimes_{\mathbb{C}} \mathbb{C}_{m\delta+k\Lambda_0}, D_0^\vee) &= \mathrm{Ext}^n_{\mathfrak{B}}(\mathbb{D}^\lambda \otimes_{\mathbb{C}} \mathbb{C}_{m\delta+k\Lambda_0}, \mathcal{D}_i^\#(D_0^\vee)) \\
&= \mathrm{Ext}^n_{\mathfrak{B}}(\mathcal{D}_i(\mathbb{D}^\lambda \otimes_{\mathbb{C}} \mathbb{C}_{m\delta+k\Lambda_0}), D_0^\vee) \\
&= \{0\}.
\end{aligned}$$

Here we used Proposition 5.1.3 in the second equality and Proposition 5.3.3 in the third equality. If λ is anti-dominant, then we have $\mathcal{D}_{w_0}(\mathbb{D}^\lambda) = W(\lambda_+) \otimes_{\mathbb{C}} \mathbb{C}_{\Lambda_0}$ for the longest element w_0 of $\mathring{W}$ by Proposition 5.3.2. Hence we have $\mathrm{Ext}^n_{\mathfrak{B}}(\mathbb{D}^\lambda \otimes_{\mathbb{C}} \mathbb{C}_{m\delta+k\Lambda_0}, D_0^\vee) = \mathrm{Ext}^n_{\mathfrak{B}}(W(\lambda_+) \otimes_{\mathbb{C}} \mathbb{C}_{m\delta+k\Lambda_0}, W(0)^\vee_{loc})$ by Theorem 4.2.3. From Theorem 4.2.7, the assertion follows in this case.
Let $s_i\mu \succ \mu$. We set $\mathbb{D}'_\lambda := \mathbb{D}_\lambda \otimes_{\mathbb{C}} \mathbb{C}_{m\delta+k\Lambda_0}$ for $\lambda \in P$. By Proposition 5.3.3, we have the following exact sequence

$$\begin{aligned} 0 \to \mathrm{Ext}^0_{\mathfrak{B}}(\mathbb{D}'_{s_i\lambda}, D_\mu^\vee) \to \mathrm{Ext}^0_{\mathfrak{B}}(\mathcal{D}_i(\mathbb{D}'_\lambda), D_\mu^\vee) \to \mathrm{Ext}^0_{\mathfrak{B}}(\mathbb{D}'_\lambda, D_\mu^\vee) \to \\ \cdots \to \mathrm{Ext}^n_{\mathfrak{B}}(\mathbb{D}'_{s_i\lambda}, D_\mu^\vee) \to \mathrm{Ext}^n_{\mathfrak{B}}(\mathcal{D}_i(\mathbb{D}'_\lambda), D_\mu^\vee) \to \mathrm{Ext}^n_{\mathfrak{B}}(\mathbb{D}'_\lambda, D_\mu^\vee) \to \cdots . \end{aligned}$$

From Theorem 5.1.2 and Proposition 5.1.3, we have

$$\begin{aligned} \mathrm{Ext}^n_{\mathfrak{B}}(\mathcal{D}_i(\mathbb{D}_\lambda \otimes_{\mathbb{C}} \mathbb{C}_{m\delta+k\Lambda_0}), D_\mu^\vee) &\cong \mathrm{Ext}^n_{\mathfrak{B}}(\mathbb{D}_\lambda \otimes_{\mathbb{C}} \mathbb{C}_{m\delta+k\Lambda_0}, \mathcal{D}_i^\#(D_\mu^\vee)) \\ &\cong \mathrm{Ext}^n_{\mathfrak{B}}(\mathbb{D}_\lambda \otimes_{\mathbb{C}} \mathbb{C}_{m\delta+k\Lambda_0}, D_{s_i\mu}^\vee). \end{aligned}$$

Therefore, the assertion follows from the induction hypothesis and the long exact sequence.

□

Corollary 5.3.5. *For each λ, $\mu \in \mathring{P}$, we have*

$$\langle \mathrm{gch}\, \mathbb{D}^\lambda, q^{\frac{(b|b)}{2}} \bar{E}_\mu(X^{-1}, q^{-1}) \rangle_{\mathrm{Ext}} = \delta_{\lambda,\mu}.$$

Proof. By Theorem 5.3.4, we have

$$\langle \mathrm{gch}\, \mathbb{D}^\lambda, \mathrm{gch}\, D_\mu \rangle_{\mathrm{Ext}} = \delta_{\lambda,\mu}.$$

Using Theorem 3.2.14, we obtain the assertion. □

Corollary 5.3.6. *For each $\lambda \in \mathring{P}$, we have*

$$\mathrm{gch}\, \mathbb{D}^\lambda = q^{\frac{(b|b)}{2}} E^\dagger_\lambda(X^{-1}, q^{-1}) / \langle \bar{E}_\lambda, E^\dagger_\lambda \rangle_{\mathrm{Ext}}.$$

Proof. Since $\{E^\dagger_\lambda(X^{-1}, q^{-1})/\langle \bar{E}_\lambda, E^\dagger_\lambda\rangle_{\mathrm{Ext}}\}_{\mu\in\mathring{P}}$ is a $\mathbb{C}(\!(q^{1/2})\!)$-basis of $\mathbb{C}(\!(q^{1/2})\!)[\mathring{P}]$, we have

$$\mathrm{gch}\ \mathbb{D}^\lambda = \sum_{\mu\in\mathring{P}} a_\mu E^\dagger_\lambda(X^{-1}, q^{-1})/\langle \bar{E}_\lambda, E^\dagger_\lambda\rangle_{\mathrm{Ext}}$$

for some $a_\mu \in \mathbb{C}(\!(q^{1/2})\!)$. Since $\langle -, -\rangle_{nonsym} = \langle -, -\rangle_{\mathrm{Ext}}$, and $\{E_\lambda(X, q, t)\}_{\lambda\in\mathring{P}}$ are orthogonal with respect to $\langle -, -\rangle'_{nonsym}$ each other, we have

$$\langle \bar{E}_\lambda(X^{-1}, q^{-1}), E^\dagger_\mu(X^{-1}, q^{-1})\rangle_{\mathrm{Ext}}/\langle \bar{E}_\lambda, E^\dagger_\lambda\rangle_{\mathrm{Ext}} = \delta_{\lambda,\mu}.$$

Hence we have $\langle \mathrm{gch}\ \mathbb{D}^\lambda, \mathrm{gch}\ D_\mu\rangle_{\mathrm{Ext}} = a_\mu$ by Theorem 3.2.14. Therefore the assertion follows from Corollary 5.3.5. $\square$

Chapter 6

Weyl modules for special current algebra of $A_{2l}^{(2)}$

In this chapter, we study the global Weyl modules of special current algebra of $A_{2l}^{(2)}$. We prove that the endomorphism rings of the global Weyl modules are isomorphic to the polynomial ring and the global Weyl modules are free over their endomorphism rings. We continue to work in the setting of the previous chapters.

6.1 Special current algebra of $A_{2l}^{(2)}$

In this section, we refer for general terminologies to [FK, Chapter 2], [FM, §2.2] and [Car, Appendix]. We set

$$\mathring{\mathfrak{h}}^{\dagger} := \bigoplus_{i=0}^{l-1} \mathbb{C}\alpha_i, \quad \mathring{\Delta}^{\dagger} := \Delta \cap \mathring{Q}^{\dagger} \ \text{ and } \ \mathring{\mathfrak{g}}^{\dagger} := (\bigoplus_{\alpha \in \mathring{\Delta}^{\dagger}} \mathfrak{g}_{\alpha}) \oplus \mathring{\mathfrak{h}}^{\dagger}.$$

Then $\mathring{\mathfrak{g}}^{\dagger}$ is a finite-dimensional simple Lie algebra of type B_l. The subalgebra $\mathring{\mathfrak{h}}^{\dagger}$ is a Cartan subalgebra of $\mathring{\mathfrak{g}}^{\dagger}$, and $\mathring{\Delta}^{\dagger}$ is the set of roots of $\mathring{\mathfrak{g}}^{\dagger}$ with respect to $\mathring{\mathfrak{h}}^\dagger$. Using the standard basis $\nu_1, ..., \nu_l$ of $\mathbb{R}^l$, we have :

$$\mathring{\Delta}^\dagger = \{\pm(\nu_i \pm \nu_j),\ \pm\nu_i \mid 1 \leq i \neq j \leq l\}.$$

and the set of positive and negative roots are

$$\mathring{\Delta}^\dagger_+ = \{\nu_i \pm \nu_j,\ \nu_i \mid 1 \leq i < j \leq l\}, \quad \mathring{\Delta}^\dagger_- = -\mathring{\Delta}^\dagger_+.$$

We denote the set of short roots of $\mathring{\mathfrak{g}}^\dagger$ by $\mathring{\Delta}^\dagger_s$ and the set of long roots of $\mathring{\mathfrak{g}}^\dagger$ by $\mathring{\Delta}^\dagger_l$. We have

$$\mathring{\Delta}^\dagger_s = \{\pm\nu_i \mid i = 1, ..., l\}$$

and

$$\mathring{\Delta}^\dagger_l = \{\pm(\nu_i \pm \nu_j) \mid 1 \leq i \neq j \leq l\}.$$

We set $\mathring{\Delta}^\dagger_{s\pm} = \mathring{\Delta}^\dagger_\pm \cap \mathring{\Delta}^\dagger_s$ and $\mathring{\Delta}^\dagger_{l\pm} = \mathring{\Delta}^\dagger_\pm \cap \mathring{\Delta}^\dagger_l$. Let $\{\alpha^\dagger_1, ..., \alpha^\dagger_l\}$ be a set of simple roots of $\mathring{\mathfrak{g}}^\dagger$. We have $\alpha^\dagger_i = -\nu_i + \nu_{i+1}$ if $i \neq l$ and $\alpha^\dagger_l = -\nu_l$. In the notation of §1.2, we have $\alpha^\dagger_i = \alpha_{l-i}$. We set $\mathring{Q}^\dagger_+ := \bigoplus_{i=1,...,l} \mathbb{Z}_{\geq 0}\alpha^\dagger_i$. We have

$$\Delta_{re} = (\mathring{\Delta}^\dagger + \mathbb{Z}\delta) \cup (2\mathring{\Delta}^\dagger_s + (2\mathbb{Z}+1)\delta).$$

The special current algebra $\mathfrak{Cg}^\dagger$ is the maximal parabolic subalgebra of $\mathfrak{g}$ that contains $\mathring{\mathfrak{g}}^\dagger$. We have $\mathfrak{Cg}^\dagger = \mathring{\mathfrak{g}}^\dagger + \mathfrak{b}_-$. We set

$$\mathfrak{Cg}^\dagger_{im} := \mathfrak{Cg}_{im}, \quad \mathfrak{Cg}^{\dagger\prime} := [\mathfrak{Cg}^\dagger, \mathfrak{Cg}^\dagger]$$

and

$$\mathfrak{Cn}^\dagger_+ := \bigoplus_{\alpha \in (\mathring{\Delta}^\dagger_+ - \mathbb{Z}_{\geq 0}\delta) \cup (2\mathring{\Delta}^\dagger_{s+} - (2\mathbb{Z}_{\geq 0}+1)\delta)} \mathfrak{g}_\alpha$$

Let $\mathring{P}^\dagger$ be the integral weight lattice of $\mathring{\mathfrak{g}}^\dagger$ and $\mathring{P}^\dagger_+$ be the set of dominant integral weights of $\mathring{\mathfrak{g}}^\dagger$. Let $\varpi^\dagger_i$ $(i = 1, ..., l)$ be the fundamental weights of $\mathring{\mathfrak{g}}^\dagger$.

We identify $\mathring{P}^\dagger$ and $\mathbb{Z}\varpi_1^\dagger \oplus \cdots \oplus \mathbb{Z}\varpi_{l-1}^\dagger \oplus \mathbb{Z}\varpi_l^\dagger$ by $\varpi_i^\dagger = \Lambda_{l-i} - \Lambda_l$ for $i \neq l$ and $\varpi_l^\dagger = \Lambda_0 - \Lambda_l/2$. We put

$$\mathring{P}_+^\dagger = \mathbb{Z}_{\geq 0}\varpi_1^\dagger \oplus \mathbb{Z}_{\geq 0}\varpi_2^\dagger \oplus \cdots \oplus \mathbb{Z}_{\geq 0}\varpi_l^\dagger.$$

Let $\mathring{W}^\dagger$ be the subgroup of W generated by $\{s_\alpha\}_{\alpha \in \mathring{\Delta}^\dagger}$.

6.2 Realization of $\mathfrak{Cg}^\dagger$

We refer to [CIK, §4.6] in this section. Let $X_{i,j}$ be a $(2l+1) \times (2l+1)$ matrix unit whose ij-entry is one. We set $H_i = X_{i,i} - X_{i+1,i+1}$ $(i = 1, ..., 2l)$. The Lie algebra $\mathfrak{sl}_{2l+1}$ is spaned by $X_{i,j}$ $(i \neq j)$ and H_i $(i = 1, ..., 2l)$. The assignment

$$X_{i,i+1} \to X_{2l+1-i,2l+2-i},\ X_{i+1,i} \to X_{2l+2-i,2l+1-i}$$

extends on $\mathfrak{sl}_{2l+1}$ as a Lie algebra automorphism. We write this automorphism by σ. Let $L(\mathfrak{sl}_{2l+1}) = \mathfrak{sl}_{2l+1} \otimes_{\mathbb{C}} \mathbb{C}[t^{\pm 1}]$ be the loop algebra corresponding to $\mathfrak{sl}_{2l+1}$ and extend σ on $L(\mathfrak{sl}_{2l+1})$ by $\sigma(X \otimes f(t)) = \sigma(X) \otimes f(-t)$. We denote the fixed point of σ in $\mathfrak{sl}_{2l+1} \otimes_{\mathbb{C}} \mathbb{C}[t]$ by $(\mathfrak{sl}_{2l+1} \otimes_{\mathbb{C}} \mathbb{C}[t])^\sigma$.

Proposition 6.2.1 (see [Kac] Theorem 8.3)**.** *The Lie algebra* $(\mathfrak{sl}_{2l+1} \otimes_{\mathbb{C}} \mathbb{C}[t])^\sigma$ *is isomorphic to* $\mathfrak{Cg}^{\dagger\prime}$.

6.3 Weyl modules for $\mathfrak{Cg}^\dagger$

Definition 6.3.1. *For each $\lambda \in \mathring{P}_+^\dagger$, the global Weyl module is a cyclic $\mathfrak{Cg}^\dagger$-module $W(\lambda)^\dagger$ generated by v_λ that satisfies the following relations:*

(1) *$hv_\lambda = \lambda(h)v_\lambda$ for each $h \in \mathfrak{h}$;*

(2) $e_{-\alpha}^{\langle\lambda,\check{\alpha}\rangle+1}v_\lambda = 0$ *for each* $\alpha \in \mathring{\Delta}_+^\dagger$*;*

(3) $\mathfrak{C}\mathfrak{n}_+^\dagger v_\lambda = 0$.

Definition 6.3.2. *For each* $\lambda \in \mathring{P}$*, the local Weyl module is a cyclic* $\mathfrak{C}\mathfrak{g}^\dagger$*-module* $W(\lambda)_{loc}^\dagger$ *generated by* v_λ *satisfies relations* (1), (2), (3) *of Definition 6.3.1 and*

(4) $Xv_\lambda = 0$ *for each* $X \in \mathfrak{C}\mathfrak{g}_{im}^\dagger$.

Theorem 6.3.3 ([FK] Corollary 6.0.1 and [FM] Corollary 2.19)**.** *For each* $\lambda \in \mathring{P}_+^\dagger$*, we have*

(1) *If* $\lambda = \sum_{i=1}^{l-1} m_i\varpi_i^\dagger + (2k-1)\varpi_l^\dagger$*, then*

$$\dim_{\mathbb{C}} W(\lambda)_{loc}^\dagger = \left(\prod_{i=1}^{l-1}\binom{2l+1}{i}^{m_i}\right)\binom{2l+1}{l}^{k-1}2^l;$$

(2) *If* $\lambda = \sum_{i=1}^{l-1} m_i\varpi_i^\dagger + 2m_l\varpi_l^\dagger$*, then*

$$\dim_{\mathbb{C}} W(\lambda)_{loc}^\dagger = \prod_{i=1}^{l}\binom{2l+1}{i}^{m_i}.$$

□

6.4 The algebra $\mathbf{A}_\lambda$

Let $\lambda \in \mathring{P}_+^\dagger$. We set

$$\mathrm{Ann}(v_\lambda) := \{X \in U(\mathfrak{C}\mathfrak{g}_{im}^\dagger) \mid Xv_\lambda = 0\} \text{ and } \mathbf{A}_\lambda := U(\mathfrak{C}\mathfrak{g}_{im}^\dagger)/\mathrm{Ann}(v_\lambda),$$

where v_λ is the cyclic vector of $W(\lambda)_{loc}^\dagger$ in Definition 6.3.1.

Proposition 6.4.1 ([CIK] §7.2)**.** *For each* $\lambda \in \mathring{P}_+^\dagger$*, the algebra* $\mathbf{A}_\lambda$ *acts on* $W(\lambda)^\dagger$ *by*

$$X.Yv_\lambda := YXv_\lambda \ \ for \ \ X \in \mathbf{A}_\lambda \ and \ Y \in U(\mathfrak{C}\mathfrak{g}^{\dagger\prime}).$$

□

6.4.1 Generators of $\mathbf{A}_\lambda$

For $i = 1, ..., l-1$, we set

$$h_{i,0} := H_i + H_{2l+1-i}, \ \ h_{i,1} := H_i - H_{2l+1-i},$$

$$x_{i,0} := X_{i,i+1} + X_{2l+1-i,2l+2-i}, \ \ x_{i,1} := X_{i,i+1} - X_{2l+1-i,2l+2-i},$$

$$y_{i,0} := X_{i+1,i} + X_{2l+2-i,2l+1-i}, \ \ y_{i,1} := X_{i+1,i} - X_{2l+2-i,2l+1-i}$$

and

$$h_{l,0} = 2(H_l + H_{l+1}), \ \ h_{l,1} = H_l - H_{l+1},$$

$$x_{l,0} := \sqrt{2}(X_{l,l+1} + X_{l+1,l+2}), \ \ x_{l,1} := -\sqrt{2}(X_{l,l+1} - X_{l+1,l+2}),$$

$$y_{l,0} := \sqrt{2}(X_{l+1,l} + X_{l+2,l+1}), \ \ y_{l,1} := -\sqrt{2}(X_{l+1,l} - X_{l+2,l+1}).$$

The Lie algebra generated by $\{x_{i,0}, \ y_{i,0}, \ h_{i,0}\}_{i=1,...,l}$ is isomorphic to the simple Lie algebra of type B_l, and $\{h_{i,0}\}_{i=1,...,l}$ is the set of its simple coroots ([Car, Theorem 9.19]). We set $z_{l,1} := \frac{1}{4}[y_{l,0}, y_{l,1}]$. As in [CFS, §3.3], we define $p_{i,r} \in U(\mathfrak{Cg}^\dagger_{im})$ ($i = 1, .., l$ *and* $r \in \mathbb{Z}_{\geq 0}$) by

$$\sum_{r \in \mathbb{Z}_{\geq 0}} p_{i,r} z^r := \exp\left(-\sum_{k=1}^{\infty} \sum_{\varepsilon=0}^{1} \frac{h_{i,\varepsilon} \otimes t^{-2k+\varepsilon}}{2k-\varepsilon} z^{2k-\varepsilon} \right)$$

for $i \neq l$ and

$$\sum_{r \in \mathbb{Z}_{\geq 0}} p_{l,r} z^r := \exp\left(-\sum_{k=1}^{\infty} \frac{h_{l,0}/2 \otimes t^{-2k}}{2k} z^{2k} + \sum_{k=1}^{\infty} \frac{h_{l,1} \otimes t^{-2k+1}}{2k-1} z^{2k-1} \right).$$

Proposition 6.4.2. *The algebra* $U(\mathfrak{Cg}^\dagger_{im})$ *is isomorphic to the polynomial ring* $\mathbb{C}[p_{i,r} | i = 1, ..., l, \ r \in \mathbb{Z}_{\geq 0}]$.

Proof. We have $\mathbb{C}[p_{i,r} | i = 1, ..., l, \ r \in \mathbb{Z}_{\geq 0}] \subset U(\mathfrak{Cg}^\dagger_{im})$. The set of generators of $U(\mathfrak{Cg}^\dagger_{im})$ is $\{h_{i,\varepsilon} \otimes t^{-2k+\varepsilon} | \ n \in \{1, .., l\}, \ k \in \mathbb{N}$ *and* $\varepsilon \in \{0, 1\}\}$. It suffices to see that $h_{n,\varepsilon} \otimes t^{-2k+\varepsilon} \in \mathbb{C}[p_{i,r} | i = 1, ..., l,\ r \in \mathbb{Z}_{\geq 0}]$ for each $i \in \{1, .., l\}$, $k \in \mathbb{N}$ and $\epsilon \in \{0, 1\}$. We have $h_{i,1} \otimes t^{-1} = p_{i,1}$ up to a constant multiple. By definition, $p_{i,2k-\varepsilon} + (h_{i,\varepsilon} \otimes t^{-2k+\varepsilon})/(2k-\varepsilon)$ is an element of $\mathbb{Q}[h_{i,s} | s < 2k-\varepsilon]$ if $i \neq l$, and $p_{l,2k-\varepsilon} - (-1)^{\varepsilon+1}(h_{l,\varepsilon}/2^{1-\varepsilon} \otimes t^{-2k+\varepsilon})/(2k-\varepsilon)$ is an element of $\mathbb{Q}[h_{l,s} | s < 2k-\varepsilon]$. The assertion follows by induction on $2k - \varepsilon$. □

Lemma 6.4.3 ([CFS] Lemma 3.2, Lemma 3.3 (iii) (b) and [CP] Lemma 1.3 (ii))**.** *Let V be a $\mathfrak{C}\mathfrak{g}^\dagger$-module and $v \in V$ be a nonzero vector such that $\mathfrak{C}\mathfrak{n}_+ v = 0$. We have the following:*

(1) *For $i \neq l$, we have $(x_{i,1} \otimes t^{-1})^{(r)}(y_{i,0})^{(r)} v = (-1)^r p_{i,r} v$ for $r \in \mathbb{N}$;*

(2) *We have $(x_{l,0})^{(2r)}(z_{l,1} \otimes t^{-1})^{(r)} v = (-1)^r p_{l,r} v$ for $r \in \mathbb{N}$.*

Proposition 6.4.4. *Let $\lambda \in \mathring{P}^\dagger_+$, $i \in \{1, ..., l-1\}$ and v_λ be cyclic vector of $W(\lambda)^\dagger$ with its weight λ. We have $p_{i,r} v_\lambda = 0$ for $r > \langle \lambda, \check{\alpha}^\dagger_i \rangle$, and $p_{l,r} v_\lambda = 0$ for $r > \lfloor \frac{\langle \lambda, \check{\alpha}^\dagger_l \rangle}{2} \rfloor$.*

Proof. Definition 6.3.1 (3) implies the set of $\mathring{\mathfrak{h}}^\dagger$-weights of $W(\lambda)^\dagger$ is the subset of $\lambda - \mathring{Q}^\dagger_+$. From Definition 6.3.1 (2) and Lemma 6.4.3 (1), we get $p_{i,r} v_\lambda = 0$ for $r > \langle \lambda, \check{\alpha}^\dagger_i \rangle$. By Definition 6.3.1 (2), $W(\lambda)^\dagger$ is an $\mathring{\mathfrak{g}}^\dagger$-integrable module. Since the set of $\mathring{\mathfrak{h}}^\dagger$-weights of $W(\lambda)^\dagger$ is contained in $\lambda - \mathring{Q}^\dagger_+$, this implies $\lambda - k\alpha^\dagger_l$ for $k > \langle \lambda, \check{\alpha}^\dagger_l \rangle$ is not a weight of a vector of $W(\lambda)^\dagger$. Since $(z_{l,1} \otimes t)$ is a root vector corresponding to $2\alpha^\dagger_l - \delta$, we obtain $p_{l,r} v_\lambda = 0$ for $r > \lfloor \frac{\langle \lambda, \check{\alpha}^\dagger_l \rangle}{2} \rfloor$. □

We set

$$\mathbf{A}'_\lambda := \mathbb{C}[p_{i,r} | 1 \leq r \leq \langle \lambda, \check{\alpha}^\dagger_i \rangle \text{ for } i \neq l,\ 1 \leq r \leq \lfloor \frac{\langle \lambda, \check{\alpha}^\dagger_l \rangle}{2} \rfloor \text{ for } i = l].$$

Corollary 6.4.5. *For each $\lambda \in \mathring{P}^\dagger_+$, there exists a $\mathbb{C}$-algebra surjection $\mathbf{A}'_\lambda \to \mathbf{A}_\lambda$.*

Proof. By Proposition 6.4.4, we have $p_{i,r}$, $p_{l,k} \in \mathrm{Ann}(v_\lambda)$ for each $r > \langle \lambda, \check{\alpha}_i^\dagger \rangle$ $(i \neq l)$ and each $k > \lfloor \frac{\langle \lambda, \check{\alpha}_l^\dagger \rangle}{2} \rfloor$. Hence we have a surjection $\mathbf{A}'_\lambda \to \mathbf{A}_\lambda$ by Proposition 6.4.2. □

We set $\mathring{P}_+^{\dagger\prime} := \{\lambda \in \mathring{P}_+^\dagger \mid \langle \lambda, \check{\alpha}_l^\dagger \rangle \in 2\mathbb{Z}_{\geq 0}\}$.

Theorem 6.4.6 ([CIK] §5.6 and Theorem 1)**.** *For each $\lambda \in \mathring{P}_+^{\dagger\prime}$ and nonzero element $f \in \mathbf{A}'_\lambda$, there exists a quotient of $W(\lambda)^\dagger$ such that f acts nontirivially on the image of the cyclic vector v_λ of $W(\lambda)^\dagger$. In particular $\mathbf{A}_\lambda \cong \mathbf{A}'_\lambda$.* □

Lemma 6.4.7 ([CIK] Lemma 5.4)**.** *For each $1 \leq s \leq k$, let V_s be representations of $\mathfrak{Cg}^\dagger$ and let v_s be vectors of V_s such that $\mathfrak{Cn}_+^\dagger v_s = 0$. We have*

$$p_{i,r}(v_1 \otimes \cdots \otimes v_k) = \sum_{r = j_1 + \cdots + j_k,\ j_i \geq 0} p_{i,j_1} v_1 \otimes \cdots \otimes p_{i,j_k} v_k$$

for all $1 \leq i \leq l$ and $r \in \mathbb{Z}_{\geq 0}$. □

6.4.2 Dimension inequalities

For each maximal ideal $\mathbf{I}$ of $\mathbf{A}_\lambda$, we define

$$W(\lambda, \mathbf{I})^\dagger := (\mathbf{A}_\lambda / \mathbf{I}) \underset{\mathbf{A}_\lambda}{\otimes} W(\lambda)^\dagger.$$

Let $U(\mathfrak{Cg}_{im})_+$ be the augmentation ideal of $U(\mathfrak{Cg}_{im})$ and $\mathbf{I}_{\lambda,0}$ be a maximal ideal of $\mathbf{A}_\lambda$ defined by $(U(\mathfrak{Cg}_{im})_+ + \mathrm{Ann}(v_\lambda))/\mathrm{Ann}(v_\lambda)$.

Proposition 6.4.8. *For each $\lambda \in \mathring{P}_+^\dagger$, we have $W(\lambda)_{loc}^\dagger \cong W(\lambda, \mathbf{I}_{\lambda,0})^\dagger$.*

Proof. The assertion follows from Definition 6.3.2 (4). □

Proposition 6.4.9 ([CIK] Proposition 6.4 and 6.5)**.** *Let $\lambda \in \mathring{P}_+^\dagger$ and let $\mathbf{I}$ be a maximal ideal of $\mathbf{A}_\lambda$.*

(1) *If* $\mu \in \mathring{P}_+^\dagger$ *satisfies* $\lambda - \mu \in \mathring{P}_+^{\dagger\prime}$, *then we have*

$$\dim_{\mathbb{C}} W(\lambda, \mathbf{I})^\dagger \geq \dim_{\mathbb{C}} W(\mu)^\dagger_{loc} \left(\prod_{i=1}^{l-1} \binom{2l+1}{i}^{(\lambda-\mu)(\check{\alpha}_i^\dagger)} \right) \binom{2l+1}{l}^{(\lambda-\mu)(\check{\alpha}_l^\dagger/2)}.$$

(2) *We have*

$$\dim_{\mathbb{C}} W(\lambda)^\dagger_{loc} \geq \dim_{\mathbb{C}} W(\lambda, \mathbf{I})^\dagger.$$

□

Corollary 6.4.10 ([CIK] Theorem 10 when $\lambda \in \mathring{P}_+^{\dagger\prime}$)**.** *For each* $\lambda \in \mathring{P}_+^\dagger$ *and each maximal ideal* $\mathbf{I}$ *of* $\mathbf{A}_\lambda$*, the dimension* $\dim_{\mathbb{C}} W(\lambda, \mathbf{I})^\dagger$ *does not depend on* $\mathbf{I}$ *and is given by Theorem 6.3.3.*

Proof. If $\lambda = \sum_{i=1}^{l-1} m_i \varpi_i^\dagger + 2m_l \varpi_l^\dagger$, then we have

$$\dim_{\mathbb{C}} W(\lambda)^\dagger_{loc} \geq \dim_{\mathbb{C}} W(\lambda, \mathbf{I})^\dagger \geq \prod_{i=1}^{l} \binom{2l+1}{i}^{m_i}$$

by Proposition 6.4.9. From Theorem 6.3.3 (2), this inequality is actually equality. If $\lambda = \sum_{i=1}^{l-1} m_i \varpi_i^\dagger + (2k-1)\varpi_l^\dagger$, then we have

$$\dim_{\mathbb{C}} W(\lambda)^\dagger_{loc} \geq \dim_{\mathbb{C}} W(\lambda, \mathbf{I})^\dagger \geq \dim_{\mathbb{C}} W(\varpi_l)^\dagger_{loc} \left(\prod_{i=1}^{l-1} \binom{2l+1}{i}^{m_i} \right) \binom{2l+1}{l}^{k-1}$$

by Proposition 6.4.9. From Theorem 6.3.3 (1), this inequality is actually equality. Hence the assertion follows. □

6.5 Freeness of $W(\lambda)^\dagger$ over $\mathbf{A}_\lambda$

In this section, we prove the following theorem

Theorem 6.5.1. *For each* $\lambda \in \mathring{P}_+^\dagger$*, the global Weyl module* $W(\lambda)^\dagger$ *is free over* $\mathbf{A}_\lambda$*.* □

To prove this theorem, we need the following preparatory result:

Theorem 6.5.2. *For each $\lambda \in \mathring{P}^{\dagger}_{+}$, the algebra $\mathbf{A}_{\lambda}$ is isomorphic to $\mathbf{A}'_{\lambda}$.* $\square$

Theorem 6.5.2 and Corollary 6.4.10 imply Theorem 6.5.1 by [Sus, Qui]. We prove Theorem 6.5.1 after proving Theorem 6.5.2.

Proof of Theorem 6.5.2. We show that the surjection $\mathbf{A}'_{\lambda} \to \mathbf{A}_{\lambda}$ is the isomorphism. We have $\dim_{\mathbb{C}} \mathbf{A}'_{\varpi_l^{\dagger}} = 1$. Since $\dim_{\mathbb{C}} \mathbf{A}_{\varpi_l^{\dagger}} \geq 1$. Hence $\mathbf{A}'_{\varpi_l^{\dagger}} \to \mathbf{A}_{\varpi_l^{\dagger}}$ is the isomorphism. If $\lambda \in \mathring{P}^{\dagger\prime}_{+}$, then the assertion is Theorem 6.4.6. We prove the assertion for $\lambda = \sum_{i=1}^{l-1} m_i \varpi_i^{\dagger} + (2m+1)\varpi_l^{\dagger}$. Let $f \in \mathbf{A}'_{\lambda}$ be a nonzero element. It is suffice to show that there exists a quotient of $W(\lambda)^{\dagger}$ such that f acts nontrivially on the image of the cyclic vector v_{λ} of $W(\lambda)^{\dagger}$. Let $\mu := \lambda - \varpi_l^{\dagger}$. We have $\mathbf{A}'_{\lambda} \cong \mathbf{A}'_{\mu}$. By checking the defining relations, we have a homomorphism of $\mathfrak{C}\mathfrak{g}^{\dagger}$-module

$$W(\lambda)^{\dagger} \to W(\varpi_l^{\dagger})^{\dagger} \otimes_{\mathbb{C}} W(\mu)^{\dagger}$$

which maps v_{λ} to $v_{\varpi_l^{\dagger}} \otimes v_{\mu}$. By Theorem 6.4.6, we have a quotient module V of $W(\mu)^{\dagger}$ such that f acts nontrivially on the image of $v_{\mu} \in W(\mu)^{\dagger}$. We have a homomorphism

$$W(\lambda)^{\dagger} \to W(\varpi_l^{\dagger})^{\dagger} \otimes_{\mathbb{C}} V \to W(\varpi_l^{\dagger})^{\dagger}_{loc} \otimes_{\mathbb{C}} V.$$

Let $v \in V$ and $w_{\varpi_l^{\dagger}} \in W(\varpi_l^{\dagger})^{\dagger}_{loc}$ be the image of v_{μ} in V and the image of $v_{\varpi_l^{\dagger}}$ in $W(\varpi_l^{\dagger})^{\dagger}_{loc}$, respectively. By Lemma 6.4.7, we have $p_{i,r}(w_{\varpi_l^{\dagger}} \otimes v) = w_{\varpi_l^{\dagger}} \otimes p_{i,r}(v)$ for each $i \in \{1, ..., l\}$ and $r \in \mathbb{Z}_{\geq 0}$. Therefore, f acts nontrivially on the highest weight vector $w_{\varpi_l^{\dagger}} \otimes v$ of $W(\varpi_l^{\dagger})^{\dagger}_{loc} \otimes V$. Hence $f v_{\lambda} \neq 0$. Hence the assertion follows. $\square$

Proof of Theorem 6.5.1. We set $N := \dim W(\lambda)^{\dagger}_{loc}$. Let $\mathfrak{m}$ be a maximal ideal of $\mathbf{A}_{\lambda}$. By Nakayama's lemma [Mat, Lemma 1.M], there exists $f \notin \mathfrak{m}$

such that $(W(\lambda)^\dagger)_f$ is generated by N elements as $(\mathbf{A}_\lambda)_f$-module, where $(W(\lambda)^\dagger)_f$ and $(\mathbf{A}_\lambda)_f$ are the localization of $W(\lambda)^\dagger$ and $\mathbf{A}_\lambda$ by f, respectively. Since $(\mathbf{A}_\lambda)_f$ is Noetherian, we have an exact sequence $(\mathbf{A}_\lambda)_f^{\oplus M} \xrightarrow{\phi} (\mathbf{A}_\lambda)_f^{\oplus N} \xrightarrow{\psi} (W(\lambda)^\dagger)_f \to 0$. For any maximal ideal $\mathfrak{n}$ such that $f \notin \mathfrak{n}$, the induced morphism $\overline{\psi} : (\mathbf{A}_\lambda)_f^{\oplus N}/\mathfrak{n}(\mathbf{A}_\lambda)_f^{\oplus N} \to (W(\lambda)^\dagger)_f/\mathfrak{n}(W(\lambda)^\dagger)_f$ is an isomorphism by Corollary 6.4.10. This implies the matrix coefficients of ϕ are contained in the Jacobson radical of $(\mathbf{A}_\lambda)_f$. Since $(\mathbf{A}_\lambda)_f$ is an integral domain and finitely generated over $\mathbb{C}$, we deduce $\phi = 0$. It follows that $(W(\lambda))^\dagger)$ is flat over $\mathbf{A}_\lambda$ by [Jot]. Since $\mathbf{A}_\lambda$ is a polynomial ring, $(W(\lambda))^\dagger)$ is a projective $\mathbf{A}_\lambda$-module. From [Qui, Sus], a finitely generated projective module over a polynomial ring is free. Hence the assertion follows. $\square$

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