ON MULTIPLIER SYSTEMS AND THETA FUNCTIONS OF HALF-INTEGRAL WEIGHT FOR THE HILBERT MODULAR GROUP $\mathrm{SL}_2(\mathfrak{o})$

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Abstract. Let F be a totally real number field and $\mathfrak{o}$ the ring of integers of F. We study theta functions which are Hilbert modular forms of half-integral weight for the Hilbert modular group $\mathrm{SL}_2(\mathfrak{o})$. We obtain an equivalent condition that there exists a multiplier system of half-integral weight for $\mathrm{SL}_2(\mathfrak{o})$. We determine the condition of F that there exists a theta function which is a Hilbert modular form of half-integral weight for $\mathrm{SL}_2(\mathfrak{o})$. The theta function is defined by a sum on a fractional ideal of F.
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1. Introduction

Put $e(z) = e^{2\pi i z}$ for $z \in \mathbb{C}$. It is known that the modular forms of $\mathrm{SL}_2(\mathbb{Z})$ of weight $1/2$ and $3/2$ are the Dedekind eta function $\eta(z)$ and its cubic power $\eta^3(z)$ up to constant, respectively. Here, $\eta(z)$ is given by

$$\eta(z) = e(z/24)\prod_{m\geq 1}(1-e(mz)) \qquad (z\in\mathfrak{h}),$$

where $\mathfrak{h}$ is the upper half plane. It is known that

$$\eta(z) = \frac{1}{2}\sum_{m\in\mathbb{Z}}\chi_{12}(m)e(m^2z/24), \qquad \eta^3(z) = \frac{1}{2}\sum_{m\in\mathbb{Z}} m\,\chi_4(m)e(m^2z/8)$$

(see [18, Corollary 1.3 and Corollary 1.4]). Here, χ_{12} and χ_4 are the primitive Dirichlet character mod 12 and mod 4, respectively. Note that $\eta(z)$ and $\eta^3(z)$ are theta functions defined by a sum on $\mathbb{Z}$.

The function $\eta(z)$ has the transformation formula with respect to modular transformations (see [1, 27, 28, 34]). Let $\left(\frac{\cdot}{\cdot}\right)$ be the Jacobi symbol. We define $\left(\frac{\cdot}{\cdot}\right)^*$ and $\left(\frac{\cdot}{\cdot}\right)_*$ by

$$\left(\frac{c}{d}\right)^* = \left(\frac{c}{|d|}\right), \qquad \left(\frac{c}{d}\right)_* = t(c,d)\left(\frac{c}{d}\right)^*, \qquad t(c,d) = \begin{cases} -1 & c,d<0 \\ 1 & \text{otherwise,}\end{cases}$$

for $c\in\mathbb{Z}\backslash\{0\}$ and $d\in 2\mathbb{Z}+1$ such that $(c,d)=1$. We understand

$$\left(\frac{0}{\pm 1}\right)^* = \left(\frac{0}{1}\right)_* = 1, \qquad \left(\frac{0}{-1}\right)_* = -1$$

(see [17, Chapter 4 §1]).

For $z\in\mathfrak{h}$, we choose $\arg z$ such that $-\pi < \arg z \leq \pi$. For $g\in\mathrm{SL}_2(\mathbb{R})$ and $z\in\mathfrak{h}$, put

$$J(g,z) = \begin{cases} \sqrt{d} & \text{if } c=0, d>0 \\ -\sqrt{d} & \text{if } c=0, d<0 \\ (cz+d)^{1/2} & \text{if } c\neq 0,\end{cases} \qquad g = \begin{pmatrix} a & b \\ c & d\end{pmatrix}. \tag{1}$$

Then we have

$$\eta(\gamma(z)) = \mathbf{v}_\eta(\gamma) J(\gamma, z)\eta(z), \qquad \gamma(z) = \frac{az+b}{cz+d} \in \mathfrak{h} \tag{2}$$

for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, where the multiplier system $\mathbf{v}_\eta(\gamma)$ is given by

$$\mathbf{v}_\eta(\gamma) = \begin{cases} \left(\dfrac{d}{c}\right)^* e\left(\dfrac{(a+d)c - bd(c^2-1) - 3c}{24}\right) & c : \text{odd} \\ \left(\dfrac{c}{d}\right)_* e\left(\dfrac{(a+d)c - bd(c^2-1) + 3d - 3 - 3cd}{24}\right) & c : \text{even.} \end{cases} \tag{3}$$

It is natural to ask the following problem. When does a Hilbert modular theta series of weight $1/2$ with respect to $\mathrm{SL}_2(\mathfrak{o})$ exist? Here, $\mathfrak{o}$ is the ring of integers of a totally real number field F. In 1983, Feng [6] studied this problem. She gave a sufficient condition for the existence of a Hilbert modular theta series of weight $1/2$ with respect to $\mathrm{SL}_2(\mathfrak{o})$ and constructed certain Hilbert modular theta series. These series are defined by a sum on $\mathfrak{o}$.

Let K be a real quadratic field and d_K the discriminant of K. Gundlach [10, p.30], [11, Remark 4.1.] showed that if $d_K \equiv 1 \bmod 8$, then there exist multiplier systems of weight $1/2$ for a Hilbert modular group belonging to a certain theta series. Naganuma [25] obtained a Hilbert modular form of level 1 for a real quadratic case with $d_K \equiv 1 \bmod 8$ and class number one, using modular imbeddings, from the theta constant with the characteristic $(1/2, 1/2, 1/2, 1/2)$ of degree 2.

In this paper, we solve the problem above completely. We consider theta functions defined by a sum on a fractional ideal of F. Let v be a place of F and F_v the completion of F at v. When v is a finite place, we write $v < \infty$. When v is an infinite place, we have $F_v \simeq \mathbb{R}$ and write $v \mid \infty$. Let $\mathbb{A}$ be the adele ring of F.

Let $n = [F : \mathbb{Q}]$ and $\iota_v : F \to F_v$ be the embedding for any v. The entry-wise embeddings of $\mathrm{SL}_2(F)$ into $\mathrm{SL}_2(F_v)$ are also denoted by ι_v. The metaplectic group of $\mathrm{SL}_2(F_v)$ is denoted by $\widetilde{\mathrm{SL}_2(F_v)}$, which is a nontrivial double covering group of $\mathrm{SL}_2(F_v)$. Set-theoretically, it is $\{[g, \tau] \mid g \in \mathrm{SL}_2(F_v), \tau \in \{\pm 1\}\}$. Its multiplication law is given by $[g, \tau][h, \sigma] = [gh, \tau\sigma c(g, h)]$ for $[g, \tau], [h, \sigma] \in \widetilde{\mathrm{SL}_2(F_v)}$, where $c(g, h)$ is the Kubota 2-cocycle on $\mathrm{SL}_2(F_v)$. Put $[g] = [g, 1]$. Note that in SL_2 case $c(\cdot, \cdot)$ is equal to the cocycle constructed by Ranga Rao (see [29]).

Let $\{\infty_1, \cdots, \infty_n\}$ be the set of infinite places of F. Put $\iota_i = \iota_{\infty_i}$ for $1 \le i \le n$. We embed $\mathrm{SL}_2(F)$ into $\mathrm{SL}_2(\mathbb{R})^n$ by $r \mapsto (\iota_1(r), \cdots, \iota_n(r))$. We denote the embedding of $\mathrm{SL}_2(F)$ into $\mathrm{SL}_2(\mathbb{A})$ by ι. Let $\mathbb{A}_f$ be the finite part of $\mathbb{A}$ and $\iota_f : \mathrm{SL}_2(F) \to \mathrm{SL}_2(\mathbb{A}_f)$ the projection of the finite part. The embedding of F into $\mathbb{A}_f$ is also denoted by ι_f.

Let $\widetilde{\mathrm{SL}_2(\mathbb{A})}$ be the adelic metaplectic group, which is a double covering of $\mathrm{SL}_2(\mathbb{A})$. Let $\tilde{H}$ be the inverse image of a subgroup H of $\mathrm{SL}_2(\mathbb{A})$ in $\widetilde{\mathrm{SL}_2(\mathbb{A})}$. It is known that $\mathrm{SL}_2(F)$ can be canonically embedded into $\widetilde{\mathrm{SL}_2(\mathbb{A})}$. The embedding $\tilde{\iota}$ is given by $g \mapsto ([\iota_v(g)])_v$ for each $g \in \mathrm{SL}_2(F)$. We define the maps $\tilde{\iota}_f : \mathrm{SL}_2(F) \to \widetilde{\mathrm{SL}_2(\mathbb{A}_f)}$ and $\tilde{\iota}_\infty : \mathrm{SL}_2(F) \to \widetilde{\mathrm{SL}_2(F_\infty)}$ by

$$\tilde{\iota}_f(g) = ([\iota_v(g)])_{v<\infty} \times ([1_2])_{v|\infty}, \qquad \tilde{\iota}_\infty(g) = ([1_2])_{v<\infty} \times ([\iota_i(g)])_{v|\infty},$$

where 1_2 is the identity matrix of size 2. Then we have $\tilde{\iota}(g) = \tilde{\iota}_f(g)\tilde{\iota}_\infty(g)$ for all $g \in \mathrm{SL}_2(F)$.

Let $\Gamma \subset \mathrm{SL}_2(\mathfrak{o})$ be a congruence subgroup. A map $\mathbf{v} : \Gamma \to \mathbb{C}^\times$ is said to be a multiplier system of half-integral weight if $\mathbf{v}(\gamma)\prod_{i=1}^n J(\iota_i(\gamma), z_i)$ is an automorphy factor for $\Gamma \times \mathfrak{h}^n$, where J is the function in (1). Let K_Γ be the closure of $\iota_f(\Gamma)$ in $\mathrm{SL}_2(\mathbb{A}_f)$ and $\tilde{K}_\Gamma$ the inverse image of K_Γ in $\widetilde{\mathrm{SL}_2(\mathbb{A}_f)}$. Let $\lambda : \tilde{K}_\Gamma \to \mathbb{C}^\times$ be a genuine character, which is defined in Section 3. Put $\mathbf{v}_\lambda(\gamma) = \lambda(\tilde{\iota}_f(\gamma))$ for $\gamma \in \Gamma$. Then $\mathbf{v}_\lambda$ is a multiplier system of half-integral weight for Γ.

Now suppose that $\mathbf{v} : \Gamma \to \mathbb{C}^\times$ is a multiplier system of half-integral weight. We obtain an equivalent condition that there exists a genuine character $\lambda : \tilde{K}_\Gamma \to \mathbb{C}^\times$ such that $\mathbf{v}_\lambda = \mathbf{v}$. Put $K_f = \prod_{v<\infty} \mathrm{SL}_2(\mathfrak{o}_v)$, which $\mathfrak{o}_v$ is the ring of integers of F_v.

Proposition 3. Let $\mathbf{v}$ be a multiplier system of half-integral weight for $\mathrm{SL}_2(\mathfrak{o})$. Then there exists a genuine character $\lambda : \tilde{K}_f \to \mathbb{C}^\times$ such that $\mathbf{v}_\lambda = \mathbf{v}$.

Corollary 2. There exists a multiplier system $\mathbf{v}$ of half-integral weight for $\mathrm{SL}_2(\mathfrak{o})$ if and only if 2 splits completely in $F/\mathbb{Q}$. There exists a genuine character of $\widetilde{\mathrm{SL}_2(\mathfrak{o}_v)}$ for all $v < \infty$, provided that this condition holds.

Now suppose that 2 splits completely in $F/\mathbb{Q}$. Let $\psi : \mathbb{A}/F \to \mathbb{C}^\times$ be an additive character such that its v-component $\psi_v(x)$ equals $e(x)$ for all $v \mid \infty$. Put $\psi_\beta(x) = \psi(\beta x)$ and $\psi_{\beta,v}(x) = \psi_v(\beta x)$ for $\beta \in F^\times$. The Schwartz space of F_v is denoted by $S(F_v)$. Let $\omega_{\psi_\beta,v}$ be the Weil representation of the metaplectic group $\widetilde{\mathrm{SL}_2(F_v)}$ on $S(F_v)$ corresponding to $\psi_{\beta,v}$.

In the case $v < \infty$, we shall determine the genuine characters of the metaplectic group $\widetilde{\mathrm{SL}_2(\mathfrak{o}_v)}$. Let λ_v be a genuine character of $\widetilde{\mathrm{SL}_2(\mathfrak{o}_v)}$. The space $(\omega_{\psi_\beta,v}, S(F_v))^{\lambda_v}$ is defined by a set of $f \in S(F_v)$ such that $\omega_{\psi_\beta,v}(g)f = \lambda_v(g)^{-1}f$ for all $g \in \widetilde{\mathrm{SL}_2(\mathfrak{o}_v)}$. We determine the space completely.

In the case $v \mid \infty$, let λ_v be a genuine character of the metaplectic group $\widetilde{\mathrm{SO}(2)}$, where $\mathrm{SO}(2)$ is a set of $\begin{pmatrix} a & b \\ -b & a \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$. The space $(\omega_{\psi_\beta,v}, S(\mathbb{R}))^{\lambda_v}$ is defined by a set of $f \in S(\mathbb{R})$ such that $\omega_{\psi_\beta,v}(g)f = \lambda_v(g)f$

for all $g \in \widetilde{\mathrm{SO}(2)}$. We have an irreducible decomposition

$$\omega_{\psi_\beta,v} = \omega^+_{\psi_\beta,v} \oplus \omega^-_{\psi_\beta,v},$$

where $\omega^+_{\psi_\beta,v}$ (resp. $\omega^-_{\psi_\beta,v}$) is an irreducible representation of the set of even (resp. odd) functions in $S(\mathbb{R})$ (see [21, Lemma 2.4.4]).

If $\beta < 0$, there exist no lowest weight vectors of $\omega^+_{\psi_\beta,v}$ or $\omega^-_{\psi_\beta,v}$. If $\beta > 0$, the vector $e(i\iota_v(\beta)x^2)$ (resp. $xe(i\iota_v(\beta)x^2)$) is the lowest weight vector of $\omega^+_{\psi_\beta,v}$ (resp. $\omega^-_{\psi_\beta,v}$) of weight $1/2$ (resp. $3/2$) (see [21, Lemma 2.4.4]). Let $\lambda_{\infty,1/2}$ be a genuine character of lowest weight $1/2$ with respect to $(\omega^+_{\psi_\beta,v}, S(\mathbb{R}))$ and $\lambda_{\infty,3/2}$ of lowest weight $3/2$ with respect to $(\omega^-_{\psi_\beta,v}, S(\mathbb{R}))$.

The set of totally positive elements of F is denoted by $F_+^\times$. Assume that $\beta \in F_+^\times$ in order that there exists a lowest weight vector of $(\omega^+_{\psi_\beta,v}, S(\mathbb{R}))$ or $(\omega^-_{\psi_\beta,v}, S(\mathbb{R}))$ for all $v < \infty$. We fix $\omega_{\psi_\beta,v}$ and λ_v for any v. Here, we assume that $\lambda_v = \lambda_{\infty,1/2}$ or $\lambda_v = \lambda_{\infty,3/2}$ for all $v \mid \infty$. Put $K = K_f \times \prod_{v|\infty} \mathrm{SO}(2)$. Let $\lambda : \tilde{K} \to \mathbb{C}^\times$ be a genuine character such that its v-component is λ_v. Let $S(\mathbb{A})$ be the Schwartz space of $\mathbb{A}$. The space $(\omega_{\psi_\beta}, S(\mathbb{A}))^\lambda$ is defined by a set of $\phi = \prod_v \phi_v \in S(\mathbb{A})$ such that $\phi_v \in (\omega_{\psi_\beta,v}, S(F_v))^{\lambda_v}$ for all v. We determine when there exists a nonzero $\phi \in (\omega_{\psi_\beta}, S(\mathbb{A}))^\lambda$.

We define the theta function Θ_ϕ by

$$\Theta_\phi(g) = \sum_{\xi \in F} \omega_{\psi_\beta}(g)\phi(\xi)$$

for $\phi \in S(\mathbb{A})$ and $g \in \widetilde{\mathrm{SL}_2(\mathbb{A})}$, where $\omega_{\psi_\beta}(g)\phi(\xi) = \prod_v \omega_{\psi_\beta,v}(g_v)\phi_v(\xi)$. The product is essentially a finite product. If $\phi \in (\omega_{\psi_\beta}, S(\mathbb{A}))^\lambda$ for λ such that the lowest weight of $(\omega_{\psi_\beta}, S(\mathbb{A}))^\lambda$ is $w \in \{1/2, 3/2\}^n$, then it is known that Θ_ϕ is a Hilbert modular form of weight w.

For $1 \le i \le n$, put $\lambda_{\infty_i} = \lambda_{\infty,w_i}$, where $w_i = 1/2$ or $3/2$. Put $S_\infty = \{\infty_i \mid w_i = 3/2\}$ and $S_2 = \{v < \infty \mid F_v = \mathbb{Q}_2\}$. Let $\mathfrak{p}_v$ be the maximal ideal of $\mathfrak{o}_v$ and q_v the order of $\mathfrak{o}_v/\mathfrak{p}_v$. Put $T_3 = \{v < \infty \mid q_v = 3\}$. We denote the order of a set S by $|S|$. Let $\mathbf{G}$ be the set of triplets $(\beta, S_3, \mathfrak{a})$ of $\beta \in F_+^\times$, a subset $S_3 \subset T_3$ and a fractional ideal $\mathfrak{a}$ of F satisfying the conditions

$$|S_2| + |S_3| + |S_\infty| \in 2\mathbb{Z}$$

and

$$(8\beta)\mathfrak{d} \prod_{v \in S_3} \mathfrak{p}_v = \mathfrak{a}^2,$$

where $\mathfrak{d}$ is the different of $F/\mathbb{Q}$. We define an equivalence relation $\sim$ on $\mathbf{G}$ by

$$(\beta, S_3, \mathfrak{a}) \sim (\beta', S_3', \mathfrak{a}') \iff S_3 = S_3',\ \beta' = \gamma^2\beta,\ \mathfrak{a}' = \gamma\mathfrak{a} \text{ for some } \gamma \in F^\times.$$

We determine when there exists a nonzero Θ_ϕ. Recall that if q_v is odd, the double covering $\widetilde{\mathrm{SL}_2(F_v)} \to \mathrm{SL}_2(F_v)$ splits on $\mathrm{SL}_2(\mathfrak{o}_v)$. We denote the image of $g \in \mathrm{SL}_2(\mathfrak{o}_v)$ under the splitting by $[g, s(g)]$. Thus if q_v is odd, there exists a genuine character $\epsilon_v : \widetilde{\mathrm{SL}_2(\mathfrak{o}_v)} \to \mathbb{C}^\times$ satisfying $\epsilon_v([g, s(g)]) = 1$ for all $g \in \mathrm{SL}_2(F_v)$.

Theorem 1. Suppose that 2 splits completely in $F/\mathbb{Q}$. Let $\beta \in F_+^\times$, S_3, $\lambda : \tilde{K} \to \mathbb{C}^\times$ and $w_1, \ldots, w_n \in \{1/2, 3/2\}$ be as above. Then there exists $\phi = \prod_v \phi_v \in (\omega_{\psi_\beta}, S(\mathbb{A}))^\lambda$ such that $\Theta_\phi \neq 0$ if and only if there exists a fractional ideal $\mathfrak{a}$ of F such that $(\beta, S_3, \mathfrak{a}) \in \mathbf{G}$.

Put

$$H = \left\{ \prod_{v \in T_3} \mathfrak{p}_v^{e_v} \mid \sum_{v \in T_3} e_v \in 2\mathbb{Z} \right\}.$$

Let Cl^+ be the narrow ideal class group of F. Put $\mathrm{Cl}^{+2} = \{\mathfrak{c}^2 \mid \mathfrak{c} \in \mathrm{Cl}^+\}$. We denote the image of the group H (resp. $\mathfrak{b} \in \mathrm{Cl}^+$) in $\mathrm{Cl}^+/\mathrm{Cl}^{+2}$ by $\bar{H}$ (resp. $[\mathfrak{b}]$).

Theorem 2. Suppose that 2 splits completely in $F/\mathbb{Q}$. Let $w_1, \ldots, w_n \in \{1/2, 3/2\}$ be as above.

(1) Suppose that $|S_2| + |S_\infty|$ is even. Then there exists $(\beta, S_3, \mathfrak{a}) \in \mathbf{G}$ if and only if $[\mathfrak{d}] \in \bar{H}$.
(2) Suppose that $|S_2| + |S_\infty|$ is odd. Then there exists $(\beta, S_3, \mathfrak{a}) \in \mathbf{G}$ if and only if $T_3 \neq \emptyset$ and $[\mathfrak{d}\mathfrak{p}_{v_0}] \in \bar{H}$. Here, v_0 is any fixed element of T_3.

Now suppose that there exists $(\beta, S_3, \mathfrak{a}) \in \mathbf{G}$. Replacing β with $\beta\gamma^2$ and $\mathfrak{a}$ with $\gamma\mathfrak{a}$ in (22), respectively, we may assume $\mathrm{ord}_v \mathfrak{a} = 0$ for $v \in S_2 \cup S_3$. For $v \in S_2 \cup S_3$, define $f_v : \mathfrak{o}_v \to \mathbb{C}$ by

$$f_v(x) = \begin{cases} 1 & \text{if } x \in 1 + 2\mathfrak{p}_v \\ -1 & \text{if } x \in -1 + 2\mathfrak{p}_v \\ 0 & \text{otherwise.} \end{cases}$$

We set

$$f = \prod_{v \in S_2 \cup S_3} f_v \times \prod_{v < \infty, v \notin S_2 \cup S_3} \mathrm{ch}\mathfrak{a}_v^{-1},$$

where $\mathfrak{a}_v = \mathfrak{a}\mathfrak{o}_v$. Put $\phi = f \times \prod_{i=1}^n f_{\infty,i}$, where $f_{\infty,i}(x) = x^{w_i - (1/2)} e(i\iota_i(\beta)x^2)$ for $x \in \mathbb{R}$ and $w_i \in \{1/2, 3/2\}$. By Theorem 1, there exists $\Theta_\phi \neq 0$ of weight $w = (w_1, \cdots, w_n)$.

Theorem 3. Let ϕ and Θ_ϕ be as above. We define a theta function $\theta_\phi : \mathfrak{h}^n \to \mathbb{C}$ by

$$\theta_\phi(z) = \sum_{\xi \in \mathfrak{a}^{-1}} f(\iota_f(\xi)) \prod_{\infty_i \in S_\infty} \iota_i(\xi) \prod_{i=1}^n e(z_i \iota_i(\beta\xi^2)).$$

for $z = (z_1, \cdots, z_n) \in \mathfrak{h}$. Then θ_ϕ is a nonzero Hilbert modular form of weight w for $\mathrm{SL}_2(\mathfrak{o})$ with respect to a multiplier system. Every theta function of weight w for $\mathrm{SL}_2(\mathfrak{o})$ with a multiplier system can be obtained in this way.

In particular, when $F = \mathbb{Q}$, we obtain $\eta(z)$ and $\eta^3(z)$ as $\theta_\phi(z)$ up to constant.

This paper is organized as follows. In section 2, we introduce Hilbert modular forms with a multiplier system and automorphic forms on the adelic metaplectic group. We describe also the modular imbedding (see [13]) and the result of Feng (see [6]). In Section 3, we determine the number of the genuine characters of the metaplectic group $\widetilde{\mathrm{SL}_2(\mathfrak{o})}$, where $\mathfrak{o}$ is the ring of integers of a finite extension F of $\mathbb{Q}_p$. Moreover, we determine the dimension of a space $(\omega_{\psi_\beta}, S(F))^\lambda$ for a genuine character λ of $\widetilde{\mathrm{SL}_2(\mathfrak{o})}$ and the Weil representation ω_{ψ_β} of $\widetilde{\mathrm{SL}_2(F)}$ on $S(F)$. In Section 4, we study the multiplier systems of half-integral weight of a congruence subgroup of $\mathrm{SL}_2(\mathfrak{o})$, where $\mathfrak{o}$ is the ring of integers of a totally real number field F. In Section 5, we define theta functions Θ_ϕ of $\widetilde{\mathrm{SL}_2(\mathbb{A})}$ and prove our main theorems. Moreover, we obtain theta functions $\theta_\phi(z)$ of $\mathfrak{h}^n$ and determine the number of the equivalence classes of the set $\mathbf{G}$. In Section 6, we give some examples in the case $F = \mathbb{Q}$ or F is a real quadratic field.

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2. The Hilbert modular forms with a multiplier system

2.1. **Beginning of the study of Hilbert modular forms.** Hilbert modular forms, which are also called Hilbert-Blumenthal modular forms, are an extension of modular forms to the several valuables case. We introduce the beginning of the study of Hilbert modular forms, referring to Mayer [23]. Hilbert wrote a manuscript from 1893-94 on the action of the modular group of a totally real field K of degree n over $\mathbb{Q}$ on the product of n upper half planes. Based on this, Blumenthal gave a detailed account of the function theory involved but his construction of a fundamental domain had a flaw. He obtained a fundamental domain with only one cusp as in the one valuable case (see [2] and [3]). Maass corrected this and showed that the number of cusps equals the class number of K (see [22] or [9]).

Blumenthal's work consists of the following three parts. First he investigates the fundamental domain of $\mathrm{GL}_2(\mathfrak{o})\backslash\mathfrak{h}^n$ for totally real number fields K of degree n with ring $\mathfrak{o}$ of integers, where $\mathfrak{h} := \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ is the upper half plane. Therefore he proves the discontinuous operation of the group $\mathrm{GL}_2(\mathfrak{o})$ on $\mathfrak{h}^n$ and investigates the fixed points of the elements of $\mathrm{GL}_2(\mathfrak{o})$ on $\mathfrak{h}^n$ and on its boundary. Then Blumenthal constructs a fundamental domain but it is based on a flaw; the existence of exactly one cusp for $\mathrm{GL}_2(\mathfrak{o})\backslash\mathfrak{h}^n$.

The second part of Blumenthal's work deals with Poincaré series. He shows their convergence and the existence of $n+1$ algebraically independent Poincaré series. He uses the result of the first part, but the proof can easily be amended by treatment of all the finitely many cusps instead of the single cusp ∞. Equivalently he shows the existence of n independent modular functions which are quotients of the $n+1$ algebraically independent Hilbert modular forms.

The third part proves the theorems of Weierstrass (see [3]), that

I) all rational functions of the fundamental domain can be algebraically expressed by n independent functions,
II) they can be rational expressed by $n+1$ appropriate functions.

This result is independent of the mistake at the beginning.

2.2. **Hilbert modular forms with multiplier systems.** We define Hilbert modular forms with multiplier systems. The contents of this subsection are mainly taken from the book of Freitag [7] and Mayer [23]. To begin with, given a subgroup Γ of $\mathrm{SL}_2(\mathbb{R})^n$, we define its operation on $\mathfrak{h}^n$ and its cusps and the notion of automorphic forms with respect to Γ.

Let $\mathfrak{h}$ denote the upper half plane $\{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ and $\mathfrak{h}^n$ the product of n upper half planes. Given a subgroup Γ of $\mathrm{SL}_2(\mathbb{R})^n$, we define its operation on $\mathfrak{h}^n$ by

$$\mathrm{SL}_2(\mathbb{R})^n \times \mathfrak{h}^n \to \mathfrak{h}^n;\ (M,\tau) \mapsto M\tau := \left(\frac{a_1\tau_1+b_1}{c_1\tau_1+d_1}, \cdots, \frac{a_n\tau_n+b_n}{c_n\tau_n+d_n}\right),$$

where

$$M = (M_1, \cdots, M_n) \text{ with } M_j = \begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix},\ \tau = (\tau_1, \cdots, \tau_n).$$

This can be continuously extended to an operation on $(\mathfrak{h} \cup \mathbb{R} \cup \{\infty\})^n$.

From now on let Γ be a discrete subgroup of $\mathrm{SL}_2(\mathbb{R})^n$. We define $\tau+\lambda$ and $\epsilon\tau+\lambda$ for $\lambda = (\lambda_1, \cdots, \lambda_n), \epsilon = (\epsilon_1, \cdots, \epsilon_n) \in \mathbb{R}^n$ and $\tau = (\tau_1, \cdots, \tau_n) \in \mathfrak{h}^n$ by

$$\tau+\lambda = (\tau_1+\lambda_1, \cdots, \tau_n+\lambda_n), \qquad \epsilon\tau+\lambda = (\epsilon_1\tau_1+\lambda_1, \cdots, \epsilon_n\tau_n+\lambda_n)$$

using the entrywise sum and the entrywise product. We define the group t_Γ of translations by

$$\{\lambda \in \mathbb{R}^n \mid \text{ there exists } M \in \Gamma \text{ such that } M\tau = \tau+\lambda \text{ for all } \tau \in \mathfrak{h}^n\}$$

and the group Λ_Γ of multipliers by the set of $\epsilon \in \mathbb{R}^n_+$ such that there exist $M \in \Gamma$ and $\lambda \in \mathbb{R}^n$ satisfying

$$M\tau = \epsilon\tau+\lambda \qquad \text{for all } \tau \in \mathfrak{h}^n,$$

where $\mathbb{R}^n_+$ is the set of $\epsilon \in \mathbb{R}^n$ with each of its entries positive. If t_Γ is isomorphic to $\mathbb{Z}_n$ and $\epsilon = (\epsilon_1, \cdots, \epsilon_n)$ is a multiplier, then Λ_Γ is a discrete subgroup of $\mathbb{R}^n_+$ and we have $N(\epsilon) := \prod_{j=1}^n \epsilon_j = 1$ (see [7, Remark I.2.3]).

We say that Γ has cusp infinity if t_Γ is isomorphic to $\mathbb{Z}^n$ and if Λ_Γ is isomorphic to $\mathbb{Z}^{n-1}$. In this case we will write Γ has cusp ∞. We say that Γ has cusp κ for some $\kappa \in (\mathbb{R} \cup \{\infty\})^n$ if there exists an $M \in \mathrm{SL}_2(\mathbb{R})$ with

$$M\kappa = (\infty, \cdots, \infty)$$

such that $M\Gamma M^{-1}$ has cusp infinity. Note that for every $\kappa \in (\mathbb{R} \cup \{\infty\})^n$, there exists an $M \in \mathrm{SL}_2(\mathbb{R})^n$ with $M\kappa = (\infty, \cdots, \infty)$ and that the definition of cusp κ is independent of the choice of M.

We define $(\mathfrak{h}^n)^* := \mathfrak{h}^n \cup (\text{the cusps of } \Gamma)$. Until the end of the subsection, suppose that

- the quotient space $(\mathfrak{h}^n)^*/\Gamma$ is compact,
- each of the projections $p_j : \Gamma \to \mathrm{SL}_2(\mathbb{R})$; $(M_1, \cdots, M_n) \mapsto M_j$ is injective.

Since $(\mathfrak{h}^n)^*/\Gamma$ is compact, there are only finitely many cusps. Let $\mathfrak{o}$ be the ring of integers of a totally real field K. Then the Hilbert modular group $\mathrm{SL}_2(\mathfrak{o})$ satisfies the suppositions above, and hence we will restrict Γ to it later.

Given $a \in \mathbb{R}^n$ and $x \in \mathbb{R}^n$, we define the trace by

$$S(ax) = a_1x_1 + \cdots + a_nx_n.$$

A discrete subgroup T of $\mathbb{R}^n$ is said to be a lattice if there exists a basis $a_1, \cdots, a_n$ of $\mathbb{R}^n$ such that $T = \mathbb{Z}a_1 + \cdots + \mathbb{Z}a_n$. This holds if and only if T is isomorphic to $\mathbb{Z}^n$. For a lattice $T \subset \mathbb{R}^n$, we define the dual lattice $T^\#$ by

$$T^\# = \{a \in \mathbb{R}^n \mid S(ax) \in \mathbb{Z} \text{ for all } x \in T\}.$$

Lemma I.4.1 [7]. Let $V \subset \mathbb{R}^n_{>0}$ be an open and connected set. Define the tube domain $D := \{\tau \in \mathfrak{h}^n \mid \mathrm{Im}(\tau) \in V\}$ corresponding to V. Let $f : D \to \mathbb{C}$ be a holomorphic function on D satisfying for some lattice $T \subset \mathbb{R}^n$

$$f(\tau + a) = f(\tau)$$

for all $a \in T$ and all $\tau \in \mathfrak{h}^n$. Then f has an unique Fourier expansion

$$f(\tau) = \sum_{g \in T^\#} a_g e^{2\pi i S(g\tau)}$$

and the series converges absolutely and uniformly on compact subsets of D.

Given $c = (c_1, \cdots, c_n), d = (d_1, \cdots, d_n) \in \mathbb{R}^n$, $r = (r_1, \cdots, r_n) \in \mathbb{Q}^n$ and $\tau \in \mathfrak{h}^n$, we define the rth power of the norm of $c\tau + d$ by

$$N(c\tau + d)^r := \prod_{j=1}^{n} (c_j\tau_j + d_j)^{r_j}$$

where the r_jth power is defined using the main branch of the logarithm $\mathbb{C}^\times \to \mathbb{R} + i(-\pi, \pi]$.

Given a holomorphic map $f : \mathfrak{h}^n \to \mathbb{C}$, $r \in \mathbb{C}^n$, a matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ and a map $\mu : \Gamma \to \mathbb{C}^\times$, we define $f|_r^\mu M : \mathfrak{h}^n \to \mathbb{C}$ by

$$\tau \mapsto \mu(M)^{-1} N(c\tau + d)^{-r} f(Mr).$$

We will write $|_k$ for $|_k^1$, where 1 is the constant map $\Gamma \to \{1\}$. Note that $N(c\tau+d)^{-r} = 1/(N(c\tau+d)^r)$ holds for every $r \in \mathbb{C}^n$ independent of the chosen branch of the complex logarithm.

Definition 1. If $f : \mathfrak{h}^n \to \mathbb{C}$ is a function satisfying the requirements in the lemma above and Γ has cusp infinity, then f is called regular at cusp ∞ if

$$a_g \neq 0 \Rightarrow g_j \geq 0 \qquad \text{for all } 1 \leq j \leq n.$$

We say that f vanishes at cusp ∞ if

$$a_g \neq 0 \Rightarrow g_j > 0 \qquad \text{for all } 1 \leq j \leq n.$$

Let κ be a cusp of Γ and $N \in \mathrm{SL}_2(\mathbb{R})^n$ be a matrix with $N^{-1}\kappa = (\infty, \cdots, \infty)$. If there exists $r \in \mathbb{Q}^n$ and a map $\mu : \Gamma \to \mathbb{C}^\times$ such that f satisfies

$$f|_r^\mu M = f \qquad \text{for all } M \in \Gamma,$$

then we say that f is regular at cusp κ (resp. vanishes at cusp κ) if $f|_r N$ has cusp ∞ with respect to the group $N^{-1}\Gamma N$ and is regular at ∞ (resp. vanishes at ∞).

Definition 2. Let Γ be a discrete subgroup of $\mathrm{SL}_2(\mathbb{R})^n$ and $\mu : \Gamma \to \mathbb{C}^\times$ be a map of finite order, which means the set $\{\mu^k \mid k \in \mathbb{N}\}$ is finite. An automorphic form of weight $r = (r_1, \cdots, r_n) \in \mathbb{Q}^n$ with respect to Γ with multiplier system μ is a holomorphic function $f : \mathfrak{h}^n \to \mathbb{C}$ with the properties

(a) $f|_r^\mu M = f \quad$ for all $M \in \Gamma$,
(b) f is regular at the cusps.

If f vanishes at all cusps, we call f a cusp form. If f is an automorphic form of weight r with multiplier system μ, we will sometimes write $f|M$ for $f|_r^\mu M$.

Freitag defined Hilbert modular forms as automorphic forms with respect to groups commensurable to the Hilbert modular group $\mathrm{SL}_2(\mathfrak{o}) \subset \mathrm{SL}_2(K)$, where two groups G, G' are said to be commensurable if $G \cap G'$ has finite index in each of the two groups(see [7]). The definition of an automorphic form is based on the one in [7], but includes multiplier systems. Freitag mentions the problem of formulating a general theory of multiplier systems. This was done by Gundlach [12] in the case of subgroups of $\mathrm{SL}_2(\mathfrak{o})$ of finite index.

Freitag showed the following facts in the case of the trivial multiplier system. However, since any multiplier system is of finite order, these facts also hold in our case (see [23]).

Proposition I.4.7 [7]. Each automorphic form f of weight $0 = (0, \cdots, 0)$ is constant.

Remark I.4.8 [7]. If f is an automorphic form, but not a cusp form, of weight $r = (r_1, \cdots, r_n)$, then we have $r_1 = \cdots = r_n$.

Corollary of Proposition I.4.9 [7]. In the case $n \geq 2$, the regularity condition (b) in the definition of an automorphic form can be omitted.

Let K be a totally real number field of degree $n := [K : \mathbb{Q}] = \dim_{\mathbb{Q}}(K)$. Then there are exactly n different embeddings of K into $\mathbb{R}$. We denote them by $K \to \mathbb{R}$; $a \mapsto a^{(j)}$ $(1 \leq j \leq n)$ and $a = a^{(1)}$ holds for all $a \in K$. We denote the ring of integers of K by $\mathfrak{o}$, which is the set

$$\{x \in K \mid F(x) = 0 \text{ for some monic polynomial } F \in \mathbb{Z}[X]\}.$$

We define the operation of $\mathrm{SL}_2(K)$ on $\mathfrak{h}^n$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tau = \left(\frac{a^{(1)}\tau_1 + b^{(1)}}{c^{(1)}\tau_1 + d^{(1)}}, \cdots, \frac{a^{(n)}\tau_n + b^{(n)}}{c^{(n)}\tau_n + d^{(n)}} \right).$$

It is the same as that on $\mathfrak{h}^n$ of the image of $\mathrm{SL}_2(K)$ with respect to the map $\mathrm{SL}_2(K) \to \mathrm{SL}_2(\mathbb{R})^n$;

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto (M^{(1)}, \cdots, M^{(n)}), \qquad M^{(j)} = \begin{pmatrix} a^{(j)} & b^{(j)} \\ c^{(j)} & d^{(j)} \end{pmatrix}. \tag{4}$$

We regard $\mathrm{SL}_2(K)$ as a subgroup of $\mathrm{SL}_2(\mathbb{R})^n$ through (4). An element λ of K is called totally positive if $\lambda^{(j)} > 0$ holds for all $1 \leq j \leq n$. We denote the set of all totally positive elements of K by $K_+^\times$.

For $\lambda \in K$, we define the norm and the trace by

$$N(\lambda) = \prod_{j=1}^{n} \lambda^{(j)}, \qquad S(\lambda) = \sum_{j=1}^{n} \lambda^{(j)}.$$

For $c, d \in K$ and $\tau \in \mathfrak{h}^n$, we define the trace by $S(c\tau) = \sum_{j=1}^{n} c^{(j)}\tau_j$ and the norm by

$$N(c\tau + d) = \prod_{j=1}^{n} (c^{(j)}\tau_j + d^{(j)}). \tag{5}$$

Moreover, for $r = (r_1, \cdots, r_n) \in \mathbb{Q}^n$, put

$$N(c\tau + d)^r := \prod_{j=1}^{n} (c^{(j)}\tau_j + d^{(j)})^{r_j},$$

where $z^{r_j} := e^{r_j \mathrm{In}\, z}$ is defined using the main branch In:$\mathbb{C}^\times \to \mathbb{R} + i(-i, i]$ of the complex logarithm. Note that for $r = (k, \cdots, k) \in \mathbb{Q}^n$, we have $N(c\tau + d)^r = N(c\tau + d)^k$.

For $\tau \in \mathfrak{h}^n$ and $\lambda \in K$, put $\tau + \lambda := (\tau_1 + \lambda^{(1)}, \cdots, \tau_n + \lambda^{(n)}) \in \mathfrak{h}^n$ and

$$\lambda\tau := (\lambda^{(1)}\tau_1, \cdots, \lambda^{(n)}\tau_n) \in \mathfrak{h}^n \quad \text{ if } \lambda \in K_+^\times.$$

Definition 3. Let $\mu : \mathrm{SL}_2(\mathfrak{o}) \to \mathbb{C}$ be a map of finite order. A Hilbert modular form for K of weight $r = (r_1, \cdots, r_n) \in \mathbb{Q}^n$ with multiplier system μ is a holomorphic function $f : \mathfrak{h}^n \to \mathbb{C}$ with the properties

(a) $f|_r^\mu M = f$ for all $M \in \mathrm{SL}_2(\mathfrak{o})$,
(b) f is regular at the cusps of $\mathrm{SL}_2(\mathfrak{o})$.

If f vanishes at all cusps, we call f a cusp form. If f has homogeneous weight $r = (k, \cdots, k) \in \mathbb{Q}^n$, we will also say that f has weight $k \in \mathbb{Q}$.

For a subgroup $\Gamma \subset \mathrm{SL}_2(\mathfrak{o})$ of finite index, we define a Hilbert modular form for K with respect to Γ as in the $\mathrm{SL}_2(\mathfrak{o})$ case.

Since $(\mathfrak{h}^n)^*/\mathrm{SL}_2(\mathfrak{o})$ is compact (see [7, Theorem I.3.6]), every Hilbert modular form is an automorphic form. If $K \neq \mathbb{Q}$, then condition (b) can be omitted (see [7, Corollary of Proposition I.4.9]).

Definition 4. Let Γ be a subgroup of $\mathrm{SL}_2(\mathfrak{o})$ of finite index. A map $\mu : \Gamma \to \mathbb{C}^\times$ is called a multiplier system if it is of finite order and there is $k \in \mathbb{Q}$ such that for all $\tau \in \mathfrak{h}^n$ and all $M_{(1)}, M_{(2)} \in \Gamma$,

$$\mu(M)N(c\tau + d)^k = \mu(M_{(1)})N(c_{(1)}M_{(2)}\tau + d_{(1)})^k \mu(M_{(2)})N(c_{(2)}\tau + d_{(2)})^k,$$

where

$$M_{(j)} = \begin{pmatrix} a_{(j)} & b_{(j)} \\ c_{(j)} & d_{(j)} \end{pmatrix} \quad (j = 1, 2), \qquad M := M_{(1)}M_{(2)} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

If f is a nonzero Hilbert modular form of weight $k \in \mathbb{Q}$ with multiplier system μ, then μ is a multiplier system in this definition. Gundlach showed that the restriction on the order of a multiplier system can be omitted (see [12]).

Lemma 1.2.11 [23]. If $\mu : \Gamma \to \mathbb{C}^\times$ is a multiplier system of integral weight k, then μ is an abelian character. In other words, we have $\mu(MN) = \mu(M)\mu(N)$ for all $M, N \in \Gamma$.

Proof. One calculates $N(c\tau + d) = N(c_{(1)}M_{(2)}\tau + d_{(1)})N(c_{(2)}\tau + d_{(2)})$, where $M_{(1)}, M_{(2)} \in \Gamma$ is as in Definition 4. This proves the assertion. $\square$

Suppose that $n > 1$. Put $\Gamma = \mathrm{SL}_2(\mathfrak{o})$ until the end of this subsection. We introduce an important example of Hilbert modular forms with trivial multiplier system. We denote the group of the elements of Γ fixing $\infty = (i\infty, \cdots, i\infty)$ by Γ_∞. Given $k \in \mathbb{N}$, we define a function $E_{2k}^H : \mathfrak{h}^n \to \mathbb{C}$ called Eisenstein series of weight $2k$ with respect to the cusp ∞ by

$$E_{2k}^H(\tau) := \sum_{M \in \Gamma_\infty \backslash \Gamma} N(c\tau + d)^{-2k} = \sum_{M \in \Gamma_\infty \backslash \Gamma} 1|_{2k} M \qquad \left(M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right).$$

Proposition I.5.8 [7]. The Eisenstein series E_{2k}^H converges absolutely for $k \geq 1$ and has the value 1 at the cusp ∞. It vanishes in all the other cusps.

Proposition I.5.10 [7]. For every Hilbert modular form f of even weight $2k \geq 2$ with trivial multiplier system, there exists an unique element E in the space spanned by all Eisenstein series of weight $2k$, such that $f - E$ is a cusp form.

2.3. **Modular imbeddings.** Siegel modular forms can be restricted to Hilbert modular forms by the modular imbedding of Hammond [13]. He gave a necessary and sufficient condition that there exist modular imbeddings for a given totally real number field. Moreover, he determined the number of equivalence classes of modular imbeddings. The contents of this subsection are mainly taken from Hammond [7] and Mayer [23].

We denote the Siegel half space by $\mathbb{H}_n$, which is the set

$$\{Z \in M_n(\mathbb{C}) : \text{ symmetric} \mid \mathrm{Im}(Z) : \text{positive definite}\}.$$

The symplectic group $Sp_n(\mathbb{R})$ is defined by

$$Sp_n(\mathbb{R}) = \left\{ M \in M_{2n}(\mathbb{R}) \mid \ {}^tMIM = I := \begin{pmatrix} -1_n & 0 \\ 0 & 1_n \end{pmatrix} \right\},$$

where 1_n is the $n \times n$ identity matrix and ${}^tM$ is the transpose of M. Note that we have $Sp_1 = \mathrm{SL}_2$.

The group $Sp_n(\mathbb{R})$ operates on $\mathbb{H}_n$ in the following way. If $\tau \in \mathbb{H}_n$ and if $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sp_n(\mathbb{R})$ for $a, b, c, d \in M_n(\mathbb{R})$, then the image $M \cdot \tau$ of τ under M is given by $M \cdot \tau = (a\tau + b)(c\tau + d)^{-1}$.

A subgroup $Sp_n(\mathbb{Z}) \subset Sp_n(\mathbb{R})$ is called the Siegel modular group of degree n. A holomorphic function f in $\mathbb{H}_n$ is a Siegel modular form of weight w if it satisfies

$$f(M \cdot \tau) = \det(c\tau + d)^w f(\tau)$$

for every $M \in Sp_g(\mathbb{Z})$ and every $\tau \in \mathfrak{h}_g$. If $g = 1$, we also require that $f(iy)$ approaches a finite limit as $y > 0$ approaches infinity.

Let $\mathfrak{o}$ be the ring of integers of a totally real number field K of degree n. A subgroup $Sp_1(\mathfrak{o}) \subset Sp_1(K)$ is called the Hilbert modular group of K. Note that $Sp_1(K)$ is regarded as a subgroup of $Sp_1(\mathbb{R})^n$ by the embedding (4). A holomorphic function f in $\mathfrak{h}^n$ is a Hilbert modular form of weight w for K if it satisfies

$$f(\mu \cdot \tau) = N(\gamma\tau + \delta)^w f(\tau)$$

for every $\mu \in Sp_1(\mathfrak{o})$ and every $\tau \in \mathfrak{h}^n$, where $\begin{pmatrix} \gamma & \delta \end{pmatrix}$ is the bottom row of μ and $N(\cdot)$ is as in (5).

For $z = (z_1, \cdots, z_n) \in \mathbb{C}^n$, let z^* be a diagonal matrix $\mathrm{diag}(z_1, \cdots, z_n)$. For $m = (m_1, \cdots, m_n) \in Sp_1(\mathbb{R})^n$ with $m_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}$, put

$$m^* = \begin{pmatrix} a^* & b^* \\ c^* & d^* \end{pmatrix} \in Sp_n(\mathbb{R}),$$

where $a = (a_1, \cdots, a_n) \in \mathbb{R}^n$ and similarly for b, c and d.

Let $\phi_0 : \mathfrak{h}^n \to \mathbb{H}^n$ be the holomorphic imbedding defined by $\tau \mapsto \tau^*$ and $\Phi_0 : Sp_1(\mathbb{R})^n \to Sp_n(\mathbb{R})$ the monomorphism defined by $m \mapsto m^*$. Then we have $\phi_0(m \cdot \tau) = \Phi_0(m) \cdot \phi_0(\tau)$.

A modular imbedding of K is a pair (ϕ, Φ) consisting of a holomorphic injection $\phi : \mathfrak{h}^n \to \mathbb{H}^n$ and a monomorphism $\Phi : Sp_1(\mathbb{R})^n \to Sp_n(\mathbb{R})$ such that

(1) There exists $N \in Sp_n(\mathbb{R})$ such that $\phi(\tau) = N \cdot \phi_0(\tau)$ for all $\tau \in \mathfrak{h}^n$ and $\Phi(m) = N\Phi_0(m)N^{-1}$ for all $m \in Sp_1(\mathbb{R})^n$,
(2) $\Phi(Sp_1(\mathfrak{o})) \subset Sp_n(\mathbb{Z})$,
(3) if f is a Siegel modular form of weight w, then the composition $f \circ \phi$ is a Hilbert modular form of weight w for K.

Proposition 2.2 [13]. The restriction (3) of the definition above can be replaced by

(3') the matrix $N = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ from (1) holds $c = 0_n$.

Let (ϕ_1, Φ_1) and (ϕ_2, Φ_2) be modular imbeddings. They are called equivalent if there exists $M \in Sp_n(\mathbb{Z})$ such that $\phi_2 = M\phi_1$ and that $\Phi_2 = M\Phi_1 M^{-1}$. Every modular imbedding is equivalent to a modular imbedding (ϕ_1, Φ_1) in which ϕ_1 is homogeneous linear. Here, ϕ_1 is homogeneous linear if there is $N = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$ for $a, d \in M_n(\mathbb{R})$ such that $\phi_1 = N\phi_0$. Then $a\,{}^t a$ and $d\,{}^t d = (a\,{}^t a)^{-1}$ are positive definite symmetric integer matrices with determinant 1.

Let (ϕ_1, Φ_1) be as above. Put $\psi(z) = az^*a^{-1}$ for $z \in \mathbb{C}^n$and $u = a\,{}^t a$. Then we have ${}^t\psi(z) = u^{-1}\psi(z)u$ for all $z \in \mathbb{C}^n$ and ψ is a normal representation of K in the sense that elements of $\mathfrak{o}$ are represented by integer matrices.

Proposition 2.6 [13]. There is one-to-one correspondence between homogeneous linear modular imbeddings (ϕ, Φ) for K and pairs (ψ, u) consisting of a non-degenerate normal representation ψ of K by rational matrices of degree n and an symmetric positive definite matrix $u \in M_n(\mathbb{Z})$ with determinant 1 such that ${}^t\psi(\rho) = u^{-1}\psi(\rho)u$ for all $\rho \in K$.

We note that (ψ_1, u_1) and (ψ_2, u_2) correspond to equivalent homogeneous linear modular imbeddings if and only if there is an unimodular matrix $v \in M_n(\mathbb{Z})$ such that $\psi_2 = v\psi_1 v^{-1}$ and that $u_2 = vu_1{}^t v$.

Theorem 2.8 (Igusa) [13]. A totally real number field K admits modular imbeddings if and only if the narrow ideal class of the different is a square in the narrow ideal class group of K.

Let A be a fractional ideal of K. We note that if A^2 is narrowly equivalent to a given ideal B, then the same thing is true for any ideal in the usual ideal class of A. Let $\pi(K)$ denote the number of usual ideal classes whose squares are narrowly equivalent to the narrow ideal class of $\mathfrak{o}'$, where $\mathfrak{o}'$ is the complementary ideal to $\mathfrak{o}$.

Theorem 2.9 (Igusa) [13]. The number of equivalence classes of modular imbeddings for a totally real number field K is the product $\pi(K)$ with the index of the subgroup of square of units in the group of totally positive units.

For a moment, we consider the real quadratic case. Let K be the real quadratic field of discriminant D. A homogeneous linear modular imbedding (ϕ, Φ) for K is said to be orthogonal if $\phi(1) = 1_n$. In the quadratic case, every modular imbedding for K is equivalent to an orthogonal modular imbedding (see [13, Theorem 3.3]).

Theorem 3.4 [13]. The orthogonal modular imbeddings for K correspond in an one-to-one manner to ordered pairs $(u, v) \in \mathbb{Z}^2$ such that $D = u^2 + v^2$ and that v is even.

Theorem 3.6 [13]. Let t be the number of primes which devide D. Modular imbeddings exist for K if and only if D is divisible by no prime of the form $4m+3$. In this case the number of equivalence classes of modular imbeddings for K is 2^{t-1}.

Müller [24] gives an explicit formulation of the modular imbedding for real quadratic fields. Let $K = \mathbb{Q}(\sqrt{D})$ where $D = u^2 + v^2$, $u, v \in \mathbb{Z}$ and v even and $\omega = (u + \sqrt{D})/2$. Then a modular imbedding is given by the pair (ψ, Ψ) defined by

$$\psi(\zeta) = \begin{pmatrix} S(\frac{\omega}{\sqrt{D}}\zeta) & S(\frac{v}{2\sqrt{D}}\zeta) \\ S(\frac{v}{2\sqrt{D}}\zeta) & S(\overline{(\frac{\omega}{\sqrt{D}})}\zeta) \end{pmatrix}, \quad \Psi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} \psi(a) & \psi(b) \\ \psi(c) & \psi(d) \end{pmatrix},$$

where $\overline{\alpha}$ is a conjugate of α and $S(\alpha\zeta) = \alpha\zeta_1 + \overline{\alpha}\zeta_2$ for $\alpha \in \mathfrak{o}$ and $\zeta \in \mathfrak{h}^2$.

Given $m' = (m'_1, m'_2)$ and $m'' = (m''_1, m''_2)$ in $\{0, 1\}^2$ with $m'_1 m''_1 + m'_2 m''_2 \in 2\mathbb{Z}$, write $m = (m_1, m_2)$ and define a function θ_m by

$$\theta_m(\tau) = \sum_{g \in \mathbb{Z}^2} \exp\left(\pi i \left({}^t(g + \frac{m'}{2})\tau(g + \frac{m'}{2}) + {}^t g m'' + {}^t m' m''/2 \right)\right),$$

where ${}^t g$ is the transpose of g and $\tau \in \mathbb{H}^2$. Additionally we put $\Theta_m = \theta_m \circ \psi$.

In case $K = \mathbb{Q}(\sqrt{17})$, Hermann [14] constructed a Hilbert modular form of half integral weight from theta products:

Theorem 3 [14]. In case $K = \mathbb{Q}(\sqrt{17})$, there exists a Hilbert modular form we denote by η_2 of weight $3/2$ with multiplier system μ_{17} such that $\mu_{17}(J) = -i$, $\mu_{17}(T) = i$ and $\mu_{17}(T_w) = e^{5\pi i/4}$ (see [14] for the definition of J, T, T_w):

$$\begin{aligned} \eta_2 \;:=\; & \Theta_{1100}\Theta_{0011}\Theta_{0000} + \Theta_{1100}\Theta_{0010}\Theta_{0001} + \Theta_{1001}\Theta_{0110}\Theta_{0000} \\ & -\Theta_{1001}\Theta_{0100}\Theta_{0010} + \Theta_{1000}\Theta_{0100}\Theta_{0011} - \Theta_{1000}\Theta_{0110}\Theta_{0001}. \end{aligned}$$

2.4. **The order of multiplier systems.** In 1985, Gundlach studied multiplier systems for Hilbert and Siegel modular groups in [11]. We introduce the Hilbert modular case.

Let K be a totally real number field of degree n and $\mathfrak{o}$ the ring of integers of K. For $v \in K$, we denote by (v) the ideal generated by v. We write $\mathfrak{h} = \mathfrak{h}_1$ for the upper half plane and $\mathfrak{h}_{-1}$ the lower half plane in $\mathbb{C}$. Put $E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in M_2(K)$. The Hilbert modular group for K is the group $\Gamma = \Gamma_K \coloneqq \mathrm{SL}_2(\mathfrak{o}) \subset \mathrm{SL}_2(K)$.

There exist n different bijections of K onto the conjugates $K^{(1)}, \cdots, K^{(n)} \subset \mathbb{R}$. We assign to each $K^{(j)}$ a complex variable $\tau^{(j)}$, the jth conjugate of τ. The canonical isomorphisms of $K(\tau)$ onto $K^{(j)}(\tau^{(j)})$ with $\tau \mapsto \tau^{(j)}$ for $j = 1, \cdots, n$ map a rational function $R(\tau) \in K(\tau)$ onto its conjugates $R^{(j)}(\tau^{(j)})$. For $R(\tau) \in K(\tau)$, trace and norm are defined by

$$TrR(\tau) = \sum j = 1^n R^{(j)}(\tau^{(j)}), \qquad N(R(\tau)) = \prod_{j=1}^{n} R^{(j)}(\tau^{(j)}).$$

For $L \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(K)$, we assign a transformation

$$\tau \mapsto L(\tau) = (a\tau + b)(c\tau + d)^{-1} = (L^{(1)}(\tau^{(1)}), \cdots, L^{(n)}(\tau^{(n)})),$$

where $L^{(j)}(\tau^{(j)}) = (a^{(j)}\tau^{(j)} + b^{(j)})(c^{(j)}\tau^{(j)} + d^{(j)})^{-1}$ for all j. A subgroup $\Lambda \subset \mathrm{SL}_2(K)$ is commensurable with Γ if $\Gamma \cap \Lambda$ has finite index in Γ and in Λ. Put $e = (e_1, \cdots, e_n)$ for $e_j \in \{\pm 1\}$. By the transformation above, a commensurable subgroup Λ with Γ acts on a product $\mathfrak{h}_e = \mathfrak{h}_{e_1} \times \cdots \times \mathfrak{h}_{e_n}$ of half planes $\mathfrak{h}_{e_j}$.

We consider only the $\mathfrak{h}_e = \mathfrak{h}^n$ case. An automorphic factor (AF) of λ on $\mathfrak{h}^n$ is a mapping $J : \lambda \times \mathfrak{h}^n \to \mathbb{C}$ such that

(1) for fixed $L \in \Lambda$, $J(L, \tau)$ is holomorphic without zeros on $\mathfrak{h}^n$.
(2) $J(LM, \tau) = J(L, M(\tau))J(M, \tau)$ for $L, M \in \Lambda$, $\tau \in \mathfrak{h}^n$.
(3) $J(-L, \tau) = J(L, \tau)$ for $L, -L \in \Lambda$, $\tau \in \mathfrak{h}^n$.

Moreover, an AF is called a classical automorphic factor (CAF) if

$$J(L, \tau) = v(L)N(c\tau + d)^r \qquad \text{for } L = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Lambda, \quad \tau \in \mathfrak{h}^n,$$

with $r \in \mathbb{C}$, the weight of J, and $v(L) \in \mathbb{C}$ depending on the choice of the branch of $\log(c^{(j)}\tau^{(j)} + d^{(j)})$ on $\mathfrak{h}$. In this case, v is called the associated multiplier system.

A suitable choice of the branch of the logarithm above on $\mathfrak{h}$ is

$$\log(\alpha z + \beta) = \log|\alpha z + \beta| + i \arg_f(\alpha z + \beta) \qquad \text{for } \alpha, \beta \in \mathbb{R}, \alpha z + \beta \neq 0$$

with

$$-\pi < \arg_1(\alpha z + \beta) \leq \pi \qquad \text{for } z \in \mathfrak{h}.$$

For $r \in \mathbb{C}$, $\tau \in \mathfrak{h}^n$ and $L = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(K)$, put

$$\mu_r(L, \tau) = N(c\tau + d)^r = \exp(r \cdot Tr \log(c\tau + d)).$$

For $L_1, L_2 \in \mathrm{SL}_2(K)$, put

$$\sigma_e^{(r)}(L_1, L_2) = \frac{\mu_r(L_1, L_2(\tau))\mu_r(L_2, \tau)}{\mu_r(L_1 L_2, \tau)}.$$

This depends on L_1, L_2, r, e, but not on τ. It is known that if $r \in \mathbb{Z}$, $\sigma_e^{(r)}(L_1, L_2) = 1$.

A multiplier system (MS) of weight r for Λ on $\mathfrak{h}^n$ is defined as a map $v : \Lambda \to \mathbb{C}^\times$ such that

(4) $v(L_1 L_2) = \sigma_e^{(r)}(L_1, L_2)v(L_1)v(L_2) \quad$ for $L_1, L_2 \in \Lambda$,
(5) $v(-E) = \exp(-\pi irn) \quad$ if $-E \in \Lambda$.

Then

$$J(L, \tau) = v(L)N(c\tau + d)^r \qquad \text{for } L \in \Lambda, \tau \in \mathfrak{h}^n$$

is a CAF of weight r for Λ on $\mathfrak{h}^n$ if and only if $v : \Lambda \to \mathbb{C}^\times$ is a MS of weight r for Λ on $\mathfrak{h}^n$.

Suppose that K is of degree $n > 1$ until the end of this subsection.

Theorem 3.1 [11]. For a subgroup Λ of $\mathrm{SL}_2(K)$, commensurable with Hilbert modular group $\Gamma = \mathrm{SL}_2(\mathfrak{o})$, acting on $\mathfrak{h}^n$, there exists a (minimal) number $g(\Lambda, e) \in \mathbb{N}$ with the following property.

- If J is a CAF of weight r for Λ on $\mathfrak{h}^n$ then

$$r \in \mathbb{Q}, \qquad g(\Lambda, e)r \in \mathbb{Z}$$

- If Λ_0 is a subgroup of finite index in Λ and J_0 a CAF of weight r_0 for Λ_0 on $\mathfrak{h}^n$ then $g(\Lambda, e)[\Lambda : \Lambda_0]r_0 \in \mathbb{Z}$.

Theorem 3.2 [11]. Under the conditions of Theorem 3.1, the MS v, associated with a CAF of Λ, satisfies $|v(L)| = 1$ for all $L \in \Lambda$ with roots of unity as values.

In particular, Gundlach studied the weight of a MS for $\Gamma = \mathrm{SL}_2(\mathfrak{o})$.

Theorem 3.3 [11]. Let v be a MS of weight r for the Hilbert modular group Γ on $\mathfrak{h}^n$.

- If $2 \mid n$, then $2rn \in \mathbb{Z}$.
- If $2 \nmid n$, then the denominator of nr has only prime factors q with $(q-1) \mid (n-1)$. For such a prime q, $l(q) \in \mathbb{Z}$ is defined by $n - 1 = (q-1)q^{l(q)}m_q$ with $q \nmid m_q$. We have

$$2nr \prod_{q:\text{prime},(q-1)|(n-1)} q^{l(q)+1} \in \mathbb{Z}.$$

The theorem above can be improved in the quadratic case.

Theorem 4.1 [11]. Let v be a MS of weight r for the Hilbert modular group Γ of a real quadratic field K on $\mathfrak{h}^2$. For special values of the discriminant d_K of K, we have

(a) If $d_K \equiv 0, 5 \bmod 8$, then $r \in \mathbb{Z}$.
(b) If $d_K \equiv 4 \bmod 8$, then $2r \in \mathbb{Z}$.

In the real quadratic case, we give the Hilbert modular forms of half-integral weight with $d_K \equiv 1 \bmod 8$ later.

2.5. **Metaplectic groups.** Let F be a totally imaginary number field over $\mathbb{Q}$ containing the nth roots of unity for a fixed $n \geq 2$. Let $\mathfrak{p}$ a place of F and $F_\mathfrak{p}$ the completion of F at $\mathfrak{p}$. Kubota [20] constructed metaplectic groups of $\mathrm{GL}_2(F_\mathfrak{p})$ and $\mathrm{GL}_2(\mathbb{A})$ explicitly using Kubota 2-cocycle, where $\mathbb{A}$ is the ring of adele of F. Note that if $n = 2$, Weil [35] discovered such groups for the first time.

We suppose that $n = 2$. Kubota mainly considered GL_2 case for a totally imaginary number field, but we consider SL_2 case for a totally real number field. This difference makes only a little change for infinite places.

Let F be a totally real number field, $\mathfrak{p}$ a place of F and $F_\mathfrak{p}$ the completion of F at $\mathfrak{p}$. The following theorem assures the existence of a covering over the local group $G_\mathfrak{p} = \mathrm{GL}_2(F_\mathfrak{p})$.

Theorem 1 [20]. Put $G_\mathfrak{p} = \mathrm{GL}_2(F_\mathfrak{p})$. For $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(F_\mathfrak{p})$, define $x(\sigma)$ by

$$x(\sigma) = \begin{cases} c & c \neq 0 \\ d & c = 0 \end{cases}$$

and put

$$a(\sigma, \tau) = (x(\sigma), x(\tau))(-x(\sigma)^{-1}x(\tau), x(\sigma\tau)) \qquad \text{for } \sigma, \tau \in \mathrm{SL}_2(F_\mathfrak{p}), \tag{6}$$

where $(\cdot, \cdot)$ is the quadratic Hilbert symbol of $F_\mathfrak{p}$. Furthermore, for $\sigma \in G_\mathfrak{p}$, define $p(\sigma) \in \mathrm{SL}_2(F_\mathfrak{p})$ by $\sigma = \begin{pmatrix} 1 & \\ & \det \sigma \end{pmatrix} p(\sigma)$ and σ^y by the matrix

$$\begin{pmatrix} 1 & \\ & y \end{pmatrix}^{-1} \sigma \begin{pmatrix} 1 & \\ & y \end{pmatrix} \qquad \text{for } y \in F_\mathfrak{p},\ y \neq 0.$$

Put

$$v\left(y, \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{cases} 1 & c \neq 0 \\ (y, d) & c = 0. \end{cases}$$

Then

$$a(\sigma, \tau) := a(p(\sigma)^{\det \tau}, p(\tau)) v(\det \tau, p(\sigma)), \qquad \sigma, \tau \in G_\mathfrak{p} \tag{7}$$

is a factor set which determines a topological covering group $\tilde{G}_\mathfrak{p}$ of $G_\mathfrak{p}$ such that $\tilde{G}_\mathfrak{p}$ is central as a group extension.

It is clear that $a(\sigma,\tau)$ in (7) is equal to one in (6) for $\sigma,\tau \in \mathrm{SL}_2(F_\mathfrak{p})$. It was proved in [19] that (6) determines a topological covering $\widetilde{\mathrm{SL}_2(F_\mathfrak{p})}$ of $\mathrm{SL}_2(F_\mathfrak{p})$ which is central as a group extension. For the proof in GL_2 case, see [20].

Let N be a positive integer divisible by 4, $\mathfrak{o}_\mathfrak{p}$ the ring of integer of $F_\mathfrak{p}$ for finite $\mathfrak{p}$ and $\mathbb{A}$ the ring of adele of F. For finite $\mathfrak{p}$, let $\mathrm{GL}_2(\mathfrak{o}_\mathfrak{p})_N$ be the group of all $\sigma \in G_\mathfrak{p}$ with $\sigma \equiv 1_2 \pmod N$, where 1_2 is the identity matrix. It is a congruence subgroup of $\mathrm{GL}_2(\mathfrak{o}_\mathfrak{p})$. Put $K_\mathfrak{p} = \mathrm{GL}_2(\mathfrak{o}_\mathfrak{p})_N$ for finite $\mathfrak{p}$ and $K_\mathfrak{p} = \mathrm{SO}(2)$ for infinite $\mathfrak{p}$. The adele group of $G_F = \mathrm{GL}_2(F)$ with usual topology will be denoted by $G_\mathbb{A}$.

The next theorem [20] explains the behavior of the factor set $a(\sigma,\tau)$ on the compact subgroup $K_\mathfrak{p}$ of $G_\mathfrak{p}$. Moreover, the theorem is useful in constructing a global covering of the adele group $G_\mathbb{A}$.

Theorem 2 [20]. Let $\mathfrak{p}$ be a finite prime of F, and let N be a natural number divisible by 4. Then the factor set $a(\sigma,\tau)$ in (7) splits on the compact subgroup $K_\mathfrak{p}$ of $G_\mathfrak{p}$. More precisely, we have

$$a(\sigma,\tau) = s(\sigma)s(\tau)s(\sigma\tau)^{-1} \qquad \text{for } \sigma,\tau \in K_\mathfrak{p},$$

with

$$s(\sigma) = \begin{cases} (c, d\det\sigma^{-1})^{-1} & \text{if } cd \neq 0 \text{ and if } \operatorname{ord} c \text{ is odd,} \\ 1 & \text{otherwise,} \end{cases}$$

for $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_\mathfrak{p}$.

Since $\sigma,\tau \in K_\mathfrak{p}$, the above definition of $s(\sigma)$ is equivalent to

$$s(\sigma) = \begin{cases} (c, d\det\sigma^{-1})^{-1} & \text{if } c \neq 0 \text{ and if } c \text{ is not a unit,} \\ 1 & \text{otherwise.} \end{cases}$$

Since we suppose that $n = 2$, we have $s(\sigma) = s(\sigma)^{-1}$ for all $\sigma \in K_\mathfrak{p}$.

Note that this was proved under the assumption that $\sigma,\tau \in \mathrm{GL}_2(\mathfrak{o}_\mathfrak{p})$ and that $\mathfrak{p}$ does not divide N. However, the number $s(\sigma)$ in the theorem is well-defined even if σ is an arbitrary element of $G_\mathfrak{p}$, or if $\mathfrak{p} \mid N$. So, we define a new factor set $b(\sigma,\tau)$ of $G_\mathfrak{p}$ by

$$b(\sigma,\tau) := a(\sigma,\tau)s(\sigma)^{-1}s(\tau)^{-1}s(\sigma\tau) \qquad \text{for } \sigma,\tau \in G_\mathfrak{p},$$

for an arbitrary finite $\mathfrak{p}$. The assertion of Theorem 2 [20] is nothing but $b(\sigma,\tau) = 1$ for all $\sigma,\tau \in K_\mathfrak{p}$, when $\mathfrak{p}$ does not divide N.

Let now g, g' be two adeles in $G_\mathbb{A}$; then $b(g_\mathfrak{p}, g'_\mathfrak{p})$, $a(g_\mathfrak{p}, g'_\mathfrak{p})$ and $s(g_\mathfrak{p})$ are all well-defined. These will be denoted by $b_\mathfrak{p}(g,g')$, $a_\mathfrak{p}(g,g')$ and $s_\mathfrak{p}(g)$, respectively. Since $b_\mathfrak{p}(g,g') = 1$ for almost all $\mathfrak{p}$, we can define a factor set $b_\mathbb{A}$ of $G_\mathbb{A}$ by

$$b_\mathbb{A}(g,g') = \prod_{\mathfrak{p}:\text{finite}} b_\mathfrak{p}(g,g'), \qquad (g, g' \in G_\mathbb{A}),$$

the product being extended over all places of F. The factor set $b_{\mathbb{A}}$ determines a central group extension $\tilde{G_{\mathbb{A}}}$ of $G_{\mathbb{A}}$. Namely, $\tilde{G_{\mathbb{A}}}$ is realized as the set of all pairs (g,ζ), $(g \in G_{\mathbb{A}},\ \zeta^2 = 1)$, with the group operation defined by

$$(g,\zeta)(g',\zeta') = (gg', b_{\mathbb{A}}(g,g')\zeta\zeta')$$

between two such pairs. We denote the element $(1,\zeta) \in \tilde{G_{\mathbb{A}}}$ by $\dot{\zeta}$. The group Z of all $\dot{\zeta}$ is contained in the center of $\tilde{G_{\mathbb{A}}}$, and $\dot{\zeta} \mapsto \zeta$ gives an isomorphism between Z and $\{\pm 1\} \subset F$.

Let N be a natural number divisible by 4, and let K be a compact subgroup of $G_{\mathbb{A}}$ defined by

$$K = \prod_{\mathfrak{p}} K_{\mathfrak{p}}.$$

For any $a \in G_{\mathbb{A}}$, its $\mathfrak{p}$-component $a_{\mathfrak{p}} = \mathrm{pr}_{\mathfrak{p}} a$ belongs to $K_{\mathfrak{p}}$ for almost all $\mathfrak{p}$. Then it follows from Theorem 2 [20] that $K \ni k \mapsto (k,1) \in \tilde{G_{\mathbb{A}}}$ is a group-theoretical isomorphism. Whenever no confusion is possible, we identify the image of the above mapping with K, and denote $(k,1)$ simply by k.

Through this identification $K \subset \tilde{G_{\mathbb{A}}}$ is given a structure of a compact topological group, and the topology coincides on $K \subset \tilde{G}_{\mathfrak{p}}$ with the previous covering topology of $\tilde{G}_{\mathfrak{p}}$ because $s(\sigma)$ in Theorem 2 [20] vanishes on a suitable neighborhood in $G_{\mathfrak{p}}$ of 1.

For $a \in G_{\mathbb{A}}$, the product of all $\mathrm{pr}_{\mathfrak{p}} a$ for finite $\mathfrak{p}$ will be denoted by $\mathrm{pr}_0 a$ and called the finite component of a. The infinite component $\mathrm{pr}_\infty a$ of a is the product of all $\mathrm{pr}_{\mathfrak{p}} a$ for infinite $\mathfrak{p}$. We put $G_0 = \mathrm{pr}_0 G_{\mathbb{A}}$, $G_\infty = \mathrm{pr}_\infty G_{\mathbb{A}}$, and more generally we write $X_0 = \mathrm{pr}_0 X$ and $X_\infty = \mathrm{pr}_\infty X$ for any subset X of $G_{\mathbb{A}}$.

Kubota constructed a global covering group $\tilde{G_{\mathbb{A}}}$ of $G_{\mathbb{A}}$ which coincides locally with the covering stated in Theorem 1 [20] (see [20] for more detail). Note that the subsets of the group K will also be identified with corresponding subsets of $\tilde{G_{\mathbb{A}}}$. Then $\tilde{G_{\mathbb{A}}} \to G_{\mathbb{A}} = \tilde{G_{\mathbb{A}}}/Z$ is an 2-fold covering map because of $K \cap Z = 1$. Since Z can be regarded as a subgroup of $\tilde{G}_{\mathfrak{p}}$ for every $\mathfrak{p}$, $\tilde{G_{\mathbb{A}}}$ is a semi-direct product of $\tilde{G}_{\mathfrak{p}}$. The covering $\tilde{G_{\mathbb{A}}} \to G_{\mathbb{A}}$ is not trivial, because it is not locally trivial at finite places (see [19]).

The group of principal adeles in $G_{\mathbb{A}}$ will be identified with G_F and it is a discrete subgroup of $G_{\mathbb{A}}$. Let $a \in G_F$ be a principal adele. Then $s_{\mathfrak{p}}(\alpha) = 1$ for almost all $\mathfrak{p}$. Therefore

$$s_{\mathbb{A}}(\alpha) = \prod_{\mathfrak{p}} s_{\mathfrak{p}}(\alpha)$$

is well-defined. Moreover, $a_{\mathfrak{p}}(\alpha,\beta) = 1$ $(\alpha,\beta \in G_F)$ for almost all $\mathfrak{p}$ and from the product formula of the norm residue symbol follows

$$\prod_{\mathfrak{p}} a_{\mathfrak{p}}(\alpha,\beta) = 1.$$

This implies $b_{\mathbb{A}}(\alpha,\beta) = s_{\mathbb{A}}(\alpha)^{-1}s_{\mathbb{A}}(\beta)^{-1}s_{\mathbb{A}}(\alpha\beta)$. So, if we put $\hat{\alpha} = (\alpha, s_{\mathbb{A}}(\alpha))$ for $\alpha \in G_F$, then

$$\hat{\alpha}\hat{\beta} = (\alpha\beta, b_{\mathbb{A}}(\alpha,\beta)s_{\mathbb{A}}(\alpha)s_{\mathbb{A}}(\beta)) = (\alpha\beta, s_{\mathbb{A}}(\alpha\beta)) = \widehat{\alpha\beta}$$

for $\alpha, \beta \in G_F$. Thus, $\alpha \mapsto \hat{\alpha}$ gives an isomorphism of G_F onto the group $\hat{G_F} \subset \tilde{G_{\mathbb{A}}}$ of all $\hat{\alpha}$; $\hat{G_F}$ is a discrete subgroup of $\tilde{G_{\mathbb{A}}}$.

In the rest of this subsection, we always assume $N > 0$ is divisible by 4. Let $\mathfrak{o}$ be the ring of integers of F and $\mathrm{GL}_2(\mathfrak{o})_N$ be the group of all $\sigma \in \mathrm{GL}_2(F)$ with $\sigma \equiv 1_2 \pmod{N}$, where 1_2 is the identity matrix.

Proposition 1 [20]. For an element $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of $\Gamma = \mathrm{GL}_2(\mathfrak{o})_N = G_F \cap K_0 G_\infty$, put

$$\chi(\sigma) = \begin{cases} \left(\dfrac{c}{d}\right) & \text{if } c \neq 0, \\ 1 & \text{if } c = 0, \end{cases}$$

where $(\frac{c}{d})$ is the quadratic residue symbol in F. Then we have $s_{\mathbb{A}}(\sigma) = \chi(\sigma)$.

Proposition 2 [20]. Let χ be as above. Then we have $\chi(\sigma\tau) = \chi(\sigma)\chi(\tau)$ for all $\sigma, \tau \in \Gamma$. In other words, χ is a character of Γ.

2.6. **Automorphic forms on the adelic metaplectic group.** In this subsection, we recall the theory of Hilbert modular forms of half-integral weight and the theory of automorphic forms on the metaplectic groups. The contents of this subsection are mainly taken from Hiraga and Ikeda [15]. For more detail, for example, see [33].

Let F be a totally real number field and ψ_1 be the nontrivial additive character of $\mathbb{A}/F$ such that the infinity component of ψ_1 is given by $x \mapsto e(x) = e^{2\pi i x}$ for every real place. Let S be a set of bad places of F, which contains all places above 2 and ∞. We also assume that S contains all non-Archimedean places v such that $c_{\psi_v} = 0$. Set

$$\mathrm{SL}_2(\mathbb{A})_S = \prod_{v \notin S} \mathrm{SL}_2(F_v) \prod_{v \in S} \mathrm{SL}_2(\mathfrak{o}_v).$$

The double covering of $\mathrm{SL}_2(\mathbb{A})_S$ defined by the 2-cocycle $\prod_{v\in S} c_v(g_{1,v}; g_{2,v})$ is denoted by $\widetilde{\mathrm{SL}_2(\mathbb{A})}_S$, where c_v is the Kubota 2-cocycle for $\mathrm{SL}_2(F_v)$. For $S \subset S_0$, we can define an embedding

$$\iota_S^{S_0} : \widetilde{\mathrm{SL}_2(\mathbb{A})}_S \to \widetilde{\mathrm{SL}_2(\mathbb{A})}_{S_0}$$

by

$$[(g_v), \zeta] \to \left[(g_v), \prod_{v \in S_0 \setminus S} s_v(g_v)\right].$$

Here, $s_v : \mathrm{SL}_2(\mathfrak{o}_v) \to \widetilde{\mathrm{SL}_2(\mathfrak{o}_v)} : g_v \mapsto [g_v, s_v(g_v)]$ is the unique splitting of the covering $\widetilde{\mathrm{SL}_2(\mathfrak{o}_v)} \to \mathrm{SL}_2(\mathfrak{o}_v)$ for $v \notin S$. The adelic metaplectic group $\widetilde{\mathrm{SL}_2(\mathbb{A})}$ is the direct limit $\operatorname{proj\,lim} \widetilde{\mathrm{SL}_2(\mathbb{A})}_S$. Then $\widetilde{\mathrm{SL}_2(\mathbb{A})}$ is a double covering of

$\mathrm{SL}_2(\mathbb{A})$ and there exists a canonical embedding $\widetilde{\mathrm{SL}_2(F_v)} \to \widetilde{\mathrm{SL}_2(\mathbb{A})}$ for each place v of F. It is well known that $\mathrm{SL}_2(F)$ can be canonically embedded into $\widetilde{\mathrm{SL}_2(\mathbb{A})}$. In fact, for each $\gamma \in \mathrm{SL}_2(F)$, the embedding is given by $\gamma \mapsto [\gamma, 1]$ for sufficiently large S.

Let $\prod_v' \widetilde{\mathrm{SL}_2(F_v)}$ be the restricted direct product with respect to $s_v(\mathrm{SL}_2(\mathfrak{o}_v))$. Then there is a canonical surjection $\prod_v' \widetilde{\mathrm{SL}_2(F_v)} \to \widetilde{\mathrm{SL}_2(\mathbb{A})}$. The image of $(g_v)_v \in \prod_v' \widetilde{\mathrm{SL}_2(F_v)}$ is also denoted by $(g_v)_v$. Note that this expression is not unique for an element of $\widetilde{\mathrm{SL}_2(\mathbb{A})}$. If $x = (x_v)_v \in \mathbb{A}$ is an adele, we define $u^{\#}(x)$ and $u^{\$}(x)$ by $u^{\#}(x) = (u^{\#}(x_v))_v$ and $u^{\$}(x) = (u^{\$}(x_v))_v$, respectively. Here, if we write $[\gamma]$ for $[\gamma, 1] \in \widetilde{\mathrm{SL}_2(F_v)}$,

$$u^{\#}(y) = \left[\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}\right], \quad u^{\$}(y) = \left[\begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix}\right] \qquad (y \in F_v).$$

Similarly, if $a = (a_v)_v \in \mathbb{A}^\times$ is an idele, then we put $m(a) = (m(a_v))_v$, where

$$m(y) = \left[\begin{pmatrix} y & 0 \\ 0 & y^{-1} \end{pmatrix}\right] \qquad (y \in F_v^\times).$$

Recall that a function f on $\widetilde{\mathrm{SL}_2(\mathbb{A})}$ is a genuine function if $f(g[1_2, \zeta]) = \zeta f(g)$ for all $g \in \widetilde{\mathrm{SL}_2(\mathbb{A})}$ and $\zeta \in \{\pm 1\}$. Suppose that a family of genuine functions f_v is given for each place v of F. We assume that there exists a set of bad primes S_0 such that $f_v(g_v) = 1$ for $g_v \in s_v(\mathrm{SL}_2(\mathfrak{o}_v))$, $v \notin S_0$. Then one can define a genuine function $\prod_v f_v$ by

$$\left(\prod_v f_v\right)((g_v)_v) = \prod_v f_v(g_v).$$

Let $\mathrm{SL}_2(\mathbb{A}_f)$ be the finite part of $\mathrm{SL}_2(\mathbb{A})$ and Γ'_f a compact open subgroup of $\mathrm{SL}_2(\mathbb{A}_f)$. The inverse image of Γ'_f in $\widetilde{\mathrm{SL}_2(\mathbb{A})}$ is denoted by $\widetilde{\Gamma'}_f$. A character $\epsilon' : \widetilde{\Gamma'}_f \to \mathbb{C}^\times$ is called a genuine character if $\epsilon'([1_2, -1]) = -1$.

Let $\{\infty_1, \cdots, \infty_n\}$ be the set of infinite places of F. The embedding $F \to \mathbb{R}$ corresponding to ∞_i is denoted by ι_i. Put $\Gamma' = \mathrm{SL}_2(F) \cap (\Gamma'_f \times \mathrm{SL}_2(\mathbb{R})^n)$. As usual, we embed $\mathrm{SL}_2(F)$ into $\mathrm{SL}_2(\mathbb{R})^n$ by $\gamma \mapsto (\iota_1(\gamma), \cdots, \iota_n(\gamma))$. Suppose that $\kappa = (\kappa_1, \cdots, \kappa_n) \in \mathbb{Z}^n$ with $\kappa_1, \cdots, \kappa_n \geq 0$. We define a factor of automorphy $J^{\epsilon', \kappa+(1/2)}(\gamma, z)$ for $\gamma \in \Gamma'$ and $z = (z_1, \cdots, z_n) \in \mathfrak{h}^n$ by

$$J^{\epsilon', \kappa+(1/2)}(\gamma, z) = \prod_{v < \infty} \epsilon'_v([\gamma, 1]) \prod_{i=1}^n \tilde{j}([\iota_i(\gamma), 1], z_i)^{2\kappa_i + 1}.$$

Let $M_{\kappa+(1/2)}(\Gamma', \epsilon')$ (respectively $S_{\kappa+(1/2)}(\Gamma', \epsilon')$) be the space of Hilbert modular forms (respectively Hilbert cusp forms) on $\mathfrak{h}^n$ with respect to the automorphy factor $J^{\epsilon', \kappa+(1/2)}(\gamma, z)$.

Thus each element $h(z) \in M_{\kappa+(1/2)}(\Gamma', \epsilon')$ satisfies

$$h(\gamma(z)) = J^{\epsilon', \kappa+(1/2)}(\gamma, z) h(z)$$

for all $\gamma \in \Gamma'$ and $z \in \mathfrak{h}^n$.

The element $h(z) \in M_{\kappa+(1/2)}(\Gamma', \epsilon')$ can be considered as an automorphic form on $\mathrm{SL}_2(F)\backslash\widetilde{\mathrm{SL}_2(\mathbb{A})}$ as follows. For each $g \in \widetilde{\mathrm{SL}_2(\mathbb{A})}$, there exist $\gamma \in \mathrm{SL}_2(F)$, $\tilde{g}_\infty \in \widetilde{\mathrm{SL}_2(\mathbb{R})}^n$ and $\tilde{g}_f \in \tilde{\Gamma}'_f$ such that $g = \gamma\tilde{g}_\infty\tilde{g}_f$ by the strong approximation theorem for $\mathrm{SL}_2(\mathbb{A})$. Then we set

$$\phi_h(g) = h(\tilde{g}_\infty(\mathbf{i})\epsilon'(\tilde{g}_f)^{-1}\prod_{i=1}^{n}(\tilde{j}(\tilde{g}_{\infty_i}, \mathbf{i})^{2\kappa_i+1})^{-1}.$$

Here, $\mathbf{i} = (\sqrt{-1}, \cdots, \sqrt{-1}) \in \mathfrak{h}^n$. Then ϕ_h can be considered as a genuine automorphic form on $\mathrm{SL}_2(F)\backslash\widetilde{\mathrm{SL}_2(\mathbb{A})}$. We set

$$A_{\kappa+(1/2)}(\mathrm{SL}_2(F)\backslash\widetilde{\mathrm{SL}_2(\mathbb{A})}; \tilde{\Gamma}'_f, \epsilon') = \{\phi_h \mid h(z) \in M_{\kappa+(1/2)}(\Gamma', \epsilon')\},$$

$$A^{\mathrm{cusp}}_{\kappa+(1/2)}(\mathrm{SL}_2(F)\backslash\widetilde{\mathrm{SL}_2(\mathbb{A})}; \tilde{\Gamma}'_f, \epsilon') = \{\phi_h \mid h(z) \in S_{\kappa+(1/2)}(\Gamma', \epsilon')\}.$$

For each $\phi_h \in A_{\kappa+(1/2)}(\mathrm{SL}_2(F)\backslash\widetilde{\mathrm{SL}_2(\mathbb{A})}; \tilde{\Gamma}'_f, \epsilon')$, $h \in M_{\kappa+(1/2)}(\Gamma)$ is recovered as follows. For $z = (z_1, \cdots, z_n) \in \mathfrak{h}^n$, there exists $g_\infty = (g_{\infty_1} \cdots, g_{\infty_n}) \in \widetilde{\mathrm{SL}_2(\mathbb{R})}^n$ such that $z = g_\infty(\mathbf{i})$. Then we have

$$h(z) = \phi_h(g_\infty)\prod_{\imath=1}^{n}\tilde{j}(g_{\infty_i}), \mathbf{i})^{2\kappa+1}.$$

We set

$$A_{\kappa+(1/2)}(\mathrm{SL}_2(F)\backslash\widetilde{\mathrm{SL}_2(\mathbb{A})}) = \cup_{(\tilde{\Gamma}'_f, \epsilon')}A_{\kappa+(1/2)}(\mathrm{SL}_2(F)\backslash\widetilde{\mathrm{SL}_2(\mathbb{A})}; \tilde{\Gamma}'_f, \epsilon'),$$

$$A^{\mathrm{cusp}}_{\kappa+(1/2)}(\mathrm{SL}_2(F)\backslash\widetilde{\mathrm{SL}_2(\mathbb{A})}) = \cup_{(\tilde{\Gamma}'_f, \epsilon')}A^{\mathrm{cusp}}_{\kappa+(1/2)}(\mathrm{SL}_2(F)\backslash\widetilde{\mathrm{SL}_2(\mathbb{A})}; \tilde{\Gamma}'_f, \epsilon'),$$

where $(\tilde{\Gamma}'_f, \epsilon')$ extends over all pairs compact open subgroups $\tilde{\Gamma}'_f \subset \mathrm{SL}_2(\mathbb{A}_f)$ and genuine characters $\epsilon' : \tilde{\Gamma}'_f \to \mathbb{C}^\times$.

Then by right translation ρ $\widetilde{\mathrm{SL}_2(\mathbb{A}_f)}$ acts on $A_{\kappa+(1/2)}(\mathrm{SL}_2(F)\backslash\widetilde{\mathrm{SL}_2(\mathbb{A})})$ and $A^{\mathrm{cusp}}_{\kappa+(1/2)}(\mathrm{SL}_2(F)\backslash\widetilde{\mathrm{SL}_2(\mathbb{A})})$. The action of $\widetilde{\mathrm{SL}_2(\mathbb{A}_f)}$ on $\cup_{(\Gamma', \epsilon')}M_{\kappa+(1/2)}(\Gamma', \epsilon')$ is also denoted by ρ. Note that the right translation ρ induces the left action of the Hecke algebra $\tilde{H}\widetilde{\mathrm{SL}_2(\mathbb{A}_f)}$ on $A_{\kappa+(1/2)}(\mathrm{SL}_2(F)\backslash\widetilde{\mathrm{SL}_2(\mathbb{A})})$ by

$$\rho(\tilde{\phi})\varphi(g) = \int_{\widetilde{\mathrm{SL}_2(\mathbb{A}_f)}/\{\pm 1\}}\tilde{\phi}(g_1)\varphi(gg_1)dg_1 \qquad (\tilde{\phi} \in \tilde{H}(\widetilde{\mathrm{SL}_2(\mathbb{A}_f)})).$$

Assume that

$$h(z) = \sum_{\xi \in F} c(\xi)\mathbf{e}(\xi z) \in M_{\kappa+(1/2)}(\Gamma', \epsilon').$$

Then one can easily show that

$$\rho(u^{\#}(x))h(z) = \sum_{\xi \in F}\psi_{1,v}(\xi x)c(\xi)\mathbf{e}(\xi z) \qquad (x \in F_v)$$

if v is a non-Archimedean place of F. Similarly, suppose that $a \in F^\times$ is a totally positive element. Denote by a_f the finite part of the principal idele $a \in F^\times$. Then we have

$$\rho(m(a_f))h(z) = a^{-\kappa-(1/2)}h(a^{-2}z),$$

where $a^{-\kappa-(1/2)} = \prod_{i=1}^n \iota_i(a)^{-\kappa_i-(1/2)}$.

For irreducible cuspidal automorphic representation ρ of $\widetilde{\mathrm{SL}_2(\mathbb{A})}$, we denote by $\rho[\kappa+(1/2)]$ the space of vectors of ρ which has weight $\kappa_i+(1/2)$ at the real place ∞_i. Then we have

$$A^{\mathrm{cusp}}_{\kappa+(1/2)}(\mathrm{SL}_2(F)\backslash\widetilde{\mathrm{SL}_2(\mathbb{A})}) = \oplus_\rho\ \rho[\kappa+(1/2)].$$

Here, ρ extends over all irreducible cuspidal representations such that its ∞_i-component is a lowest weight representation with lowest weight $\kappa_i+(1/2)$.

For each pair of fractional ideals $\mathfrak{a}$ and $\mathfrak{b}$ such that $\mathfrak{ab} \subset \mathfrak{o}_F$, we define a congruence subgroup $\Gamma[\mathfrak{a},\mathfrak{b}] \subset \mathrm{SL}_2(F)$ by

$$\Gamma[\mathfrak{a},\mathfrak{b}] = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(F) \;\middle|\; a,d \in \mathfrak{o}_F,\ b \in \mathfrak{a},\ c \in \mathfrak{b} \right\}.$$

Similarly, if v is a non-Archimedean place, we define a compact open subgroup $\Gamma_v[\mathfrak{a}_v,\mathfrak{b}_v] \subset \mathrm{SL}_2(F_v)$ by

$$\Gamma_v[\mathfrak{a}_v,\mathfrak{b}_v] = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(F_v) \;\middle|\; a,d \in \mathfrak{o}_v,\ b \in \mathfrak{a}_v,\ c \in \mathfrak{b}_v \right\}.$$

Put $\Gamma = \Gamma[\mathfrak{d}_F^{-1}, 4\mathfrak{d}_F]$ and $\Gamma_v = \Gamma_v[\mathfrak{d}_v^{-1}, 4\mathfrak{d}_v]$, where $\mathfrak{d}_F$ is the different of $F/\mathbb{Q}$.

Suppose that $\kappa = (\kappa_1,\cdots,\kappa_n) \in \mathbb{Z}^n$, $\kappa_1,\cdots,\kappa_n \geq 0$. Let $\eta \in \mathfrak{o}^\times$ be a unit such that $N_{F/\mathbb{Q}}(\eta) = \prod_{i=1}^n(-1)^{\kappa_i}$. We fix such a unit η once and for all. Put $\psi(x) = \psi_1(\eta x)$. In this setting, we have $\mathfrak{c}_{\psi_v} = \mathfrak{d}_v = \mathfrak{d}\mathfrak{o}_v$ for every non-Archimedean place v. There exists a genuine character $\epsilon_v : \Gamma_v \to \mathbb{C}^\times$ such that $\omega_{\psi_v}(g_v)\phi_{0,v} = \epsilon_v(g_v^{-1})\phi_{0,v}$ for each non-Archimedean place v by the following lemma [15].

Lemma 1.1 [15]. Suppose that F is non-Archimedean. Let $\mathfrak{o}$ be the ring of integers of F, ψ a nontrivial additive character of F and c_ψ the order of ψ. Put $\mathfrak{c} = \mathfrak{c}_\psi = \mathfrak{p}^{c_\psi}$ and $\Gamma = \Gamma[\mathfrak{c}^{-1}, 4\mathfrak{c}]$. Let $\phi_0 \in S(F)$ be the characteristic function of $\mathfrak{o}$. There exists a genuine character $\epsilon : \tilde{\Gamma} \to \mathbb{C}^\times$ such that

$$\omega_\psi(g)\phi_0 = \epsilon^{-1}(g)\phi_0 \qquad \text{for all } g \in \tilde{\Gamma}.$$

Proof. One shall show that

$$\omega_\psi(u^\#(b)m(a))\phi_0 = \frac{\alpha_\psi(1)}{\alpha_\psi(a)}\phi_0 \quad (a \in \mathfrak{o}^\times,\ b \in \mathfrak{c}^{-1}),$$

$$\omega_\psi(u^\$(c))\phi_0 = \phi_0 \quad (c \in 4\mathfrak{c}).$$

Here, $\alpha_\psi(\cdot)$ is the Weil constant. The first equation is easy. The second equation follows from the fact that

$$\omega_\psi\left(\left[\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\right]\right)\phi_0(t) = \overline{\alpha_\psi(1)}|2|^{1/2}\phi_0(2t)$$

is invariant under $\{u^\$(c) \mid c \in 4\mathfrak{c}\}$. Note that $\phi_0(2t)$ is the characteristic function of $2^{-1}\mathfrak{o}$. Since $\tilde{\Gamma}$ is generated by these elements (modulo the center), the lemma follows. □

Here, $\phi_{0,v} \in S(F_v)$ is the characteristic function of $\mathfrak{o}_v$. We define a factor of automorphy $j^{\kappa+(1/2),\eta}(\gamma, z)$ for $\gamma \in \Gamma$ and $z = (z_1, \cdots, z_n) \in \mathfrak{h}^n$ by

$$j^{\kappa+(1/2),\eta}(\gamma, z) = \prod_{v<\infty} \epsilon_v([\gamma, 1]) \prod_{i=1}^{n} \tilde{j}([\iota_i(\gamma), 1], z_i)^{2\kappa_i+1}.$$

We simply write $j^{\kappa+(1/2)}(\gamma, z)$ for $j^{\kappa+(1/2),\eta}(\gamma, z)$ when there is no fear of confusion.

Let Γ, κ and η be as above. We denote by $M_{\kappa+(1/2)}(\Gamma)$ the space of Hilbert modular forms for Γ with respect to the factor of automorphy $j^{\kappa+(1/2)}(\gamma, z)$. We also denote by $S_{\kappa+(1/2)}(\Gamma)$ the subspace of $M_{\kappa+(1/2)}(\Gamma)$ which consists of all cusp forms.

Note that when $\kappa_1 = \cdots = \kappa_n = 0$ and $\eta = 1$, the automorphy factor $j^{1/2}(\gamma, z)$ satisfies the formula

$$\theta_0(\gamma(z)) = j^{1/2}(\gamma, z)\theta_0(z), \tag{8}$$

where $\theta_0(z)$ is the basic theta function given by

$$\theta_0(z) = \sum_{\xi \in \mathfrak{o}_F} \mathbf{e}(\xi^2 z).$$

When $F = \mathbb{Q}$, the definition of $j^{\kappa+(1/2)}(\gamma, z)$ agrees with classical definition. For a proof of the formula (8), one can consult Shimura [32], although the normalization of theta function in [15] is different from that given in [32]. In particular, $\theta_0(z)$ is $\theta(2z, 0, l_0)$ in Shimura's notation.

2.7. **An analog of the Dedekind eta function in Hilbert modular case.** In 1983, Feng constructed an analog of the Dedekind eta function $\eta(z)$ in the Hilbert modular case. She constructed Hilbert modular forms of weight $1/2$ with respect to the full Hilbert modular group $\mathrm{SL}_2(\mathfrak{o})$, where $\mathfrak{o}$ is the ring of integers of a totally real number field. The contents of this subsection are mainly taken from Feng [6] and Shimura [31].

Put $e(z) = e^{2\pi i z}$ for $z \in \mathbb{C}$. Let $\mathfrak{h}$ be the complex upper half plane. The Dedekind eta function $\eta(z)$ defined by

$$\eta(z) = e(1/24) \prod_{n=1}^{\infty} (1 - e(nz)) \qquad (z \in \mathfrak{h})$$

is a modular form of weight 1/2. This is a theta function;

$$\eta(z) = \sum_{m=1}^{\infty} \chi(m) e(m^2 z/24),$$

where $\chi(m) = \left(\frac{3}{m}\right)$ is the Dirichlet character modulo 12.

Let F be a totally real number field of degree n and $\mathfrak{o}$ the ring of integers of F. Assume that $n > 1$. There is an embedding $a \mapsto (a^{(1)}, \cdots, a^{(n)})$ from F into $\mathbb{R}^n$, where $a \mapsto a^{(k)} \in \mathbb{R}$ for $k = 1, 2, \cdots, n$ are all the isomorphisms of F into $\mathbb{R}$. It induces the embedding $\alpha \mapsto (\alpha^{(1)}, \cdots, \alpha^{(n)})$ from $\mathrm{SL}_2(\mathfrak{o})$ into $\mathrm{SL}_2(\mathbb{R})^n$, where $\alpha \mapsto \alpha^{(k)}$ for $1 \leq k \leq n$ are the entrywise embeddings from $\mathrm{SL}_2(\mathfrak{o})$ into $\mathrm{SL}_2(\mathbb{R})$. An element a of F is said to be totally positive if $a^{(k)} > 0$ for all k. In this case, we write $a \gg 0$.

The group $\mathrm{SL}_2(\mathfrak{o})$ acts on $\mathfrak{h}^n$ by the rule

$$\alpha(z_1, \cdots, z_n) = (\alpha^{(1)}(z_1), \cdots, \alpha^{(n)}(z_n)),$$

where

$$\alpha^{(k)}(z_k) = \frac{a^{(k)} z_k + b^{(k)}}{c^{(k)} z_k + d^{(k)}} \quad \text{for } \alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathfrak{o}), \quad (z_1, \cdots, z_n) \in \mathfrak{h}^n.$$

By the embedding above, we may consider every $a \in F$ as an element of $\mathbb{R}^n$. For $z = (z_1, \cdots, z_n) \in \mathbb{C}^n$, set $e_F(az) = e(a^{(1)} z_1 + \cdots + a^{(n)} z_n)$. If $z \in \mathfrak{h}^n$, put

$$\theta(z) = \sum_{u \in \mathfrak{o}} e_F(u^2 z/2)$$

and

$$(\mathbf{cz} + \mathbf{d})^{1/2} = \prod_{k=1}^{n} (c^{(k)} z_k + d^{(k)})^{1/2} \quad \text{for } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathfrak{o}).$$

From the result of Shimura [31], we know that

$$\theta(\gamma z) = j(\gamma, z)\theta(z) \quad \text{for every } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0, \quad z \in \mathfrak{h}^n,$$

where

$$\Gamma_0 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathfrak{o}) \mid b \in 2\mathfrak{d}^{-1}, c \in 2\mathfrak{d} \right\}$$

and $j(\gamma, z) = \epsilon(\gamma)(\mathbf{cz} + \mathbf{d})^{1/2}$ with a root of unity $\epsilon(\gamma)$.

Let 1_2 be the identity matrix of $\mathrm{SL}_2(\mathfrak{o})$. For an integral ideal $\mathfrak{a}$ of F, put

$$\Gamma_{\mathfrak{a}} = \{\gamma \in \mathrm{SL}_2(\mathfrak{o}) \mid \gamma \equiv 1_2 \bmod \mathfrak{a}\}.$$

Let $\mathfrak{c}$ be an integral ideal of F, $\mathfrak{d}$ the different of $F/\mathbb{Q}$ and ω a primitive ideal character of F with conductor $\mathfrak{c}P$, where P is the product of some archimedean primes of F. Suppose that for $b \in \mathfrak{o}$,

$$\omega(b\mathfrak{o}) = \mathrm{sgn}(b)^r := \prod_{k=1}^{n} \mathrm{sgn}(b^{(k)})^{r_k} \quad \text{if } b \equiv 1 \bmod \mathfrak{c},$$

where $r = (r_1, \cdots, r_n) \in \{0,1\}^n$. For $b \in F^\times$, set

$$\omega_0(b) = \begin{cases} \omega(b\mathfrak{o})\mathrm{sgn}(b)^r & \text{if } (b, \mathfrak{c}) = 1 \\ 0 & \text{otherwise.} \end{cases}$$

This is a primitive character of $\mathfrak{o}/\mathfrak{c}$. Put $|r| = r_1 + \cdots + r_n$.

We give a precise definition of a Hilbert modular form. Let Γ be a congruence subgroup of Γ_0. A holomorphic function g on $\mathfrak{h}^n$ is said to be a Hilbert modular form of weight 1/2 with respect to Γ if we have

$$g(\gamma, z) = j(\gamma, z)g(z) \qquad \text{for every } \gamma \in \Gamma.$$

We prepare the following lemmas to introduce Theorem 1 [6] and its proof.

Lemma 4 [6]. Let ρ be an element of $F^\times$ such that $\rho\mathfrak{c}\mathfrak{d} + \mathfrak{c} = \mathfrak{o}$. The Gauss sum of ω is defined by

$$\begin{aligned} \tau(\omega) &= \sum_{a \in \mathfrak{o}/\mathfrak{c}} \mathrm{sgn}(\rho a)^r \omega(\rho a \mathfrak{c}\mathfrak{d}) e_F(\rho a) \\ &= \mathrm{sgn}(\rho)^r \omega(\rho\mathfrak{c}\mathfrak{d}) \sum_{a \in \mathfrak{o}/\mathfrak{c}} \omega_0(a) e_F(\rho a). \end{aligned}$$

Here, for $u \in \mathfrak{o}$, put $e_F(u) = \exp(2\pi i \mathrm{Tr}(u))$.

If $\omega^2 = 1$, we have $\tau(\omega) = i^{|r|} N(\mathfrak{c})^{1/2}$.

Lemma 5 [6]. For $u \in F$ and an ideal $\mathfrak{a}$ of $\mathfrak{o}$, we have

$$\sum_{a \in \mathfrak{a}} e_F[(a+u)^2 z/2] = \mu(\mathfrak{a})^{-1} N(-iz)^{-1/2} \sum_{b \in \tilde{\mathfrak{a}}} e_F(-b^2/(2z)) e_F(bu),$$

where $z \in \mathfrak{h}^n$, $N(-iz) = \prod_{k=1}^n (-iz_k)$, $\tilde{\mathfrak{a}} = \mathfrak{d}^{-1}\mathfrak{a}^{-1}$ and $\mu(\mathfrak{a})$ is the volume $\mathrm{Vol}(\mathbb{R}^n/\mathfrak{a})$ of $\mathbb{R}^n/\mathfrak{a}$.

Theorem 1 [6]. Let ω, ω_0, and $\mathfrak{c}$ be the same as above. Suppose that

(1) $\omega^2 = 1$;
(2) $\mathfrak{c}\mathfrak{d}$ is a principal ideal generated by a totally positive number δ;
(3) $\mathfrak{c}$ is divisible by every prime factor of 2;
(4) we have $u^2 \equiv 1 \bmod 2\mathfrak{c}$ for every u prime to $\mathfrak{c}$.

Then

$$f(z, \omega) = \sum_{u \in \mathfrak{o}} \omega_0(u) e_F\left(\frac{u^2 z}{2\delta}\right)$$

is a Hilbert modular form of weight 1/2 with respect to a congruence subgroup $\Gamma_{\mathfrak{a}} \subset \Gamma_0$ for some $\mathfrak{a}$. Moreover, for every $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathfrak{o})$ we have

$$f(\gamma(z), \omega) = j^*(\gamma, z) f(z, \omega),$$

where $j^*(\gamma,z)=\epsilon^*(\gamma)(\mathbf{cz}+\mathbf{d})^{1/2}$ with a root of unity $\epsilon^*(\gamma)$. In fact, we have

$$j^*(\gamma,z)=\begin{cases}(-i)^{|r|}N(-iz)^{1/2} & \text{if } \gamma=\begin{pmatrix}0&1\\-1&0\end{pmatrix}\\ e_F(b\delta^{-1}/2) & \text{if } \gamma=\begin{pmatrix}1&b\\0&1\end{pmatrix}\ (b\in\mathfrak{o})\\ j(\gamma,z) & \text{if } \gamma\in\Gamma_{\mathfrak{a}},\end{cases}$$

where $N(-iz)=\prod_{k=1}^{n}(-iz_k)$ for $z=(z_1,\cdots,z_n)\in\mathfrak{h}^n$.

Proof. Put

$$h(z,\omega)=\sum_{u\in\mathfrak{o}}\omega_0(u)e_F(u^2z/2).$$

By Lemma 5 [6], we have

$$\begin{aligned}h(z,\omega)&=\sum_{u\in\mathfrak{o}/\mathfrak{c}}\omega_0(u)\sum_{x-u\in\mathfrak{c}}e_F(x^2z/2)\\&=\mu(\mathfrak{c})^{-1}N(-iz)^{-1/2}\sum_{b\in\bar{\mathfrak{c}}}e_F(-b^2/(2z))\sum_{u\in\mathfrak{o}/\mathfrak{c}}\omega_0(u)e_F(ub)\\&=\mu(\mathfrak{c})^{-1}N(-iz)^{-1/2}\tau(\omega)\sum_{b\in(\delta^{-1})}\operatorname{sgn}(b)^r\omega(b\delta\mathfrak{o})e_F(-b^2/(2z)).\end{aligned}$$

Set $b\delta=v$ with δ of assumption (2); then

$$\begin{aligned}\sum_{b\in(\delta^{-1})}\operatorname{sgn}(b)^r\omega(b\delta\mathfrak{o})e_F(-b^2/(2z))&=\sum_{v\in\mathfrak{o}}\operatorname{sgn}(v)^r\omega(v\mathfrak{o})e_F(-v^2/(2z\delta^2))\\&=h(-1/(\delta^2z),\omega).\end{aligned}$$

From Lemma 4 [6], we get

$$h(z,\omega)=i^{|r|}N(\delta)^{-1/2}N(-iz)^{-1/2}h(-1/(\delta^2z),\omega).$$

Since $f(z,\omega)=h(z/\delta,\omega)$, we have

$$f(-1/z,\omega)=(-i)^{|r|}N(-iz)^{1/2}f(z,\omega).$$

By assumption (3), we have

$$\operatorname{Tr}\left(\frac{u^2-1}{2\delta}\right)\equiv 0 \mod \mathbb{Z}$$

and hence $f(z+b,\omega)=e_F(b/(2\delta))f(z,\omega)$. Note that $e_F(b/(2\delta))$ is a root of unity.

Now we write

$$f(z,\omega)=\sum_{u_0\in\mathfrak{o}/\mathfrak{c}}\omega_0(u_0)\sum_{u-u_0\in\mathfrak{c}}e_F(u^2z/(2\delta)).$$

We observe that

$$\sum_{u-u_0\in\mathfrak{c}}e_F(u^2z/(2\delta)) \tag{9}$$

is a special case of function of [31, Proposition 7.1] with $S = P = 1$, $\chi = 1$, $L = c$, $g = u_0$, $h = 0$, $a = \delta^{-1}$ and $a \gg 0$. So from [31, Proposition 7.1], we know that (9) is a Hilbert modular form of weight $1/2$ with respect to a congruence subgroup $\Gamma_{\mathfrak{a}}$ of Γ_0 for some $\mathfrak{a} \subset 2\mathfrak{d}$ and

$$\sum_{u-u_0 \in \mathfrak{c}} e_F(u^2\gamma(z)/(2\delta)) = j(\gamma, z) \sum_{u-u_0 \in \mathfrak{c}} e_F(u^2 z/(2\delta))$$

for every $\gamma \in \Gamma_{\mathfrak{a}}$. Hence

$$f(\gamma(z), \omega) = j(\gamma, z) f(z, \omega) \qquad \text{for every } \gamma \in \Gamma_{\mathfrak{a}}.$$

From assumption (4) of the theorem, it is easy to see the only odd divisor of $N(\mathfrak{c})$ is 3. □

Feng applied this to the real quadratic field F. If $a \in F$, denote by $\bar{a}$ the conjugate of a and put $N(a) = a\bar{a}$.

Lemma 6 [6]. Let u be an algebraic integer of $\mathbb{Q}(\sqrt{d})$ with a square-free positive integer d.

(1) If $d \equiv 1 \bmod 8$ and $(u, 2) = 1$, then $u^2 \equiv 1 \bmod 8$.
(2) If $d \equiv 1 \bmod 24$ and $(u, 6) = 1$, then $u^2 \equiv 1 \bmod 24$.

Theorem 2 [6]. Put $F = \mathbb{Q}(\sqrt{d})$ with a squarefree integer $d > 1$. Suppose that F has a unit $\lambda > 0$ such that $N(\lambda) = -1$. For $k = 1, 2$, set

$$f_k(z) = \sum_{u \in \mathfrak{o}} \omega_{k0}(u) e_F(u^2 z/(2c_k\lambda\sqrt{d})),$$

where $c_1 = 4, c_2 = 12$, and

$$\omega_{10}(u) = \begin{cases} (-1)^{[N(u)-1]/2} & \text{for } (u, 2) = 1 \\ 0 & \text{otherwise}, \end{cases}$$

$$\omega_{20}(u) = \begin{cases} \left(\frac{3}{N(u)}\right) & \text{for } (u, 6) = 1 \\ 0 & \text{otherwise}. \end{cases}$$

If $d \equiv 1 \bmod 8$, f_1 is a Hilbert modular form of the type described in Theorem 1 [6]. If further $d \equiv 1 \bmod 24$, the same is true for f_2.

Proof. The different of $\mathbb{Q}(\sqrt{d})/\mathbb{Q}$ is $(\sqrt{d})$. Then the theorem follows from Lemma 6 [6] and Theorem 1 [6]. □

3. Local theory, genuine characters of $\widetilde{\mathrm{SL}_2(\mathfrak{o})}$

Let F be a finite extension of $\mathbb{Q}_p$ until the end of this section. Let $\mathfrak{o}$ be the ring of integers of F and $\mathfrak{p}$ the maximal ideal of $\mathfrak{o}$. Let q be the order of the residue field $\mathfrak{o}/\mathfrak{p}$ and $\mathfrak{d}$ the different of $F/\mathbb{Q}_p$.

For $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(F)$, put $x(g) = c$ if $c \neq 0$ and $x(g) = d$ if $c = 0$. The Kubota 2-cocycle on $\mathrm{SL}_2(F)$ is defined by

$$c(g,h) = \langle x(g)x(gh), x(h)x(gh)\rangle_F$$

for $g, h \in \mathrm{SL}_2(F)$, where $\langle \cdot, \cdot \rangle_F$ is the quadratic Hilbert symbol for F. Let $\widetilde{\mathrm{SL}_2(F)}$ be the metaplectic group of $\mathrm{SL}_2(F)$. Set-theoretically, it is

$$\{[g,\tau] \mid g \in \mathrm{SL}_2(F), \tau \in \{\pm 1\}\}.$$

Its multiplication law is given by $[g,\tau][h,\sigma] = [gh, \tau\sigma c(g,h)]$. This is a nontrivial double covering group of $\mathrm{SL}_2(F)$. Put $[g] = [g,1]$. For a subgroup H of $\mathrm{SL}_2(F)$, the inverse image of H in $\widetilde{\mathrm{SL}_2(F)}$ is denoted by $\tilde{H}$. A function $\epsilon_F : \widetilde{\mathrm{SL}_2(\mathfrak{o})} \to \mathbb{C}$ is genuine if $\epsilon_F([1_2, -1]\gamma) = -\epsilon_F(\gamma)$ for all $\gamma \in \widetilde{\mathrm{SL}_2(\mathfrak{o})}$. We determine the number of genuine characters of $\widetilde{\mathrm{SL}_2(\mathfrak{o})}$.

For $a \in F^\times, \tau = \pm 1$ and $b, c \in F$, put

$$m(a,\tau) = \left[\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, \tau\right], \qquad u^+(b) = \left[\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\right],$$
$$u^-(c) = \left[\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}\right], \qquad N = \left[\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right].$$

For $k \in \mathbb{Z}$ such that $k \geq 0$, we define the subgroups $U^+(\mathfrak{p}^k)$, $U^-(\mathfrak{p}^k)$ and $\tilde{A}$ of $\widetilde{\mathrm{SL}_2(\mathfrak{o})}$ by $U^+(\mathfrak{p}^k) = \{u^+(b) \mid b \in \mathfrak{p}^k\}$, $U^-(\mathfrak{p}^k) = \{u^-(c) \mid c \in \mathfrak{p}^k\}$ and $\tilde{A} = \{m(a,\tau) \mid a \in \mathfrak{o}^\times\}$, respectively. Note that $\widetilde{\mathrm{SL}_2(\mathfrak{o})}$ is generated by $U^+(\mathfrak{o})$, N and $m(1,-1)$.

Lemma 1. Put $M = \min\{\mathrm{ord}(a^2-1) \mid a \in \mathfrak{o}^\times\}$. Then we have

$$M = \begin{cases} 0 & (q \geq 4) \\ 1 & (q = 3) \\ 2 & (q = 2, F \neq \mathbb{Q}_2) \\ 3 & (F = \mathbb{Q}_2). \end{cases}$$

Proof. Let π be a prime element of F. If $q \geq 4$, then there exists $a \in \mathfrak{o}^\times$ such that $a^2 - 1 \in \mathfrak{o}^\times$. Thus we have $M = 0$. If $q = 3$, then $a^2 - 1 \in \mathfrak{p}$ for all $a \in \mathfrak{o}^\times$. Since $(\pi+1)^2 - 1 = \pi(\pi+2) \notin \mathfrak{p}^2$, we have $M = 1$.

In the case $q = 2$, $a^2 - 1 = (a-1)(a+1) \in \mathfrak{p}^2$ for all $a \in \mathfrak{o}^\times$. If $F \neq \mathbb{Q}_2$, then we have $2 \in \mathfrak{p}^2$. Since $(\pi+1)^2 - 1 = \pi(\pi+2) \notin \mathfrak{p}^3$, we have $M = 2$. It is well-known that $M = 3$ if $F = \mathbb{Q}_2$. $\square$

The derived group of a group G is denoted by $D(G)$. Since $[m(a,\tau), u^+(b)] = u^+((a^2-1)b)$ and $[m(a,\tau), u^-(c)] = u^-((a^{-2}-1)c)$ hold, we have

$$U^+(\mathfrak{p}^M), U^-(\mathfrak{p}^M) \subset D(\widetilde{\mathrm{SL}_2(\mathfrak{o})}) \tag{10}$$

by Lemma 1.

Lemma 2. Suppose that q is even. Then there exists $r \in 1+4\mathfrak{o}$ such that $\langle r, x\rangle_F = (-1)^{\mathrm{ord}x}$ for $x \in F^\times$.

Proof. Put $r = 1+4c$ for $c \in \mathfrak{o}$. We show that there exists c such that $F(\sqrt{r})/F$ is an unramified quadratic extension. We denote the residue field of a local field L by $k(L)$ and the image of an element u of the ring of integers of $F(\sqrt{r})$ in $k(F(\sqrt{r}))$ by $\bar{u}$.

We define a map $\mathbf{p} : k(F) \to k(F)$ by $\mathbf{p}(t) = t^2 - t$ for $t \in k(F)$. We have $\mathbf{p}(t) = \mathbf{p}(1-t) \neq \mathbf{p}(s)$ for all $s \in k(F)\backslash\{t, 1-t\}$. Since $[k(F) : \mathbf{p}(k(F))] = 2$, there exists c such that $\bar{c} \notin \mathbf{p}(k(F))$. Then it is known that a polynomial $X^2 - X - \bar{c}$ is irreducible over $k(F)$. Put $y = (1-\sqrt{r})/2$ and $f(X) = X^2 - X - c \in \mathfrak{o}[X]$. Since $f(y) = 0$ and $f'(y) = 2y - 1 = -\sqrt{r} \neq 0$, $k(F)(\bar{y})/k(F)$ is a quadratic extension and $k(F(\sqrt{r})) = k(F(y))$ equals $k(F)(\bar{y})$. Therefore $F(\sqrt{r})/F$ is an unramified quadratic extension (see [26, §32:6]). $\square$

Lemma 3. Suppose that q is even and $F \neq \mathbb{Q}_2$. Then there exist no genuine characters of $\widetilde{\mathrm{SL}_2(\mathfrak{o})}$.

Proof. Let $b, c \in \mathfrak{o}$ such that $r = 1 - bc \in \mathfrak{o}^\times$ and put $\zeta = \langle r, b\rangle_F$. We have

(11)
$$[u^-(c), u^+(b)] = \left[\begin{pmatrix} r & b^2c \\ -bc^2 & 1+bc+b^2c^2 \end{pmatrix}\right] = u^-(-r^{-1}bc^2)m(r,\zeta)u^+(r^{-1}b^2c).$$

When $F/\mathbb{Q}_2$ is a ramified extension, we choose $b, c \in 2\mathfrak{o}$ such that r satisfies the condition in Lemma 2. We have $U^+(\mathfrak{p}^2), U^-(\mathfrak{p}^2) \subset D(\widetilde{\mathrm{SL}_2(\mathfrak{o})})$ by (10). Then we have $m(r,\zeta) \in D(\widetilde{\mathrm{SL}_2(\mathfrak{o})})$ by (11). Let π be a prime element of F. Set $b' = b\pi$ and $c' = c\pi^{-1}$, which lie in $\mathfrak{p}$. We have $\langle 1 - b'c', b'\rangle_F = \langle r, b\pi\rangle_F = -\zeta$. Thus we have $m(1,-1) \in D(\widetilde{\mathrm{SL}_2(\mathfrak{o})})$ and there exist no genuine characters of $\widetilde{\mathrm{SL}_2(\mathfrak{o})}$.

Next assume that $F/\mathbb{Q}_2$ is an unramified extension and that $F \neq \mathbb{Q}_2$. We have $U^+(\mathfrak{o})$, $U^-(\mathfrak{o}) \subset D(\widetilde{\mathrm{SL}_2(\mathfrak{o})})$ by (10). Substituting 1 for c in (11), we have $m(1-b,\zeta) \in D(\widetilde{\mathrm{SL}_2(\mathfrak{o})})$, whenever $1-b \in \mathfrak{o}^\times$. Since $\zeta = \langle 1-b, b\rangle_F$ equals 1, we have $m(1-b,1) \in D(\widetilde{\mathrm{SL}_2(\mathfrak{o})})$. Similarly, substituting -1 for c and replacing b with $-b$ in Equation (11), we have $m(1-b, \langle 1-b,-1\rangle_F) \in D(\widetilde{\mathrm{SL}_2(\mathfrak{o})})$. Thus it suffices to show that there exists b such that $\langle 1-b,-1\rangle_F$ equals -1.

Since $F/\mathbb{Q}_2$ is unramified, $F(\sqrt{-1})/F$ is a ramified extension. Thus there exists $u \in \mathfrak{o}^\times$ such that $\langle u,-1\rangle_F = -1$. Since $[\mathfrak{o}^\times : 1+\mathfrak{p}] = q-1$ is odd, we may assume that $u \in 1+\mathfrak{p}$. Then there exists $b \in \mathfrak{p}$ such that $u = 1-b$ satisfies $\langle u,-1\rangle_F = -1$. $\square$

An additive character $\mathbf{e}_p$ of $\mathbb{Q}_p$ is defined by $\mathbf{e}_p(x) = e(-x)$ for all $x \in \mathbb{Z}[1/p]$. We define a nontrivial additive character ψ_β of F by $x \mapsto \mathbf{e}_p(\mathrm{Tr}_{F/\mathbb{Q}_p}(\beta x))$ for $\beta \in F^\times$. The order of ψ_β is denoted by $\mathrm{ord}\psi_\beta \in \mathbb{Z}$, which is defined by $\psi_\beta(\mathfrak{p}^{-\mathrm{ord}\psi_\beta}) = 1$ and $\psi_\beta(\mathfrak{p}^{-\mathrm{ord}\psi_\beta - 1}) \neq 1$. We have $\mathrm{ord}\psi_\beta = \mathrm{ord}\mathfrak{d} + \mathrm{ord}\beta$.

Let $S(F)$ be the Schwartz space of F. The Fourier transformation $\hat{\phi}$ of $\phi \in S(F)$ is defined by $\hat{\phi}(x) = \int_F \phi(y)\psi_\beta(xy)dy$. Here, dy is self-dual on the Fourier transformation. In other words, dy is the Haar measure such that the Plancherel's formula $\int_F |\phi(y)|^2 dy = \int_F |\hat{\phi}(y)|^2 dy$ holds, where $|\cdot|$ is the absolute value on $\mathbb{C}$.

We denote the characteristic function of a subset A of a set X by ch A. In the case $F = \mathbb{Q}_p$, we have the volume $\mathrm{vol}(p^m\mathbb{Z}_p)$ of $p^m\mathbb{Z}_p$ equals $p^{-m-(\mathrm{ord}\beta/2)}$ and $\widehat{\mathrm{ch}\, p^m\mathbb{Z}_p} = \mathrm{vol}(p^m\mathbb{Z}_p)\,\mathrm{ch}\, p^{-(m+\mathrm{ord}\beta)}\mathbb{Z}_p$ for $m \in \mathbb{Z}$.

Put $A_\phi = \int_F \phi(x)\psi_\beta(ax^2)dx$ and $B_\phi = \int_F \hat{\phi}(x)\psi_\beta(-x^2/4a)dx$ for $a \in F^\times$ and $\phi \in S(F)$. Now let $|\cdot|$ be the absolute value on F. There exists a constant $\alpha_{\psi_\beta}(a) \in \mathbb{C}$ called the Weil constant such that $A_\phi = \alpha_{\psi_\beta}(a)|2a|^{-1/2}B_\phi$ holds. It is known that $\alpha_{\psi_\beta}(ab^2) = \alpha_{\psi_\beta}(a)$ for $a, b \in F^\times$ and that $\alpha_{\psi_\beta}(a) = \alpha_\psi(a\beta)$, where $\psi = \psi_1$. Moreover, we have $\alpha_{\psi_\beta}(-a) = \overline{\alpha_{\psi_\beta}(a)}$ and $\alpha_{\psi_\beta}(a)^8 = 1$ (see [29, 35]).

The Weil representation ω_{ψ_β} is a representation of $\widetilde{\mathrm{SL}_2(F)}$ on $S(F)$. For $\phi \in S(F)$, we have

$$\begin{cases} \omega_{\psi_\beta}(m(a,\tau))\phi(x) = \tau\alpha_{\psi_\beta}(1)\alpha_{\psi_\beta}(-a)|a|^{1/2}\phi(ax) \\ \omega_{\psi_\beta}(u^+(b))\phi(x) = \psi_\beta(bx^2)\phi(x) \\ \omega_{\psi_\beta}(N)\phi(x) = |2|^{1/2}\alpha_{\psi_\beta}(-1)\hat{\phi}(-2x). \end{cases} \tag{12}$$

Since $\widetilde{\mathrm{SL}_2(F)}$ is generated by the above elements, ω_{ψ_β} is determined by these formulas. In particular, we have $\omega_{\psi_\beta}([g,\tau])\phi = \tau\omega_{\psi_\beta}([g])\phi$.

We define a map $s : \mathrm{SL}_2(\mathfrak{o}) \to \{\pm 1\}$ by

$$s(g) = \begin{cases} 1 & c \in \mathfrak{o}^\times \\ \langle c, d\rangle_F & c \in \mathfrak{p}\backslash\{0\} \\ \langle -1, d\rangle_F & c = 0 \end{cases} \quad \text{for } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathfrak{o}). \tag{13}$$

If q is odd, we have

$$s(g) = \begin{cases} 1 & cd = 0 \\ \langle c, d\rangle_F^{\mathrm{ord}\, c} & cd \neq 0. \end{cases}$$

Recall that the double covering $\widetilde{\mathrm{SL}_2(F)} \to \mathrm{SL}_2(F)$ splits on $\mathrm{SL}_2(\mathfrak{o})$ if and only if q is odd. The splitting is given by $g \mapsto [g, s(g)]$. Thus if q is odd, a map $\epsilon_F : \widetilde{\mathrm{SL}_2(\mathfrak{o})} \to \mathbb{C}^\times$ defined by $\epsilon_F([g,\tau]) = \tau\, s(g)$ is a genuine character.

Lemma 4. Suppose that q is odd. Let $\epsilon_F : \widetilde{\mathrm{SL}_2(\mathfrak{o})} \to \mathbb{C}^\times$ be the genuine character defined above. It satisfies $\omega_{\psi_\beta}([g,\tau])\mathrm{ch}\ \mathfrak{o} = \epsilon_F([g,\tau])^{-1}\mathrm{ch}\ \mathfrak{o}$ if $\mathrm{ord}\psi_\beta = 0$. If $q \geq 5$, then it is the unique genuine character of $\widetilde{\mathrm{SL}_2(\mathfrak{o})}$.

Proof. The first part of this lemma follows from straightforward computation. If $q \geq 5$, then by (10) we have $U^+(\mathfrak{o})$, $U^-(\mathfrak{o}) \subset D(\widetilde{\mathrm{SL}_2(\mathfrak{o})})$. Since $N = u^+(-1)u^-(1)u^+(-1)$, $\widetilde{\mathrm{SL}_2(\mathfrak{o})}^{ab}$ is trivial. Thus there exists a unique genuine character ϵ_F of $\widetilde{\mathrm{SL}_2(\mathfrak{o})}$. □

For $a \in \mathbb{Z}$, we define a subset $S(\mathfrak{p}^a)$ of $S(F)$ by $S(\mathfrak{p}^a) = \{f \in S(F) \mid \mathrm{Supp}\, f \subset \mathfrak{p}^a\}$. For $a \leq b \in \mathbb{Z}$, put $S(\mathfrak{p}^a/\mathfrak{p}^b) = \{f \in S(\mathfrak{p}^a) \mid f(x+t) = f(x)$ for all $t \in \mathfrak{p}^b\}$. For $f \in S(F)\backslash\{0\}$, there exists a pair (a, b) such that $f \in S(\mathfrak{p}^a/\mathfrak{p}^b)$.

Lemma 5. Suppose that $q = 3$ (resp. $F = \mathbb{Q}_2$). If $\mathrm{ord}\psi_\beta = -1$ (resp. -3), then the group $\widetilde{\mathrm{SL}_2(\mathfrak{o})}$ preserves $S(\mathfrak{o}/2\mathfrak{p})$ with respect to ω_{ψ_β}. We define $f \in S(\mathfrak{o}/2\mathfrak{p})$ by

$$f(x) = \begin{cases} 1 & \text{if } x \in 1 + 2\mathfrak{p} \\ -1 & \text{if } x \in -1 + 2\mathfrak{p} \\ 0 & \text{otherwise.} \end{cases} \tag{14}$$

Then the subspace of odd functions in $S(\mathfrak{o}/2\mathfrak{p})$ is $\mathbb{C}f$ and there exists a genuine character μ_β of $\widetilde{\mathrm{SL}_2(\mathfrak{o})}$ such that $\omega_{\psi_\beta}([g, \tau])f = \mu_\beta([g, \tau])^{-1}f$.

In the case $q = 3$, there exist three genuine characters of $\widetilde{\mathrm{SL}_2(\mathfrak{o})}$, ϵ_F and μ_β, where μ_β extends over all elements β such that $\mathrm{ord}\psi_\beta = -1$. Moreover, the value $\mu_\beta(u^+(1)) = \psi_\beta(-1)$ is a primitive 3rd root of unity, which determines μ_β.

Proof. Suppose that $F = \mathbb{Q}_2$ and that $\mathrm{ord}\psi_\beta = \mathrm{ord}\beta = -3$. It is clear that $\widetilde{\mathrm{SL}_2(\mathfrak{o})}$ preserves $S(\mathfrak{o}/2\mathfrak{p})$ with respect to ω_{ψ_β}. If $\phi \in S(\mathfrak{o}/2\mathfrak{p})$ is an odd function, then ϕ satisfies $\phi(x) = \phi(-x) = -\phi(x)$ for all $x \in \mathfrak{p}$. Thus we have $\phi(\mathfrak{p}) = 0$. Since $F = \mathbb{Q}_2$, we have $\mathfrak{o}^\times = (1 + 2\mathfrak{p}) \cup (-1 + 2\mathfrak{p})$ and then $\phi(x) = \phi(1)f \in \mathbb{C}f$.

Thus there exists a genuine character μ_β of $\widetilde{\mathrm{SL}_2(\mathfrak{o})}$ such that $\omega_{\psi_\beta}([g, \tau])f = \mu_\beta([g, \tau])^{-1}f$. Since $u^-(-1) = Nu^+(1)N^{-1}$ and $N = u^+(-1)u^-(1)u^+(-1)$, the value $\mu_\beta(u^+(1))$ determines μ_β.

Suppose that $q = 3$ and that $\mathrm{ord}\psi_\beta = -1$. Then we prove the first part of the lemma similar to the case above. By [16, §2.10], $\widetilde{\mathrm{SL}_2(\mathfrak{o})}^{ab}$ has order 3. Thus there exist three genuine characters of $\widetilde{\mathrm{SL}_2(\mathfrak{o})}$. We have $f(x) \neq 0$ if and only if $x \in \mathfrak{o}^\times$. By (12), we have $\omega_{\psi_\beta}(u^+(b))f(x) = \psi_\beta(bx^2)f(x)$ for $b \in \mathfrak{o}$. If $f(x) \neq 0$, since we have $x^2 - 1 \in \mathfrak{p}$ by Lemma 1, $\mu_\beta(u^+(b))^{-1} = \psi_\beta(bx^2) = \psi_\beta(b)$. In particular, $\mu_\beta(u^+(1)) = \psi_\beta(-1)$ is a primitive 3rd root of unity.

For $\gamma \in F$ such that $\mathrm{ord}\psi_\gamma = -1$, if $\mu_\beta = \mu_\gamma$, then we have $\psi_\beta(1) = \psi_\gamma(1)$ and then $\beta/\gamma \in 1+\mathfrak{p}$. Since we have $[\mathfrak{o}^\times : (1+\mathfrak{p})] = 2$, the genuine characters of $\widetilde{\mathrm{SL}_2(\mathfrak{o})}$ are ϵ_F and μ_β, where μ_β extends over all elements β such that $\mathrm{ord}\psi_\beta = -1$. □

Lemma 6. In Lemma 5, suppose that $F = \mathbb{Q}_2$ and that $\mathrm{ord}\psi_\beta = -3$. Then the value $\mu_\beta(u^+(1)) = e(\beta)$ is a primitive 8th root of unity, which determines μ_β. Moreover, there exist four genuine characters μ_β of $\widetilde{\mathrm{SL}_2(\mathbb{Z}_2)}$, where μ_β extends over all elements β such that $\mathrm{ord}\beta = -3$.

Proof. If $F = \mathbb{Q}_2$, we have $\mathrm{ord}\psi_\beta = \mathrm{ord}\beta$. By Lemma 5, there exists a genuine character of $\widetilde{\mathrm{SL}_2(\mathbb{Z}_2)}$. Since $\widetilde{\mathrm{SL}_2(\mathbb{Z}_2)}^{ab}$ has order 4 by [16, §2.10], the number of genuine characters of $\widetilde{\mathrm{SL}_2(\mathbb{Z}_2)}$ is 4. We have $\omega_{\psi_\beta}(u^+(b))f(x) = e(-\beta b x^2)f(t)$ for f in (14) and $b \in \mathbb{Z}_2$ by (12). If $f(x) \neq 0$, then we have $x^2 - 1 \in 8\mathbb{Z}_2$ and $e(-\beta b x^2) = e(-\beta b)$. In particular, $\mu_\beta(u^+(1)) = e(\beta)$ is a primitive 8th root of unity.

For $\gamma \in \mathbb{Q}_2$ such that $\mathrm{ord}\gamma = -3$, if $\mu_\beta = \mu_\gamma$, then we have $e(\beta) = e(\gamma)$ and then $\beta/\gamma \in 1 + 8\mathbb{Z}_2$. Since we have $[\mathbb{Z}_2^\times : (1 + 8\mathbb{Z}_2)] = 4$, there exist four genuine characters μ_β of $\widetilde{\mathrm{SL}_2(\mathbb{Z}_2)}$, where μ_β extends over all elements β such that $\mathrm{ord}\beta = -3$. $\square$

Given β, let μ_β be the nontrivial genuine character given in Lemma 5 or Lemma 6. Then we have

$$\mu_\beta([g]) = s(g)\kappa(\beta, g) \qquad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathfrak{o}),$$

where $\kappa(\beta, g)$ is a continuous function for g. In the case $F = \mathbb{Q}_2$ and $\mathrm{ord}\,\beta = -3$, we have

$$\kappa(\beta, g) = \begin{cases} \psi_\beta(-(a+d)c + 3c) & c \in \mathbb{Z}_2^\times \\ \psi_\beta((c-b)d - 3(d-1)) & c \in 2\mathbb{Z}_2. \end{cases} \tag{15}$$

In the case $q = 3$ and $\mathrm{ord}\psi_\beta = -1$, we have

$$\kappa(\beta, g) = \psi_\beta(-(a+d)c + bd(c^2 - 1)). \tag{16}$$

Remark. Suppose that K' is a compact open subgroup of $\mathrm{SL}_2(\mathfrak{o})$. Let $\lambda' : \tilde{K'} \to \mathbb{C}^\times$ be a genuine character. Then one can show that there exists a continuous function κ' on K' such that $\lambda'([g]) = s(g)\kappa'(g)$ for all $g \in K'$. As we do not need this result for the rest of this paper, we omit its proof.

Put $K = \mathrm{SL}_2(\mathfrak{o})$ and $G = \mathrm{SL}_2(F)$. It is known that K (resp. $\tilde{K}$) is a compact open subgroup of G (resp. $\tilde{G}$). Let (π, V) be an irreducible smooth representation of $\tilde{G}$. For a character λ of $\tilde{K}$, we define a set $(\pi, V)^\lambda$ by

$$(\pi, V)^\lambda = \{f \in V \mid \pi(g)f = \lambda(g)^{-1} f \text{ for all } g \in \tilde{K}\}.$$

In particular, we consider $(\omega_{\psi_\beta}, S(F))^\lambda$ for a genuine character $\lambda : \tilde{K} \to \mathbb{C}^\times$ such that $\lambda(u^+(1)) \neq 1$. Since $\lambda(u^+(1)) \neq 1$, λ is one of the characters μ_β in Lemma 5 or Lemma 6. In particular, we have $q = 3$ or $F = \mathbb{Q}_2$. When $q = 3$ (resp. $F = \mathbb{Q}_2$), we have that $\mathrm{ord}\psi_\beta = -1$ (resp. -3).

Proposition 1. The representation $c\text{-Ind}_{\tilde{K}}^{\tilde{G}}\lambda$ is irreducible supercuspidal. We have

$$\dim_{\mathbb{C}}(\pi, V)^{\lambda} = \begin{cases} 1 & \pi = c\text{-Ind}_{\tilde{K}}^{\tilde{G}}\lambda \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Put $\lambda^g(x) = \lambda(gxg^{-1})$ for $g \in \tilde{G}$. We shall prove that

$$\mathrm{Hom}_{g^{-1}\tilde{K}g\cap\tilde{K}}(\lambda^g, \lambda) = 0 \quad \text{for } g \notin \tilde{K}. \tag{17}$$

Put $m(a) = \mathrm{diag}(a, a^{-1})$ for $a \in F^{\times}$. Since it is known that

$$\mathrm{SL}_2(F) = \bigcup_{n=0}^{\infty} Km(\pi^n)K,$$

we have only to consider the case $g = m(\pi^n)$ for $n > 0$. Since $u^+(1) \in g^{-1}Kg \cap K$, we have $\lambda(u^+(1)) \neq \lambda^g(u^+(1))$, which proves (17).

It is known that (17) implies the first assertion (see [4, §11.4]). Although [4] treated the GL_2 case, the proof is also valid in our case. By [4, §2.5], we have $\mathrm{Hom}_{\tilde{G}}(c\text{-Ind}_{\tilde{K}}^{\tilde{G}}\lambda, \pi) \simeq \mathrm{Hom}_{\tilde{K}}(\lambda, \pi)$, which completes the proof. $\square$

It is known that

$$\omega_{\psi_\beta} = \omega_{\psi_\beta}^{+} \oplus \omega_{\psi_\beta}^{-},$$

where $\omega_{\psi_\beta}^{+}$ (resp. $\omega_{\psi_\beta}^{-}$) is the restriction of ω_{ψ_β} to the even (resp. odd) functions. Note that these restrictions are irreducible but not isomorphic, because $\omega_{\psi_\beta}^{+}$ is not supercuspidal. We have $\omega_{\psi_\beta}^{-} \simeq \text{c-Ind}_{\tilde{K}}^{\tilde{G}}\mu_\beta$ by Proposition 1. Since $\lambda(u^+(1)) \neq 1$, we have $\dim(\omega_{\psi_\beta}^{+}, S(F)^+)^{\lambda} = 0$ by Proposition 1, where $S(F)^+$ is the subspace of the even functions in $S(F)$.

If $q = 3$ and $\mathrm{ord}\psi_\beta = -1$ or if $F = \mathbb{Q}_2$ and $\mathrm{ord}\beta = -3$, then we have

$$\dim_{\mathbb{C}}(\omega_{\psi_\beta}, S(F))^{\lambda} = \begin{cases} 1 & \lambda = \mu_\beta \\ 0 & \text{otherwise.} \end{cases} \tag{18}$$

Now we assume that q is odd. By Lemma 4, there exists a genuine character $\epsilon_F : \widetilde{\mathrm{SL}_2(\mathfrak{o})} \to \mathbb{C}^{\times}$ and we have $\omega_{\psi_\beta}([g, \tau])\,\mathrm{ch}\ \mathfrak{o} = \epsilon_F([g, \tau])^{-1}\mathrm{ch}\ \mathfrak{o}$, where $\mathrm{ord}\psi_\beta = 0$.

Lemma 7. Put $T_\beta = (\omega_{\psi_\beta}, S(F))^{\epsilon_F}$. Then we have

$$T_\beta = \begin{cases} \mathbb{C}\ \mathrm{ch}\ \mathfrak{p}^{-\mathrm{ord}\psi_\beta/2} & \text{if ord } \psi_\beta \equiv 0 \bmod 2 \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Put $D = \mathrm{ord}\psi_\beta$. Suppose that $D = 0$. Then we have $\epsilon_F(U^+(\mathfrak{o})) = 1$ and it is clear that $\widetilde{\mathrm{SL}_2(\mathfrak{o})}$ preserves $S(\mathfrak{o}/\mathfrak{o}) = \mathbb{C}\ \mathrm{ch}\ \mathfrak{o}$ with respect to ω_{ψ_β}. Then we have $T_\beta = \mathbb{C}\ \mathrm{ch}\ \mathfrak{o} = \mathbb{C}\ \mathrm{ch}\ \mathfrak{p}^{-D/2}$. Since we have $\omega_{\psi_{\beta t^2}}(g) = \omega_{\psi_\beta}(m(t,1)g\,m(t,1)^{-1})$ for $t \in F^{\times}$ and $g \in \widetilde{\mathrm{SL}_2(\mathfrak{o})}$, the same is true for $T_{\beta t^2}$ for all $t \in F^{\times}$. Thus we have covered the case where $\mathrm{ord}\psi_\beta$ is even.

Next assume $\mathrm{ord}\psi_\beta$ is odd. By the same argument as in the previous paragraph, it is enough to consider the case $D = 1$. Then we have $\omega_{\psi_\beta}(h)\phi = \epsilon_F(h)^{-1}\phi$. By (12), we have $\psi_\beta(bx^2) = 1$ for all $b \in \mathfrak{o}$ when $\phi(x) \neq 0$. In particular, since $\mathrm{ord}bx^2 \geq \mathrm{ord}x^2 \geq -1$, we have $\phi \in S(\mathfrak{o})$.

We assume $\phi \in S(\mathfrak{p}^a/\mathfrak{p}^b)$ such that $a \geq 0$ is maximal and b is minimal. A calculation of the Fourier transformation shows that $\hat{\phi} \in S(\mathfrak{p}^{-1-b}/\mathfrak{p}^{-1-a})$. Since $\alpha_{\psi_\beta}(-1)\hat{\phi}(-2x) = \phi(x)$ by (12), we have $\hat{\phi} \in S(\mathfrak{p}^a/\mathfrak{p}^b)$. Then $a = -1-b$ is less than 0, which contradicts $a \geq 0$. $\square$

Set $F^{\times 2} = \{x^2 \mid x \in F^\times\}$. Assume that $q = 3$ or $F = \mathbb{Q}_2$. By Lemma 5 and Lemma 6, there exist genuine characters μ_β of $\widetilde{\mathrm{SL}_2(\mathfrak{o})}$.

Lemma 8. When $q = 3$ (resp. $F = \mathbb{Q}_2$), we put $T_\beta = (\omega_{\psi_\beta}, S(F))^{\mu_\gamma}$, where $\gamma \in F^\times$ such that $\mathrm{ord}\psi_\gamma = -1$ (resp. -3). Then we have

$$\dim T_\beta = \begin{cases} 1 & \text{if } \beta/\gamma \in F^{\times 2} \\ 0 & \text{otherwise.} \end{cases} \tag{19}$$

In particular, when $\beta/\gamma \in 1 + \mathfrak{p}$ (resp. $1 + 8\mathbb{Z}_2$), we have $T_\beta = \mathbb{C}f$, where f is the function in (14).

Proof. We prove the lemma in the case $q = 3$. The proof for $F = \mathbb{Q}_2$ is similar. Put $D = \mathrm{ord}\psi_\beta$. Then we may assume that $D \in \{0, -1\}$ in the same way as the proof of Lemma 7. By Proposition 1 and (18), we have $\dim T_\beta = 0$ when $D = 0$. Suppose that $D = -1$ and that $\phi \in T_\beta$ is nonzero. Then by Lemma 5, we have $\phi \in \mathbb{C}f$. Lemma 5 shows that f lies in T_β if and only if $\beta/\gamma \in 1 + \mathfrak{p}$. We have $1 + \mathfrak{p} \subset F^{\times 2}$, which completes the proof. $\square$

4. Multiplier systems for $\mathrm{SL}_2(\mathfrak{o})$

From now on, let F be a totally real number field such that $[F : \mathbb{Q}] = n$. Let v be a place of F and $\mathbb{A}$ the adele ring of F. We denote the completion of F at v by F_v. If v is an infinite place, we write $v \mid \infty$. Otherwise, we write $v < \infty$. For $v < \infty$, let $\mathfrak{o}_v$, $\mathfrak{p}_v$ and $\mathfrak{d}_v$ be the ring of integers of F_v, the maximal ideal of $\mathfrak{o}_v$ and the different of $F_v/\mathbb{Q}_p$, respectively.

For any v, let $\iota_v : F \to F_v$ be the embedding. The entrywise embeddings of $\mathrm{SL}_2(F)$ into $\mathrm{SL}_2(F_v)$ are also denoted by ι_v. Let $\{\infty_1, \cdots, \infty_n\}$ be the set of infinite places of F. Put $\iota_i = \iota_{\infty_i}$ for $1 \leq i \leq n$. We embed $\mathrm{SL}_2(F)$ into $\mathrm{SL}_2(\mathbb{R})^n$ by $r \mapsto (\iota_1(r), \cdots, \iota_n(r))$.

We define the metaplectic group $\widetilde{\mathrm{SL}_2(\mathbb{R})}$ of $\mathrm{SL}_2(\mathbb{R})$ similar to the case $F_v/\mathbb{Q}_p$. Let $\mathfrak{S}$ be a finite set of places of F, which contains all places above 2 and ∞. Set

$$\mathrm{SL}_2(\mathbb{A})_{\mathfrak{S}} = \prod_{v \in \mathfrak{S}} \mathrm{SL}_2(F_v) \times \prod_{v \notin \mathfrak{S}} \mathrm{SL}_2(\mathfrak{o}_v).$$

The double covering of $\mathrm{SL}_2(\mathbb{A})_{\mathfrak{S}}$ defined by the 2-cocycle $\prod_{v \in \mathfrak{S}} c_v(g_{1,v}, g_{2,v})$ is denoted by $\widetilde{\mathrm{SL}_2(\mathbb{A})}_{\mathfrak{S}}$, where c_v is the Kubota 2-cocycle for $\mathrm{SL}_2(F_v)$.

Let $s_v : \mathrm{SL}_2(\mathfrak{o}_v) \to \{\pm 1\}$ is the map s in (13) for $v < \infty$. For a finite set $\mathfrak{S}'$ of places of F such that $\mathfrak{S} \subset \mathfrak{S}'$, we can define an embedding $\iota_{\mathfrak{S}}^{\mathfrak{S}'}$: $\widetilde{\mathrm{SL}_2(\mathbb{A})}_{\mathfrak{S}} \to \widetilde{\mathrm{SL}_2(\mathbb{A})}_{\mathfrak{S}'}$ by

$$[(g_v), \zeta] \mapsto \Big[(g_v), \quad \zeta \prod_{v \in \mathfrak{S}' \setminus \mathfrak{S}} s_v(g_v)\Big].$$

For $v < \infty$, a map $\mathbf{s}_v : \mathrm{SL}_2(\mathfrak{o}_v) \to \widetilde{\mathrm{SL}_2(\mathfrak{o}_v)}$ is given by $\mathbf{s}_v(\gamma) = [\gamma, s_v(\gamma)]$ for $\gamma \in \mathrm{SL}_2(\mathfrak{o}_v)$. The adelic metaplectic group $\widetilde{\mathrm{SL}_2(\mathbb{A})}$ is the direct limit $\varinjlim \widetilde{\mathrm{SL}_2(\mathbb{A})}_{\mathfrak{S}}$. It is a double covering of $\mathrm{SL}_2(\mathbb{A})$ and there exists a canonical embedding $\widetilde{\mathrm{SL}_2(F_v)} \to \widetilde{\mathrm{SL}_2(\mathbb{A})}$ for each v. Let $\prod_v' \widetilde{\mathrm{SL}_2(F_v)}$ be the restricted direct product with respect to $\mathbf{s}_v(\mathrm{SL}_2(\mathfrak{o}_v))$. Then there is a canonical surjection $\prod_v' \widetilde{\mathrm{SL}_2(F_v)} \to \widetilde{\mathrm{SL}_2(\mathbb{A})}$. The image of $(g_v)_v \in \prod_v' \widetilde{\mathrm{SL}_2(F_v)}$ is also denoted by $(g_v)_v$. Note that for a given $g \in \widetilde{\mathrm{SL}_2(\mathbb{A})}$, the expression $g = (g_v)_v$ is not unique.

We denote the embedding of $\mathrm{SL}_2(F)$ into $\mathrm{SL}_2(\mathbb{A})$ by ι. The finite part of $\mathrm{SL}_2(\mathbb{A})$ is denoted by $\mathrm{SL}_2(\mathbb{A}_f)$. Let $\iota_f : \mathrm{SL}_2(F) \to \mathrm{SL}_2(\mathbb{A}_f)$ be the projection of the finite part and $\iota_\infty : \mathrm{SL}_2(F) \to \mathrm{SL}_2(F_\infty) = \mathrm{SL}_2(\mathbb{R})^n$ that of the infinite part. Then we have $\iota(g) = \iota_f(g)\iota_\infty(g)$ for all $g \in \mathrm{SL}_2(F)$. The embedding of F into $\mathbb{A}_f$ is also denoted by ι_f.

It is known that $\mathrm{SL}_2(F)$ can be canonically embedded into $\widetilde{\mathrm{SL}_2(\mathbb{A})}$. The embedding $\tilde{\iota}$ is given by $g \mapsto ([\iota_v(g)])_v$ for each $g \in \mathrm{SL}_2(F)$. We define the maps $\tilde{\iota}_f : \mathrm{SL}_2(F) \to \widetilde{\mathrm{SL}_2(\mathbb{A}_f)}$ and $\tilde{\iota}_\infty : \mathrm{SL}_2(F) \to \widetilde{\mathrm{SL}_2(F_\infty)}$ by

$$\tilde{\iota}_f(g) = ([\iota_v(g)])_{v<\infty} \times ([1_2])_{v|\infty}, \qquad \tilde{\iota}_\infty(g) = ([1_2])_{v<\infty} \times ([\iota_i(g)])_{v|\infty}.$$

Then we have $\tilde{\iota}(g) = \tilde{\iota}_f(g)\tilde{\iota}_\infty(g)$ for all $g \in \mathrm{SL}_2(F)$.

For $z \in \mathfrak{h}$, we choose $\arg z$ such that $-\pi < \arg z \leq \pi$. For $\gamma = [g, \tau] \in \widetilde{\mathrm{SL}_2(\mathbb{R})}$, $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $z \in \mathfrak{h}$, $\tilde{j} : \widetilde{\mathrm{SL}_2(\mathbb{R})} \times \mathfrak{h} \to \mathbb{C}$ is an automorphy factor given by

$$\tilde{j}(\gamma, z) = \begin{cases} \tau\sqrt{d} & \text{if } c = 0, d > 0, \\ -\tau\sqrt{d} & \text{if } c = 0, d < 0, \\ \tau(cz+d)^{1/2} & \text{if } c \neq 0. \end{cases} \tag{20}$$

Note that $\tilde{j}([g, \tau], z)$ is the unique automorphy factor such that $\tilde{j}([g, \tau], z)^2 = j(g, z)$, where $j(g, z)$ is the usual automorphy factor on $\mathrm{SL}_2(\mathbb{R}) \times \mathfrak{h}$ (see [15, §7]). Note that $\tilde{j}([g], z) = J(g, z)$, where $J(g, z)$ is defined in (1).

Definition 1. Let $\Gamma \subset \mathrm{SL}_2(\mathfrak{o})$ be a congruence subgroup. the map $\mathbf{v} = \mathbf{v}(\gamma) : \Gamma \to \mathbb{C}^\times$ is said to be a multiplier system of half-integral weight if $\mathbf{v}(\gamma) \prod_{i=1}^n \tilde{j}([\iota_i(\gamma)], z_i)$ is an automorphy factor for $\Gamma \times \mathfrak{h}^n$, where $\tilde{j}$ is the automorphy factor in (20).

We have $\tilde{j}(\gamma_1, \gamma_2(z))\tilde{j}(\gamma_2, z) = \tilde{j}(\gamma_1\gamma_2, z)$ for $\gamma_1, \gamma_2 \in \widetilde{\mathrm{SL}_2(\mathbb{R})}$. Replacing γ_i with $[g_i]$ for $i = 1, 2$, we have

$$\tilde{j}([g_1], g_2(z))\tilde{j}([g_2], z) = c_{\mathbb{R}}(g_1, g_2)\tilde{j}([g_1g_2], z), \tag{21}$$

where $c_{\mathbb{R}}(\cdot, \cdot)$ is the Kubota 2-cocycle at infinite places.

Lemma 9. A function $\mathbf{v} : \Gamma \to \mathbb{C}^\times$ is a multiplier system of half-integral weight if and only if we have

$$\mathbf{v}(\gamma_1)\mathbf{v}(\gamma_2) = c_\infty(\gamma_1, \gamma_2)\mathbf{v}(\gamma_1\gamma_2) \qquad \gamma_1, \gamma_2 \in \Gamma,$$

where $c_\infty(\gamma_1, \gamma_2) = \prod_{i=1}^n c_{\mathbb{R}}(\iota_i(\gamma_1), \iota_i(\gamma_2))$.

Proof. We have $\iota_i(\gamma_1)\iota_i(\gamma_2) = \iota_i(\gamma_1\gamma_2)$ for all i. Thus (21) and Definition 1 prove the lemma. $\square$

Let $K_\Gamma \subset \mathrm{SL}_2(\mathbb{A}_f)$ be the closure of $\iota_f(\Gamma)$ in $\mathrm{SL}_2(\mathbb{A}_f)$. Then K_Γ is a compact open subgroup and we have $\iota_f^{-1}(K_\Gamma) = \Gamma$. Let $\tilde{K}_\Gamma$ be the inverse image of K_Γ in $\widetilde{\mathrm{SL}_2(\mathbb{A}_f)}$.

Lemma 10. Let $\lambda : \tilde{K}_\Gamma \to \mathbb{C}^\times$ be a genuine character. Put $\mathbf{v}_\lambda(\gamma) = \lambda(\tilde{\iota}_f(\gamma))$ for $\gamma \in \Gamma$. Then $\mathbf{v}_\lambda$ is a multiplier system of half-integral weight for Γ.

Proof. For $\gamma_1, \gamma_2 \in \Gamma$, we have $\tilde{\iota}(\gamma_1)\tilde{\iota}(\gamma_2) = \tilde{\iota}(\gamma_1\gamma_2)$. The left-hand side equals $\tilde{\iota}_f(\gamma_1)\tilde{\iota}_\infty(\gamma_1)\tilde{\iota}_f(\gamma_2)\tilde{\iota}_\infty(\gamma_2)$. Since $\tilde{\iota}_\infty(g) = ([\iota_i(g)])_{i=1,\cdots,n}$ for $g \in \mathrm{SL}_2(F)$ and $\tilde{\iota}_\infty(\gamma_1)$ commutes with $\tilde{\iota}_f(\gamma_2)$, we have

$$\tilde{\iota}_f(\gamma_1)\tilde{\iota}_f(\gamma_2) = \tilde{\iota}_f(\gamma_1\gamma_2)[1_2, c_\infty(\gamma_1, \gamma_2)].$$

Since λ is genuine, Lemma 9 proves the lemma. $\square$

For $v < \infty$, the map $\mathbf{s}_v$ is the splitting on $K_1(4)_v$, where

$$K_1(4)_v = \{\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathfrak{o}_v) \mid c \equiv 0, d \equiv 1 \bmod 4\}.$$

If $K_\Gamma \subset K_1(4)_f = \prod_{v<\infty} K_1(4)_v$, we may define a splitting $\mathbf{s} : K_\Gamma \to \widetilde{\mathrm{SL}_2(\mathbb{A})}$ by

$$\mathbf{s}(\gamma) = (\mathbf{s}_v(\iota_v(\gamma)))_{v<\infty} \times ([1_2])_{v|\infty}.$$

The map $\mathbf{s}$ is a homomorphism. Then we have $\tilde{K}_\Gamma = \mathbf{s}(K_\Gamma) \cdot \{[1_2, \pm 1]\}$. Note that $\mathbf{s}(K_\Gamma) \subset \widetilde{\mathrm{SL}_2(\mathbb{A}_f)}$ is a compact open subgroup.

For any congruence subgroup Γ, a map $\mathbf{v}_0 : \Gamma \to \mathbb{C}^\times$ is defined by $\mathbf{v}_0(\gamma) = \prod_{v<\infty} s_v(\iota_v(\gamma))$, which is not always a multiplier system of half-integral weight for Γ.

Corollary 1. If $\Gamma \subset \Gamma_1(4) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathfrak{o}) \mid c \equiv 0, d \equiv 1 \bmod 4\right\}$, then $\mathbf{v}_0$ is a multiplier system of half-integral weight for Γ.

Proof. Since $\Gamma \subset \Gamma_1(4)$, we have $K_\Gamma \subset K_1(4)_f$. We define a genuine character $\lambda : \tilde{K}_\Gamma \to \mathbb{C}^\times$ by

$$\lambda(\mathbf{s}(k)[1_2, \tau]) = \tau, \qquad k \in K_\Gamma, \tau \in \{\pm 1\}.$$

Put $\mathbf{v}_\lambda(\gamma) = \lambda(\tilde{\iota}_f(\gamma))$ for $\gamma \in \Gamma$. Since $\mathbf{s}(\gamma) = ([\iota_v(\gamma), s_v(\iota_v(\gamma))])_{v<\infty}$, we have

$$\mathbf{v}_\lambda(\gamma) = \lambda(\mathbf{s}(\gamma)[1_2, \mathbf{v}_0(\gamma)]) = \mathbf{v}_0(\gamma).$$

Therefore Lemma 10 proves the corollary. $\square$

Now suppose that $\Gamma \subset \mathrm{SL}_2(\mathfrak{o})$ is a congruence subgroup and that $\mathbf{v} : \Gamma \to \mathbb{C}^\times$ is a multiplier system of half-integral weight.

Lemma 11. There exists a genuine character $\lambda : \tilde{K}_\Gamma \to \mathbb{C}^\times$ such that $\mathbf{v}_\lambda = \mathbf{v}$ if and only if there exists a congruence subgroup $\Gamma' \subset \Gamma \cap \Gamma_1(4)$ such that $\mathbf{v}(\gamma) = \mathbf{v}_0(\gamma)$ for all $\gamma \in \Gamma'$.

Proof. Suppose that there exists a genuine character $\lambda : \tilde{K}_\Gamma \to \mathbb{C}^\times$ such that $\mathbf{v}_\lambda = \mathbf{v}$. Since $\mathrm{Ker}\ \lambda$ and $\mathbf{s}(\Gamma_1(4))$ are open in $\widetilde{\mathrm{SL}_2(\mathbb{A}_f)}$, the intersection is also open. We denote its image in $\mathrm{SL}_2(\mathbb{A}_f)$ by K'. Then we have $\tilde{K}' = \mathbf{s}(K') \times \{[1_2, \pm 1]\}$. Put $\Gamma' = \iota_f^{-1}(K')$. Then we have $\mathbf{v}(\gamma) = \mathbf{v}_\lambda(\gamma) = \mathbf{v}_0(\gamma)$ for all $\gamma \in \Gamma'$.

Conversely, suppose that there exists a congruence subgroup $\Gamma' \subset \Gamma \cap \Gamma_1(4)$ such that $\mathbf{v}(\gamma) = \mathbf{v}_0(\gamma)$ for all $\gamma \in \Gamma'$. Then the closure $K_{\Gamma'}$ of $\iota_f(\Gamma')$ in $\mathrm{SL}_2(\mathbb{A}_f)$ is a compact open subgroup. Since $\iota_f(\Gamma)$ is dense and $K_{\Gamma'}$ is open in K_Γ, we have $K_\Gamma = \iota_f(\Gamma) \cdot K_{\Gamma'}$. For $k \in \tilde{K}_\Gamma$, there exist $\gamma \in \Gamma$, $k' \in K_{\Gamma'}$ and $\tau \in \{\pm 1\}$ such that $k = \tilde{\iota}_f(\gamma)\mathbf{s}(k')[1_2, \tau]$.

We assume that k also equals $\tilde{\iota}_f(\gamma_0)\mathbf{s}(k'_0)[1_2, \tau_0]$ for $\gamma_0 \in \Gamma$, $k'_0 \in K_{\Gamma'}$ and $\tau_0 \in \{\pm 1\}$. Put $\omega = \gamma_0^{-1}\gamma$. Then we have $\omega \in \Gamma'$ and

$$\tilde{\iota}_f(\gamma) = \tilde{\iota}_f(\gamma_0)\tilde{\iota}_f(\omega)[1_2, c_\infty(\gamma_0, \omega)], \qquad \tilde{\iota}_f(\omega) = \mathbf{s}(\iota_f(\omega))[1_2, \mathbf{v}_0(\omega)].$$

Then we have $k = \tilde{\iota}_f(\gamma)\mathbf{s}(k')[1_2, \tau] = \tilde{\iota}_f(\gamma_0)\mathbf{s}(\iota_f(\omega)k')[1_2, \tau\mathbf{v}_0(\omega)c_\infty(\gamma_0, \omega)]$. Thus we have $k'_0 = \iota_f(\omega)k'$ and $\tau_0 = \tau\mathbf{v}_0(\omega)c_\infty(\gamma_0, \omega)$. Since $\mathbf{v} = \mathbf{v}_0$ in Γ' and $\mathbf{v}(\gamma) = \mathbf{v}(\gamma_0)\mathbf{v}(\omega)c_\infty(\gamma_0, \omega)$ by Lemma 9, we have $\mathbf{v}(\gamma_0)\tau_0 = \mathbf{v}(\gamma)\tau$. Then the function $\lambda(k) = \mathbf{v}(\gamma)\tau$ is well-defined.

Since $\lambda(k[1_2, \sigma]) = \mathbf{v}(\gamma)\tau\sigma = \sigma\lambda(k)$ for $\sigma \in \{\pm 1\}$, λ is genuine. It suffices to show that $\lambda(k_1k_2) = \lambda(k_1)\lambda(k_2)$ for all $k_1, k_2 \in \tilde{K}_\Gamma$. There exist $\gamma_i \in \Gamma$, $k'_i \in K_{\Gamma'}$ and $\tau_i \in \{\pm 1\}$ such that $k_i = \tilde{\iota}_f(\gamma_i)\mathbf{s}(k'_i)[1_2, \tau_i]$ for $i = 1, 2$. Then we have $\lambda(k_1)\lambda(k_2) = \mathbf{v}(\gamma_1)\mathbf{v}(\gamma_2)\tau_1\tau_2$. Replacing $K_{\Gamma'}$ with its sufficiently small subgroup, we may assume that $\mathbf{s}(K_{\Gamma'})$ is a normal subgroup of $\tilde{K}_\Gamma$. Then we have

$$\tilde{\iota}_f(\gamma_2)^{-1}\mathbf{s}(k'_1)\tilde{\iota}_f(\gamma_2) = \mathbf{s}(\iota_f(\gamma_2)^{-1}k'_1\iota_f(\gamma_2)) \in \mathbf{s}(K_{\Gamma'}).$$

Since

$$\tilde{\iota}_f(\gamma_1)\tilde{\iota}_f(\gamma_2) = \tilde{\iota}_f(\gamma_1\gamma_2)[1_2, c_\infty(\gamma_1, \gamma_2)],$$

$\lambda(k_1k_2)$ equals $\mathbf{v}(\gamma_1\gamma_2)c_\infty(\gamma_1, \gamma_2)\tau_1\tau_2$. By Lemma 9, we have $\lambda(k_1k_2) = \lambda(k_1)\lambda(k_2)$, which proves the lemma. $\square$

Proposition 2. If $F \neq \mathbb{Q}$, then any multiplier system $\mathbf{v}$ of half-integral weight of any congruence subgroup $\Gamma \subset \mathrm{SL}_2(\mathfrak{o})$ is obtained from a genuine character of $\tilde{K}_\Gamma$.

Proof. By Lemma 11, it suffices to show that there exists a congruence subgroup $\Gamma' \subset \Gamma \cap \Gamma_1(4)$ such that $\mathbf{v}(\gamma) = \mathbf{v}_0(\gamma)$ for all $\gamma \in \Gamma'$. We assume that a congruence subgroup Γ satisfies $\Gamma \subset \Gamma_1(4)$ by replacing Γ with $\Gamma \cap \Gamma_1(4)$. Let $D(G)$ be the derived subgroup of a group G. Since $\mathbf{v}_0(\gamma)/\mathbf{v}(\gamma)$ is a character of Γ, we have $\mathbf{v}_0(\gamma)/\mathbf{v}(\gamma) = 1$ for all $\gamma \in D(\Gamma)$. By the congruence subgroup property, $D(\Gamma)$ contains a congruence subgroup Γ' (see [30, Corollary 3 of Theorem 2] or [16, §3]). Thus we have $\mathbf{v}(\gamma) = \mathbf{v}_0(\gamma)$ for all $\gamma \in \Gamma'$, which proves this proposition. $\square$

By Lemma 11 and Proposition 2, the multiplier system of half-integral weight of a congruence subgroup Γ associated with an automorphy factor in the sense of Shimura [33] is obtained from a genuine character of $\tilde{K}_\Gamma$.

Lemma 12. If $F = \mathbb{Q}$, then we have

$$\mathbf{v}_0(g) = \begin{cases} \left(\dfrac{d}{c}\right)^* & c: \text{ odd} \\ \left(\dfrac{c}{d}\right)_* & c: \text{ even}, \end{cases} \qquad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}).$$

Proof. In the case $(c,d) = (\pm 1, 0)$, we have $\left(\dfrac{0}{c}\right)^* = \mathbf{v}_0(g) = 1$. In the case $(c,d) = (0,1)$ (resp. $(0,-1)$), we have $\left(\dfrac{0}{d}\right)_* = \mathbf{v}_0(g) = 1$(resp. -1). If $c \neq 0$ and $d \in 2\mathbb{Z}+1$ satisfy $(c,d) = 1$, we have

$$\left(\frac{c}{d}\right)^* = \left(\frac{c}{|d|}\right), \qquad \left(\frac{c}{d}\right)_* = t(c,d)\left(\frac{c}{|d|}\right), \qquad t(c,d) = \begin{cases} -1 & c,d<0 \\ 1 & \text{otherwise.} \end{cases}$$

Suppose that $cd \neq 0$. Put $u = c \cdot 2^{-\mathrm{ord}_2 c}$. Then we have $(u,d) = (c,d) = 1$. Put $t_0(x,y) = (-1)^{(x-1)(y-1)/4}$ for $x, y \in 2\mathbb{Z}+1$. If a prime p satisfies $p \mid c$, we have

$$\langle c, d\rangle_p = \begin{cases} \left(\dfrac{d}{p}\right)^{\mathrm{ord}_p c} & p \geq 3 \\ t_0(u,d)\left(\dfrac{2}{|d|}\right)^{\mathrm{ord}_2 c} & p = 2. \end{cases}$$

If c is odd, then we have $\left(\dfrac{d}{c}\right)^* = \prod_{p|c}\left(\dfrac{d}{p}\right)^{\mathrm{ord}_p c} = \mathbf{v}_0(g)$. If c is even, then we have $\left(\dfrac{u}{|d|}\right)\left(\dfrac{d}{|u|}\right) = t(c,d)t_0(u,d)$ (see [17, p.51]). Thus we have

$$\left(\frac{c}{d}\right)_* = t(c,d)\left(\frac{2}{|d|}\right)^{\mathrm{ord}_2 c}\left(\frac{u}{|d|}\right) = t_0(u,d)\left(\frac{2}{|d|}\right)^{\mathrm{ord}_2 c}\left(\frac{d}{|u|}\right) = \mathbf{v}_0(g).$$

□

Put

$$K_f = \prod_{v<\infty} \mathrm{SL}_2(\mathfrak{o}_v).$$

Then K_f is a compact open group of $\mathrm{SL}_2(\mathbb{A}_f)$. The inverse image of K_f in $\widetilde{\mathrm{SL}_2(\mathbb{A}_f)}$ is denoted by $\tilde{K}_f$. We have $\mathrm{SL}_2(\mathfrak{o}) = \mathrm{SL}_2(F) \cap K_f \cdot \mathrm{SL}_2(F_\infty)$.

Proposition 3. Let $\mathbf{v}$ be a multiplier system of half-integral weight for $\mathrm{SL}_2(\mathfrak{o})$. Then there exists a genuine character $\lambda : \tilde{K}_f \to \mathbb{C}^\times$ such that $\mathbf{v}_\lambda = \mathbf{v}$.

Proof. If $F \neq \mathbb{Q}$, the assertion is proved by Proposition 2. If $F = \mathbb{Q}$, let $\mathbf{v}_\eta$ be the multiplier system of $\eta(z)$ in (3). Put

$$\Gamma(12) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \mid a \equiv d \equiv 1, b \equiv c \equiv 0 \bmod 12 \right\}.$$

By Lemma 12, we have $\mathbf{v}_\eta(\gamma) = \mathbf{v}_0(\gamma)$ for $\gamma \in \Gamma(12)$. Since $\mathbf{v}_\eta(\gamma)/\mathbf{v}(\gamma) = 1$ for all $\gamma \in D(\mathrm{SL}_2(\mathbb{Z}))$, we have $\mathbf{v}(\gamma) = \mathbf{v}_0(\gamma)$ for all $\gamma \in D(\mathrm{SL}_2(\mathbb{Z})) \cap \Gamma(12)$, which is a congruence subgroup. By Lemma 11, there exists a genuine character $\lambda : \tilde{K}_f \to \mathbb{C}^\times$ such that $\mathbf{v}_\lambda = \mathbf{v}$. □

Corollary 2. There exists a multiplier system $\mathbf{v}$ of half-integral weight for $\mathrm{SL}_2(\mathfrak{o})$ if and only if 2 splits completely in $F/\mathbb{Q}$. There exists a genuine character of $\widetilde{\mathrm{SL}_2(\mathfrak{o}_v)}$ for all $v < \infty$, provided that this condition holds.

Proposition 4. Suppose that 2 splits completely in $F/\mathbb{Q}$. Let $\mathbf{v}_\lambda$ be a multiplier system of half-integral weight of $\mathrm{SL}_2(\mathfrak{o})$, where $\lambda = \prod_{v<\infty} \lambda_v$ is a genuine character of $\tilde{K}_f$. Put $S_2 = \{v < \infty \mid F = \mathbb{Q}_2\}$ and $T_3 = \{v < \infty \mid q_v = 3\}$. If q_v is odd, let ϵ_v be the genuine character of $\widetilde{\mathrm{SL}_2(\mathfrak{o}_v)}$ from Lemma 4. We set $S_3 = \{v \in T_3 \mid \lambda_v \neq \epsilon_v\}$. Let β_v be a element of $F^\times$ such that $\lambda_v = \mu_{\beta_v}$ for $v \in S_2 \cup S_3$. Then we have

$$\mathbf{v}_\lambda(\gamma) = \mathbf{v}_0(\gamma) \prod_{v \in S_2 \cup S_3} \kappa_v(\beta_v, \iota_v(\gamma)) \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathfrak{o}).$$

Here, if $v \in S_2$,

$$\kappa_v(\beta_v, g) = \begin{cases} \psi_{\beta_v}(-(a+d)c + 3c) & c \in \mathbb{Z}_2^\times \\ \psi_{\beta_v}((c-b)d - 3(d-1)) & c \in 2\mathbb{Z}_2 \end{cases}$$

and if $v \in S_3$,

$$\kappa_v(\beta_v, g) = \psi_{\beta_v}(-(a+d)c + bd(c^2 - 1))$$

for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathfrak{o}_v)$. Note that $\kappa_v(\beta_v, \iota_v(\gamma))$ is a continuous function on γ.

Proof. We have $\mathbf{v}_\lambda(\gamma) = \lambda(\tilde{\iota}_f(\gamma)) = \prod_{v<\infty} \lambda_v([\iota_v(\gamma)])$. If $v \notin S_2 \cup S_3$, then we have $\epsilon_v([g]) = s_v(g)$ for all $g \in \mathrm{SL}_2(\mathfrak{o}_v)$. If $v \in S_2$ (resp. S_3), we have $\mu_{\beta_v}([g]) = s_v(g)\kappa_v(\beta_v, g)$ by (15) (resp. (16)) for all $g \in \mathrm{SL}_2(\mathfrak{o}_v)$. This proves the proposition. □

5. The condition of the existence of a theta function

Suppose that 2 splits completely in $F/\mathbb{Q}$. By Lemma 3, there exists a genuine character $\lambda_v : \widetilde{\mathrm{SL}_2(\mathfrak{o}_v)} \to \mathbb{C}^\times$ for all $v < \infty$. If $v < \infty$, put $K_v = \mathrm{SL}_2(\mathfrak{o}_v)$. If $v \mid \infty$, put $K_v = \mathrm{SO}(2)$. Then K_v is a maximal compact subgroup of $\mathrm{SL}_2(F_v)$ for all v. Let β be an element of $F^\times$ and ψ_β the character of $\mathbb{A}/F$ as in Section 1. For any v, we denote the Weil representation of $\widetilde{\mathrm{SL}_2(F_v)}$ by $\omega_{\psi_\beta,v}$.

Let $\{\infty_1, \cdots, \infty_n\}$ be the set of $v \mid \infty$ and $S(\mathbb{R})$ the Schwartz space of $\mathbb{R}$. We have an irreducible decomposition

$$\omega_{\psi_\beta,v} = \omega^+_{\psi_\beta,v} \oplus \omega^-_{\psi_\beta,v},$$

where $\omega^+_{\psi_\beta,v}$ (resp. $\omega^-_{\psi_\beta,v}$) is the irreducible representation of $\widetilde{\mathrm{SL}_2(\mathbb{R})}$ on the set of even (resp. odd) functions in $S(\mathbb{R})$ (see [21, Lemma 2.4.4]).

The group $\widetilde{\mathrm{SL}_2(\mathbb{R})}$ has a maximal compact subgroup $\widetilde{\mathrm{SO}(2)}$, which is the inverse image of $\mathrm{SO}(2)$ in $\widetilde{\mathrm{SL}_2(\mathbb{R})}$. It is known that if $\lambda_v : \widetilde{\mathrm{SO}(2)} \to \mathbb{C}^\times$ is a genuine character, $\dim_{\mathbb{C}}(\omega_{\psi_\beta,v}, S(\mathbb{R}))^{\lambda_v}$ is at most 1. Let $\lambda_{\infty,1/2}$ be a genuine character of lowest weight 1/2 with respect to $(\omega^+_{\psi_\beta,v}, S(\mathbb{R}))$ and $\lambda_{\infty,3/2}$ of lowest weight 3/2 with respect to $(\omega^-_{\psi_\beta,v}, S(\mathbb{R}))$. For $\beta > 0$, $(\omega^+_{\psi_\beta,v}, S(\mathbb{R}))^{\lambda_{\infty,1/2}} = \mathbb{C}\, e(i\iota_v(\beta)x^2)$ and $(\omega^-_{\psi_\beta,v}, S(\mathbb{R}))^{\lambda_{\infty,3/2}} = \mathbb{C}\, xe(i\iota_v(\beta)x^2)$ are spaces of lowest weight vectors. If $\beta < 0$, there exist no lowest weight vectors with respect to $(\omega^+_{\psi_\beta,v}, S(\mathbb{R}))$ or $(\omega^-_{\psi_\beta,v}, S(\mathbb{R}))$.

Note that $\lambda_v(\mathbf{s}_v(\mathrm{SL}_2(\mathfrak{o}_v))) = 1$ for all but finitely many places $v < \infty$. Then a genuine character $\lambda_f : \tilde{K}_f \to \mathbb{C}^\times$ is given by $\lambda_f(g) = \prod_{v<\infty} \lambda_v(g_v)$ for $g = (g_v)_v \in \tilde{K}_f$. Put $w = (w_1, \cdots, w_n) \in \{1/2, 3/2\}^n$. We define an automorphy factor $j^{\lambda_f,w}(\gamma, z)$ for $\gamma \in \mathrm{SL}_2(\mathfrak{o})$ and $z = (z_1, \cdots, z_n) \in \mathfrak{h}^n$ by

$$j^{\lambda_f,w}(\gamma, z) = \prod_{v<\infty} \lambda_v([\iota_v(\gamma)]) \prod_{i=1}^{n} \tilde{j}([\iota_i(\gamma)], z_i)^{2w_i},$$

where $\tilde{j}$ is given by (20).

In particular, we have $j^{\lambda_f,w}(-1_2, z) = \prod_{v<\infty} \lambda_v([-1_2]) \times (-\sqrt{-1})^{\sum 2w_i}$. If it does not equal 1, the space of Hilbert modular forms of weight w for $\mathrm{SL}_2(\mathfrak{o})$ is $\{0\}$.

Put $K = K_f \times \prod_{v|\infty} \mathrm{SO}(2)$. There exists a genuine character $\lambda : \tilde{K} \to \mathbb{C}^\times$ such that its v-component equals λ_v, where λ_{∞_i} is $\lambda_{\infty,1/2}$ or $\lambda_{\infty,3/2}$ for $1 \leq i \leq n$. Then we have an automorphy factor $j^{\lambda_f,w}(\gamma, z)$ corresponding to λ such that $\lambda_{\infty_i} = \lambda_{\infty,w_i}$.

Let $M_w(\mathrm{SL}_2(\mathfrak{o}),\lambda_f)$ be the space of Hilbert modular forms on $\mathfrak{h}^n$ with respect to $j^{\lambda_f,w}(\gamma,z)$. A holomorphic function $h(z)$ of $\mathfrak{h}^n$ belongs to the space $M_w(\mathrm{SL}_2(\mathfrak{o}),\lambda_f)$ if and only if

$$h(\gamma(z))=j^{\lambda_f,w}(\gamma,z)h(z),$$

where $\gamma(z)=(\iota_1(\gamma)(z_1),\cdots,\iota_n(\gamma)(z_n))$ for $\gamma\in\mathrm{SL}_2(\mathfrak{o})$ and $z\in\mathfrak{h}^n$. (When $F=\mathbb{Q}$, the usual cusp condition is also required.)

For each $g\in\widetilde{\mathrm{SL}_2(\mathbb{A})}$, there exist $\gamma\in\mathrm{SL}_2(F)$, $g_\infty\in\widetilde{\mathrm{SL}_2(\mathbb{R})}^n$ and $g_f\in\tilde{K}_f$ such that $g=\gamma g_\infty g_f$ by the strong approximation theorem for $\mathrm{SL}_2(\mathbb{A})$. Put $\mathbf{i}=(\sqrt{-1},\cdots,\sqrt{-1})\in\mathfrak{h}^n$. For $h\in M_w(\mathrm{SL}_2(\mathfrak{o}),\lambda_f)$, put

$$\varphi_h(g)=h(g_\infty(\mathbf{i}))\lambda_f(g_f)^{-1}\prod_{i=1}^n\tilde{j}(g_{\infty_i},\sqrt{-1})^{-2w_i}.$$

Then φ_h is an automorphic form on $\mathrm{SL}_2(F)\backslash\widetilde{\mathrm{SL}_2(\mathbb{A})}$.

Let $\mathcal{A}_w(\mathrm{SL}_2(F)\backslash\widetilde{\mathrm{SL}_2(\mathbb{A})},\lambda_f)$ be the space of automorphic forms φ on $\mathrm{SL}_2(F)\backslash\widetilde{\mathrm{SL}_2(\mathbb{A})}$ satisfying the following conditions (1), (2), and (3).

(1) $\varphi(gk_\infty)=\varphi(g)\prod_{i=1}^n\tilde{j}(k_{\infty,i},\sqrt{-1})^{-2w_i}$ for all $g\in\widetilde{\mathrm{SL}_2(\mathbb{A})}$ and $k_\infty=(k_{\infty,1},\ldots,k_{\infty,n})\in\widetilde{\mathrm{SO}(2)}^n$.

(2) φ is a lowest weight vector with respect to the right translation of $\widetilde{\mathrm{SL}_2(\mathbb{R})}^n$.

(3) $\varphi(gk)=\lambda_f(k)^{-1}\varphi(g)$ for all $g\in\widetilde{\mathrm{SL}_2(\mathbb{A})}$ and $k\in\tilde{K}_f$.

Then $\Phi: h\mapsto\varphi_h$ gives rise to an isomorphism

$$M_w(\mathrm{SL}_2(\mathfrak{o}),\lambda_f)\xrightarrow{\sim}\mathcal{A}_w(\mathrm{SL}_2(F)\backslash\widetilde{\mathrm{SL}_2(\mathbb{A})},\lambda_f).$$

For $\varphi\in\mathcal{A}_w(\mathrm{SL}_2(F)\backslash\widetilde{\mathrm{SL}_2(\mathbb{A})},\lambda_f)$, put $h=\Phi^{-1}(\varphi)$. Then we have

$$h(z)=\varphi(g_\infty)\prod_{i=1}^n\tilde{j}(g_{\infty_i},\sqrt{-1})^{2w_i},\qquad g_\infty\in\widetilde{\mathrm{SL}_2(\mathbb{R})}^n,\ g_\infty(\mathbf{i})=z.$$

Now suppose that $v<\infty$. Let $\mathfrak{d}$ be the different of $F/\mathbb{Q}$ and q_v the order of the residue field $\mathfrak{o}_v/\mathfrak{p}_v$. When q_v is odd, let ϵ_v be the genuine character ϵ_F from Lemma 4. Put $S_2=\{v\mid F_v=\mathbb{Q}_2\}$, $T_3=\{v<\infty\mid q_v=3\}$ and $S_3=\{v\in T_3\mid\lambda_v\neq\epsilon_v\}$. Since 2 splits completely in $F/\mathbb{Q}$, we have $|S_2|=n$. If $\lambda_v=\epsilon_v$, by Lemma 7, $(\omega_{\psi_\beta,v},S(F_v))^{\lambda_v}$ is not 0 if and only if we have

$$\mathrm{ord}_v\psi_{\beta,v}\equiv 0 \bmod 2.$$

Otherwise, by Lemma 8, $(\omega_{\psi_\beta,v},S(F_v))^{\lambda_v}$ is not 0 only if we have

$$\mathrm{ord}_v\psi_{\beta,v}\equiv 1 \bmod 2.$$

Then, if $(\omega_{\psi_\beta,v}, S(F_v))^{\lambda_v} \neq 0$ for all $v < \infty$, there exists a fractional ideal $\mathfrak{a}$ such that

$$(8\beta)\mathfrak{d} \prod_{v \in S_3} \mathfrak{p}_v = \mathfrak{a}^2. \tag{22}$$

Replacing β with $\beta\gamma^2$ and $\mathfrak{a}$ with $\gamma\mathfrak{a}$ in (22) for $\gamma \in F^\times$, we may assume $\mathrm{ord}_v \mathfrak{a} = 0$ for $v \in S_2 \cup S_3$. Then we have $\mathrm{ord}_v \psi_{\beta,v} = -1$ (resp. -3) for $v \in S_3$ (resp. S_2).

Conversely, suppose that there exists a fractional ideal $\mathfrak{a}$ satisfying (22) for a subset $S_3 \subset T_3$. For $v < \infty$, put

$$\lambda_v = \begin{cases} \epsilon_v & \text{if } \mathrm{ord}_v \psi_{\beta,v} \equiv 0 \mod 2 \\ \mu_\beta & \text{if } \mathrm{ord}_v \psi_{\beta,v} \equiv 1 \mod 2, \end{cases}$$

where μ_β is a genuine character in Lemma 5 or Lemma 6. By Lemma 7 and Lemma 8, we have $(\omega_{\psi_\beta,v}, S(F_v))^{\lambda_v} \neq 0$ for all $v < \infty$.

Let $\lambda : \tilde{K} \to \mathbb{C}^\times$ be a genuine character such that its v-component equals λ_v for $v < \infty$, and $\lambda_{\infty_i} = \lambda_{\infty,w_i}$, depending on $w_i \in \{1/2, 3/2\}$. Put $S_\infty = \{\infty_i \mid w_i = 3/2\}$. For $v \mid \infty$, recall that if $\iota_v(\beta) > 0$ (resp. $\iota_v(\beta) < 0$), $(\omega^+_{\psi_\beta,v}, S(\mathbb{R}))^{\lambda_{\infty,1/2}} = \mathbb{C}\, e(i\iota_v(\beta)x^2)$ (resp. $\{0\}$) and $(\omega^-_{\psi_\beta,v}, S(\mathbb{R}))^{\lambda_{\infty,3/2}} = \mathbb{C}\, xe(i\iota_v(\beta)x^2)$ (resp. $\{0\}$) are spaces of lowest weight vectors.

Then from now on, suppose that $\beta \in F_+^\times$, which is the set of totally positive elements of F. Let $S(\mathbb{A})$ be the Schwartz space of $\mathbb{A}$ and $(\omega_{\psi_\beta}, S(\mathbb{A}))^\lambda$ the set of functions $\phi = \prod_v \phi_v \in S(\mathbb{A})$ such that $\phi_v \in (\omega_{\psi_\beta,v}, S(F_v))^{\lambda_v}$ for all v. For $\phi \in S(\mathbb{A})$, we define the theta function Θ_ϕ by

$$\Theta_\phi(g) = \sum_{\xi \in F} \omega_{\psi_\beta}(g)\phi(\xi) \qquad g = (g_v) \in \widetilde{\mathrm{SL}_2(\mathbb{A})}, \tag{23}$$

where $\omega_{\psi_\beta}(g)\phi(\xi) = \prod_v \omega_{\psi_\beta,v}(g_v)\phi_v(\iota_v(\xi))$ is essentially a finite product. We have $\Theta_\phi(gk) = \lambda(k)^{-1}\Theta_\phi(g)$ for all $g \in \widetilde{\mathrm{SL}_2(\mathbb{A})}$ and $k \in \tilde{K}_f$. If $\phi \in (\omega_{\psi_\beta}, S(\mathbb{A}))^\lambda$, then Θ_ϕ is a Hilbert modular form of weight $w = (w_1, \cdots, w_n)$.

It is known that

$$\omega_{\psi_\beta} = \bigoplus_S \omega_{\psi_\beta,S}, \qquad \omega_{\psi_\beta,S} = \left(\bigotimes_{v \in S} \omega^-_{\psi_\beta,v}\right) \otimes \left(\bigotimes_{v \notin S} \omega^+_{\psi_\beta,v}\right),$$

where S ranges over all finite subsets of places of F (see [8, §3.4]). We define a map Θ from ω_{ψ_β} to the space of automorphic forms on $\widetilde{\mathrm{SL}_2(\mathbb{A})}$ by $\Theta(\phi)(g) = \Theta_\phi(g)$. Then it is known that

$$\mathrm{Im}(\Theta) \simeq \bigoplus_{|S|:\text{even}} \omega_{\psi_\beta,S}, \tag{24}$$

(see [8, Proposition 3.1]).

Fix a choice of S_∞, or equivalently the values $w_1, \cdots, w_n \in \{1/2, 3/2\}$. Let $\mathbf{G}$ be the set of triplets $(\beta, S_3, \mathfrak{a})$ of $\beta \in F_+^\times$, a subset $S_3 \subset T_3$ and a fractional ideal $\mathfrak{a}$ of F satisfying (22) and the condition (A),

$$|S_2| + |S_3| + |S_\infty| \in 2\mathbb{Z}. \tag{A}$$

We define an equivalence relation $\sim$ on $\mathbf{G}$ by

$$(\beta, S_3, \mathfrak{a}) \sim (\beta', S_3', \mathfrak{a}') \Longleftrightarrow S_3 = S_3',\ \beta' = \gamma^2\beta,\ \mathfrak{a}' = \gamma\mathfrak{a} \text{ for some } \gamma \in F^\times.$$

Theorem 1. Suppose that 2 splits completely in $F/\mathbb{Q}$. Let $\beta \in F_+^\times$, $\lambda : \tilde{K} \to \mathbb{C}^\times$ and $w_1, \ldots, w_n \in \{1/2, 3/2\}$ be as above. Let S_3 be determined by λ as above. Then there exists $\phi = \prod_v \phi_v \in (\omega_{\psi_\beta}, S(\mathbb{A}))^\lambda$ such that $\Theta_\phi \neq 0$ if and only if there exists a fractional ideal $\mathfrak{a}$ of F such that $(\beta, S_3, \mathfrak{a}) \in \mathbf{G}$.

Proof. Let $\lambda_v : \widetilde{\mathrm{SL}_2(\mathfrak{o}_v)} \to \mathbb{C}^\times$ be the v-component of λ for any $v < \infty$. We already proved that there exists $\prod_{v<\infty} \phi_v \neq 0$ such that $\phi_v \in (\omega_{\psi_\beta,v}, S(F_v))^{\lambda_v}$ for all $v < \infty$ if and only if there exists a fractional ideal $\mathfrak{a}$ of F satisfying (22). Suppose that these equivalent conditions hold. Since we have $(\omega^+_{\psi_\beta,v}, S(\mathbb{R}))^{\lambda_{\infty,1/2}} = \mathbb{C}\, e(i\iota_v(\beta)x^2)$ and $(\omega^-_{\psi_\beta,v}, S(\mathbb{R}))^{\lambda_{\infty,3/2}} = \mathbb{C}\, xe(i\iota_v(\beta)x^2)$ for all $v \mid \infty$, there exists a nonzero $\phi = \prod_v \phi_v \in (\omega_{\psi_\beta}, S(\mathbb{A}))^\lambda$. It is clear that if there exists a nonzero $\phi = \prod_v \phi_v \in (\omega_{\psi_\beta}, S(\mathbb{A}))^\lambda$, $\prod_{v<\infty} \phi_v \neq 0$ satisfies $\phi_v \in (\omega_{\psi_\beta,v}, S(F_v))^{\lambda_v}$ for all $v < \infty$.

Suppose there exists a nonzero $\phi = \prod_v \phi_v \in (\omega_{\psi_\beta}, S(\mathbb{A}))^\lambda$. Note that $|S_2| + |S_3| + |S_\infty|$ is the number of v such that ϕ_v is an odd function. Then $|S|$ in (24) is $|S_2| + |S_3| + |S_\infty|$. By (24), it is clear that $\Theta_\phi \neq 0$ if and only if the condition (A) holds. □

Let H be a group of fractional ideals that consists of all elements of the form

$$\prod_{v \in T_3} \mathfrak{p}_v^{e_v}, \qquad \sum_v e_v \in 2\mathbb{Z}.$$

Let Cl^+ be the narrow ideal class group of F. Put $\mathrm{Cl}^{+2} = \{\mathfrak{c}^2 \mid \mathfrak{c} \in \mathrm{Cl}^+\}$. We denote the image of the group H (resp. $\mathfrak{b} \in \mathrm{Cl}^+$) in $\mathrm{Cl}^+/\mathrm{Cl}^{+2}$ by $\bar{H}$ (resp. $[\mathfrak{b}]$).

Theorem 2. Suppose that 2 splits completely in $F/\mathbb{Q}$. Let $w_1, \ldots, w_n \in \{1/2, 3/2\}$ be as above.

(1) Suppose that $|S_2| + |S_\infty|$ is even. Then there exists $(\beta, S_3, \mathfrak{a}) \in \mathbf{G}$ if and only if $[\mathfrak{d}] \in \bar{H}$.
(2) Suppose that $|S_2| + |S_\infty|$ is odd. Then there exists $(\beta, S_3, \mathfrak{a}) \in \mathbf{G}$ if and only if $T_3 \neq \emptyset$ and $[\mathfrak{d}\mathfrak{p}_{v_0}] \in \bar{H}$. Here, v_0 is any fixed element of T_3.

Proof. We prove the theorem in case (1). The proof for case (2) is similar.

If $[\mathfrak{d}] \in \bar{H}$, we have $(8\beta)\mathfrak{d}\prod_{v\in T_3} \mathfrak{p}_v^{e_v} = \mathfrak{a}'^2$ such that $\sum_v e_v$ is even for a fractional ideal $\mathfrak{a}'$ and $\beta \in F_+^\times$. Put $S_3 = \{v \in T_3 \mid e_v : \text{odd}\}$. Since

$|S_2| + |S_3| + |S_\infty|$ is even, we have $(\beta, S_3, \mathfrak{a}) \in \mathbf{G}$, where

$$\mathfrak{a} = \prod_{v \in T_3 \setminus S_3} \mathfrak{p}_v^{-e_v/2} \mathfrak{a}'.$$

Conversely, if there exists $(\beta, S_3, \mathfrak{a}) \in \mathbf{G}$, it satisfies (22) and $|S_3|$ is even. Then we have $[\mathfrak{d}] = \prod_{v \in S_3} [\mathfrak{p}_v] \in \bar{H}$. $\square$

Let w_i be $1/2$ or $3/2$ for $1 \leq i \leq n$. Suppose that there exists $(\beta, S_3, \mathfrak{a}) \in \mathbf{G}$. Replacing $(\beta, S_3, \mathfrak{a})$ with an equivalent element of $\mathbf{G}$, we may assume $\mathrm{ord}_v \mathfrak{a} = 0$ for $v \in S_2 \cup S_3$. Let f_v be the function f in (14) and put

$$f = \prod_{v \in S_2 \cup S_3} f_v \times \prod_{v < \infty, v \notin S_2 \cup S_3} \mathrm{ch} \mathfrak{a}_v^{-1},$$

where $\mathfrak{a}_v = \mathfrak{a}\mathfrak{o}_v$. Put $\phi = f \times \prod_{i=1}^n f_{\infty,i}$, where $f_{\infty,i}(x) = x^{w_i - (1/2)} e(i\iota_i(\beta)x^2)$ for $x \in \mathbb{R}$. By Theorem 1, there exists $\Theta_\phi \neq 0$ of weight $w = (w_1, \cdots, w_n)$.

Put $z = (z_1, \cdots, z_n), \mathbf{i} = (\sqrt{-1}, \cdots, \sqrt{-1}) \in \mathfrak{h}^n$. We define $x_i, y_i \in \mathbb{R}$ by $z_i = x_i + \sqrt{-1} y_i$ for $1 \leq i \leq n$. Then we have $z = g_\infty(\mathbf{i})$, where $g_\infty = (g_{\infty_1}, \cdots, g_{\infty_n}) \in \mathrm{SL}_2(\mathbb{R})^n$, $g_{\infty_i} = \begin{pmatrix} y_i^{1/2} & y_i^{1/2} x_i \\ 0 & y_i^{-1/2} \end{pmatrix}$. Since $\lambda_v([1_2]) = 1$ for $v < \infty$, we have

$$\Theta_\phi(g_\infty) = \sum_{\xi \in \mathfrak{a}^{-1}} f(\iota_f(\xi)) \prod_{i=1}^n \omega_{\psi_\beta, \infty_i}([g_{\infty_i}]) f_{\infty,i}(\iota_i(\xi)).$$

Theorem 3. Let ϕ and Θ_ϕ be as above. We define a theta function $\theta_\phi : \mathfrak{h}^n \to \mathbb{C}$ by

$$\theta_\phi(z) = \sum_{\xi \in \mathfrak{a}^{-1}} f(\iota_f(\xi)) \prod_{\infty_i \in S_\infty} \iota_i(\xi) \prod_{i=1}^n e(z_i \iota_i(\beta\xi^2)).$$

Then θ_ϕ is a nonzero Hilbert modular form of weight w for $\mathrm{SL}_2(\mathfrak{o})$ with respect to a multiplier system.

Every theta function of weight w for $\mathrm{SL}_2(\mathfrak{o})$ with a multiplier system may be obtained in this way.

Proof. Since

$$\omega_{\psi_\beta, \infty_i}([g_{\infty,i}]) f_{\infty,i}(\iota_i(\xi)) = y_i^{w_i/2} \iota_i(\xi)^{w_i - (1/2)} e(z_i \iota_i(\beta\xi^2)),$$

we have $\theta_\phi(z) = \Theta_\phi(g_\infty) \times \prod_{i=1}^n y_i^{-w_i/2}$. Then θ_ϕ is nonzero. Note that

$$\tilde{j}([g_{\infty_i}], \sqrt{-1})^{2w_i} = y_i^{-w_i/2}.$$

Since $\phi \in (\omega_{\psi_\beta}, S(\mathbb{A}))^\lambda$, we have $\Theta_\phi \in \mathcal{A}_w(\mathrm{SL}_2(F) \backslash \widetilde{\mathrm{SL}_2(\mathbb{A})}, \lambda_f)$. Then we have $\theta_\phi = \Phi^{-1}(\Theta_\phi) \in M_w(\mathrm{SL}_2(\mathfrak{o}), \lambda_f)$. The multiplier system of θ_ϕ is $\mathbf{v}_\lambda$ given by

$$\mathbf{v}_\lambda(\gamma) = \mathbf{v}_0(\gamma) \prod_{v \in S_2 \cup S_3} \kappa_v(\beta, \gamma) \qquad \gamma \in \mathrm{SL}_2(\mathfrak{o}),$$

where κ_v for $v \in S_2 \cup S_3$ is the function in Proposition 4.

By Proposition 3, if θ is a theta function of weight w for $\mathrm{SL}_2(\mathfrak{o})$ with a multiplier system $\mathbf{v}$, we have a genuine character λ_f of $\tilde{K}_f$ such that $\mathbf{v} = \mathbf{v}_{\lambda_f}$. Let $\lambda = \lambda_f \times \prod_{i=1}^n \lambda_{\infty,w_i}$ be a genuine character of $\tilde{K}$. Then there exists nonzero $\phi \in (\omega_{\psi_\beta}, S(\mathbb{A}))^\lambda$ such that $\theta = \theta_\phi$ up to constant, which completes the proof. $\square$

Proposition 5. Let Cl be the usual ideal class group of F. Let $Sq : \mathrm{Cl} \to \mathrm{Cl}^+$ be the homomorphism given by $[\mathfrak{a}] \mapsto [\mathfrak{a}^2]$ for a fractional ideal $\mathfrak{a}$ of F. The number of equivalence classes of $\mathbf{G}$ is equal to

$$[E^+ : E^2] \sum_{\substack{S_3 \subset T_3 \\ \text{(A)}}} |Sq^{-1}([\mathfrak{d} \prod_{v \in S_3} \mathfrak{p}_v])|,$$

where S_3 ranges over all subsets of T_3 satisfying (A). Here, E^+ is the group of totally positive units of F and E^2 is the subgroup of squares of units of F.

Proof. We follow the argument of Hammond [13] Theorem 2.9. For given S_3 satisfying (A), the number of ideal classes $[\mathfrak{a}]$ such that $\mathfrak{a}^2$ is narrowly equivalent to $\mathfrak{d} \prod_{v \in S_3} \mathfrak{p}_v$ is equal to $|Sq^{-1}([\mathfrak{d} \prod_{v \in S_3} \mathfrak{p}_v])|$. Then for a given fractional ideal $\mathfrak{a}$ such that $\mathfrak{a}^2$ is narrowly equivalent to $\mathfrak{d} \prod_{v \in S_3} \mathfrak{p}_v$, the number of equivalence classes of triplets of the form $(\beta, S_3, \mathfrak{a})$ such that $\beta \in F_+^\times$ satisfying (22) is equal to $[E^+ : E^2]$. $\square$

6. The case $F = \mathbb{Q}$ or F is a real quadratic field

Suppose that $F = \mathbb{Q}$. If $S_\infty = \emptyset$, the equivalence class of $\mathbf{G}$ is

$$\{(1/24, \{3\}, \mathbb{Z})\}.$$

The theta function obtained by $\{(1/24, \{3\}, \mathbb{Z})\}$ equals $2\eta(z)$. Then its multiplier system equals $\mathbf{v}_\eta$ in (3). If $S_\infty = \{\infty\}$, then the equivalence class of $\mathbf{G}$ is $\{(1/8, \emptyset, \mathbb{Z})\}$. The theta function obtained by $\{(1/8, \emptyset, \mathbb{Z})\}$ equals $2\eta^3(z)$. Then its multiplier system equals the cubic power of $\mathbf{v}_\eta$.

Now suppose that $F = \mathbb{Q}(\sqrt{D})$, where $D > 1$ is a square-free integer. When there exists $(\beta, S_3, \mathfrak{a}) \in \mathbf{G}$, one of the followings holds.

(C1) $(8\beta)\mathfrak{d} = \mathfrak{a}^2$ and $S_3 = \emptyset$.
(C2) $(8\beta)\mathfrak{d}\mathfrak{p} = \mathfrak{a}^2$ such that $N_{F/\mathbb{Q}}(\mathfrak{p}) = 3$ and $S_3 = \{\mathfrak{p}\}$.
(C3) $(8\beta)\mathfrak{d}\mathfrak{p}\bar{\mathfrak{p}} = \mathfrak{a}^2$ such that $N_{F/\mathbb{Q}}(\mathfrak{p}) = N_{F/\mathbb{Q}}(\bar{\mathfrak{p}}) = 3$ and $S_3 = \{\mathfrak{p}, \bar{\mathfrak{p}}\}$.

If $|S_\infty|$ is even, (C1) or (C3) holds. If $|S_\infty|$ is odd, (C2) holds.

Also suppose that $D \equiv 1 \bmod 8$. Then 2 splits in $F/\mathbb{Q}$ and we have $\mathfrak{d} = (\sqrt{D})$.

Lemma 13. Let N be a positive square-free integer. Put $L = \mathbb{Q}(\sqrt{-1})$ (resp. $L = \mathbb{Q}(\sqrt{-3})$). Then the following statements are equivalent.

(a) N is a norm of an element of $L^\times$.
(b) N is a norm of an integer of L.

(c) No prime factor of N are inert in $L/\mathbb{Q}$.
(d) There exist integers u and v such that $N = u^2 + v^2$ (resp. $N = 3u^2 + v^2$).

Proof. The statements (b), (c) and (d) are equivalent by [5, §68 and §70]. If $L = \mathbb{Q}(\sqrt{-1})$, although [5, §68] treated the case N is odd but the proof is valid in general case. If $L = \mathbb{Q}(\sqrt{-3})$, [5, §70] treated the case N is odd and not divisible by 3, but the proof is valid in general case. If (b) holds, then clearly (a) holds.

It suffices to show that if (a) holds, then (c) holds. Suppose that $\alpha_N \in L^\times$ satisfies $N = N_{L/\mathbb{Q}}(\alpha_N)$. If a prime p is inert in $L/\mathbb{Q}$, it is a prime element of L and we have $N_{L/\mathbb{Q}}(p) = p^2$. Then if $p \mid N$, we have $p \mid \alpha_N$ and $p^2 \mid N$, which contradicts that N is square-free. □

We consider an analogy of the following: if K is a real quadratic field, then a necessary and sufficient condition that the narrow ideal class of the different of $K/\mathbb{Q}$ be a square is that the discriminant D of K be the sum of two integer squares (see [13] Proposition 3.1).

Lemma 14. A necessary and sufficient condition that the narrow ideal class of $\mathfrak{d}\mathfrak{p}$ is a square for a prime ideal $\mathfrak{p}$ which has norm 3 is that D is of the form $3u^2 + v^2$ for some $u, v \in \mathbb{N}$.

Proof. Suppose that the narrow ideal class of $\mathfrak{d}\mathfrak{p}$ is a square with $N_{F/\mathbb{Q}}(\mathfrak{p}) = 3$. Then there exists $\sigma \in F_+^\times$ and a fractional ideal $\mathfrak{a}$ of F such that $(\sigma)\mathfrak{d}\mathfrak{p} = \mathfrak{a}^2$. Taking the norm of both sides, we have $3N_{F/\mathbb{Q}}(\sigma)D = A^2$, where A is the norm of $\mathfrak{a}$. Put $\sigma = s + t\sqrt{D}$ for $s, t \in \mathbb{Q}$ such that $s > 0$. Since $3(s^2 - t^2 D)D = A^2$, we have

$$D = \left(\frac{tD}{s}\right)^2 + 3\left(\frac{A}{3s}\right)^2.$$

Put $L = \mathbb{Q}(\sqrt{-3})$. Then we have $D \in N_{L/\mathbb{Q}}(L^\times)$. Lemma 13 implies that $D = 3u^2 + v^2$ for some $u, v \in \mathbb{N}$.

We assume that there exists $u, v \in \mathbb{N}$ such that $D = 3u^2 + v^2$. Since $D \equiv 1 \bmod 8$, we have $u' = u/2 \in \mathbb{Z}$. Put $\rho = (v + \sqrt{D})/2$. Then we have $N_{F/\mathbb{Q}}(\rho) = (v^2 - D)/4 = -3u'^2$. Let $\mathfrak{q} = \mathfrak{q}_Q$ be a prime ideal which divides ρ, where Q is a rational prime which is divisible by $\mathfrak{q}$. Since $\rho/Q \notin \mathfrak{o}$, we have $N_{F/\mathbb{Q}}(\mathfrak{q}) = Q$ and $\mathrm{ord}_{\mathfrak{q}}\rho = \mathrm{ord}_Q 3u'^2$. Therefore if $Q \neq 3$, $\mathrm{ord}_{\mathfrak{q}}\rho$ is even.

Since $N_{F/\mathbb{Q}}(\rho) = -3u'^2$, there exists a prime ideal $\mathfrak{q}_3$ of F which divides both 3 and ρ. Since 3 splits or ramifies in $F/\mathbb{Q}$, $\mathrm{ord}_{\mathfrak{q}_3}\rho$ is odd. Put

$$\mathfrak{a} = \prod_{\mathfrak{q} \nmid 3} \mathfrak{q}^{(\mathrm{ord}_{\mathfrak{q}}\rho)/2} \times \mathfrak{q}_3^{(\mathrm{ord}_{\mathfrak{q}_3}\rho + 1)/2}. \tag{25}$$

Then we have $(\sqrt{D}\rho)\mathfrak{q}_3 = \mathfrak{d}\mathfrak{a}^2$. Since $\sqrt{D}\rho \in F_+^\times$, we have $\mathfrak{d}\mathfrak{q}_3 = (\mathfrak{d}\mathfrak{a})^2$ in Cl^+. □

Proposition 6. Suppose that $F=\mathbb{Q}(\sqrt{D})$, where $D>1$ is a square-free integer such that $D\equiv 1 \bmod 8$.

(1) There exist $\beta\in F_+^\times$ and a fractional ideal $\mathfrak{a}$ satisfying (C1) if and only if $p\equiv 1 \bmod 4$ for all primes $p\mid D$.
(2) There exist $\beta\in F_+^\times$ and a fractional ideal $\mathfrak{a}$ satisfying (C2) if and only if $p\equiv 0$ or $1 \bmod 3$ for all primes $p\mid D$.
(3) There exists $\beta\in F_+^\times$ and a fractional ideal $\mathfrak{a}$ satisfying (C3) if and only if $D\equiv 1 \bmod 24$ and $p\equiv 1 \bmod 4$ for all primes $p\mid D$.

Proof. For a prime ideal $\mathfrak{p}$ such that $N_{F/\mathbb{Q}}(\mathfrak{p})=3$, the equation $(8\beta)\mathfrak{d}\mathfrak{p}=\mathfrak{a}^2$ implies that the narrow ideal class of $\mathfrak{d}\mathfrak{p}$ is a square. Note that a positive integer x is of the form $3u^2+v^2$ for some $u,v\in\mathbb{N}$ if and only if all primes p which divides x satisfies $p\equiv 0$ or $1 \bmod 3$. Then Lemma 14 proves the second assertion.

The equation $(8\beta)\mathfrak{d}=\mathfrak{a}^2$ implies that the narrow ideal class of $\mathfrak{d}$ is a square. Note that a positive integer x is of the form u^2+v^2 for some $u,v\in\mathbb{N}$ if and only if all primes p which divides x satisfies $p\equiv 1 \bmod 4$. Then [13] Proposition 3.1 proves the first assertion.

There exist two distinct prime ideal $\mathfrak{p}$ and $\bar{\mathfrak{p}}$ such that such that $N_{F/\mathbb{Q}}(\mathfrak{p})=N_{F/\mathbb{Q}}(\bar{\mathfrak{p}})=3$ if and only if 3 splits in $F/\mathbb{Q}$. This condition holds if and only if $D\equiv 1 \bmod 24$. In the case $D\equiv 1 \bmod 24$, we have $\mathfrak{p}\bar{\mathfrak{p}}=(3)$. Then the equation $(8\beta)\mathfrak{d}\mathfrak{p}\bar{\mathfrak{p}}=\mathfrak{a}^2$ implies that the narrow ideal class of $\mathfrak{d}$ is a square. Thus, similarly to the first assertion, [13] Proposition 3.1 proves the third assertion. □

Example 1. Put $D=73$. Then Cl^+ is trivial and the fundamental unit $\epsilon=1068+125\sqrt{D}$ of F has norm -1. By Proposition 6, there exist $\beta\in F_+^\times$ and a fractional ideal $\mathfrak{a}$ satisfying every condition of (C1), (C2) or (C3). We write $\bar{\xi}$ for the conjugate of $\xi\in F$.

Set $\beta=1/(8\epsilon\sqrt{D})$. We have $(8\beta)\mathfrak{d}=(\epsilon^{-1})=\mathfrak{o}$, where $\mathfrak{o}$ is the ring of integers of F. Suppose that $S_\infty=\emptyset$ and that $\iota_1=\mathrm{id}$. Then $(\beta,\emptyset,\mathfrak{o})\in\mathbf{G}$ satisfies (C1) and we have

$$\theta_\phi(z)=\sum_{\xi\in\mathfrak{o}} f(\iota_f(\xi))e(\beta\xi^2 z_1)e(\overline{\beta\xi^2}z_2),$$

where

$$f(\iota_f(\xi))=\prod_{v\in S_2} f_v(\iota_v(\xi)) \qquad \text{for } \xi\in\mathfrak{o}$$

with $S_2=\{v_2,\bar{v}_2 \mid \mathfrak{p}_{v_2}=(2,(1-\sqrt{D})/2), \mathfrak{p}_{\bar{v}_2}=(2,(1+\sqrt{D})/2)\}$. Assume that $\iota_{v_2}=\mathrm{id}$ and $\iota_{\bar{v}_2}(\xi)=\bar{\xi}$. Then, for $\xi\in\mathfrak{o}$, we have

$$f(\iota_f(\xi))=f_{v_2}(\xi)f_{\bar{v}_2}(\bar{\xi})=\begin{cases}1 & \text{if } \xi\in 2\mathbb{Z}+2\sqrt{D}\mathbb{Z}+1\\ -1 & \text{if } \xi\in 2\mathbb{Z}+2\sqrt{D}\mathbb{Z}+\sqrt{D}\\ 0 & \text{otherwise.}\end{cases}$$

Computing the norm of ξ shows that θ_ϕ corresponds to the function f_1 in Theorem 2 [6].

Example 2. Put $D = 793 = 13 \cdot 61$. Then Cl^+ has order 8 and the fundamental unit $4393 + 156\sqrt{D}$ of F has norm 1. By Proposition 6, there exist $\beta \in F_+^\times$ and a fractional ideal $\mathfrak{a}$ satisfying every condition of (C1), (C2) or (C3). We write $\bar{\xi}$ for the conjugate of $\xi \in F$. Put $T_3 = \{v < \infty \mid q_v = 3\}$.

(a) Let $\mathfrak{q}_7$ be a prime ideal $(7, 3+\sqrt{D})$, which divides (7) and $\mathfrak{q}_2$ a prime ideal $(2, (1-\sqrt{D})/2)$, which divides (2). Put $\rho = (3+\sqrt{D})/2$, $\beta = \rho/(8\sqrt{D}) \in F^\times$ and $\mathfrak{a} = \mathfrak{q}_2\mathfrak{q}_7$. Then we have $(8\beta)\mathfrak{d} = (\rho) = \mathfrak{a}^2$. Suppose that $S_\infty = \emptyset$ and that $\iota_1 = \mathrm{id}$. Then $(\beta, \emptyset, \mathfrak{a}) \in \mathbf{G}$ satisfies (C1). Since

$$S_2 = \{v_2, \bar{v}_2 \mid \mathfrak{p}_{v_2} = \mathfrak{q}_2, \mathfrak{p}_{\bar{v}_2} = (2)/\mathfrak{q}_2 = (2, (1+\sqrt{D})/2)\},$$

we have

$$\theta_\phi(z) = \sum_{\xi \in \mathfrak{a}^{-1}} f(\iota_f(\xi)) e(\beta\xi^2 z_1) e(\overline{\beta\xi^2} z_2),$$

where $f(\iota_f(\xi)) = f_{v_2}(\iota_{v_2}(\xi)) f_{\bar{v}_2}(\iota_{\bar{v}_2}(\xi))$.

Assume that $\iota_{v_2} = \mathrm{id}$ and $\iota_{\bar{v}_2}(\xi) = \bar{\xi}$, and we have $f(\iota_f(\xi)) = f_{v_2}(\xi) f_{\bar{v}_2}(\bar{\xi})$. Since $D = 793 \equiv 25 \bmod 128$, we assume $\sqrt{D} \equiv -5 \bmod 64\mathbb{Z}_2$ in $\mathbb{Q}_2$. For $\xi \in \mathfrak{a}^{-1} = 14^{-1}(14, (3-\sqrt{D})/2)$, we have

$$f(\iota_f(\xi)) = \begin{cases} 1 & \text{if } \xi \in 2\Delta\mathbb{Z} + 2\sqrt{D}\mathbb{Z} + \Delta + \sqrt{D} \\ -1 & \text{if } \xi \in 2\Delta\mathbb{Z} + 2\sqrt{D}\mathbb{Z} + \sqrt{D} \\ 0 & \text{otherwise,} \end{cases}$$

where $\Delta = (1 - 5\sqrt{D})/7$.

(b) Now put $\rho = 3(3+\sqrt{D})/2$, $\beta = \rho/(8\sqrt{D})$ and $\mathfrak{a} = 3\mathfrak{q}_2\mathfrak{q}_7$. Let $\mathfrak{q}_3 = (3, 1-\sqrt{D})$ and $\mathfrak{q}_3' = (3, 1+\sqrt{D})$ be the prime ideals which divides (3). Since $\mathfrak{q}_3\mathfrak{q}_3' = (3)$, we have $(8\beta)\mathfrak{d}\mathfrak{q}_3\mathfrak{q}_3' = \mathfrak{a}^2$. Suppose that $S_\infty = \{\infty_1, \infty_2\}$ and that $\iota_1 = \mathrm{id}$. Put

$$S_3 = T_3 = \{v_3, \bar{v}_3 \mid \mathfrak{p}_{v_3} = \mathfrak{q}_3, \mathfrak{p}_{\bar{v}_3} = \mathfrak{q}_3'\}.$$

Then $(\beta, S_3, \mathfrak{a}) \in \mathbf{G}$ satisfies (C3) and we have

$$\theta_\phi(z) = \sum_{\xi \in \mathfrak{a}^{-1}} f(\iota_f(\xi)) \xi\bar{\xi} e(\beta\xi^2 z_1) e(\overline{\beta\xi^2} z_2),$$

where

$$f(\iota_f(\xi)) = \prod_{v \in S_2 \cup S_3} f_v(\iota_v(\xi)) \qquad \text{for } \xi \in \mathfrak{a}^{-1}.$$

Assume that $\iota_v = \mathrm{id}$ for $v \in \{v_2, v_3\}$ and that $\iota_v(\xi) = \bar{\xi}$ for $v \in \{\bar{v}_2, \bar{v}_3\}$, and we have

$$f(\iota_f(\xi)) = f_{v_2}(\xi) f_{\bar{v}_2}(\bar{\xi}) f_{v_3}(\xi) f_{\bar{v}_3}(\bar{\xi}).$$

For $\xi \in \mathfrak{a}^{-1} = 42^{-1}(14, (3-\sqrt{D})/2)$, we have $f(\iota_f(\xi)) \in \{\pm 1\}$ if $\xi \in \mathbb{Z}_2^\times \cap \mathbb{Z}_3^\times$ and $f(\iota_f(\xi)) = 0$ if not. Since $D = 793 \equiv 1 \bmod 9$, we assume $\sqrt{D} \equiv 1$

mod $9\mathbb{Z}_3$. Then we have

$$f(\iota_f(\xi)) = \begin{cases} 1 & \text{if } \xi \in 6\Delta\mathbb{Z} + 6\sqrt{D}\mathbb{Z} \pm \{\sqrt{D}, 1\} \\ -1 & \text{if } \xi \in 6\Delta\mathbb{Z} + 6\sqrt{D}\mathbb{Z} \pm \{2\Delta + \sqrt{D}, 3\Delta + \sqrt{D}\} \\ 0 & \text{otherwise,} \end{cases}$$

where $\Delta = (1 - 5\sqrt{D})/7$.

(c) Now put $\rho = (5 + \sqrt{D})/2$. Put $\mathfrak{q}_2' = (2)/\mathfrak{q}_2 = (2, (1 + \sqrt{D})/2)$. Since $N_{F/\mathbb{Q}}(\rho) = -2^6 \cdot 3$, we have $(\rho) = \mathfrak{q}_2'^6 \mathfrak{q}_3$. Put $\beta = \rho/(8\sqrt{D}) \in F_+^\times$ and $\mathfrak{a} = \mathfrak{q}_2'^3 \mathfrak{q}_3$. Suppose that $S_\infty = \{\infty_1\}$ and that $\iota_1 = \mathrm{id}$. Put

$$S_3 = \{v_3 \mid \mathfrak{p}_{v_3} = \mathfrak{q}_3\} \subset T_3.$$

Then $(\beta, S_3, \mathfrak{a}) \in \mathbf{G}$ satisfies (C2):

$$(8\beta)\mathfrak{d}\mathfrak{q}_3 = \mathfrak{a}^2$$

and we have

$$\theta_\phi(z) = \sum_{\xi \in \mathfrak{a}^{-1}} f(\iota_f(\xi)) \xi e(\beta\xi^2 z_1) e(\overline{\beta\xi^2} z_2),$$

where

$$f(\iota_f(\xi)) = \prod_{v \in S_2 \cup S_3} f_v(\iota_v(\xi)) \qquad \text{for } \xi \in \mathfrak{a}^{-1}.$$

Assume that $\iota_v = \mathrm{id}$ for $v \in \{v_2, v_3\}$ and that $\iota_v(\xi) = \bar{\xi}$ for $v \in \{\bar{v}_2\}$ and we have

$$f(\iota_f(\xi)) = f_{v_2}(\xi) f_{\bar{v}_2}(\bar{\xi}) f_{v_3}(\xi).$$

Then, for $\xi \in \mathfrak{a}^{-1} = 24^{-1}(24, (5 - \sqrt{D})/2)$, we have

$$f(\iota_f(\xi)) = \begin{cases} 1 & \text{if } \xi \in 2(1 - \sqrt{D})\mathbb{Z} + 6\sqrt{D}\mathbb{Z} + \{1, -\sqrt{D}\} \\ -1 & \text{if } \xi \in 2(1 - \sqrt{D})\mathbb{Z} + 6\sqrt{D}\mathbb{Z} + \{-1, \sqrt{D}\} \\ 0 & \text{otherwise.} \end{cases}$$

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