Percolation on crystal lattices and covering monotonicity of percolation clusters

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Chapter 1

Introduction

1.1 Background

1.1.1 Bond percolation model

Percolation theory is a branch of probability theory that describes the behavior of clusters, which are connected components of randomly obtained objects. The theory has its origin in applied problems. One of the most famous mathematical formulations is the modeling of immersion in porous stone, which is represented by the following model: Let $\mathbb{L}^d = (\mathbb{Z}^d, \mathbb{E}^d)$ be the d-dimensional cubic lattice, where $\mathbb{Z}^d$ is the set of all d-tuples $x = (x_1, \ldots, x_d)$ of integers x_i, and the edge set $\mathbb{E}^d$ is the set of all unordered pairs $\{x, y\}$ of $\mathbb{Z}^d$ with $\|x - y\|_1 = 1$ (Here, $\| \cdot \|_q$ represents the L_q-norm on $\mathbb{R}^d$). For a fixed $p \in [0, 1]$, each edge (bond) in $\mathbb{L}^d$ is assumed to be open with probability p, and closed otherwise, independently of all other edges. Open edges correspond to voids randomly generated in the stone, and the probability p means the proportion of voids in the stone. Let $C = C(0)$ be the cluster containing the origin $0 \in \mathbb{Z}^d$ in the subgraph consisting of all open edges. The immersion of water into the center of the stone is expressed as the infiniteness of the cluster C (Figure 1.1). This model is called the *bond percolation model*.

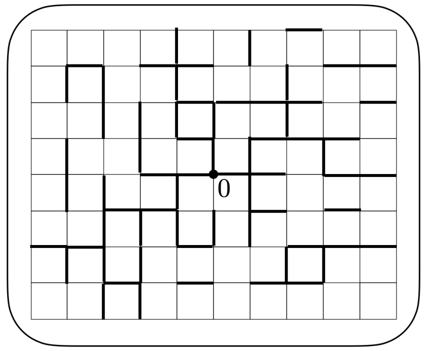

Figure 1.1: Sketch of the model of porous stone. The thick lines indicate open edges.

For each $p \in [0,1]$, the *percolation probability* $\theta(p)$ is defined by

$$\theta(p) = \mathbb{P}_p(C \text{ is an infinite cluster}),$$

where $\mathbb{P}_p$ expresses the probability measure constructed as the product measure of those from all bonds. The percolation probability $\theta(p)$ is a non-decreasing function with respect to p and is positive when $p \in [0,1]$ is greater than a certain value $p_c(\mathbb{L}^d)$. This value $p_c(\mathbb{L}^d)$ is called the *critical probability*, which is formally defined by

$$p_c(\mathbb{L}^d) := \sup\{p \in [0,1] \,:\, \theta(p) = 0\}.$$

One of the remarkable properties of this model is the *phase transition*, which appears as the difference in the connectivity of two vertices. Here, we denote by $x \leftrightarrow y$ the event that two points $x,\, y \in \mathbb{Z}^d$ are connected by an open path.

Theorem 1.1. If $p > p_c(\mathbb{L}^d)$, then there exists a constant $c := c(p) > 0$ such that

$$\mathbb{P}_p(x \leftrightarrow y) \geq c \text{ for any } x, y \in \mathbb{Z}^d. \tag{1.1}$$

If $p < p_c(\mathbb{L}^d)$, then there exists $\sigma := \sigma(p) > 0$ such that

$$\mathbb{P}_p(x \leftrightarrow y) \leq \exp(-\sigma\|x-y\|_1) \text{ for any } x, y \in \mathbb{Z}^d. \tag{1.2}$$

The estimate (1.1) follows from the uniqueness of the infinite cluster [1]. The exponential decay (1.2) follows from the paper [25] by Menshikov. By combining (1.2) with a basic discussion, we can see that the limit

$$\xi(p) := -\lim_{n\to\infty} \frac{n}{\log \mathbb{P}_p(0 \leftrightarrow na_1)} \tag{1.3}$$

exists for $0 < p < p_c(\mathbb{L}^d)$, where $a_1 = (1, 0, \dots, 0) \in \mathbb{Z}^d$. The limit $\xi(p)$, which is called the *correlation length*, represents the "connectability" of two distant vertices. The correlation length $\xi(p)$ is strictly increasing on $(0, p_c(\mathbb{L}^d))$ and satisfies

$$\xi(p) \longrightarrow \begin{cases} 0 & \text{as } p \downarrow 0, \\ \infty & \text{as } p \uparrow p_c(\mathbb{L}^d). \end{cases}$$

A natural alternative model to the bond percolation model is the *site percolation model*, in which each vertex (site) of $\mathbb{L}^d$ is assumed to be open with the same probability $p \in [0, 1]$ independently. The critical probability $p_c^{\mathrm{site}}(\mathbb{L}^d)$ is defined in the same way as in the bond percolation model.

1.1.2 Critical probability

The bond and site percolation models can be defined on an arbitrary graph which is connected, infinite, and locally finite (i.e., every vertex has finite degree). In this thesis, we denote by $p_c(X)$ (resp. $p_c^{\mathrm{site}}(X)$) the critical probability of the bond (resp. site) percolation model on a graph X. In general, the comparison $p_c(X) \leq p_c^{\mathrm{site}}(X)$ holds for X (see, e.g., [15, Section 1.6]). Because of the phase transition represented by Theorem 1.1, there have been a lot of interest in calculating the critical probability.

In the textbook [19] on percolation theory, Kesten discusses prototypes of the graphs with which we shall work, and formulates a *periodic graph*.

Definition 1.2. A connected graph X with no loops is called a *periodic graph* if the following conditions hold:

(1) X is embedded in $\mathbb{R}^d$ in such a way that each coordinate vector of $\mathbb{R}^d$ is a period for the image;

(2) there exists a constant $K < \infty$ such that the degree of each vertex is at most K; and

(3) all edges of X have finite length. Every compact set of $\mathbb{R}^d$ intersects only finitely many edges of X.

The cubic lattice $\mathbb{L}^d$, the triangular lattice $\mathbb{T}$ and the honeycomb lattice $\mathbb{H}$ are examples of periodic graphs. Except for certain two-dimensional lattices, the exact value of the critical probability has not been obtained. As exceptional cases, Kesten [20] showed that

$$p_c(\mathbb{L}^2) = \frac{1}{2}. \tag{1.4}$$

The critical probabilities of $\mathbb{T}$ and $\mathbb{H}$ are obtained by [28] as

- $p_c(\mathbb{T}) = 2\sin(\pi/18)$ and
- $p_c(\mathbb{H}) = 1 - p_c(\mathbb{T}) = 1 - 2\sin(\pi/18)$.

For the calculation of these critical probabilities, the concept of "duality" plays an important role. Here, the dual graph X_{dual} of a plane graph X is the graph whose vertices are faces of X. The adjacency relation of two vertices of X_{dual} is determined by that of two faces of X (Figure 1.2).

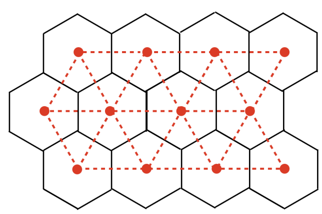

Figure 1.2: Honeycomb lattice (black) and its dual graph (red).

Under certain conditions of symmetry on X, it can be shown that

$$p_c(X) + p_c(X_{\text{dual}}) = 1. \tag{1.5}$$

This is actually used to obtain the critical probabilities $p_c(\mathbb{T})$ and $p_c(\mathbb{H})$. Also, by combining (1.5) with the self-duality of $\mathbb{L}^2$, we obtain (1.4). By a similar argument as for obtaining (1.5), the equality

$$p_c^{\text{site}}(\mathbb{T}) = \frac{1}{2}$$

can also be obtained.

While there are several techniques to calculate the critical probabilities of two-dimensional graphs, there is a dearth of such techniques for graphs with three or more dimensions. In fact, the exact value of the critical probability of such graphs has not been obtained, and only an evaluation has been given. One simple way of obtaining an estimate is to focus on the inclusion relation of graphs. For example, the two-dimensional cubic lattice $\mathbb{L}^2$ may be embedded in $\mathbb{L}^3$ in a natural way. With this embedding, if the origin of $\mathbb{L}^2$ belongs to an infinite cluster for a particular value of p, then it is also an infinite cluster of $\mathbb{L}^3$. Thus we have

$$p_c^{\mathrm{site}}(\mathbb{L}^3) \leq p_c^{\mathrm{site}}(\mathbb{L}^2). \tag{1.6}$$

As a stronger result than (1.6), Campanino and Russo [6] give the comparison

$$p_c^{\mathrm{site}}(\mathbb{L}^3) < p_c^{\mathrm{site}}(\mathbb{T}) = \frac{1}{2},$$

by focusing on the "covering" relation of $\mathbb{L}^3$ and $\mathbb{T}$. This idea is formulated in [4] in a more general context as follows: Let $G \curvearrowright X$ be a free action of a group G on a graph X. The quotient graph $X_1 := X/G$ is defined as the graph whose vertices are G-orbits, and an edge $\{Gx, Gy\}$ appears in X_1 if there are representatives $x_0 \in Gx$, $y_0 \in Gy$ that are neighbors in X (Figure 1.3).

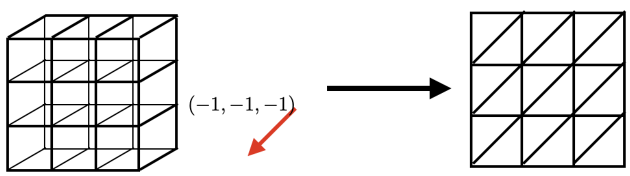

Figure 1.3: Example of a quotient graph. The triangular lattice $\mathbb{T}$ is the quotient graph by the action $\mathbb{Z} \curvearrowright \mathbb{L}^3$ generated by translation by the vector $(-1, -1, -1)$.

In this setting, the comparison

$$p_c(X) \leq p_c(X_1) \tag{1.7}$$

of two critical probabilities holds. This result implies that the percolation cluster in a graph is in some sense larger than that in its quotient graph. It is known from [24] that the strictness of (1.7) is established as follows.

Proposition 1.3 ([24, Theorem 0.1])**.** If X and X_1 are quasi-transitive and $p_c(X) < 1$, then $p_c(X) < p_c(X_1)$.

The idea of covering is also applied to the numerical study [30], which gives the upper bounds for the critical probabilities $p_c(\mathbb{L}^3)$ and $p_c(\mathbb{B})$ as

$$p_c(\mathbb{L}^3) \leq 0.347926 \text{ and } p_c(\mathbb{B}) \leq 0.292893,$$

where $\mathbb{B}$ represents the body-centered cubic lattice.

1.1.3 First passage percolation model and shape theorem

In order to observe the relationship between graphs and percolation clusters in detail, this thesis applies the idea of covering to the *first passage percolation* (FPP) model. The FPP model, which was introduced in 1965 by Hammersley and Welsh [17], is a time evolution version of the bond percolation model: each edge $e \in \mathbb{E}^d$ of the d-dimensional cubic lattice is independently assigned a random nonnegative time $t_e \geq 0$ according to a fixed distribution ν. The distribution ν, which is a Borel probability measure on $[0, \infty)$, is called the *time distribution*. The passage time $T(\gamma)$ of a path $\gamma = (e_1, \ldots, e_r)$ is defined as the sum $T(\gamma) := \sum_{i=1}^{r} t_{e_i}$. For two points x, $y \in \mathbb{R}^d$, we denote by $T(x, y)$ the first passage time from x to y, that is,

$$T(x, y) := \inf\{T(\gamma) \ : \ \gamma \text{ is a path from } x' \text{ to } y'\},$$

where x' and y' are the closest lattice points of x and y, respectively, with a deterministic rule to break ties. As a counterpart of the cluster C in the bond percolation model, we consider the *percolation region* $B(t)$, which is defined as

$$B(t) := \{x \in \mathbb{R}^d \ : \ T(0, x) \leq t\}$$

for a time $t \geq 0$.

Cox and Durrett [8] showed the *shape theorem*, namely a "law of large numbers" for the percolation region under the moment condition

$$\mathbb{E} \min\{t_1, \ldots, t_{2d}\}^d < \infty. \tag{1.8}$$

Here, $\mathbb{E}$ denotes the expectation and random variables $t_1, \ldots, t_{2d} \sim \nu$ are independent copies of random times.

Theorem 1.4 ([8])**.** Suppose that the time distribution ν satisfies the condition (1.8). Then either of the following holds:

(a) There exists a deterministic, convex, compact set $\mathcal{B}$ in $\mathbb{R}^d$ such that for each $\epsilon > 0$, it holds almost surely that

$$(1 - \epsilon)\mathcal{B} \subset \frac{B(t)}{t} \subset (1 + \epsilon)\mathcal{B} \text{ for all large } t.$$

(b) For all $R > 0$, it holds almost surely that

$$\{x \in \mathbb{R}^d \ : \ \|x\|_1 \leq R\} \subset \frac{B(t)}{t} \text{ for all large } t.$$

From the additional discussion shown in the textbook [18], whether the case (a) or (b) occurs is determined by the probability $\nu(0)$ that a random time is equal to 0: the case (a) occurs when $\nu(0) < p_c(\mathbb{L}^d)$, while the case (b) occurs when $\nu(0) \geq p_c(\mathbb{L}^d)$.

The set $\mathcal{B} \subset \mathbb{R}^d$ in the case (a) is called the *limit shape*. Let us observe the shape of $\mathcal{B}$. In this case, the limit shape $\mathcal{B}$ is given by the unit ball with respect to some norm μ on $\mathbb{R}^d$, which is the reason for the convexity of $\mathcal{B}$. Let $R := \mu(a_1)^{-1}$ with $a_1 := (1, 0, \ldots, 0) \in \mathbb{Z}^d$. Combining the convexity with the symmetric property of $\mathcal{B}$, we obtain the following inclusions (Figure 1.4):

$$\{x \in \mathbb{R}^d \ : \ \|x\|_1 \leq R\} \subset \mathcal{B}, \tag{1.9}$$

$$\mathcal{B} \subset \{x \in \mathbb{R}^d \ : \ \|x\|_\infty \leq R\}. \tag{1.10}$$

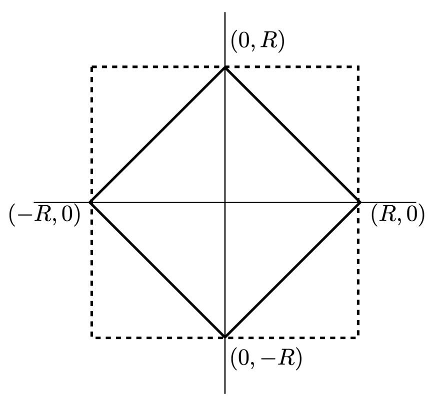

Figure 1.4: Case $d = 2$. The limit shape $\mathcal{B}$ lies between the solidly drawn diamond and the dashed square.

The equality of (1.9) holds when the time distribution ν is a Dirac measure δ_1 at 1, which means that each random time t_e has the value 1 almost surely. Indeed, for each $t \in \mathbb{Z}_{\geq 0}$, the percolation region $B(t)$ consists of the vertices $x \in \mathbb{Z}^d$ with $\|x\|_1 \leq t$. Thus we can observe that the limit shape $\mathcal{B}$ is the unit ball with respect to the norm $\|\cdot\|_1$ (Figure 1.5).

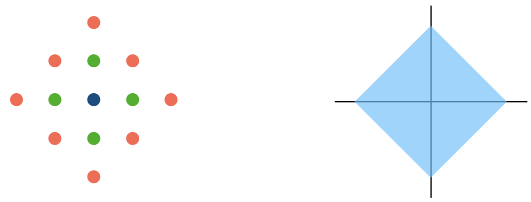

Figure 1.5: Sketch of $B(2)$ (left) and the limit shape $\mathcal{B}$ (right).

If the time distribution ν is not a Dirac measure, then the percolation region $B(t)$ tends to spread out in the diagonal direction, which is intuitively inferred from the following observation: Let us focus on two vertices $x, y \in \mathbb{Z}^2$ with $\|x\|_1 = \|y\|_1$ such that x is on the diagonal and y is on the axis. In this case, the number of shortest paths (in terms of graph distance) from the origin to x is greater than that from the origin to y. Thus the first passage time $T(0, x)$ tends to be smaller than $T(0, y)$.

In fact, the simulation study [2] of the FPP model on $\mathbb{L}^2$ shows that the limit shape comes closer to the Euclidean ball for time distributions with larger variability. However, it is unlikely that the limit shape is really the Euclidian ball since [9] shows that for any ν with finite first moment, the limit shape is not the Euclidean ball for sufficiently large d.

It is an open question as to which compact convex sets are realizable as the limit shape, and even the following has not yet been proved:

- there exist no time distributions ν such that the equality of the inclusion (1.10) hold.

Except for a few studies on the triangular lattice model (e.g., [10, 29]), most of the work on the FPP model on periodic graphs has been done as the cubic lattice model.

1.1.4 Crystal lattices

Several studies have been made on the formulation of periodic graphs in the context of percolation theory as exemplified by Kesten [19], which we have reviewed in the previous section. Grimmett [16] gives a formulation of a "d-dimensional lattice" by group action on a graph and summarizes its basic properties. Despite such studies, few results have been published on percolation models formulated on general periodic graphs.

This thesis uses the formulation of "crystal lattices," which was introduced by Kotani and Sunada [21]. Though the definition of a crystal lattice itself is equivalent to those above (see Section 2.1), it has several interesting properties regarding the study of percolation theory.

A d-dimensional *crystal lattice* is a regular covering graph X over a finite graph X_0 whose covering transformation group L is a free abelian group with rank d. This definition is equivalent to the following: a graph X is a d-dimensional crystal lattice if there exists a free action $L \curvearrowright X$ of a free abelian group L with rank d such that the quotient graph $X_0 = X/L$ is finite. The finite graph X_0 is called a *base graph.* Figure 1.6 shows examples of crystal lattices.

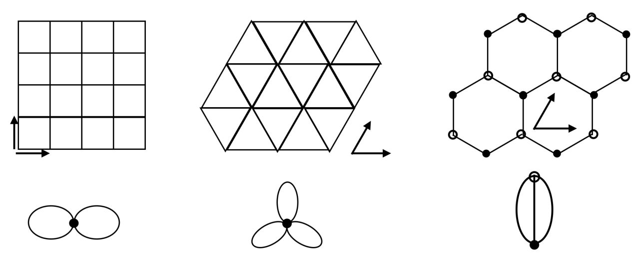

Figure 1.6: Cubic lattice (left), triangular lattice (center) and honeycomb lattice (right). The arrows indicate a basis of the action on each lattice, and the graphs below are their base graphs.

The "shape" of a d-dimensional crystal lattice $X = (V, E)$ is defined as a map from X to $\mathbb{R}^d$. A map $\mathbf{\Phi} : X \to \mathbb{R}^d$ is called a *periodic realization* if there exists an injective homomorphism $\rho : L \to \mathbb{R}^d$ satisfying the following conditions:

- the image $\Gamma := \rho(L)$ is a lattice group of $\mathbb{R}^d$; and
- $\Phi(\sigma x) = \Phi(x) + \rho(\sigma)$ holds for any $x \in V$ and $\sigma \in L$.

Note that the realized crystal $\mathbf{\Phi}(X)$ is invariant under translation by any vector $\mathbf{b} \in \Gamma = \rho(L)$. The homomorphism $\rho : L \to \mathbb{R}^d$ is called the *period homomorphism* of the realization $\mathbf{\Phi} : X \to \mathbb{R}^d$. Though the period ρ is determined uniquely from $\mathbf{\Phi}$, we often write a periodic realization as the pair $(\mathbf{\Phi}, \rho)$ in order to emphasize that ρ represents the period of the realization. Figure 1.7 shows examples of periodic realizations of the honeycomb lattice.

Here, the arrows show the basis of the lattice group. Note that the left and center realizations have the same lattice group and thus the same period.

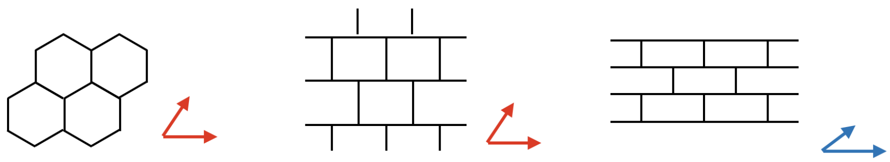

Figure 1.7: Three examples of periodic realizations of the honeycomb lattice.

Periodic realization of crystal lattices has an interesting property with respect to covering, which can be stated as follows: Let $\Phi : X \to \mathbb{R}^d$ be a periodic realization of a d-dimensional crystal lattice X. By identifying some suitable d_1-dimensional subspace of $\mathbb{R}^d$ with $\mathbb{R}^{d_1}$, we observe that the orthogonal projection $P(\Phi(X))$ onto $\mathbb{R}^{d_1}$ coincides with the image of a periodic realization $\Phi_1 : X_1 \to \mathbb{R}^{d_1}$ of a d_1-dimensional crystal lattice X_1. We also see that X_1 can be expressed by X/G for some group action $G \curvearrowright X$, and that the following commutative property holds:

$$\begin{tikzcd} X \arrow[r, "\Phi"] \arrow[d, "\omega"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathbb{R}^d \arrow[d, "P"] \\ X_1 \arrow[r, "\Phi_1"'] & \mathbb{R}^{d_1}, \end{tikzcd} \tag{1.11}$$

where $\omega : X \to X_1$ is the quotient map and $P : \mathbb{R}^d \to \mathbb{R}^{d_1}$ is the orthogonal projection (Figure 1.8).

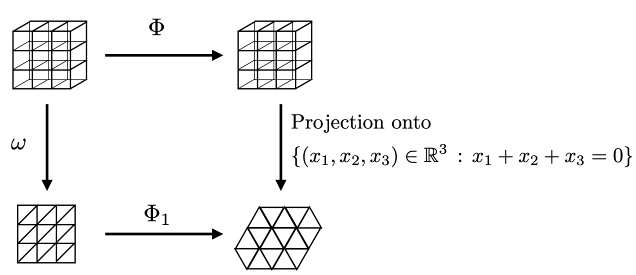

Figure 1.8: Example of a projective relation. By the orthogonal projection $P : \mathbb{R}^3 \to \mathbb{R}^2 \simeq \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 + x_2 + x_3 = 0\}$, the realization of $\mathbb{L}^3$ is projected onto that of $\mathbb{T}$.

The study of crystal lattices comes from discrete geometric analysis (see, e.g., [27]) and is rich in considerations of the shape of lattices. A typical example is the concept of the *standard realization* [21], which is the periodic realization with maximal symmetry among all realizations. When considering percolation models on the triangular or the cubic lattices, it is often implicitly assumed that these lattices are naturally realized in $\mathbb{R}^d$, which is equivalent to a standard realization. Thus, the formulation of crystal lattices is suitable for studying the relationship between the shape of the percolation clusters and the lattices.

1.2 Contributions

This thesis studies the bond percolation and FPP models defined on crystal lattices. The aim of this thesis is to study the relationship between the shapes of percolation clusters and crystal lattices. In particular, motivated by the comparison (1.7), we show the covering monotonicity of percolation clusters.

The main results of this thesis can be divided into the following three main categories. The precise definitions of some terminologies will be given in the later chapters.

1.2.1 Generalization of the shape theorem

Fix a periodic realization (Φ, ρ) of a d-dimensional crystal lattice $X = (V, E)$. Let $(t_e : e \in E)$ be a family of i.i.d. random times with a time distribution ν. The percolation region $B(t)$ is defined in the same way as the cubic lattice model. We assume the following moment condition:

$$\mathbb{E} \min\{t_1, \ldots, t_{l_X}\}^d < \infty, \tag{1.12}$$

where l_X is the edge connectivity of X and $t_1, \ldots, t_{l_X} \sim \nu$ are independent copies of t_e. This is a generalization of the moment condition (1.8) for the cubic lattice model. The first main theorem of this thesis is given by the following, which is a generalization of the shape theorem.

Theorem 1.5. Let (Φ, ρ) be a periodic realization of a d-dimensional crystal lattice X. Suppose that the time distribution ν satisfies (1.12). Then the following hold:

(a) If $\nu(0) < p_c(X)$, then there exists a deterministic, convex, compact set $\mathcal{B} \subset \mathbb{R}^d$ such that for each $\epsilon > 0$, it holds almost surely that

$$(1-\epsilon)\mathcal{B} \subset \frac{B(t)}{t} \subset (1+\epsilon)\mathcal{B} \text{ for all large } t.$$

(b) If $\nu(0) \geq p_c(X)$, then for all $R > 0$, it holds almost surely that

$$\{x \in \mathbb{R}^d \ : \ \|x\|_1 \leq R\} \subset \frac{B(t)}{t} \text{ for all large } t.$$

Here, the limit shape $\mathcal{B}$ is given by the unit ball with respect to the norm μ in Proposition 3.3. Throughout this thesis, the limit shape $\mathcal{B}$ is assumed to be the whole space $\mathbb{R}^d$ in the case (b).

Some basic properties of the limit shape $\mathcal{B}$ are given in Section 3.2.2. One remarkable property is that the limit shape $\mathcal{B}$ depends only on X, ν and the period ρ. For example, when we consider the FPP models on the periodic realizations of the honeycomb lattice shown in Figure 1.7, the limit shapes obtained from the left and center realizations are the same, although the realizations are different.

1.2.2 Covering monotonicity of the limit shapes

The second result is derived from the comparison (1.7). By using the projective relation (1.11) of two crystal lattices, this thesis gives the covering monotonicity of the limit shapes.

Theorem 1.6. Let $\Phi : X \to \mathbb{R}^d$ and $\Phi_1 : X_1 \to \mathbb{R}^{d_1}$ be periodic realizations of crystal lattices X and X_1 satisfying the projective relation (1.11). Assume that ν satisfies the moment condition (1.12) for X and X_1. Then the following holds for the limit shapes $\mathcal{B}$ and $\mathcal{B}_1$:

$$\mathcal{B}_1 \subset P(\mathcal{B}).$$

This result gives insights regarding the limit shape of the cubic FPP model: To observe the shape of $\mathcal{B}$ of the cubic lattice $\mathbb{L}^d$, we can consider the projection $P : \mathbb{R}^d \to \mathbb{R}^{d_1}$ in some suitable direction and obtain a periodic realization of a covered lattice X_1. Then we obtain $\mathcal{B}_1 \subset P(\mathcal{B})$ for the limit shape $\mathcal{B}_1$ of X_1, implying that the projection of the limit shape $\mathcal{B}$ of $\mathbb{L}^d$ to the space $\mathbb{R}^{d_1}$ is bounded below by the limit shape $\mathcal{B}_1$ of X_1 (Figure 1.9). An application of Theorem 1.6 is given in Section 3.3.1.

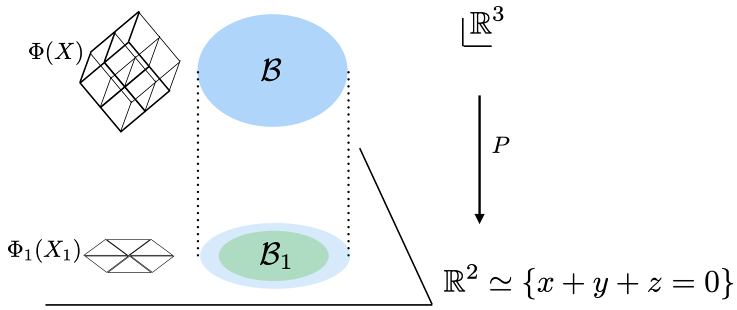

Figure 1.9: Limit shapes $\mathcal{B}$ and $\mathcal{B}_1$.

Note that Theorem 1.5 implies that the limit shape $\mathcal{B}$ (resp. $\mathcal{B}_1$) is bounded if and only if $\nu(0) < p_c(X)$ (resp. $\nu(0) < p_c(X_1)$). From this, we can see that for two crystal lattices X, X_1 in Theorem 1.6, the comparison (1.7) easily follows from Theorem 1.6.

Corollary 1.7. $p_c(X) \leq p_c(X_1)$.

Proof. For each $p \in [0,1]$, the time distribution $\nu := p\delta_0 + (1-p)\delta_1$, where δ_a is the Dirac measure at $a \in \mathbb{R}$, satisfies the condition (1.12) for X and X_1. Thus we have

$$\begin{aligned} p = \nu(0) < p_c(X) &\Longleftrightarrow \mathcal{B} \text{ is bounded} \\ &\Longrightarrow \mathcal{B}_1 \text{ is bounded} \Longleftrightarrow p < p_c(X_1), \end{aligned}$$

which implies Corollary 1.7. Here, the second implication follows from Theorem 1.6. □

1.2.3 Covering monotonicity of the inverse correlation length

An analogy of Theorem 1.6 can be obtained in the subcritical bond percolation model. Consider the bond percolation model on a periodic realization $\Phi : X \to \mathbb{R}^d$ of a crystal lattice X with probability $0 < p < p_c(X)$. The correlation length (1.3) also exists in this model. We can show that the inverse correlation length, which is defined as the limit

$$\varphi_X(x) = -\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}_p(0 \leftrightarrow nx) \qquad (x \in \mathbb{R}^d),$$

is a norm on $\mathbb{R}^d$. We will state in Proposition 4.7 that the norm $\varphi_X(\cdot)$ depends only on X, p and the period ρ as well as the limit shape in Theorem 1.5.

As an analogue of the limit shape $\mathcal{B}$ in Theorem 1.5, we define C as the unit ball with respect to the norm φ_X. This ball represents the region where the cluster $C = C(0)$ can spread within a certain probability cost (Remark 4.13). This thesis shows that the analogue of Theorem 1.6 holds for this unit ball C.

Theorem 1.8. Let $\Phi : X \to \mathbb{R}^d$ and $\Phi_1 : X_1 \to \mathbb{R}^{d_1}$ be periodic realizations of crystal lattices X and X_1 satisfying the projective relation (1.11). Suppose $0 < p < p_c(X) \leq p_c(X_1)$. Then the following holds for the unit balls C, C_1 with respect to the norms φ_X, φ_{X_1}, respectively:

$$C_1 \subset P(C).$$

For the proof of Theorem 1.8, this thesis gives a generalization of the large deviation result for the cluster in $\mathbb{L}^d$ [7, 22] to a crystal lattice model, which is stated in Theorem 4.11.

Chapter 2

Preliminaries

2.1 Crystal lattices

A crystal lattice, which is the main object of this thesis, is defined as a covering graph over a finite graph. In this section, we review the concept of covering graphs and crystal lattices. We refer to [27] for a more detailed description.

2.1.1 Graphs

A *graph* is an ordered pair $X = (V, E)$ of disjoint sets V and E with two maps $i : E \to V \times V$ and $\iota : E \to E$ satisfying

$$\begin{aligned} &\iota^2 = I_E \text{ (the identity map of } E\text{) and} \\ &\iota(e) \neq e,\ i(\iota(e)) = \tau(i(e)) \end{aligned}$$

for any $e \in E$, where $\tau : V \times V \to V \times V$ is the map defined by $\tau(x, y) = (y, x)$. Elements of V are called *vertices* of X, and elements of E are called *edges* of X. We call i and ι the *incident map* and the *inversion map* of X, respectively. We put $i(e) = (o(e), t(e))$ and call $o(e)$ and $t(e)$ the *origin* and the *terminus*, respectively. The edge $\iota(e)$ is called the *inversion* of e and is sometimes written as $\bar{e}$. For $x \in V$, we write $E_x := \{e \in E\ :\ o(e) = x\}$.

For two graphs $X_1 = (V_1, E_1)$ and $X_2 = (V_2, E_2)$, a *morphism* $f : X_1 \to X_2$ is a pair $f = (f_V, f_E)$ of two maps $f_V : V_1 \to V_2$, $f_E : E_1 \to E_2$ satisfying

$$\begin{aligned} &i(f_E(e)) = (f_V(o(e)), f_V(t(e))), \\ &f_E(\bar{e}) = \overline{f_E(e)}. \end{aligned}$$

When both f_V and f_E are bijective, the morphism f is called an *isomorphism.* We abbreviate f_V and f_E as f when there is no confusion. For a graph X, we denote by $\mathrm{Aut}(X)$ the automorphism group of X.

A graph $X = (V, E)$ is said to be *transitive* if for any vertices $x, y \in V$, there exists an automorphism $\sigma \in \mathrm{Aut}(X)$ with $\sigma x = y$. More generally, X is said to be *quasi-transitive* if V can be divided into a finite number of vertex sets $V_1, \ldots, V_m$ such that for any $k \in \{1, ..., m\}$ and $x, y \in V_k$, there exists an automorphism $\sigma \in \mathrm{Aut}(X)$ with $\sigma x = y$.

An *action* $G \curvearrowright X$ of a group G on a graph X is a group homomorphism $h : G \to \mathrm{Aut}(X)$, which naturally gives rise to actions $G \curvearrowright V$ and $G \curvearrowright E$ by $gx := h(g)(x)$ for $x \in V$ and $ge := h(g)(e)$ for $e \in E$, respectively.

We say that G acts on X *freely* when the action $G \curvearrowright V$ is free and $ge \neq \bar{e}$ for any $g \in G$ and $e \in E$. For a free action $G \curvearrowright X$, we define the *quotient graph* X/G of X as the pair $X/G = (V/G, E/G)$ of the orbit spaces V/G and E/G whose incident and inversion maps are induced from those of X.

Remark 2.1. This formulation of quotient graphs is more inclusive than that introduced in Section 1.1.2, in the sense that it allows graphs with parallel edges and loops.

2.1.2 Covering graphs

Let $X = (V, E)$ and $X_0 = (V_0, E_0)$ be connected graphs. A morphism $\omega : X \to X_0$ is called a *covering map* if

- ω_V is surjective; and
- for every $x \in V$, the restriction $\omega_{E}\restriction_{E_x} : E_x \to E_{0,\omega(x)}$ is bijective.

The graph X is called a *covering graph* of X_0. The *covering transformation group* $G(\omega)$ of a covering map ω is the set of automorphisms $\sigma \in \mathrm{Aut}(X)$ with $\omega \circ \sigma = \omega$. We say that a covering map $\omega : X \to X_0$ is *regular* if for any $x, y \in V$ with $\omega(x) = \omega(y)$, there exists a transformation $\sigma \in G(\omega)$ such that $\sigma x = y$.

A *path* γ in a graph X is a sequence $\gamma = (e_1, e_2, \ldots, e_r)$ of edges with $o(e_{i+1}) = t(e_i)$ for $i = 1, 2, \ldots, r-1$. One of the most basic properties of a covering map $\omega : X \to X_0$ is the *unique path-lifting property*: for any path $\gamma_0 = (e_{0,1}, e_{0,2}, \ldots, e_{0,r})$ in X_0 and a vertex $x \in X$ with $\omega(x) = o(e_{0,1})$, there exists a unique path $\gamma = (e_1, e_2, \ldots, e_r)$ in X, called a *lifting* of γ_0, such that $o(e_1) = x$ and $\omega(e_i) = e_{0,i}$ for $i = 1, 2, \ldots, r$.

Taking a quotient can be characterized by a covering map. We can see that for a regular covering map $\omega : X \to X_0$, the action $G(\omega) \curvearrowright X$ of the covering transformation group $G(\omega)$ is free and its quotient graph $X/G(\omega)$ is isomorphic to X_0. On the other hand, the following theorem holds.

Theorem 2.2 ([27, Theorem 5.2])**.** Suppose that a group G acts freely on a graph X. Then the canonical morphism $\omega : X \to X/G$ is a regular covering map whose covering transformation group is G.

We give a proposition for the composition of covering maps. Note that, for two covering maps $\omega : X \to X_1$ and $\omega' : X_1 \to X_0$, we can check that the composition $\tilde{\omega} := \omega' \circ \omega : X \to X_0$ is also a covering map, and that the covering transformation group $G(\omega)$ is a subgroup of $G(\tilde{\omega})$.

Proposition 2.3 ([27, Theorem 5.5])**.** Let $\omega : X \to X_1$, $\omega' : X_1 \to X_0$ be covering maps. Suppose that ω and the composition $\tilde{\omega} = \omega' \circ \omega$ are regular and that $G(\omega)$ is a normal subgroup of $G(\tilde{\omega})$. Then the covering map ω' is regular and the covering transformation group $G(\omega')$ is isomorphic to the factor group $G(\tilde{\omega})/G(\omega)$.

In the above proposition, the isomorphism $G(\tilde{\omega})/G(\omega) \simeq G(\omega')$ is obtained from the map $\pi : G(\tilde{\omega}) \to G(\omega')$ defined as

$$\pi(\sigma)\omega(x) = \omega(\sigma x) \quad (x \in V) \tag{2.1}$$

and

$$\pi(\sigma)\omega(e) = \omega(\sigma e) \quad (e \in E)$$

for $\sigma \in G(\tilde{\omega})$. We refer to the proof of [27, Theorem 5.5] for the well-definedness of this map.

Tree-lifting property

From the unique path-lifting property, we can consider a lifting of trees as follows: Let $\omega : X \to X_0$ be a regular covering over a finite graph X_0. Let $\mathcal{T}_0$ be a spanning tree of X_0. Fix two vertices $x_0 \in X_0$ and $x \in X$ with $\omega(x) = x_0$. Then there exists a unique subtree $\mathcal{T} \subset X$, which we call a *lifting* of $\mathcal{T}_0$, satisfying $x \in \mathcal{T}$ and the restriction $\omega_{\restriction \mathcal{T}} : \mathcal{T} \to \mathcal{T}_0$ of ω is an isomorphism. Indeed, we can construct $\mathcal{T}$ as

$$\mathcal{T} := \bigcup_i \gamma_i,$$

where each path γ_i is the lifting of the unique path in $\mathcal{T}_0$ from x_0 to each leaf y_0^i, a vertex of $\mathcal{T}_0$ with degree 1. The uniqueness follows from the unique path-lifting property. For this tree $\mathcal{T}$ and $\sigma \in G(\omega)$, the translation $\sigma(\mathcal{T})$, denoted by $\mathcal{T}_\sigma$, is the unique lifting of $\mathcal{T}_0$ containing σx. The following proposition states that the vertex set of a covering graph can be represented as the array of a spanning tree of X_0.

Proposition 2.4. Let $\omega : X = (V, E) \to X_0$ be a regular covering over a finite graph X_0. For a spanning tree $\mathcal{T}_0 \subset X_0$ and its lifting $\mathcal{T} \subset X$, the following holds:

$$V = \bigsqcup_{\sigma \in G(\omega)} \mathcal{T}_\sigma.$$

Proof. Take $y \in V$ arbitrarily. For $y_0 := \omega(y)$, we can find $y' \in \mathcal{T}$ with $\omega(y') = y_0$ since $\omega_{\restriction \mathcal{T}}$ is a bijection. From the regularity of ω, there exists $\sigma \in G(\omega)$ such that $y = \sigma(y')$, which implies $y \in \mathcal{T}_\sigma$. Thus $V \subset \cup_{\sigma \in G(\omega)} \mathcal{T}_\sigma$. The disjointness of the right hand side follows from the uniqueness of the lifting tree. $\square$

2.1.3 Crystal lattices

A *crystal lattice* X is a regular covering graph over a finite connected graph X_0 whose transformation group L is a free abelian group. The graph X_0 is called a *base graph* of X and the transformation group L is called an *abstract period lattice*. The *dimension* $\dim X$ of a crystal lattice X is defined to be the rank of L.

Theorem 2.2 gives an alternative description of a crystal lattice. Namely, a graph X is a crystal lattice if and only if there exists a free action $L \curvearrowright X$ of a free abelian group L, and the quotient graph $X_0 := X/L$ is finite. The following are examples of crystal lattices.

Example 2.5. The d-dimensional cubic lattice $\mathbb{L}^d = (\mathbb{Z}^d, \mathbb{E}^d)$ is a crystal lattice with dimension d. Indeed, the free abelian group $\mathbb{Z}^d$ acts naturally on $\mathbb{L}^d$ by translation, and the quotient graph is a bouquet (Figure 1.6).

Example 2.6. The honeycomb lattice $\mathbb{H}$ and the triangular lattice $\mathbb{T}$ are also crystal lattices with dimension 2. Indeed, we can give the action of $\mathbb{Z}^2$ by translation in such a way that the basis of $\mathbb{Z}^2$ comprises the translations shown in Figure 1.6.

Remark 2.7. Grimmett [16] gives a formulation of a "d-dimensional lattice" as a locally finite graph X such that the vertex set has finitely many orbits under a free action $\mathbb{Z}^d \curvearrowright X$. This definition is essentially the same as that of crystal lattices.

From Proposition 2.4, the vertex set V of X can be divided into the liftings of a spanning tree $\mathcal{T}$ of the base graph X_0:

$$V = \bigsqcup_{\sigma \in \mathbb{Z}^d} \mathcal{T}_\sigma. \tag{2.2}$$

Here, we identify the free abelian group L with $\mathbb{Z}^d$ by taking some $\mathbb{Z}$-basis of L. This implies that a d-dimensional crystal lattice can be considered as a d-dimensional array of spanning trees of the base graph.

Periodic realization

The "shape" of a crystal lattice is defined as a map to the space $\mathbb{R}^d$. Let $X = (V, E)$ be a d-dimensional crystal lattice over a finite graph $X_0 = (V_0, E_0)$, and let L be its abstract period lattice. A *realization* of X into $\mathbb{R}^d$ is a map $\Phi : V \to \mathbb{R}^d$, where the edges of X are realized as the segments connecting their endpoints. We often write a realization as $\Phi : X \to \mathbb{R}^d$, which is said to be *periodic* if there exists an injective homomorphism $\rho : L \to \mathbb{R}^d$ satisfying the following conditions:

- the image $\Gamma := \rho(L)$ is a lattice group of $\mathbb{R}^d$, that is, there exists a basis $(a_1, \dots, a_d)$ of $\mathbb{R}^d$ such that

$$\Gamma = \{\lambda_1 a_1 + \cdots + \lambda_d a_d \ : \ \lambda_i \in \mathbb{Z}\}; \text{ and}$$

- for any vertex $x \in V$ and $\sigma \in L$,

$$\Phi(\sigma x) = \Phi(x) + \rho(\sigma). \tag{2.3}$$

The equation (2.3) implies that the realized crystal $\Phi(X)$ is invariant under translation by any vector $\mathbf{b} \in \Gamma$. From this, the periodic structure of Φ is considered to be represented by $\Gamma = \rho(L)$. The homomorphism $\rho : L \to \mathbb{R}^d$ is called the *period homomorphism* of the realization $\Phi : X \to \mathbb{R}^d$. Though the period ρ is determined uniquely from Φ, we often write a periodic realization as the pair (Φ, ρ) in order to emphasize that ρ represents the period of the realization.

Let $\omega : X \to X_0$ be the covering map. The *building block* of Φ is the E_0-indexed family $(\mathbf{v_0}(e_0))_{e_0 \in E_0}$ of vectors defined by

$$\mathbf{v_0}(e_0) := \Phi(t(e)) - \Phi(o(e)), \tag{2.4}$$

where $e \in E$ is an edge with $\omega(e) = e_0$. This definition does not depend on the choice of $e \in E$. Indeed, for any $f \in E$ with $\omega(f) = e_0$, the regularity of ω implies the existence of $\sigma \in L$ with $\sigma f = e$. We thus have

$$\begin{aligned}\Phi(t(e)) - \Phi(o(e)) =& \Phi(t(\sigma f)) - \Phi(o(\sigma f)) \\ =& [\Phi(t(f)) + \rho(\sigma)] - [\Phi(o(f)) + \rho(\sigma)] \\ =& \Phi(t(f)) - \Phi(o(f)),\end{aligned}$$

where the second equality follows from the definition of graph morphism and (2.3).

The following theorem states that a crystal lattice X can be realized with an arbitrary period.

Theorem 2.8 ([27, Theorem 7.2])**.** Let X be a d-dimensional crystal lattice over X_0 whose abstract period lattice is L. Then for any injective homomorphism $\rho : L \to \mathbb{R}^d$ such that $\rho(L)$ is a lattice group of $\mathbb{R}^d$, there exists a periodic realization Φ whose period homomorphism is ρ.

Remark 2.9. The definition of periodic graphs by Kesten (Definition 1.2) is essentially equivalent to that of crystal lattices. Indeed, Theorem 2.8 implies that any d-dimensional crystal lattice X can be realized in such a way that the period $\Gamma = \rho(L)$ is equal to $\mathbb{Z}^d$, which is nothing but the first item of Definition 1.2. On the other hand, for any graph X satisfying the three conditions in Definition 1.2, we can consider the free action $\mathbb{Z}^d \curvearrowright X$ by translation in the coordinate vectors, and obtain the finite quotient graph.

Projective relation

Let X be a d-dimensional crystal lattice whose base graph is X_0. We consider a periodic realization $\Phi : X \to \mathbb{R}^d$ of X with the period homomorphism $\rho : L \to \mathbb{R}^d$. An orthogonal projection $P : \mathbb{R}^d \to W$ onto some d_1-dimensional subspace W is said to be a *rational projection* if the image $P(\rho(L))$ is a lattice group of W. We identify W with $\mathbb{R}^{d_1}$. We put $L_1 := L/\mathrm{Ker}(P \circ \rho)$, which is a free abelian group with rank d_1. We also write the quotient graph $X/\mathrm{Ker}(P \circ \rho)$ as $X_1 = (V_1, E_1)$. From Theorem 2.2, the quotient map $\omega : X \longrightarrow X_1$ is a regular covering map whose covering transformation group is $\mathrm{Ker}(P \circ \rho)$. Since $\mathrm{Ker}(P \circ \rho)$ is a normal subgroup of L, it follows from Proposition 2.3 that X_1 is a regular covering graph over X_0 whose transformation group is L_1. In particular, X_1 is a crystal lattice with dimension d_1.

We observe that $\omega(x) = \omega(y)$ implies $P \circ \Phi(x) = P \circ \Phi(y)$. Indeed, from the regularity of ω, we can take $\sigma \in \mathrm{Ker}(P \circ \rho)$ such that $x = \sigma y$, and obtain

$$P(\Phi(x)) = P(\Phi(\sigma y)) = P(\Phi(x)) + P(\rho(\sigma)) = P(\Phi(x)).$$

Thus we can find a map $\Phi_1 : X_1 \longrightarrow \mathbb{R}^{d_1}$ such that the commutative diagram (1.11) holds. We last check that Φ_1 is a periodic realization. Let $\pi : L \longrightarrow L_1$ be the canonical homomorphism. The homomorphism $P \circ \rho : L \longrightarrow \mathbb{R}^{d_1}$ induces the injective homomorphism $\rho_1 : L_1 \longrightarrow \mathbb{R}^{d_1}$ with $\rho_1 \circ \pi = P \circ \rho$. For $x_1 \in V_1$ and $\sigma_1 \in L_1$, we can take $x \in V$ and $\sigma \in L$ such that $\omega(x) = x_1$ and $\pi(\sigma) = \sigma_1$. We then have $\sigma_1 x_1 = \omega(\sigma x)$ from (2.1) and

$$\begin{aligned}\Phi_1(\sigma_1 x_1) &= \Phi_1(\omega(\sigma x)) = P(\Phi(\sigma x)) = P(\Phi(x)) + P(\rho(\sigma)) \\ &= \Phi_1(x_1) + \rho_1(\sigma_1).\end{aligned}$$

Thus Φ_1 is a periodic realization whose period homomorphism is ρ_1.

Maximal abelian covering graphs

For a finite connected graph $X_0 = (V_0, E_0)$, we can define the "maximal" crystal lattice whose base graph is X_0. Let $C_1(X_0, \mathbb{Z})$ be the group of 1-chains with coefficients in $\mathbb{Z}$, that is,

$$C_1(X_0, \mathbb{Z}) := \Big\{\alpha = \sum_{e \in E_0} a_e e \ : \ a_e \in \mathbb{Z}\Big\}.$$

The group of 0-chains is also defined as

$$C_0(X_0, \mathbb{Z}) := \Big\{\alpha = \sum_{x \in V_0} a_x x \ : \ a_x \in \mathbb{Z}\Big\}.$$

The boundary operator $\partial : C_1(X_0, \mathbb{Z}) \longrightarrow C_0(X_0, \mathbb{Z})$ is defined by the extension of

$$\partial e = t(e) - o(e) \quad (e \in E_0).$$

The first homology group $H_1(X_0, \mathbb{Z})$ of graph X_0 is defined to be the kernel of ∂:

$$H_1(X_0, \mathbb{Z}) := \{\alpha \in C_1(X_0, \mathbb{Z}) \,:\, \partial\alpha = 0\}. \tag{2.5}$$

From the discussion in [27, Section 6.1], there exists a covering graph, denoted by $X_0^{\mathrm{ab}} = (V_0^{\mathrm{ab}}, E_0^{\mathrm{ab}})$, over X_0 whose covering transformation group is $H_1(X_0, \mathbb{Z})$. The graph X_0^{ab} is called the *maximal abelian covering graph* over X_0.

Let $\omega^{\mathrm{ab}} : X_0^{\mathrm{ab}} \to X_0$ be the covering map. The following theorem states that ω^{ab} is maximal among all abelian covering maps, i.e., regular covering maps whose covering transformation group is abelian.

Theorem 2.10 ([27, Theorem 6.1]). Let $\omega_1 : X_1 \to X_0$ be an abelian covering graph over X_0. There exists a regular covering map $\omega : X_0^{\mathrm{ab}} \to X_1$ such that $\omega^{\mathrm{ab}} = \omega_1 \circ \omega$.

Since $H_1(X_0, \mathbb{Z})$ is a free abelian group, X_0^{ab} is a crystal lattice with dimension $d := \operatorname{rank} H_1(X_0, \mathbb{Z})$. From (2.2), we identify X_0^{ab} as the d-dimensional array of a spanning tree $\mathcal{T}$ of X_0:

$$V_0^{\mathrm{ab}} = \bigsqcup_{\sigma \in \mathbb{Z}^d} \mathcal{T}_\sigma. \tag{2.6}$$

Moreover, it follows from the construction of the maximal abelian covering graph (see [27, Sections 5.5, 6.1]) that we can identify $H_1(X_0, \mathbb{Z})$ with $\mathbb{Z}^d$ such that the equivalence

$$\begin{aligned} &\|\sigma - \tau\|_1 = 1 \iff \\ &\quad \text{there exists a unique edge } e \text{ with } t(e) \in \mathcal{T}_\sigma \text{ and } o(e) \in \mathcal{T}_\tau \end{aligned} \tag{2.7}$$

holds for any σ, $\tau \in \mathbb{Z}^d$ (Figure 2.1).

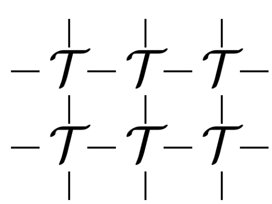

Figure 2.1: Sketch of maximal abelian covering graph with dimension 2. The lines indicate the edges that connect adjacent trees.

Example 2.11. The cubic lattice $\mathbb{L}^d$ and the honeycomb lattice $\mathbb{H}$ shown in Figure 2.2 are maximal abelian covering graphs, with the base graphs also shown below.

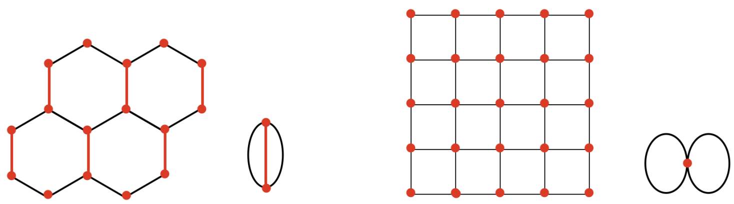

Figure 2.2: Honeycomb lattice (left) and cubic lattice (right). The red lines and points indicate a spanning tree of each base graph.

A diamond lattice $\mathbb{D}$ can be regarded as a higher-dimensional version of the honeycomb lattice in the sense that $\mathbb{D}$ is the maximal abelian covering graph over the finite graph consisting of two points and four parallel edges connecting them.

2.2 FKG inequality and a remark on the critical probability

The FPP model and the bond percolation model, which we introduced in Section 1.1, can be formulated on the product space indexed by a countable set S. When we consider the FPP model with time distribution ν, we assume the probability space $(\boldsymbol{\Omega}, \mathcal{F}, \mathbb{P})$ defined by

$$\boldsymbol{\Omega} := [0, \infty)^S, \quad \mathbb{P} := \nu^{\otimes S} \tag{2.8}$$

and the Borel σ-algebra $\mathcal{F}$. Here $\nu^{\otimes S}$ stands for the product measure on $\boldsymbol{\Omega}$. Each element $\mathbf{t} = (t_s : s \in S) \in \boldsymbol{\Omega}$ is called a *configuration*. A partial order $\leq$ on the configuration space $\boldsymbol{\Omega}$ is defined by

$$\mathbf{t} \leq \mathbf{t}' \overset{\text{def}}{\iff} t_s \leq t'_s \text{ for any } s \in S$$

for two configurations $\mathbf{t} = (t_s : s \in S)$, $\mathbf{t}' = (t'_s : s \in S) \in \boldsymbol{\Omega}$. An event $A \in \mathcal{F}$ is called *increasing* if $\mathbf{t}' \in A$ whenever $\mathbf{t} \in A$ and $\mathbf{t} \leq \mathbf{t}'$. For the cubic lattice $\mathbb{L}^d = (\mathbb{Z}^d, \mathbb{E}^d)$, we can consider the FPP model by setting $S = \mathbb{E}^d$. The following are examples of increasing events:

- $\{T(\gamma) > t\}$ for a path γ in $\mathbb{L}^d$ and $t > 0$.
- $\{T(x, y) > t\}$ for two points $x, y \in \mathbb{R}^d$ and $t > 0$.

The probability space of the bond percolation model is formulated by replacing $[0, \infty)$ by $\{0, 1\}$. That is, for the probability $p \in [0, 1]$, we consider the probability space $(\Omega, \mathcal{F}, \mathbb{P}_p)$ defined by

$$\Omega := \{0, 1\}^S, \quad \mathbb{P}_p = [(1-p)\delta_0 + p\delta_1]^{\otimes S} \tag{2.9}$$

and the σ-algebra $\mathcal{F}$ generated by finite-dimensional cylinder sets[1]. Here we assume that a bond e is open (resp. closed) if $t_e = 1$ (resp. $t_e = 0$). We consider the bond percolation model on a graph $X = (V, E)$ by setting $S = E$. The following are some examples of increasing events of the bond percolation model on a graph X:

- $\{x \leftrightarrow y\}$ for two points $x,\ y \in V$.
- $\{x \leftrightarrow \infty\}$ for $x \in V$.

Here, $x \leftrightarrow y$ means that two points $x,\ y$ are connected by a path of open edges. We also write $x \leftrightarrow \infty$ when x is contained in an infinite cluster.

In this section, we introduce the FKG inequality [14], which is commonly used in percolation theory. The simplest form of the FKG inequality is the following.

Theorem 2.12 (FKG inequality)**.** Let A and B be two increasing events in the probability space (2.8). Then

$$\mathbb{P}(A \cap B) \geq \mathbb{P}(A)\mathbb{P}(B). \tag{2.10}$$

The same result also holds for the probability space (2.9). From the FKG inequality, we can check as follows that the critical probability of an arbitrary connected graph $X = (V, E)$ does not depend on the choice of the origin $0 \in V$, which is trivial for graphs such as $\mathbb{L}^d$, $\mathbb{T}$ or $\mathbb{H}$ introduced in Section 1.1.2.

Let $p > 0$. For a vertex $x \in V$, let $\theta(p, x)$ be the percolation probability at x:

$$\theta(p, x) := \mathbb{P}_p(x \leftrightarrow \infty).$$

[1] A finite-dimensional cylinder set is a set $\{\mathbf{t} \in \Omega : t_{e_i} = \epsilon_i \text{ for } i = 1, 2, \ldots, n\}$ for some $n \in \mathbb{Z}_{\geq 0}$, $e_i \in S$ and $\epsilon_i \in \{0, 1\}$.

Let $y \neq x \in V$ be another vertex and suppose $\theta(p, y) > 0$. Then we obtain the inclusion

$$\{x \leftrightarrow \infty\} \supset \{x \leftrightarrow y\} \cap \{y \leftrightarrow \infty\},$$

which implies

$$\theta(p, x) \geq \mathbb{P}_p(\{x \leftrightarrow y\} \cap \{y \leftrightarrow \infty\}).$$

From the FKG inequality, the right hand side is bounded below by

$$\mathbb{P}_p(x \leftrightarrow y)\theta(p, y) > 0,$$

which implies $\theta(p, x) > 0$. The same argument holds for the opposite direction and we obtain

$$\theta(p, x) > 0 \Longleftrightarrow \theta(p, y) > 0. \tag{2.11}$$

Thus we can define the critical probability $p_c(X)$ of X as

$$p_c(X) := \sup\{p \in [0, 1] \,:\, \theta(p, x) = 0\},$$

which does not depend on the choice of $x \in V$.

Notation and assumptions

Throughout this thesis, we deal with a crystal lattice X as an *undirected graph*, whose edge set is given by the orbit space $E/\mathbb{Z}_2$ of the action $\mathbb{Z}_2 := \mathbb{Z}/2\mathbb{Z} \curvearrowright E$ defined by $e \mapsto \bar{e}$. Here we simply denote by E the orbit space $E/\mathbb{Z}_2$. We write $x \in X$ for a vertex x of a graph X. We also assume that X has the "origin", denoted by $0 \in X$.

When we consider the FPP (resp. bond percolation) model on a crystal lattice $X = (V, E)$, we always assume the probability space defined as (2.8) (resp. (2.9)) with $S = E$. A configuration of Ω is denoted by $\mathbf{t} = (t_e : e \in E) \in \Omega$. We denote by $\mathbb{E}$ the expectation with respect to this probability space.

When we consider a periodic realization $\Phi : X \to \mathbb{R}^d$, we write by $\rho : L \to \mathbb{R}^d$ its period homomorphism and by $\Gamma := \rho(L)$ the lattice group. A vertex $x \in X$ is always assumed to be realized by Φ in $\mathbb{R}^d$. We abbreviate $\Phi(x) \in \mathbb{R}^d$ by $x \in \mathbb{R}^d$. We denote by $\mathcal{D}$ the "rational points" of Γ, that is,

$$\mathcal{D} = \{q_1 a_1 + \cdots + q_d a_d \ : \ q_i \in \mathbb{Q}\},$$

where $(a_1, \ldots, a_d)$ is a basis of the lattice group Γ. Note that $\mathcal{D}$ coincides with the set of points $x \in \mathbb{R}^d$ such that $nx \in \Gamma$ for some $n \in \mathbb{N}$, and thus $\mathcal{D}$ does not depend on the choice of a basis of Γ.

In several places in this thesis we have to deal with another crystal lattice X_1. Expressions with respect to X_1 are given with subscripts, such as Φ_1, Γ_1 and $\mathcal{D}_1$.

The following are also used throughout this thesis:

- $d_X(x, y)$: the graph distance of two vertices x, $y \in X$.
- I_A : the indicator function of the event $A \in \mathcal{F}$.
- $a_n \approx b_n$ means that $\lim_{n\to\infty} \frac{\log a_n}{\log b_n} = 1$.

Unless otherwise noted, a periodic realization $\Phi : X \to \mathbb{R}^d$ of a crystal lattice $X = (V, E)$ is assumed to satisfy the following conditions:

- The origin $0 \in X$ is assumed to be mapped to $0 \in \mathbb{R}^d$.
- A periodic realization Φ is *nondegenerate*, that is, the map $\Phi : V \to \mathbb{R}^d$ is injective. Later we will remark that this assumption is actually not essential (Remarks 3.12 and 4.9).

Chapter 3

First passage percolation model on crystal lattices

This chapter studies the FPP model on crystal lattices. In Section 3.1, we first give some observations for the asymptotic behavior of the first passage time. In Section 3.2, we give a proof of the generalized shape theorem (Theorem 1.5) and some properties of the limit shape. The covering monotonicity of the limit shapes (Theorem 1.6) is proved in Section 3.3.

3.1 Asymptotic speed of first passage time

Let $\Phi : X \to \mathbb{R}^d$ be a periodic realization of a d-dimensional crystal lattice $X = (V, E)$. Fix a time distribution ν. For a path $\gamma = (e_1, \ldots, e_r)$ in X, the passage time $T(\gamma)$ is the random variable defined by

$$T(\gamma) := \sum_{i=1}^{r} t_{e_i}.$$

For two points x, $y \in \mathbb{R}^d$, which may not be realized vertices of X, we denote by $T(x, y)$ the first passage time between x and y, that is,

$$T(x, y) := \inf\{T(\gamma) \,:\, \gamma \text{ is a path from } x' \text{ to } y'\},$$

where x', $y' \in X$ are the closest realized points of x and y, respectively, with a deterministic rule to break ties. The percolation region $B(t)$ is defined by

$$B(t) := \{x \in \mathbb{R}^d \,:\, T(0, x) \le t\}$$

for a time $t \geq 0$.

Remark 3.1. We can easily see that $T(x, y)$ and $T(x + \mathbf{b}, y + \mathbf{b})$ have the same distribution for any vector $\mathbf{b} \in \Gamma$.

The *edge connectivity* of X is the minimum number $l_X \in \mathbb{N}$ such that there exists a set $\{e_1, \dots, e_{l_X}\}$ of edges that separates X. Menger's theorem (see, e.g., [11]) gives an alternative description of the edge connectivity as follows:

$$\begin{aligned} l_X = \max\{l' \in \mathbb{N} : \text{ for any } x \neq y \in V, \text{ there exist} \\ l' \text{ edge-disjoint paths from } x \text{ to } y\}. \end{aligned}$$

From this remark and a basic discussion for the passage time (see, e.g., [3]), the following lemma holds.

Lemma 3.2. Let $t_1, \dots, t_{l_X}$ be independent copies of t_e and let $k \geq 1$. Then $\mathbb{E} \min\{t_1, \dots, t_{l_X}\}^k < \infty$ holds if and only if $\mathbb{E}T(0, x)^k < \infty$ holds for all $x \in X$.

Proof. From the definition of the edge connectivity l_X, there exist l_X edges which are incident to the origin 0. Thus, we obtain

$$T(0, x) \geq \min\{t_1, \dots, t_{l_X}\}$$

for $x \neq 0 \in X$. Hence we can see that $\mathbb{E}T(0, x)^k < \infty$ implies $\mathbb{E} \min\{t_1, \dots, t_{l_X}\}^k < \infty$.

Suppose $\mathbb{E} \min\{t_1, \dots, t_{l_X}\}^k < \infty$ and we take a vertex $x \neq 0 \in X$ arbitrarily. Then there exist l_X paths $\gamma_1, \gamma_2, \dots, \gamma_{l_X}$ which are edge-disjoint and join 0 and x. For these paths, we have

$$T(0, x) \leq \min\{T(\gamma_1), \dots, T(\gamma_{l_X})\},$$

which implies that

$$\mathbb{P}(T(0, x)^k > s) \leq \mathbb{P}(\min\{T(\gamma_1)^k, \dots, T(\gamma_{l_X})^k\} > s) \tag{3.1}$$

holds for any $s \geq 0$. Since $T(\gamma_1), \dots, T(\gamma_{l_X})$ are independent, the right hand side is equal to

$$\mathbb{P}\left(\bigcap_{i=1}^{l_X} \{T(\gamma_i)^k > s\}\right) = \prod_{i=1}^{l_X} \mathbb{P}(T(\gamma_i)^k > s). \tag{3.2}$$

Without loss of generality, we may assume that $\gamma_1 = (e_1, e_2, \ldots, e_r)$ is the longest path. Hölder's inequality implies

$$T(\gamma_1) = t_{e_1} + t_{e_2} + \cdots + t_{e_r} \leq r^{1/q}(t_{e_1}^k + t_{e_2}^k + \cdots + t_{e_r}^k)^{1/k}, \tag{3.3}$$

where q is the real number with $1/k + 1/q = 1$. Let $B := r^{k/q}$. Then (3.3) implies

$$\mathbb{P}(T(\gamma_1)^k > s) \leq \mathbb{P}(t_{e_1}^k + t_{e_2}^k + \cdots + t_{e_r}^k > s/B) \leq r\mathbb{P}(t_1^k > s/rB), \tag{3.4}$$

where t_1 is an independent copy of a random time in Lemma 3.2. The same estimate holds for $T(\gamma_2), \ldots, T(\gamma_{l_X})$. Combining (3.2) with (3.4), the right hand side of (3.1) is bounded above by

$$r^{l_X} \prod_{i=1}^{l_X} \mathbb{P}(t_i^k > s/rB) = r^{l_X}\mathbb{P}(\min\{t_1, \ldots, t_{l_X}\}^k > s/rB),$$

where the equality follows from the independence of t_i. By integrating with respect to s from 0 to ∞, we obtain

$$\mathbb{E}T(0,x)^k \leq r^{l_X+1}B\mathbb{E}\min\{t_1, \ldots, t_{l_X}\}^k < \infty,$$

which completes the proof of Lemma 3.2. $\square$

Hereafter, we assume that the time distribution ν satisfies

$$\mathbb{E}\min\{t_1, \ldots, t_{l_X}\} < \infty. \tag{3.5}$$

Similarly to the cubic model, we first prove the following proposition.

Proposition 3.3. Suppose that the time distribution satisfies (3.5). Then for each $x \in \mathcal{D}$, the limit

$$\mu(x) := \lim_{n\to\infty} \frac{T(0,nx)}{n} \tag{3.6}$$

exists almost surely. Moreover, the function $\mu : \mathcal{D} \to \mathbb{R}$ depends only on X, ν and the period ρ.

The following theorem ([23, Theorem 1.10]) is essential for the proof of Proposition 3.3.

Theorem 3.4 (Subadditive ergodic theorem)**.** Suppose that a sequence $(X_{m,n})_{0\le m<n}$ of random variables satisfies the following conditions:

- $X_{0,n} \le X_{0,m} + X_{m,n}$ for all $0 < m < n$;
- the joint distributions of the two sequences

$$(X_{m,m+k})_{k\ge1} \text{ and } (X_{m+1,m+k+1})_{k\ge1}$$

  are the same for all $m \ge 0$;
- for each $k \ge 1$, the sequence $(X_{nk,(n+1)k})_{n\ge0}$ is stationary and ergodic; and
- $\mathbb{E}X_{0,1} < \infty$ and $\mathbb{E}X_{0,n} > -cn$ for some finite constant $c < \infty$.

Then

$$\lim_{n\to\infty} \frac{X_{0,n}}{n} = \inf_n \frac{\mathbb{E}X_{0,n}}{n} = \lim_{n\to\infty} \frac{\mathbb{E}X_{0,n}}{n}$$

holds almost surely and in L_1.

We now turn to the proof of Proposition 3.3.

Proof of Proposition 3.3. Take the minimum number $N \in \mathbb{Z}_{>0}$ with $Nx \in \Gamma$. We can easily see that the array $(T(mNx, nNx))_{0\le m<n}$ of random variables satisfies the conditions of Theorem 3.4. Note that the integrability of $T(0, Nx)$ follows from the assumption (3.5) and Lemma 3.2. Thus we see that the limit

$$\lim_{k\to\infty} \frac{T(0, kNx)}{k}$$

exists almost surely and is constant. We set

$$\mu(x) := \lim_{k\to\infty} \frac{T(0, kNx)}{kN}.$$

Since each kNx is a vertex on the lattice group $\Gamma = \rho(L)$, the limit $\mu(x)$ depends only on x, ν, X and the period ρ. Take $j = 1, 2, \ldots, N-1$ arbitrarily. From the triangle inequality, we have

$$|T(0, (kN + j)x) - T(0, kNx)| \le T((kN + j)x, kNx)$$

and thus, for any $\epsilon > 0$,

$$\begin{aligned}&\sum_{k=1}^{\infty} \mathbb{P}(|T(0,(kN+j)x) - T(0,kNx)| > \epsilon k)\\ \le &\sum_{k=1}^{\infty} \mathbb{P}(T((kN+j)x, kNx) > \epsilon k)\\ = &\sum_{k=1}^{\infty} \mathbb{P}(T(jx,0) > \epsilon k) < \infty.\end{aligned}$$

Here, the equality follows from Remark 3.1, and the finiteness is due to the integrability of $T(jx,0)$. Then it follows from the Borel–Cantelli lemma that

$$\mathbb{P}(\limsup_{k\to\infty}\{|T(0,(kN+j)x) - T(0,kNx)| > \epsilon k\}) = 0.$$

By taking the complementary event, we have

$$\mathbb{P}\left(\bigcup_{n}\bigcap_{k\ge n}\{|T(0,(kN+j)x) - T(0,kNx)| \le \epsilon k\}\right) = 1,$$

which implies the almost sure convergence

$$\lim_{k\to\infty}\frac{1}{kN}|T(0,(kN+j)x) - T(0,kNx)| = 0.$$

Therefore, we obtain that

$$\lim_{k\to\infty}\frac{1}{kN+j}T(0,(kN+j)x) = \mu(x)$$

holds almost surely. Since $j = 1, 2, \dots, N-1$ is taken arbitrarily, the proof of Proposition 3.3 is completed. □

We summarize the basic properties of μ.

Proposition 3.5. The following hold:

(1) $\mu(x+y) \le \mu(x) + \mu(y)$ for any x, $y \in \mathcal{D}$.

(2) $\mu(cx) = |c|\mu(x)$ for any $c \in \mathbb{Q}$ and $x \in \mathcal{D}$.

Proof. Fix $x, y \in \mathcal{D}$. Let N be the minimum number with $Nx, Ny \in \Gamma$. It follows from Remark 3.1 and Proposition 3.3 that

$$\frac{T(kNx, kNy)}{kN} \sim \frac{T(0, kN(y-x))}{kN} \to_d \mu(y-x) \tag{3.7}$$

as $k \to \infty$. Here, $\sim$ means that the distributions are the same, and $\to_d$ represents the convergence in distribution. By the definition of $T(\cdot,\cdot)$, we have the following triangle inequality

$$\frac{T(0, kN(x+y))}{kN} - \frac{T(0, kNx)}{kN} \leq \frac{T(kNx, kN(x+y))}{kN}.$$

The left hand side converges in distribution to $\mu(x+y) - \mu(x)$. By replacing $y \in \mathcal{D}$ with $x + y \in D$ in (3.7), we can see that the right hand side converges to $\mu(y)$, and obtain the first item.

Let $c \in \mathbb{Q}_{\geq 0}$. Then we have

$$\mu(cx) = \lim_{n\to\infty} \frac{T(0, ncx)}{n} = c \lim_{n\to\infty} \frac{T(0, ncx)}{nc} = c\mu(x).$$

The symmetry of T and (3.7) implies that $\mu(x) = \mu(-x)$. Thus the second item also holds for all $c \in \mathbb{Q}$. □

From Proposition 3.5, we can see that the function $\mu : \mathcal{D} \to \mathbb{R}$ has continuity, that is, $\mu(x_n) \to 0$ for any sequence $(x_n)_{n=1,2,\ldots}$ in $\mathcal{D}$ with $x_n \to 0$. Indeed, by setting $x_n = \sum_{i=1}^{d} q_i^{(n)} a_i$ for $q_i^{(n)} \in \mathbb{Q}$, where $(a_1, \ldots, a_d)$ is a basis of the lattice group Γ, we obtain

$$\mu(x_n) \leq \sum_{i=1}^{d} \mu(q_i^{(n)} a_i) = \sum_{i=1}^{d} |q_i^{(n)}| \mu(a_i) \to 0 \tag{3.8}$$

as $n \to \infty$. Here, the inequality and equality follow from (1) and (2) of Proposition 3.5, respectively. From this continuity, we can define $\mu(x)$ for any $x \in \mathbb{R}^d$ as the limit

$$\mu(x) = \lim_{n\to\infty} \mu(x_n), \tag{3.9}$$

where $(x_n)_{n=1,2,\ldots}$ is a sequence in $\mathcal{D}$ that converges to x. The existence of the limit follows from the triangle inequality ((1) of Proposition 3.5)

$$|\mu(x_n) - \mu(x_m)| \leq \mu(x_n - x_m).$$

From (3.8), the right hand side converges to 0 as $n, m \to 0$, which implies that $(\mu(x_n))_{n=1,2,\ldots}$ is a Cauchy sequence. In the same way, we can check that the limit (3.9) does not depend on the choice of a sequence $(x_n)_{n=1,2,\ldots}$.

Hereinafter, we think of μ as a function defined on the space $\mathbb{R}^d$ by continuous expansion. The following proposition states that the positivity of $\mu(x)$ depends on the probability $\nu(0)$ that the random time is equal to 0.

Proposition 3.6. The following hold:

(a) If $\nu(0) < p_c(X)$, then $\mu(x) > 0$ for all $x \in \mathbb{R}^d \setminus \{0\}$.

(b) If $\nu(0) \geq p_c(X)$, then $\mu(x) = 0$ for all $x \in \mathbb{R}^d$.

The proof of this proposition is similar to that of [18, Theorem 6.1][1]. From Propositions 3.5 and 3.6, the limit μ is a norm on $\mathbb{R}^d$ whenever $\nu(0) < p_c(X)$.

3.2 Generalization of the shape theorem

In this section, we give a proof of a general version of the shape theorem (Theorem 1.5) and summarize the basic properties of the limit shape.

3.2.1 Proof

We first note that the assumption (1.12) of Theorem 1.5 is stronger than (3.5) for the existence of the limit μ. By using Proposition 3.6, we rewrite Theorem 1.5 with respect to the limit μ as follows.

Theorem 1.5′. Let (Φ, ρ) be a periodic realization of a d-dimensional crystal lattice X. Suppose that the time distribution ν satisfies (1.12). Then the following hold:

(a) If μ is a norm on $\mathbb{R}^d$, then for each $\epsilon > 0$, it holds almost surely that

$$(1-\epsilon)\mathcal{B} \subset \frac{B(t)}{t} \subset (1+\epsilon)\mathcal{B} \text{ for all large } t,$$

[1] Although [18, Theorem 6.1] is stated by using another critical probability $p_T(X)$, the critical point at which the expected value of the cluster size becomes infinite, the uniqueness $p_c(X) = p_T(X)$ can be shown for an arbitrary crystal lattice X in the same way as for $\mathbb{L}^d$ (see, e.g., [15]).

where $\mathcal{B} \subset \mathbb{R}^d$ is the unit ball

$$\mathcal{B} := \{x \in \mathbb{R}^d \,:\, \mu(x) \le 1\}. \tag{3.10}$$

(b) If $\mu \equiv 0$, then for all $R > 0$, it holds almost surely that

$$\{x \in \mathbb{R}^d \,:\, \|x\|_1 \le R\} \subset \frac{B(t)}{t} \text{ for all large } t.$$

The proof of Theorem 1.5′ reduces to showing the following convergence, which states that the convergence $T(0, nx)/n \to \mu(x)$ is uniform on the directions.

Proposition 3.7. Suppose that the time distribution satisfies (1.12). Then, the convergence

$$\lim_{x\in\mathcal{D},\, \|x\|_1\to\infty} \left(\frac{T(0,x)}{\|x\|_1} - \mu\left(\frac{x}{\|x\|_1}\right)\right) = 0$$

holds almost surely.

Proof of Theorem 1.5′. We first show the case (a). Since μ is a norm on $\mathbb{R}^d$, there exists a constant $C > 0$ such that

$$C^{-1}\mu(x) \le \|x\|_1 \le C\mu(x)$$

holds for any $x \in \mathbb{R}^d$. Fix $\epsilon > 0$ arbitrarily. We take $\delta > 0$ sufficiently small that

$$\frac{1}{1-\delta C} \le 1+\epsilon \text{ and } 1-\epsilon \le \frac{1}{1+\delta C}$$

hold. From Proposition 3.7, we can take a constant K such that

$$\left| \frac{T(0,x)}{\|x\|_1} - \mu\left(\frac{x}{\|x\|_1}\right)\right| < \delta \tag{3.11}$$

holds for any $x \in \mathcal{D}$ with $\|x\|_1 \ge K$. We set

$$M := \max\left\{ CK, \sup_{\|y\|_1\le K} T(0,y)\right\} < \infty.$$

We now check that the following inclusion holds for all $t > M$:

$$(1-\epsilon)t\mathcal{B} \subset B(t) \subset (1+\epsilon)t\mathcal{B}. \tag{3.12}$$

For a point $x \in \mathcal{D}$ with $\|x\|_1 \leq K$, we have $T(0,x) \leq M \leq t$ and $\mu(x) \leq C\|x\|_1 \leq CK \leq t$. Thus the inclusion (3.12) holds. We check the inclusion (3.12) for $x \in \mathcal{D}$ with $\|x\|_1 \geq K$.

If $x \in (1-\epsilon)t\mathcal{B}$, then by (3.11), we have

$$T(0,x) \leq \mu(x) + \delta\|x\|_1 \leq (1-\epsilon)t + \delta C\mu(x).$$

Again by $\mu(x) \leq (1-\epsilon)t$, the last expression is bounded above by

$$(1-\epsilon)t \times (1+\delta C) \leq t.$$

Thus we obtain $x \in B(t)$, which implies the first inclusion of (3.12). Suppose $x \in B(t)$, then it follows from (3.11) that

$$\mu(x) \leq T(0,x) + \delta\|x\|_1 \leq t + \delta C\mu(x),$$

which implies

$$\mu(x) \leq \frac{t}{1-\delta C} \leq (1+\epsilon)t.$$

Thus we obtain the second inclusion of (3.12).

We next show the case (b). Fix $R > 0$ arbitrarily. We take K such that

$$\frac{T(0,x)}{\|x\|_1} < \frac{1}{R} \tag{3.13}$$

holds for any $x \in \mathcal{D}$ with $\|x\|_1 \geq K$. It suffices to check that

$$\{x \in \mathbb{R}^d \ : \ \|x\|_1 \leq tR\} \subset B(t)$$

holds for any $t > \sup_{\|y\|_1 \leq K} T(0,y)$. Suppose $\|x\|_1 \leq tR$. If $\|x\|_1 \leq K$, then $T(0,x) < t$ and we have $x \in B(t)$. Otherwise, from (3.13), we have

$$T(0,x) < \frac{\|x\|_1}{R} \leq \frac{tR}{R} = t$$

and obtain $x \in B(t)$. □

We will now give a proof of Proposition 3.7. If $d = 1$, then Proposition 3.7 is the same as Proposition 3.3. Hereafter we assume $d \geq 2$. For the proof, we need to check the following stochastic estimate for the passage time $T(x, y)$.

Lemma 3.8. Let $d \geq 2$. Suppose that the time distribution ν satisfies (1.12). Then there exists a constant $\kappa < \infty$ such that

$$\mathbb{P}\left(\sup_{x \in X, x \neq 0} \frac{T(0,x)}{\|x\|_1} \leq \kappa\right) > 0. \tag{3.14}$$

The case $X = \mathbb{L}^d$ of this lemma is given by [3, Lemma 2.20]. For the generalization to a crystal lattice model, the d-dimensional structure (2.2) of X, which we reviewed in Section 2.1.3, plays an important role.

Proof. We first check that Lemma 3.8 follows from the estimate

$$\sum_{x \in X} \mathbb{P}(T(0,x) \geq C\|x\|_1) < \infty \tag{3.15}$$

for some constant $C > 0$. Let $\{x_1, x_2, \ldots\}$ be an ordering of the vertex set $X \setminus \{0\}$. From (3.15), we have

$$\mathbb{P}\left(\bigcup_{n \geq N} \{T(0,x_n) \geq C\|x_n\|_1\}\right) \leq \sum_{n \geq N} \mathbb{P}(T(0,x_n) \geq C\|x_n\|_1) < 1/3$$

for a large number N. By taking the complement, we obtain

$$\mathbb{P}\left(\sup_{n \geq N} \frac{T(0,x_n)}{\|x_n\|_1} < C\right) > 2/3,$$

and there exists $\kappa' > 0$ such that

$$\mathbb{P}\left(\max_{n=1,\ldots,N-1} \frac{T(0,x_n)}{\|x_n\|_1} \leq \kappa'\right) > 2/3.$$

Letting $\kappa := \max\{\kappa', C\}$ implies Lemma 3.8.

We now turn to the proof of (3.15). Let us recall the division (2.2) of X into trees. We denote by $0_{\mathbb{Z}^d}$ the identity element of $\mathbb{Z}^d$ in order to distinguish it from the origin $0 \in X$. We can assume that the origin $0 \in X$ is in the tree $\mathcal{T}_{0_{\mathbb{Z}^d}}$. Denote by $|\sigma|$ the L_1-norm of $\sigma \in \mathbb{Z}^d$ and let $\sigma \sim \tau$ mean $|\sigma - \tau| = 1$.

Let l_X be the edge connectivity of X. For a number $R \in \mathbb{Z}_{>0}$ and $\sigma \in \mathbb{Z}^d$, we define the box $\Lambda(\sigma) \subset \mathbb{Z}^d$ as

$$\Lambda(\sigma) := 2R\sigma + (-R, R]^d \cap \mathbb{Z}^d.$$

We set R sufficiently large so that the following condition holds (Figure 3.1): for any element $\sigma \sim 0_{\mathbb{Z}^d}$, there exist l_X edge-disjoint paths $\gamma_1^{(\sigma)}, \ldots, \gamma_{l_X}^{(\sigma)}$ satisfying that

- all vertices of the paths are in $\bigsqcup_{z \in \Lambda\left(0_{\mathbb{Z}^d}\right) \cup \Lambda(\sigma)} \mathcal{T}_z$; and
- each path connects $0 \in X$ and $(2R\sigma)0 \in X$.

The existence of this R follows from the periodic structure of the graph X.

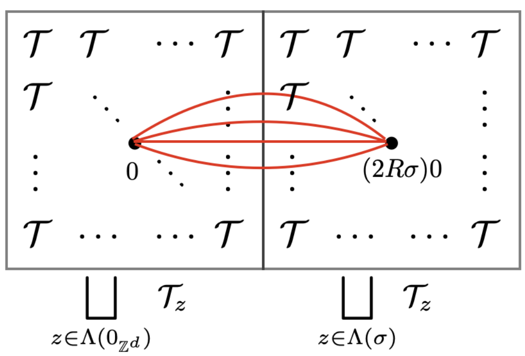

Figure 3.1: l_X paths $\gamma_1^{(\sigma)}, \ldots, \gamma_{l_X}^{(\sigma)}$ from 0 to $(2R\sigma)0$ (red).

For two elements σ, τ with $\sigma \sim \tau$, let $T(\sigma, \tau)$ be the minimum passage time of the l_X paths $(2R\sigma)\gamma_1^{(\tau-\sigma)}, \ldots, (2R\sigma)\gamma_{l_X}^{(\tau-\sigma)}$, which connect $(2R\sigma)0$ and $(2R\tau)0$. For a self-avoiding path $\pi = (0_{\mathbb{Z}^d} = \sigma_0, \sigma_1, \ldots, \sigma_m = \sigma)$ of length m in $\mathbb{Z}^d$, we have

$$T(0, (2R\sigma)0) \leq \sum_{i=0}^{m-1} T(\sigma_i, \sigma_{i+1}). \tag{3.16}$$

We let $T_i := T(\sigma_i, \sigma_{i+1})$ and denote by $T(\pi)$ the right hand side of (3.16). By Lemma 3.2 and the assumption (1.12), each T_i has a finite dth moment. Since we assume $d \geq 2$, the variance $\mathrm{Var}(T_i)$ of T_i is finite. We set

$$\mathrm{Var}_{\max} := \max_{0_{\mathbb{Z}^d} \sim \tau} \mathrm{Var}(T(0_{\mathbb{Z}^d}, \tau)),$$

$$\mathbb{E}_{\max} := \max_{0_{\mathbb{Z}^d} \sim \tau} \mathbb{E}T(0_{\mathbb{Z}^d}, \tau).$$

Since T_i and T_j are independent whenever $|i-j| > 1$, the variance $\mathrm{Var}(T(\pi))$ of $T(\pi)$ is equal to

$$\mathrm{Var}(T(\pi)) = \sum_{i=0}^{m-1} \mathbb{E}(T_i - \mathbb{E}T_i)^2 + 2\sum_{i=0}^{m-2} \mathbb{E}(T_i - \mathbb{E}T_i)(T_{i+1} - \mathbb{E}T_{i+1}).$$

This variance is bounded above by $3m\mathrm{Var}_{\max}$. Indeed, it follows from the Cauchy-Schwartz inequality that

$$\mathbb{E}|(T_i - \mathbb{E}T_i)(T_{i+1} - \mathbb{E}T_{i+1})| \le \mathrm{Var}(T_i)^{1/2}\mathrm{Var}(T_{i+1})^{1/2} \le \mathrm{Var}_{\max}.$$

Chebyshev's inequality implies that

$$\begin{aligned}
\mathbb{P}\left(T(\pi) \ge m(\mathbb{E}_{\max} + 1)\right) &\le \mathbb{P}\left(\sum_{i=0}^{m-1}(T_i - \mathbb{E}T_i) \ge m\right) \\
&\le \frac{1}{m^2}\mathrm{Var}(T(\pi)) \\
&\le \frac{3}{m}\mathrm{Var}_{\max}. \qquad (3.17)
\end{aligned}$$

To prove (3.15), we need to improve this estimate. For each $\sigma \in \mathbb{Z}^d \setminus \{0_{\mathbb{Z}^d}\}$, we can take $2d$ paths $\pi_1(\sigma), \pi_2(\sigma), \ldots, \pi_{2d}(\sigma)$ in $\mathbb{Z}^d$ that connect $0_{\mathbb{Z}^d}$ and σ and satisfy the following conditions:

- they are vertex-disjoint except for $0_{\mathbb{Z}^d}$ and σ; and
- the length of each path is less than $|\sigma| + K_d$.

Here, the constant K_d, depending on d, is the cost of making a detour in order that these paths do not overlap. We set $\sigma \in \mathbb{Z}^d \setminus \{0_{\mathbb{Z}^d}\}$ and let $\pi_1, \ldots, \pi_{2d}$ be paths satisfying the above conditions. We consider the separation $\pi_j = \pi_j^1 + \pi_j^2 + \pi_j^3$ of each path π_j, where π_j^1 (resp. π_j^3) is the first (resp. last) step of π_j. Let

$$U := \max_{j=1,2,\ldots,2d} T(\pi_j^1) \text{ and } U' := \max_{j=1,2,\ldots,2d} T(\pi_j^3).$$

Then we obtain $T(0,(2R\sigma)0) \le U + \min_{j=1,2,\dots,2d} T(\pi_j^2) + U'$, and

$$\begin{aligned}&\mathbb{P}\big(T(0,(2R\sigma)0) \ge (\mathbb{E}_{\max}+1)(|\sigma|+K_d)+2|\sigma|\big)\\ \le&\mathbb{P}(U \ge |\sigma|) + \mathbb{P}(U' \ge |\sigma|) + \mathbb{P}\big(\min_{j=1,2,\dots,2d} T(\pi_j^2) \ge (\mathbb{E}_{\max}+1)(|\sigma|+K_d)\big).\end{aligned}$$

The first and second terms of the right hand side are bounded above by

$$2d \max_{0_{\mathbb{Z}^d}\sim\tau} \mathbb{P}(T(0_{\mathbb{Z}^d},\tau) \ge |\sigma|).$$

This is summable in $\sigma \in \mathbb{Z}^d$ since $T(0_{\mathbb{Z}^d},\tau)$ has a finite dth moment. The independence of $T(\pi_1^2), T(\pi_2^2), \dots, T(\pi_{2d}^2)$ implies that the last term is equal to

$$\prod_{j=1}^{2d} \mathbb{P}\big(T(\pi_j^2) \ge (\mathbb{E}_{\max}+1)(|\sigma|+K_d)\big).$$

Since the length of each path π_j^2 is less than $|\sigma|+K_d$, it follows from (3.17) that this is bounded above by

$$3^{2d}\left(\frac{\mathrm{Var}_{\max}}{|\sigma|+K_d}\right)^{2d},$$

which is summable in $\sigma \in \mathbb{Z}^d$. Now we have proved that

$$\sum_{\sigma\in\mathbb{Z}^d} \mathbb{P}(T(0,(2R\sigma)0) \ge C_1|\sigma|) < \infty, \tag{3.18}$$

where C_1 is a constant satisfying $C_1|\sigma| \ge (\mathbb{E}_{\max}+1)(|\sigma|+K_d)+2|\sigma|$ for all $\sigma \ne 0_{\mathbb{Z}^d}$.

Finally, we consider the passage time for all vertices that do not necessarily coincide with $(2R\sigma)0$. Since the number of vertices in each box $\bigsqcup_{z\in\Lambda(\sigma)} \mathcal{T}_z$ is finite, it follows from Lemma 3.2 that the random variable

$$S(\sigma) := \max\left\{T((2R\sigma)0,x) \ : \ x \in \bigsqcup_{z\in\Lambda(\sigma)} \mathcal{T}_z\right\}$$

has a finite dth moment. For any vertex $x \in X$, the first passage time $T(0,x)$ is bounded above by

$$T(0,(2R\sigma)0) + S(\sigma),$$

where $\sigma = \sigma_x \in \mathbb{Z}^d$ is the unique element satisfying $x \in \bigsqcup_{z\in\Lambda(\sigma)} \mathcal{T}_z$. We take a constant C_2 such that $C_2\|x\|_1 \geq |\sigma_x|$ holds for any $x \in X$ and set $C := 2C_1C_2$. Then we obtain

$$\begin{aligned}
&\sum_{x\in X} \mathbb{P}(T(0,x) \geq C\|x\|_1) \\
&\leq \sum_{\sigma\in\mathbb{Z}^d} \sum_{x\in\bigsqcup_{z\in\Lambda(\sigma)} \mathcal{T}_z} \left[\mathbb{P}(T(0,(2R\sigma)0) \geq C_1|\sigma|) + \mathbb{P}\left(S(\sigma) \geq C_1|\sigma|\right)\right] \\
&= \left|\bigsqcup_{z\in\Lambda(\sigma)} \mathcal{T}_z\right| \sum_{\sigma\in\mathbb{Z}^d} \left[\mathbb{P}(T(0,(2R\sigma)0) \geq C_1|\sigma|) + \mathbb{P}\left(S(\sigma) \geq C_1|\sigma|\right)\right] \\
&= \left|\bigsqcup_{z\in\Lambda(\sigma)} \mathcal{T}_z\right| \left[\sum_{\sigma\in\mathbb{Z}^d} \mathbb{P}(T(0,(2R\sigma)0) \geq C_1|\sigma|) + \sum_{\sigma\in\mathbb{Z}^d} \mathbb{P}\left(S(0_{\mathbb{Z}^d}) \geq C_1|\sigma|\right)\right],
\end{aligned}$$

where $\left|\bigsqcup_{z\in\Lambda(\sigma)} \mathcal{T}_z\right|$ denotes the number of vertices in $\bigsqcup_{z\in\Lambda(\sigma)} \mathcal{T}_z$. In the last equality, we use the fact that the distribution of $S(\sigma)$ does not depend on $\sigma \in \mathbb{Z}^d$. In the last expression, it follows from (3.18) that the first sum is finite. The finiteness of the second sum follows from the fact that $S(0_{\mathbb{Z}^d})$ has a finite dth moment. This completes the proof of (3.15). □

The remaining discussion for the proof of Proposition 3.7 is similar to that of [3, Theorem 2.16]. Let $\kappa < \infty$ be a constant satisfying (3.14) in Lemma 3.8. We say that a vertex $z \in X$ is *good* if the inequality

$$T(z,x) \leq \kappa\|z-x\|_1$$

holds for any $x \in X$ with $z \neq x$. Lemma 3.8 implies that

$$\mathbb{P}(z \text{ is good}) > 0$$

for any $z \in X$.

Proof of Proposition 3.7. Let $z \in \mathcal{D}$. We take the minimum number $M \in \mathbb{Z}_{>0}$ such that $Mz \in \Gamma$ and consider the sequence $(mMz)_{m=0,1,\ldots}$ in Γ. The ergodic theorem implies that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n} I_{\{iMz \text{ is good}\}} = \mathbb{P}(0 \text{ is good}) \tag{3.19}$$

holds almost surely. Let Ξ_1 be the event that (3.19) holds for any $z \in \mathcal{D}$. We also define Ξ_2 as the event that the convergence (3.6) in Proposition 3.3 holds for any $x \in \mathcal{D}$. Since $\mathcal{D}$ is a countable set, we can see $\mathbb{P}(\Xi_1) = 1 = \mathbb{P}(\Xi_2)$.

Suppose that Proposition 3.7 does not hold. Then, there exists $\epsilon > 0$ such that $\mathbb{P}(D_\epsilon) > 0$, where D_ϵ is the event that there exists and a sequence $(x_n)_{n=1,2,\ldots}$ of $\mathcal{D}$ such that $\|x_n\|_1 \to \infty$ and

$$|T(0, x_n) - \mu(x_n)| \geq \epsilon \|x_n\|_1 \tag{3.20}$$

holds for any n. Since $\mathbb{P}(\Xi_1) = \mathbb{P}(\Xi_2) = 1$, we have $\mathbb{P}(\Xi_1 \cap \Xi_2 \cap D_\epsilon) > 0$. Fix a configuration $\mathbf{t} \in \Xi_1 \cap \Xi_2 \cap D_\epsilon$. We show that a contradiction occurs for the configuration $\mathbf{t}$.

By taking a subsequence, we can assume that the normalized sequence $x'_n := x_n / \|x_n\|_1$ converges to some point $y \in \mathbb{R}^d$ with $\|y\|_1 = 1$. Fix $\delta > 0$ arbitrarily. Then there exists a number N such that

$$\|x'_n - y\|_1 < \delta \tag{3.21}$$

holds for any $n > N$. Since μ is a norm on $\mathbb{R}^d$ (or $\mu \equiv 0$), there exists some constant $C > 0$ such that $\mu(\cdot) \leq C \| \cdot \|_1$, which implies

$$\begin{aligned} |\mu(x_n) - \|x_n\|_1 \mu(y)| &\leq \mu(x_n - \|x_n\|_1 y) \\ &\leq C \|x_n\|_1 \|x'_n - y\|_1 \\ &< C \|x_n\|_1 \delta. \end{aligned} \tag{3.22}$$

Combining (3.20) with (3.22), we obtain

$$|T(0, x_n) - \|x_n\|_1 \mu(y)| > \epsilon \|x_n\|_1 / 2 \tag{3.23}$$

for sufficiently small $\delta > 0$.

We can find some $z \in \mathcal{D}$ such that $\|z - y\|_1 < \delta$ and $\|z\|_1 = 1$. Fix the minimum number M with $Mz \in \Gamma$. Since $\mathbf{t} \in \Xi_1$, the convergence (3.19) holds. In particular, there exist infinite number of good vertices in $(mMz)_{m=0,1,\ldots}$ and we can take a subsequence $(m_k Mz)_{k=0,1,\ldots}$ such that $m_k Mz$ is a good vertex. For this subsequence, we have

$$\frac{k}{m_k} = \frac{1}{m_k} \sum_{i=0}^{m_k} I_{\{iMz \text{ is good}\}} \to \mathbb{P}(0 \text{ is good}) > 0$$

as $k \to \infty$, which implies

$$\frac{m_{k+1}}{m_k} = \frac{m_{k+1}}{k+1}\frac{k}{m_k}\frac{k+1}{k} \to 1.$$

For any number n, let $k = k(n)$ be the number satisfying

$$m_k M \le \|x_n\|_1 < m_{k+1} M.$$

Fix $K > 0$ such that

$$m_{k+1} < (1+\delta) m_k \tag{3.24}$$

and

$$\left| \frac{T(0, m_k M z)}{m_k M} - \mu(z) \right| < \delta \tag{3.25}$$

hold for all $k > K$. We now let n be large enough that $k(n) > K$. Then we have

$$\begin{aligned}\left| \frac{T(0,x_n)}{\|x_n\|_1} - \mu(y) \right| \le& \left| \frac{T(0,x_n) - T(0, m_k M z)}{\|x_n\|_1} \right| + \frac{T(0, m_k M z)}{m_k M}\left(1 - \frac{m_k M}{\|x_n\|_1}\right) \\ &+ \left| \frac{T(0, m_k M z)}{m_k M} - \mu(z) \right| + |\mu(z) - \mu(y)|. \end{aligned} \tag{3.26}$$

We give an upper bound for each term of the right hand side of (3.26) to obtain the contradiction with (3.23).

Term 1. The triangle inequality for the first passage time implies

$$|T(0,x_n) - T(0, m_k M z)| \le T(x_n, m_k M z) \le \kappa \|x_n - m_k M z\|_1, \tag{3.27}$$

where the second inequality follows from the goodness of $m_k M z$. We also have

$$\begin{aligned} &\|x_n - m_k M z\|_1 \\ \le& \|x_n - \|x_n\|_1 y\|_1 + \big\| \|x_n\|_1 y - \|x_n\|_1 z \big\|_1 + \big\| \|x_n\|_1 z - m_k M z \big\|_1 \\ =& \|x_n\|_1 \left(\|x_n' - y\|_1 + \|y - z\|_1 + \left| 1 - \frac{m_k M}{\|x_n\|_1} \right| \right). \end{aligned}$$

From (3.24) and the setting of $k = k(n)$, we have

$$\left| 1 - \frac{m_k M}{\|x_n\|_1} \right| \le 1 - \frac{1}{1+\delta}. \tag{3.28}$$

By combining it with (3.21) and (3.27), the first term is bounded above by

$$\kappa\left(2\delta + 1 - \frac{1}{1+\delta}\right).$$

Term 2. From (3.25) and (3.28), we have the upper bound

$$(\mu(z)+\delta)\left(1 - \frac{1}{1+\delta}\right).$$

Term 3. From (3.25), we have the upper bound δ.
Term 4. Since z is chosen to satisfy $\|y - z\|_1 < \delta$, Term 4 is bounded above by $C\delta$.

These four estimates imply that the left hand side of (3.26) converges to 0 as $\delta \to 0$, which contradicts (3.23). □

3.2.2 Properties of the limit shape

In this subsection, we summarize basic relations between a periodic realization (Φ, ρ) and the limit shape, denoted by $\mathcal{B}_\Phi$.

First, we discuss the case where the time distribution ν is the Dirac measure δ_1. As we have seen in Section 1.1.3, this case is almost equivalent to considering the graph distance, and the limit shape is given by [27, Section 9.7] as follows: Let $H_1(X_0, \mathbb{Z})$ be the homology group of X_0, defined by (2.5) in Section 2.1.3. Let $\hat{\mathbf{v}}_0 : H_1(X_0, \mathbb{Z}) \to \mathbb{R}^d$ be the homomorphism defined by

$$\alpha = \sum_{e \in E_0} a_e e \longmapsto \sum_{e \in E_0} a_e \mathbf{v_0}(e),$$

where $\mathbf{v_0}(e)$ is the building block defined by (2.4) in Section 2.1.3. We consider the linear extension $\hat{\mathbf{v}}_{0\mathbb{R}} : H_1(X_0, \mathbb{R}) \to \mathbb{R}^d$. Let $|\cdot|_1$ be the norm on $H_1(X_0, \mathbb{R})$ defined as

$$|\alpha|_1 = \sum_{e \in E_0} |a_e|$$

for 1-chain $\alpha = \sum_{e \in E_0} a_e$. The limit shape $\mathcal{B}_\Phi$ is obtained as follows.

Proposition 3.9 ([27])**.** If the time distribution ν is the Dirac measure δ_1, then

$$\mathcal{B}_\Phi = \hat{\mathbf{v}}_{0\mathbb{R}}(D),$$

where D is the unit ball in $H_1(X_0, \mathbb{R})$ with respect to the norm $|\cdot|_1$.

In general, the relations between a periodic realization (Φ, ρ) and the limit shape $\mathcal{B}_\Phi$ are given as follows:

Proposition 3.10. The following hold:

(1) The limit shape $\mathcal{B}$ depends only on X, ν and the period ρ, that is, for two periodic realizations (Φ, ρ), (Φ', ρ'), we have $\mathcal{B}_\Phi = \mathcal{B}_{\Phi'}$ whenever $\rho = \rho'$.

(2) $\mathcal{B}_\Phi = \mathcal{B}_{\Phi+\mathbf{b}}$ for any $\mathbf{b} \in \mathbb{R}^d$, where $\Phi + \mathbf{b}$ is the periodic realization obtained by the map $x \mapsto \Phi(x) + \mathbf{b}$.

(3) $\mathcal{B}_{A\circ\Phi} = A\mathcal{B}_\Phi$ for any $A \in GL_d(\mathbb{R})$, where $GL_d(\mathbb{R})$ is the general linear group of degree d (note that $A \circ \Phi$ is also a periodic realization, whose period homomorphism is given by $A \circ \rho$).

Proof. The first item follows from Proposition 3.3 and (3.10).

We show the second item. We fix $\mathbf{b} \in \mathbb{R}^d$ arbitrarily. Let $T'(\cdot, \cdot)$ and $B'(t)$ be the first passage time and the percolation region with respect to the realization $\Phi + \mathbf{b}$. We then have

$$T'(0, x) = T(-\mathbf{b}, x - \mathbf{b}) \le T(0, -\mathbf{b}) + T(0, x - \mathbf{b}),$$

which implies the inclusion

$$B(t) + \mathbf{b} \subset B'(t + T(0, -\mathbf{b})).$$

By dividing by t and letting $t \to \infty$, we obtain $\mathcal{B}_\Phi \subset \mathcal{B}_{\Phi+\mathbf{b}}$. By replacing $\mathbf{b}$ with $-\mathbf{b}$, we obtain the opposite inclusion. Thus, the proof of the second item is completed.

For the third item, we denote by $\Gamma', T'(\cdot, \cdot), \mu'(\cdot)$ the characters with respect to $A \circ \Phi$. Then we have $\Gamma' = A\Gamma$ and the relation

$$T(0, x) = T'(0, Ax)$$

holds for any $x \in \Gamma$. This implies that $\mu(x) = \mu'(Ax)$ holds for any $x \in \mathbb{R}^d$. From (3.10), the proof is completed. □

Example 3.11. In Figure 1.7, since the realizations shown on the left and the center have the same period homomorphism ρ, the limit shapes obtained from the FPP model are the same, although the realizations are different.

Remark 3.12. At the end of Chapter 2, we assumed that the realization $\mathbf{\Phi}$ is nondegenerate in order to formulate the FPP model on a crystal lattice. Proposition 3.10 implies that this assumption is not essential for the limit shapes. Indeed, for any crystal lattice X with a degenerate realization $(\mathbf{\Phi}, \rho)$, we can obtain a nondegenerate realization $(\mathbf{\Phi}', \rho')$ with $\rho = \rho'$ by shifting the vertices in the same orbit of the action $L \curvearrowright X$ in the same direction.

Proposition 3.10 gives the symmetric property of the limit shape. Let $\mathrm{Sym}(\mathbf{\Phi}(X))$ be the symmetric group of the image $\mathbf{\Phi}(X)$, that is,

$$\mathrm{Sym}(\mathbf{\Phi}(X)) = \{g \in M(d) \; : \; g\mathbf{\Phi}(X) = \mathbf{\Phi}(X)\},$$

where $M(d)$ is the group of congruent transformations of $\mathbb{R}^d$. We write $M(d)$ as the semi-product $M(d) = \mathbb{R}^d \rtimes O(d)$ of the translation $\mathbb{R}^d$ and the rotation $O(d)$. Let $p : \mathrm{Sym}(\mathbf{\Phi}(X)) \to O(d)$ be the group homomorphism defined by

$$(\mathbf{b}, A) \mapsto A.$$

Then we obtain the following.

Proposition 3.13. For any $A \in \mathrm{Im}(p)$, $A\mathcal{B}_{\mathbf{\Phi}} = \mathcal{B}_{\mathbf{\Phi}}$. In other words, the limit shape $\mathcal{B}_{\mathbf{\Phi}}$ has the symmetry given by $\mathrm{Im}(p)$.

Proof. For any $(\mathbf{b}, A) \in \mathrm{Sym}(\mathbf{\Phi}(X))$, we obtain

$$A \circ \mathbf{\Phi}(X) + \mathbf{b} = \mathbf{\Phi}(X),$$

which implies

$$\mathcal{B}_{\mathbf{\Phi}} = \mathcal{B}_{A\circ\mathbf{\Phi}+\mathbf{b}} = \mathcal{B}_{A\circ\mathbf{\Phi}} = A\mathcal{B}_{\mathbf{\Phi}}.$$

Here we use the second and third items of Proposition 3.10 for the second and third equalities, respectively. □

Example 3.14. In Figure 1.7, the honeycomb lattice on the left has rotational symmetry, which implies the same symmetry of the limit shape. Note that the limit shape obtained from the lattice in the center is the same as that obtained from the one on the left. Thus, the lattice in the center also has rotational symmetry.

3.3 Covering monotonicity of the limit shapes

In this section, we give some examples and an application of Theorem 1.6 and then prove the theorem.

For two crystal lattices $X = (V, E)$ and $X_1 = (V_1, E_1)$ in Theorem 1.6, we consider the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ defined by (2.8) with $S = E \sqcup E_1$. We assume that the covering map $\omega : X \to X_1$ maps the origin $0 \in X$ to the origin of X_1. We write $\omega(0)$ by 0 for short. The same notation as in the previous sections is used concerning the FPP model on X. For the covered graph X_1, we use subscripts, such as T_1, $B_1(t)$, μ_1 and $\mathcal{B}_1$.

3.3.1 Examples and an application

We first remark on the strictness of the inclusion in Theorem 1.6.

Remark 3.15. When the time distribution ν is the Dirac measure δ_1, the equality

$$\mathcal{B}_1 = P(\mathcal{B})$$

of Theorem 1.6 holds. Indeed, the limit shape $\mathcal{B}$ of X is the image $\hat{\mathbf{v}}_{0\mathbb{R}}(D)$ as shown in Proposition 3.9. From the projective relation (1.11), we can see that the building block of $\Phi_1 : X_1 \to \mathbb{R}^{d_1}$ is $(P \circ \mathbf{v}_0(e_0))_{e_0 \in E_0}$. This implies that the limit shape $\mathcal{B}_1$ is given by the image $P \circ \mathbf{v}_{0\mathbb{R}}(D)$, which is equal to $P(\mathcal{B})$.

Remark 3.16. Since a crystal lattice is quasi-transitive, it follows from Proposition 1.3 that the strict inequality

$$p_c(X) < p_c(X_1)$$

holds for two lattices X, X_1 in Theorem 1.6. Therefore, for the time distribution ν with $p_c(X) < \nu(0) < p_c(X_1)$, we can see that $\mathcal{B} = \mathbb{R}^d$ and $\mathcal{B}_1$ is compact. In particular, the strict inequality $\mathcal{B}_1 \subsetneq P(\mathcal{B})$ holds.

The following are examples and an application of Theorem 1.6.

Example 3.17. As shown in Figure 1.8, the cubic lattice $\mathbb{L}^3$ is projected by the orthogonal projection $P : \mathbb{R}^3 \to \mathbb{R}^2 \simeq \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 + x_2 + x_3 = 0\}$ onto the triangular lattice $\mathbb{T}$. Theorem 1.6 implies that the projection of the limit shape $\mathcal{B}$ of $\mathbb{L}^3$ to the plane $\{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 + x_2 + x_3 = 0\}$ is bounded below by the limit shape $\mathcal{B}_1$ of $\mathbb{T}$.

Example 3.18. For the cubic lattice $\mathbb{L}^2$, we can consider a "coarse" action of $\mathbb{Z}^2$ as shown in Figure 3.2, and obtain the base graph X_0, which consists of two points and four parallel edges.

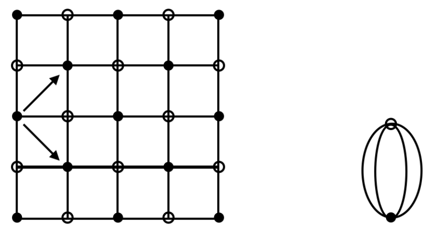

Figure 3.2: An action on $\mathbb{L}^2$ (left) and the base graph X_0 (right).

As mentioned in Example 2.11, the maximal abelian covering graph of X_0 is the diamond lattice $\mathbb{D}$. Theorem 2.10 implies that there exists a covering $\omega : \mathbb{D} \to \mathbb{L}^2$. Moreover, the explicit construction of periodic realizations of $\mathbb{D}$ and $\mathbb{L}^2$ satisfying the projective relation (1.11) is given in [27, Section 8.1]. In this setting, Theorem 1.6 gives an upper bound on the limit shape of $\mathbb{L}^2$.

Example 3.19. As an application of Theorem 1.6, we present an evaluation of the norm μ with respect to the diagonal direction.

Let X be the cubic lattice $\mathbb{L}^2$. By the orthogonal projection $P : \mathbb{R}^2 \to W$ onto the subspace $W := \{(x_1, x_2) \in \mathbb{R}^2 \ : \ x_2 = x_1\}$, we obtain a periodic realization of the quotient graph X_1, defined as the one-dimensional line with parallel edges (Figure 3.3). From the law of large numbers, we can easily see that

$$\mu_1((1,1)) = 2\mathbb{E}\min\{t_1, t_2\}$$

for a suitable time distribution ν. From Theorem 1.6, we can see that

$$\{x \in W \ : \ \mu_1(x) \leq R\} \subset P(\{y \in \mathbb{R}^2 \ : \ \mu(y) \leq R\})$$

for any $R > 0$. By setting $R := \mu_1((1,1))$, we can see that there exists $y \in P^{-1}((1,1))$ satisfying

$$\mu(y) \leq \mu_1((1,1)). \tag{3.29}$$

From the symmetric property of $\mathbb{L}^2$, the symmetric point y' of y with respect to the line W also satisfies (3.29). Thus, we obtain

$$\mu((1,1)) = \mu((y + y')/2) \leq \frac{1}{2}(\mu(y) + \mu(y')) \leq \mu_1((1,1)),$$

and we have the upper estimate

$$\mu((1,1)) \leq 2\mathbb{E}\min\{t_1, t_2\}.$$

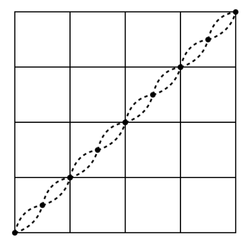

Figure 3.3: The cubic lattice $\mathbb{L}^2$ (black lines) and its quotient graph X_1 (dots) realized in W.

3.3.2 Proof

We prove Theorem 1.6 in this subsection by giving some lemmas. The first lemma is a generalization of [18, Proposition 1.14], stating that the asymptotic speed "from point to line" is equal to that "from point to point." Here, for a point $x \in \mathbb{R}^d$ and a closed subset $A \subset \mathbb{R}^d$, we denote by $T(x, A)$ the first passage time from x to A:

$$T(x, A) = \inf_{y \in A} T(x, y). \tag{3.30}$$

Lemma 3.20. Suppose that the time distribution ν satisfies (1.12) for X and let A be a closed subset of $\mathbb{R}^d$. Then there exists a point $x \in A$ such that

$$\mu(x) = \lim_{n \to \infty} \frac{T(0, nA)}{n} \tag{3.31}$$

holds almost surely.

Proof. This is clear for the case $0 \in A$ or $\nu(0) \geq p_c(X)$ since we can see that both sides of (3.31) is equal to 0. Consider the case $0 \notin A$ and $\nu(0) < p_c(X)$. Let $r > 0$ satisfy the closed subset rA being tangent to the limit shape $\mathcal{B} = \{x \in \mathbb{R}^d \,:\, \mu(x) \leq 1\}$, that is,

- $\mu(x) \geq 1$ for any $x \in rA$; and
- there exists $y \in rA$ such that $\mu(y) = 1$.

Fix a point $y \in rA$ with $\mu(y) = 1$. From the definition of the first passage time, we can easily see that

$$\limsup_{n\to\infty} \frac{T(0, nrA)}{n} \le \mu(y) = 1$$

holds almost surely. We prove that

$$1 \le \liminf_{n\to\infty} \frac{T(0, nrA)}{n} \tag{3.32}$$

holds almost surely. Suppose that (3.32) does not hold almost surely. Then there exists $\delta > 0$ with $\mathbb{P}(\Xi) > 0$, where Ξ is the event defined as

$$\Xi := \left\{ \liminf_{n\to\infty} \frac{T(0, nrA)}{n} \le 1 - 4\delta \right\}. \tag{3.33}$$

Consider a configuration in the event Ξ. By (3.33), we can take a subsequence $\{n_k\}_k$ such that

$$\frac{T(0, n_k rA)}{n_k} \le 1 - 3\delta$$

holds for all large k. From (3.30), we can take a sequence $y_1, y_2, \ldots$ of points with $y_k \in n_k rA$ such that

$$\frac{T(0, y_k)}{n_k} \le \frac{T(0, n_k rA)}{n_k} + \delta.$$

Thus, we have

$$\frac{T(0, y_k)}{n_k} \le 1 - 2\delta,$$

which is equivalent to

$$y_k \in B(n_k(1 - 2\delta)).$$

Take $\epsilon > 0$ small enough to satisfy $(1+\epsilon)(1-2\delta) \le 1-\delta$. Then, Theorem 1.5 implies that

$$\frac{y_k}{n_k} \in (1 - 2\delta)\frac{B(n_k(1 - 2\delta))}{n_k(1 - 2\delta)} \subset (1 - 2\delta)(1 + \epsilon)\mathcal{B} \subset (1 - \delta)\mathcal{B}$$

holds for all large k almost surely. This leads to $\mu(y_k/n_k) \le 1 - \delta$. Since $y_k/n_k \in rA$, this contradicts the assumption that rA is tangent to $\mathcal{B}$.

From the above discussion, we obtain that

$$\mu(y) = \lim_{n\to\infty} \frac{T(0, nrA)}{n} = r \lim_{n\to\infty} \frac{T(0, nA)}{n}$$

holds almost surely. Setting $x := y/r \in A$, we have

$$\mu(x) = \lim_{n\to\infty} \frac{T(0, nA)}{n},$$

which completes the proof of Lemma 3.20. □

In the next lemma, we compare two passage times T_1 and T. Note that the following lemma itself does not assume any lattice structures of X and X_1.

Lemma 3.21. For any vertex $x_1 \in X_1$ and $t \geq 0$, the inequality

$$\mathbb{P}(T_1(0, x_1) \geq t) \geq \mathbb{P}(T(0, \widetilde{x}_1) \geq t \text{ for any } \widetilde{x}_1 \in \omega^{-1}(x_1)) \tag{3.34}$$

holds.

Theorem 1.6 follows from Lemmas 3.20 and 3.21.

Proof of Theorem 1.6. Take $x_1 \in \mathcal{D}_1$ with $\mu_1(x_1) \leq 1$ arbitrarily, and fix $N \in \mathbb{Z}_{>0}$ with $Nx_1 \in \Gamma_1$. Note that kNx_1 coincides with a realized vertex for any $k = 1, 2, \ldots$, and it follows from (3.34) that

$$\mathbb{P}(T_1(0, kNx_1) \geq t) \geq \mathbb{P}(T(0, y) \geq t \text{ for any } y \in \omega^{-1}(kNx_1)).$$

The projective relation (1.11) implies that the right hand side is bounded below by

$$\mathbb{P}(T(0, P^{-1}(kNx_1)) \geq t)$$

since any points $y \in \omega^{-1}(kNx_1)$ are in the subspace $P^{-1}(kNx_1)$. By integrating with respect to t from 0 to ∞, we obtain

$$\mathbb{E}T_1(0, kNx_1) \geq \mathbb{E}T(0, P^{-1}(kNx_1)). \tag{3.35}$$

From Lemma 3.20, we can find a point $y \in P^{-1}(x_1)$ such that

$$\mu(y) = \lim_{k\to\infty} \frac{T(0, kNP^{-1}(x_1))}{kN}. \tag{3.36}$$

The sequence $\left(\frac{T(0,kNP^{-1}(x_1))}{kN}\right)_{k=1,2,\ldots}$ of random variables is uniformly integrable. Indeed, for some vertex $x \in \omega^{-1}(Nx_1)$, we have

$$\frac{T(0,kNP^{-1}(x_1))}{kN} \leq \frac{T(0,kx)}{kN} \leq \frac{1}{kN}\sum_{i=0}^{k-1} T(ix,(i+1)x).$$

From the assumption (1.12), the dth moment of the right hand side is bounded above by some constant, which does not depend on k. Thus, by taking the expectation of (3.36), we obtain

$$\mu(y) = \lim_{k\to\infty} \frac{\mathbb{E}T(0,kNP^{-1}(x_1))}{kN}. \tag{3.37}$$

By combining (3.35) with (3.37), we obtain

$$1 \geq \mu_1(x_1) = \lim_{k\to\infty} \frac{\mathbb{E}T_1(0,kNx_1)}{kN} \geq \lim_{k\to\infty} \frac{\mathbb{E}T(0,kNP^{-1}(x_1))}{kN} = \mu(y),$$

which implies $y \in \mathcal{B}$. Here, the first equality follows from Theorem 3.4.

We now obtain $\mathcal{B}_1 \cap \mathcal{D}_1 \subset P(\mathcal{B})$. Since the projection $P(\mathcal{B})$ of the limit shape is closed, the proof of Theorem 1.6 is completed. □

We next prove Lemma 3.21. The key idea of the proof is based on the FKG inequality (Theorem 2.12), which we reviewed in Section 2.2. We note that the right hand side of the FKG inequality (2.10) can be regarded as the probability $\mathbb{P}(A' \cap B')$ of the intersection $A' \cap B'$, where A', B' are independent copies of A, B. Here, we can roughly expect that the probability $\mathbb{P}(A \cap B)$ decreases as the correlation of A and B decreases. For two paths γ_1, γ_2 in X_1 and their liftings $\widetilde{\gamma}_1$, $\widetilde{\gamma}_2$, the "correlation" of $\widetilde{\gamma}_1$, $\widetilde{\gamma}_2$ is less than that of γ_1, γ_2. Thus, by regarding the event $\{T_1(0,x_1) \geq t\}$ as the intersection $\bigcap_\gamma \{T(\gamma) \geq t\}$ of the events $\{T(\gamma) \geq t\}$ for all paths from 0 to x_1 and comparing it with $\bigcap_\gamma \{T(\widetilde{\gamma}) \geq t\}$, we have obtained a proof of Lemma 3.21.

For a rigorous proof of Lemma 3.21, we require two more lemmas. Let A and B be two increasing events which depend only on the family $(t_1, t_2, \ldots, t_m)$ of i.i.d. random variables with the distribution ν. Let t_a and t_b be independent copies of t_m. We define A' (resp. B') as the event obtained from A (resp. B) by replacing t_m with t_a (resp. t_b). Then the following holds.

Lemma 3.22.

$$\mathbb{P}(A \cap B) \geq \mathbb{P}(A' \cap B').$$

Proof. Let $[m] := \{1, 2, \ldots, m\}$. We denote by $\mathbb{P}_\bullet := \nu^{\otimes \bullet}$ the product measure on $[0, \infty)^\bullet$. Since A and B do not depend on t_a, t_b, we can identify these events with the Borel subsets of $[0, \infty)^{[m]}$, and we have

$$\begin{aligned}
&\mathbb{P}(A \cap B) \\
&= \int_{[0,\infty)^{[m]}} I_{A\cap B} d\mathbb{P}_{[m]} \\
&= \int_{[0,\infty)^{[m-1]}} \int_{[0,\infty)^{\{m\}}} I_{A\cap B} d\mathbb{P}_{\{m\}} d\mathbb{P}_{[m-1]} \\
&= \int_{[0,\infty)^{[m-1]}} \mathbb{P}_{\{m\}}\big(A_{(t_1,\ldots,t_{m-1})} \cap B_{(t_1,\ldots,t_{m-1})}\big) d\mathbb{P}_{[m-1]}. \qquad (3.38)
\end{aligned}$$

Here, we denote by $A_{(t_1,\ldots,t_{m-1})}$ the set of $t_m \in [0, \infty)^{\{m\}}$ with $I_A(t_1, \ldots, t_m) = 1$. We can easily see that

$$\mathbb{P}_{\{m\}}\big(A_{(t_1,\ldots,t_{m-1})} \cap B_{(t_1,\ldots,t_{m-1})}\big) \geq \mathbb{P}_{\{m\}}\big(A_{(t_1,\ldots,t_{m-1})}\big)\mathbb{P}_{\{m\}}\big(B_{(t_1,\ldots,t_{m-1})}\big). \quad (3.39)$$

Indeed, since both $A_{(t_1,\ldots,t_{m-1})}$ and $B_{(t_1,\ldots,t_{m-1})}$ are increasing subsets of the half line $[0, \infty)^{\{m\}}$, one is included in the other. Thus, the left hand side of (3.39) is equal to one of the two probabilities of the right hand side.

We identify the events A', B' with the Borel subsets of $[0, \infty)^{[m-1]\sqcup\{a,b\}}$. We define the events $A'_{(t_1,\ldots,t_{m-1})}$, $B'_{(t_1,\ldots,t_{m-1})}$ on $[0, \infty)^{\{a,b\}}$ as the sets of $(t_a, t_b) \in [0, \infty)^{\{a,b\}}$ with $I_{A'}(t_1, \ldots, t_{m-1}, t_a, t_b) = 1$, $I_{B'}(t_1, \ldots, t_{m-1}, t_a, t_b) = 1$, respectively. Then the right hand side of (3.39) is equal to

$$\mathbb{P}_{\{a,b\}}\big(A'_{(t_1,\ldots,t_{m-1})}\big)\mathbb{P}_{\{a,b\}}\big(B'_{(t_1,\ldots,t_{m-1})}\big) = \mathbb{P}_{\{a,b\}}\big(A'_{(t_1,\ldots,t_{m-1})} \cap B'_{(t_1,\ldots,t_{m-1})}\big). \quad (3.40)$$

By combining (3.38) with (3.39) and (3.40), we obtain

$$\begin{aligned}
\mathbb{P}(A \cap B) &\geq \int_{[0,\infty)^{[m-1]}} \mathbb{P}_{\{a,b\}}\big(A'_{(t_1,\ldots,t_{m-1})} \cap B'_{(t_1,\ldots,t_{m-1})}\big) d\mathbb{P}_{[m-1]} \\
&= \int_{[0,\infty)^{[m-1]}} \int_{[0,\infty)^{\{a,b\}}} I_{A'\cap B'} d\mathbb{P}_{\{a,b\}} d\mathbb{P}_{[m-1]} \\
&= \mathbb{P}(A' \cap B'),
\end{aligned}$$

which completes the proof of Lemma 3.22. □

Lemma 3.23. Let $\gamma_1, \dots \gamma_n$ be arbitrary self-avoiding paths in X_1 which start from the origin 0 and let $\widetilde{\gamma}_1, \dots, \widetilde{\gamma}_n$ in X be their liftings. Then the inequality

$$\mathbb{P}\left(\bigcap_{i=1}^{n}\{T_1(\gamma_i) \geq t\}\right) \geq \mathbb{P}\left(\bigcap_{i=1}^{n}\{T(\widetilde{\gamma}_i) \geq t\}\right) \tag{3.41}$$

holds for any $t \geq 0$.

Proof. We set $\gamma_i = (e_{i,1}, \dots, e_{i,r_i})$ and $\widetilde{\gamma}_i = (\widetilde{e}_{i,1}, \dots, \widetilde{e}_{i,r_i})$ for $i = 1, 2, \dots, n$. Note that $\widetilde{e}_{i,j} \in E$ is mapped to $e_{i,j} \in E_1$ by the covering map $\omega : X \to X_1$.

Let $\mathcal{I} := \{(i, j) : i = 1, 2, \dots, n \text{ and } j = 1, 2, \dots, r_i\}$ be the index set. For a partition $\mathcal{S} = \{S_1, \dots, S_m\}$ of $\mathcal{I}$, we denote by $\pi_{\mathcal{S}} : \mathcal{I} \to \mathcal{S}$ the canonical map, which is defined by $\pi_{\mathcal{S}}(i, j) = S_k$ when $(i, j) \in S_k$. We define the probability $\mathbb{P}^{(\mathcal{S})} \in [0, 1]$ by

$$\mathbb{P}^{(\mathcal{S})} := \mathbb{P}\left(\bigcap_{i=1}^{n}\left\{\sum_{j=1}^{r_i} t_{\pi_{\mathcal{S}}(i,j)} \geq t\right\}\right), \tag{3.42}$$

where $(t_S : S \in \mathcal{S})$ is the $\mathcal{S}$-indexed family of i.i.d random variables with the distribution ν.

We set the partition $\mathcal{S}$ (resp. $\widetilde{\mathcal{S}}$) of $\mathcal{I}$ by the equivalence relation

$$(i, j) \sim (i', j') \overset{\text{def}}{\iff} e_{i,j} = e_{i',j'}$$
$$(\text{resp. } (i, j) \sim (i', j') \overset{\text{def}}{\iff} \widetilde{e}_{i,j} = \widetilde{e}_{i',j'}).$$

The inequality (3.41) can be rewritten as

$$\mathbb{P}^{(\mathcal{S})} \geq \mathbb{P}^{(\widetilde{\mathcal{S}})}. \tag{3.43}$$

Since $\widetilde{e}_{i,j} = \widetilde{e}_{i',j'}$ implies $e_{i,j} = e_{i',j'}$, the partition $\widetilde{\mathcal{S}}$ is finer than $\mathcal{S}$. We show that a further division of the partition $\mathcal{S}$ decreases the probability (3.42).

Let $\mathcal{S}'$ be a partition obtained from $\mathcal{S} := \{S_1, \dots, S_m\}$ by splitting some element, say $S_m \in \mathcal{S}$, into two nonempty subsets S_a, S_b. For the $\mathcal{S}$-indexed family $(t_S : S \in \mathcal{S})$ of i.i.d random variables, we take two independent copies t_{S_a}, t_{S_b} of t_{S_m} and obtain the $\mathcal{S}'$-indexed family $(t_S : S \in \mathcal{S}')$. Let $p : \mathcal{I} \to \{1, 2, \dots, n\}$ be the projection defined by $(i, j) \mapsto i$. We write the event in (3.42) as the intersection $A \cap B$ of the two events A, B, where

$$A = \bigcap_{i \in p(S_a)}\left\{\sum_{j=1}^{r_i} t_{\pi_{\mathcal{S}}(i,j)} \geq t\right\} \text{ and } B = \bigcap_{i \notin p(S_a)}\left\{\sum_{j=1}^{r_i} t_{\pi_{\mathcal{S}}(i,j)} \geq t\right\}.$$

We also set

$$A' = \bigcap_{i \in p(S_a)} \left\{ \sum_{j=1}^{r_i} t_{\pi_{\mathcal{S}'}(i,j)} \geq t \right\} \text{ and } B' = \bigcap_{i \notin p(S_a)} \left\{ \sum_{j=1}^{r_i} t_{\pi_{\mathcal{S}'}(i,j)} \geq t \right\}.$$

Here, we note that the event A' does not depend on t_{S_b}. Indeed, the assumption that each γ_i is self-avoiding implies that the restriction $p_{\upharpoonright S_k}$ of the map p to some element $S_k \in \mathcal{S}$ is injective. Since S_a and S_b are disjoint subsets of the same element $S_m \in \mathcal{S}$, we have $p(S_a) \cap p(S_b) = \emptyset$.

Therefore, the event A' (resp. B') can also be obtained from A (resp. B) by replacing t_{S_m} with t_{S_a} (resp. t_{S_b}). From Lemma 3.22, we obtain

$$\mathbb{P}^{(\mathcal{S})} \geq \mathbb{P}^{(\mathcal{S}')}.$$

We can take a finite sequence $\mathcal{S} = \mathcal{S}^{(0)}, \mathcal{S}^{(1)}, \dots, \mathcal{S}^{(K)} = \widetilde{\mathcal{S}}$ of partitions of $\mathcal{I}$ such that $\mathcal{S}^{(k+1)}$ is obtained from $\mathcal{S}^{(k)}$ by splitting an element of $\mathcal{S}^{(k)}$ into two nonempty sets. By replacing $\mathcal{S}$, $\mathcal{S}'$ with $\mathcal{S}^{(k)}$, $\mathcal{S}^{(k+1)}$ and iterating the above discussion for $k = 0, 1, \dots, K-1$, we obtain (3.43). This completes the proof of (3.41). $\square$

We now turn to the proof of Lemma 3.21.

Proof of Lemma 3.21. Let Λ_R be the ball with radius R:

$$\Lambda_R = \{y_1 \in X_1 \, : \, d_{X_1}(0, y_1) \leq R\}.$$

Letting R be sufficiently large that Λ_R includes x_1, we set the restricted first passage time as

$$T_1^R(0, x_1) := \inf\{T_1(\gamma) \, : \, \gamma \text{ is a path in } \Lambda_R \text{ from } 0 \text{ to } x_1\}.$$

Let $\{\gamma_1, \gamma_2, \dots, \gamma_n\}$ be the finite set of all self-avoiding paths in Λ_R that go from 0 to x_1. We then have

$$T_1^R(0, x_1) := \min_{i=1,\dots,n} T_1(\gamma_i),$$

and it follows from (3.41) that

$$\mathbb{P}(T_1^R(0, x_1) \geq t) = \mathbb{P}\left(\bigcap_{i=1}^{n} \{T_1(\gamma_i) \geq t\}\right) \geq \mathbb{P}\left(\bigcap_{i=1}^{n} \{T(\widetilde{\gamma}_i) \geq t\}\right),$$

where each $\widetilde{\gamma}_i$ is the lifting of γ_i which starts from 0. Since the terminus of each path $\widetilde{\gamma}_i$ is in $\omega^{-1}(x_1)$, the last expression is bounded below by

$$\mathbb{P}(T(0, \widetilde{x}_1) \geq t \text{ for any } \widetilde{x}_1 \in \omega^{-1}(x_1)).$$

Letting $R \to \infty$ completes the proof of Lemma 3.21. $\square$

Chapter 4

Bond percolation on crystal lattices

This chapter studies the bond percolation model on crystal lattices. First, in Section 4.1, we summarize some observations obtained from the graph structure of a crystal lattice. In Section 4.2, we give the inverse correlation length, which is a norm on $\mathbb{R}^d$ induced from the bond percolation model, and present a large deviation result for the cluster C (Theorem 4.11). As an application of this result, the covering monotonicity of the inverse correlation length (Theorem 1.8) is proved in Section 4.3.

4.1 Results obtained from the graph structure of a crystal lattice

4.1.1 Phase transition

In this subsection, we check that the phase transition represented by Theorem 1.1 for the cubic lattice model also holds for a crystal lattice model.

Let $X = (V, E)$ be a crystal lattice. We first note that the exponential decay (1.2) in Theorem 1.1 also holds for a crystal lattice model. Namely, for $p < p_c(X)$, there exists a constant $\sigma := \sigma(p) > 0$ such that

$$\mathbb{P}_p(x \leftrightarrow y) \leq \exp(-\sigma d_X(x, y)) \tag{4.1}$$

holds for any $x, y \in X$. The proof can be obtained in the same manner as for [15, Theorem 5.4] by replacing the L_1-norm in [15] with the graph distance.

For the proof of (1.1) in Theorem 1.1, we need to show the *uniqueness of the infinite cluster*: if there exist infinite clusters in X with positive probability, then the number of them is equal to one almost surely.

Let $\theta(p,x) := \mathbb{P}_p(x \leftrightarrow \infty)$ be the percolation probability defined in Section 2.2. For the origin $0 \in X$, we abbreviate $\theta(p,0)$ by $\theta(p)$. Note that, we have already seen in (2.11) that whether $\theta(p)$ is positive or not does not depend on the choice of the origin.

Proposition 4.1. Let X be a d-dimensional crystal lattice. Then for any $p \in (0,1)$ such that $\theta(p) > 0$, there exists a unique infinite cluster in X almost surely.

In the case of $X = \mathbb{L}^d$, Proposition 4.1 has already been obtained by [1]. In order to generalize it to a crystal lattice model, we use the result of [5]. For a graph $Y = (V_Y, E_Y)$, the *isoperimetric constant* $\kappa(Y)$ is defined by

$$\kappa(Y) = \inf_{W \subset V_Y} \frac{|\partial W|}{|W|},$$

where the infimum runs over all finite connected subsets $W \subset V_Y$. Here, the boundary ∂W is defined as the set of all vertices of W with at least one neighbor in $V_Y \setminus W$, and $|\cdot|$ denotes the number of vertices. A graph Y is said to be *amenable* if the isoperimetric constant $\kappa(Y)$ is equal to 0. The general result of uniqueness is stated as follows.

Proposition 4.2 ([5])**.** Assume that Y is connected, transitive and amenable. Then for any $p \in (0,1)$ such that $\theta(p) > 0$, there exists a unique infinite cluster in Y almost surely.

We note that in Proposition 4.2, the condition that Y is transitive can be relaxed to being quasi-transitive. We now turn to the proof of Proposition 4.1.

Proof of Proposition 4.1. Since a crystal lattice is connected and quasi-transitive, we only need to check that X is amenable. Recall that, from (2.2) in Section 2.1.3, the vertex set V of X can be considered as an array of finite trees:

$$V = \bigsqcup_{\sigma \in \mathbb{Z}^d} \mathcal{T}_\sigma.$$

Moreover, from the periodic structure of X, we can see that there exists a constant K such that

$$\mathcal{T}_\sigma \sim \mathcal{T}_\tau \Longrightarrow \|\sigma - \tau\|_1 \leq K \tag{4.2}$$

holds for any $\sigma, \tau \in \mathbb{Z}^d$. Here, $\mathcal{T}_\sigma \sim \mathcal{T}_\tau$ means that there exist vertices $x \in \mathcal{T}_\sigma$ and $y \in \mathcal{T}_\tau$ such that x is adjacent to y.

For $R \in \mathbb{Z}_{\geq 0}$, we define the finite connected subset

$$W_R := \bigsqcup_{\|\sigma\|_1 \leq R} \mathcal{T}_\sigma \subset V.$$

Then, from (4.2), we can see that the boundary ∂W_R consists of vertices of $\mathcal{T}_\sigma$ with $\|\sigma\|_1 \geq R - K + 1$. Thus, we have

$$\frac{|\partial W_R|}{|W_R|} \leq \frac{(2R+1)^d - (2(R-K)+1)^d}{(2R+1)^d} \to 0$$

as $R \to \infty$. This implies that X is amenable. □

From these observations, we obtain the following phase transition.

Theorem 4.3. Let X be a crystal lattice. If $p > p_c(X)$, then there exists a constant $c := c(p) > 0$ such that

$$\mathbb{P}_p(x \leftrightarrow y) \geq c \text{ for any } x, y \in X. \tag{4.3}$$

If $p < p_c(X)$, then there exists $\sigma := \sigma(p) > 0$ such that

$$\mathbb{P}_p(x \leftrightarrow y) \leq \exp(-\sigma d_X(x, y)) \text{ for any } x, y \in X. \tag{4.4}$$

Proof. We have already seen the exponential decay (4.4) in (4.1). We now show (4.3). From Proposition 4.1 we can see that two vertices x, y that belong to an infinite cluster must be connected almost surely. Therefore, we have

$$\mathbb{P}_p(x \leftrightarrow y) \geq \mathbb{P}_p(\{x \leftrightarrow \infty\} \cap \{y \leftrightarrow \infty\}).$$

It follows from the FKG inequality (2.10) that the right hand side is bounded below by

$$\theta(p, x)\theta(p, y) > 0,$$

which does not depend on the distance between x and y. □

4.1.2 Estimate of the critical probability of the maximal abelian covering graph

In Section 1.1.2, we have seen the critical probabilities of the cubic lattice $\mathbb{L}^2$ and the honeycomb lattice $\mathbb{H}$. From these values, we can obtain the comparison

$$p_c(\mathbb{L}^2) < p_c(\mathbb{H}). \tag{4.5}$$

In this subsection, we reconsider this inequality from the viewpoint of maximal abelian covering graphs, which we introduced in Section 2.1.3. This observation provides a direct proof of a generalization of (4.5).

Let X_0 be a finite connected graph and $X_0^{\mathrm{ab}} = (V_0^{\mathrm{ab}}, E_0^{\mathrm{ab}})$ be the maximal abelian covering graph over X_0. Recall that, from (2.6) in Section 2.1.3, the vertex set V_0^{ab} of X_0^{ab} can be represented by the array

$$V_0^{\mathrm{ab}} = \bigsqcup_{\sigma \in \mathbb{Z}^d} \mathcal{T}_\sigma$$

of a spanning tree $\mathcal{T}$ of X_0, with the relation (2.7). We further assume that

$$\begin{gathered}\text{there exist two or more vertices in } \mathcal{T}_0 \\ \text{are adjacent to different trees, respectively.}\end{gathered} \tag{4.6}$$

This condition excludes crystal lattices that are essentially the same as the cubic lattice as shown in Figure 4.1.

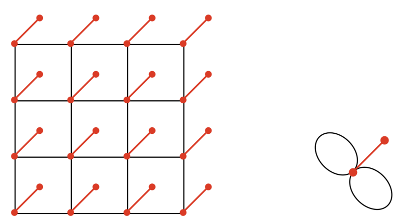

Figure 4.1: Maximal abelian covering graph (left) over a finite graph (right). This graph does not satisfy the condition (4.6).

A generalization of (4.5) is stated as follows.

Proposition 4.4. Let X_0^{ab} be the maximal abelian covering over some finite graph X_0 with dimension d. If X_0^{ab} satisfies the condition (4.6), then

$$p_c(\mathbb{L}^d) < p_c(X_0^{\mathrm{ab}}).$$

Proof. We use the "enhancement" technique (see, e.g., [15, Section 3.3] for a detailed formulation) to X_0^{ab}. Consider the bond percolation model on X_0^{ab} with probability $p \in [0, 1]$. We add the following enhancement on this model:

$$\begin{aligned}&\text{any two vertices in the same tree } \mathcal{T}_\sigma \\ &\text{are assumed to be connected with probability one.}\end{aligned} \tag{4.7}$$

From condition (4.6), this enhancement is "essential" in the sense that it contributes to the creation of infinite clusters. We write by $p_c^{\mathrm{enh}}(X_0^{\mathrm{ab}})$ the critical probability of X_0^{ab} under this enhancement. Then [15, Theorem 3.16] implies that

$$p_c^{\mathrm{enh}}(X_0^{\mathrm{ab}}) < p_c(X_0^{\mathrm{ab}}). \tag{4.8}$$

Incidentally, the bond percolation model on X_0^{ab} with the enhancement (4.7) is essentially the same as the normal percolation model on the cubic lattice $\mathbb{L}^d$. Namely, for two vertices σ, $\tau \in \mathbb{Z}^d$, we have

$$\mathbb{P}_p(\mathcal{T}_\sigma \overset{\mathrm{enh}}{\longleftrightarrow} \mathcal{T}_\tau \text{ in } X_0^{\mathrm{ab}}) = \mathbb{P}_p(\sigma \leftrightarrow \tau \text{ in } \mathbb{L}^d). \tag{4.9}$$

Here, $\mathcal{T}_\sigma \overset{\mathrm{enh}}{\longleftrightarrow} \mathcal{T}_\tau$ means that two trees $\mathcal{T}_\sigma$ and $\mathcal{T}_\tau$ are connected by a path of open edges in the setting of this enhancement. In the right hand side of (4.9), we identify σ, $\tau \in \mathbb{Z}^d$ with the vertices of $\mathbb{L}^d$. From (4.9), we easily obtain

$$p_c^{\mathrm{enh}}(X_0^{\mathrm{ab}}) = p_c(\mathbb{L}^d). \tag{4.10}$$

Combining (4.8) with (4.10), the proposition follows. □

Example 4.5. We can apply Proposition 4.4 to the honeycomb lattice, which is a maximal abelian covering graph, to obtain (4.5). Recall that the diamond lattice $\mathbb{D}$ is also a maximal abelian covering graph with dimension 3. Thus we obtain the strict inequality

$$p_c(\mathbb{L}^3) < p_c(\mathbb{D})$$

as a higher-dimensional version of (4.5).

4.2 Inverse correlation length and large deviation result for percolation clusters

In this section, we fix a periodic realization $\Phi : X \to \mathbb{R}^d$ of a d-dimensional crystal lattice X. We write by $A \leftrightarrow B$ the connection of two sets A, $B \subset \mathbb{R}^d$:

$$A \leftrightarrow B \overset{\text{def}}{\Longleftrightarrow} \text{two vertices } x', y' \in X$$
$$\text{are connected by a path of open edges,}$$

where x', $y' \in X$ are the nearest vertices of A, B, respectively. For brevity, we write $\{x\} \leftrightarrow \{y\}$ by $x \leftrightarrow y$ as in the previous section.

Remark 4.6. We can easily see that $\mathbb{P}_p(x \leftrightarrow y) = \mathbb{P}_p(x + \mathbf{b} \leftrightarrow y + \mathbf{b})$ holds for any vector $\mathbf{b} \in \Gamma$ and points x, $y \in \mathbb{R}^d$.

In this section, we present a large deviation result for the percolation cluster in a crystal lattice model (Theorem 4.11). In the cubic lattice model, this result has been obtained by [7, 22]. Since the proof in [7] does not mention the graph structure of the cubic lattice, it can also be applied to a crystal lattice model.

To state the large deviation result, we need to check that the *inverse correlation length*, which is defined as the limit

$$\varphi_X(x) = -\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}_p(0 \leftrightarrow nx), \tag{4.11}$$

exists and is a norm on $\mathbb{R}^d$ in the subcritical phase $p < p_c(X)$. We first prove the existence of the inverse correlation length.

Proposition 4.7. Let $p > 0$. The limit (4.11) exists for any $x \in \mathcal{D}$. Moreover, $\varphi_X(\cdot)$ depends only on X, p and the period ρ.

For the proof of Proposition 4.7, we use the subadditive limit theorem, which is simple but useful for percolation theory. We give the statement of it with a proof.

Proposition 4.8 (Subadditive limit theorem)**.** Let $(a_n)_{n=1,2,\dots}$ be a sequence of real numbers. Suppose that $(a_n)_{n=1,2,\dots}$ is subadditive, that is, $a_{n+m} \le a_n + a_m$ holds for all n, m. Then the limit

$$\alpha = \lim_{n\to\infty} \frac{a_n}{n}$$

exists. Furthermore, the limit α satisfies $a_n \ge \alpha n$ for all n.

Proof. Fix $m \in \mathbb{Z}_{>0}$. For each n, we write $n = km + r$ with $0 \le r < m$. Then the subadditivity implies $a_n \le ka_m + a_r$ and we have

$$\frac{a_n}{n} \le \frac{km}{km+r}\frac{a_m}{m} + \frac{a_r}{n}.$$

By taking the limit superior as $n \to \infty$, we obtain

$$\limsup_{n\to\infty} \frac{a_n}{n} \le \frac{a_m}{m}. \tag{4.12}$$

Taking the limit inferior as $m \to \infty$ of (4.12) implies the existence of the limit $\alpha = \lim_{n\to\infty} a_n/n$. The condition $a_n \ge \alpha n$ also follows from (4.12). □

We now turn to the proof of Proposition 4.7.

Proof of Proposition 4.7. Let $N \in \mathbb{Z}_{>0}$ be the minimum number with $Nx \in \Gamma$. We consider the sequence $(a_k)_{k=1,2,\dots}$ defined by

$$a_k := -\log \mathbb{P}_p(0 \leftrightarrow kNx).$$

For k, $l \in \mathbb{Z}_{>0}$, the event $\{0 \leftrightarrow (k+l)Nx\}$ includes $\{0 \leftrightarrow kNx\} \cap \{kNx \leftrightarrow (k+l)Nx\}$ and we have

$$\begin{aligned}
\mathbb{P}_p(0 \leftrightarrow (k+l)Nx) &\ge \mathbb{P}_p(\{0 \leftrightarrow kNx\} \cap \{kNx \leftrightarrow (k+l)Nx\}) \\
&\ge \mathbb{P}_p(0 \leftrightarrow kNx)\mathbb{P}_p(kNx \leftrightarrow (k+l)Nx) \\
&= \mathbb{P}_p(0 \leftrightarrow kNx)\mathbb{P}_p(0 \leftrightarrow lNx). \qquad (4.13)
\end{aligned}$$

Here we use the FKG inequality for the second inequality. The equality follows from Remark 4.6. This implies the subaddtivity $a_{k+l} \le a_k + a_l$ of the sequence $(a_k)_{k=1,2,\dots}$. Proposition 4.8 implies the existence of the limit

$$\lim_{k\to\infty} \frac{a_k}{k} = -\lim_{k\to\infty} \frac{1}{k} \log \mathbb{P}_p(0 \leftrightarrow kNx).$$

We set

$$\varphi_X(x) = -\lim_{k\to\infty} \frac{1}{kN} \log \mathbb{P}_p(0 \leftrightarrow kNx).$$

Since each kNx is a vertex on the lattice group $\Gamma = \rho(L)$, the limit $\varphi_X(x)$ depends only on x, p, X and the period ρ. We take $j = 1, 2, \dots, N-1$ arbitrarily. In the same way as for (4.13), we have

$$\mathbb{P}_p(0 \leftrightarrow (kN+j)x) \ge \mathbb{P}_p(0 \leftrightarrow kNx)\mathbb{P}_p(0 \leftrightarrow jx),$$

$$\mathbb{P}_p(0 \leftrightarrow kNx) \geq \mathbb{P}_p(0 \leftrightarrow (kN+j)x)\mathbb{P}_p(0 \leftrightarrow -jx).$$

These inequalities imply

$$\begin{aligned}\frac{a_k}{kN} - \frac{-\log \mathbb{P}_p(0 \leftrightarrow -jx)}{kN} &\leq \frac{-\log \mathbb{P}_p(0 \leftrightarrow (kN+j)x)}{kN} \\ &\leq \frac{a_k}{kN} + \frac{-\log \mathbb{P}_p(0 \leftrightarrow jx)}{kN}.\end{aligned}$$

By taking $k \to \infty$, we have

$$-\lim_{k\to\infty} \frac{1}{kN} \log \mathbb{P}_p(0 \leftrightarrow (kN+j)x) = \varphi_X(x).$$

Since j is taken arbitrary, the proof of Proposition 4.7 is completed. □

Remark 4.9. As noted at the end of Chapter 2, the realization $\mathbf{\Phi}$ is assumed to be nondegenerate. Proposition 4.7 implies that this assumption is not essential for the inverse correlation length, as well as the case of the FPP model (Remark 3.12).

We next summarize the basic properties of $\varphi_X(\cdot)$ in the subcritical phase $p < p_c(X)$.

Proposition 4.10. Let $0 < p < p_c(X)$. For the inverse correlation length $\varphi_X(\cdot)$, the following hold:

(1) $\varphi_X(x) > 0$ for all $x \neq 0 \in \mathcal{D}$.

(2) $\varphi_X(x+y) \leq \varphi_X(x) + \varphi_X(y)$ for all $x, y \in \mathcal{D}$.

(3) $\varphi(cx) = |c|\varphi(x)$ for all $x \in \mathcal{D}$ and $c \in \mathbb{Q}$.

Proof. The first item follows from the exponential decay (4.4) in Theorem 4.3. Indeed, since the realization $\mathbf{\Phi}$ is periodic, we can find a constant $\delta > 0$ such that

$$d_X(0,z) \geq \delta \|z\|_1$$

for all $z \in X$. Thus for any $x \neq 0 \in \mathcal{D}$ and $N \in \mathbb{Z}_{>0}$ with $Nx \in \Gamma$, we have

$$\mathbb{P}_p(0 \leftrightarrow kNx) \leq \exp(-\sigma d_X(0, kNx)) \leq \exp\left(-\sigma\delta kN\|x\|_1\right),$$

which implies

$$\varphi_X(x) = -\lim_{k\to\infty} \frac{\log \mathbb{P}_p(0 \leftrightarrow kNx)}{kN} \geq \sigma\delta\|x\|_1 > 0.$$

The second item can be proved by using the FKG inequality as follows: Let N be the minimum number with Nx, $Ny \in \Gamma$. Then we have

$$\begin{aligned}\mathbb{P}_p(0 \leftrightarrow kN(x+y)) &\geq \mathbb{P}_p(\{0 \leftrightarrow kNx\} \cap \{kNx \leftrightarrow N(x+y)\}) \\ &\geq \mathbb{P}_p(0 \leftrightarrow kNx)\mathbb{P}_p(kNx \leftrightarrow kN(x+y)) \\ &= \mathbb{P}_p(0 \leftrightarrow kNx)\mathbb{P}_p(0 \leftrightarrow kNy).\end{aligned}$$

Here, we use the FKG inequality for the second inequality. The equality follows from Remark 4.6. By taking the logarithm, multiplying by $-\frac{1}{kN}$ and taking the limit as $k \to \infty$ of both sides, we obtain the second item.

We next prove the third item. If $c \geq 0$, we have

$$\varphi_X(cx) = -\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}_p(0 \leftrightarrow ncx) = -c \lim_{n\to\infty} \frac{1}{nc} \log \mathbb{P}_p(0 \leftrightarrow ncx) = c\varphi_X(x).$$

For any $c \in \mathbb{Q}$ with $c < 0$, we can take a number N with $Ncx \in \Gamma$ and obtain

$$\mathbb{P}_p(0 \leftrightarrow Ncx) = \mathbb{P}_p(-Ncx \leftrightarrow 0) = \mathbb{P}_p(0 \leftrightarrow -Ncx)$$

from Remark 4.6. Thus the third item also follows for $c < 0$. $\square$

In a similar way as for the norm μ in Section 3.1, the inverse correlation length $\varphi_X(\cdot)$ can be continuously expanded to the function $\varphi_X : \mathbb{R}^d \to \mathbb{R}$. From Proposition 4.10, the function $\varphi_X(\cdot)$ is a norm on $\mathbb{R}^d$ in the subcritical phase $0 < p < p_c(X)$.

At the end of this section, we summarize the notation and state the large deviation result. Fix $0 < p < p_c(X)$. The norm $\varphi_X(\cdot)$ in Proposition 4.7 induces the one-dimensional Hausdorff measure $\mathcal{H}_{\varphi_X}$ on $\mathbb{R}^d$ as follows: For a nonempty subset $U \subset \mathbb{R}^d$, we denote by $\varphi_X(U) := \sup\{\varphi_X(x-y) \,:\, x, y \in U\}$ the φ_X-diameter of U. For $\delta > 0$, a family $(U_i)_{i=1,2,\ldots}$ of subsets $U_i \subset \mathbb{R}^d$ is called a δ-cover of $E \subset \mathbb{R}^d$ if

$$E \subset \bigcup_{i=1}^{\infty} U_i \text{ and } \varphi_X(U_i) < \delta \text{ for } i = 1, 2, \ldots.$$

We set

$$\mathcal{H}_{\varphi_X,\delta}(E) := \inf\left\{\sum_{i=1}^{\infty} \varphi_X(U_i) \,:\, (U_i)_{i=1,2,\ldots} \text{ is a } \delta\text{-cover of } E\right\}$$

and the one-dimensional Hausdorff measure of E is defined by

$$\mathcal{H}_{\varphi_X}(E) := \lim_{\delta\to 0} \mathcal{H}_{\varphi_X,\delta}(E).$$

Let $\mathcal{K}_c$ be the set of all compact and connected subsets of $\mathbb{R}^d$. The Hausdorff distance $D_H(K_1, K_2)$ of two elements K_1, $K_2 \in \mathcal{K}_c$ is defined by

$$D_H(K_1, K_2) := \max\Big\{\max_{x\in K_1} d(x, K_2), \max_{y\in K_2} d(K_1, y)\Big\},$$

where $d(x, A) := \inf\{||x - y||_2 \,:\, y \in A\}$. We define the equivalence relation on $\mathcal{K}_c$ by translation: two elements K_1, $K_2 \in \mathcal{K}_c$ are equivalent if and only if there is some vector $\mathbf{b} \in \mathbb{R}^d$ such that $K_1 = K_2 + \mathbf{b}$. Let $\bar{\mathcal{K}}_c$ be the quotient with respect to this relation. We write the equivalence class of $K \in \mathcal{K}_c$ as $\bar{K}$ and define its measure $\mathcal{H}_{\varphi_X}(\bar{K})$ as $\mathcal{H}_{\varphi_X}(K)$ for some $K \in \bar{K}$. The distance $\bar{D}_H(\bar{K}_1, \bar{K}_2)$ of $\bar{K}_1$, $\bar{K}_2 \in \bar{\mathcal{K}}_c$ is defined by

$$\bar{D}_H(\bar{K}_1, \bar{K}_2) := \inf_{\mathbf{b}_1,\mathbf{b}_2\in\mathbb{R}^d} D_H(K_1 + \mathbf{b}_1, K_2 + \mathbf{b}_2).$$

As we introduced in Section 1.1.1, we denote by $C = C(0)$ the cluster (connected component) containing the origin $0 \in X$ in the subgraph consisting of all open edges. Recall that, by the periodic realization $\Phi : X \to \mathbb{R}^d$, the edges of X are assumed to be realized as the segments connecting their endpoints. Here, we suppose that the cluster C is realized as the connected union of the segments. If $p < p_c(X)$, we can see that $C \in \mathcal{K}_c$ almost surely.

We consider the equivalence class $\bar{C}$ of the cluster C to be the $\bar{\mathcal{K}}_c$-valued random variable. The stochastic shape of the cluster C is represented by a large deviation principle for the shrinking sequence $(\bar{C}/n)_{n=1,2,\ldots}$ of $\bar{C}$.

Theorem 4.11. Let $0 < p < p_c(X)$. The sequence $(\bar{C}/n)_{n=1,2,\ldots}$ satisfies a large deviation principle with good rate function $\mathcal{H}_{\varphi_X}$: for any Borel subset $\bar{\mathcal{U}}$ of $\bar{\mathcal{K}}_c$, the following inequality holds:

$$\begin{aligned}
-\inf\{\mathcal{H}_{\varphi_X}(\bar{U}) \,:\, \bar{U} \in \mathrm{interior}(\bar{\mathcal{U}})\} &\leq \liminf_{n\to\infty} \frac{1}{n} \log \mathbb{P}_p(\bar{C}/n \in \bar{\mathcal{U}}) \\
&\leq \limsup_{n\to\infty} \frac{1}{n} \log \mathbb{P}_p(\bar{C}/n \in \bar{\mathcal{U}}) \\
&\leq -\inf\{\mathcal{H}_{\varphi_X}(\bar{U}) \,:\, \bar{U} \in \mathrm{closure}(\bar{\mathcal{U}})\}.
\end{aligned}$$

For a proof of Theorem 4.11, we refer to the proof of [7, Theorem 1.1], which shows the case of $X = \mathbb{L}^d$.

4.3 Covering monotonicity of the inverse correlation length

This section is devoted to the proof of Theorem 1.8. For two crystal lattices $X = (V, E)$ and $X_1 = (V_1, E_1)$ in Theorem 1.8, we consider the probability space $(\Omega, \mathcal{F}, \mathbb{P}_p)$ defined by (2.9) with $S = E \sqcup E_1$. As in Section 3.3, we assume that the covering map $\omega : X \to X_1$ maps the origin $0 \in X$ to the origin of X_1. We write $\omega(0)$ by 0 for short.

We prove Theorem 1.8 by giving two lemmas. The first lemma, which is obtained from Theorem 4.11, is an analogy of Lemma 3.20.

Lemma 4.12. Suppose $0 < p < p_c(X)$. Let $A \subset \mathbb{R}^d$ be a closed subset of $\mathbb{R}^d$. Then there exists a point $x \in A$ such that

$$\varphi_X(x) = -\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}_p(0 \leftrightarrow nA)$$

holds.

Proof. Let $x_0 \in A$ be the point with $\varphi_X(x_0) = \min\{\varphi_X(x) : x \in A\}$. The inclusion $\{0 \leftrightarrow nA\} \supset \{0 \leftrightarrow nx_0\}$ of events implies

$$\liminf_{n\to\infty} \frac{1}{n} \log \mathbb{P}_p(0 \leftrightarrow nA) \geq \lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}_p(0 \leftrightarrow nx_0) = -\varphi_X(x_0).$$

Thus, the proof of Lemma 4.12 suffices to show the following inequality:

$$\limsup_{n\to\infty} \frac{1}{n} \log \mathbb{P}_p(0 \leftrightarrow nA) \leq -\varphi_X(x_0). \tag{4.14}$$

We set the subset $\bar{\mathcal{U}} \subset \bar{\mathcal{K}}_c$ as

$$\bar{\mathcal{U}} := \{\bar{U} \in \bar{\mathcal{K}}_c \ : \ \exists U \in \bar{U} \text{ s.t. } 0 \in U \text{ and } U \cap A \neq \emptyset\}.$$

We first check that $\bar{\mathcal{U}}$ is closed (and thus is a Borel subset of $\bar{\mathcal{K}}_c$). Let $(\bar{U}_k)_{k=1,2,\ldots}$ be a sequence in $\bar{\mathcal{U}}$ which converges to $\bar{U} \in \bar{\mathcal{K}}_c$. For each k, we can take $U_k \in \bar{U}_k$ such that $0 \in U_k$ and $U_k \cap A \neq \emptyset$. Since $0 \in U_k$ and $\bar{U}_k$ converges, we can see that all U_k's are in a bounded area of $\mathbb{R}^d$. Thus, it follows from the Blaschke selection theorem (see, e.g., [13, Theorem 3.16]) that we can take a subsequence $(U_{k_l})_{l=1,2,\ldots}$ that converges to some element

$U' \in \mathcal{K}_c$. Since A is closed, we can see that $0 \in U'$ and $U' \cap A \neq \emptyset$. Moreover, we have

$$\bar{D}_H(\bar{U}_{k_l}, \bar{U}') \leq D_H(U_{k_l}, U') \to 0$$

as $l \to \infty$, which implies $\bar{U}'$ is equal to $\bar{U}$. Since $\bar{U}' \in \bar{\mathcal{U}}$, we have $\bar{U} \in \bar{\mathcal{U}}$. Thus $\bar{\mathcal{U}}$ is closed.

For this $\bar{\mathcal{U}}$, the inclusion $\{0 \leftrightarrow nA\} \subset \{\bar{C}/n \in \bar{\mathcal{U}}\}$ holds and we obtain

$$\limsup_{n\to\infty} \frac{1}{n} \log \mathbb{P}_p(0 \leftrightarrow nA) \leq \limsup_{n\to\infty} \frac{1}{n} \log \mathbb{P}_p(\bar{C}/n \in \bar{\mathcal{U}}). \tag{4.15}$$

The upper bound of Theorem 4.11 implies

$$\begin{aligned}\limsup_{n\to\infty} \frac{1}{n} \log \mathbb{P}_p(\bar{C}/n \in \bar{\mathcal{U}}) &\leq -\inf\{\mathcal{H}_{\varphi_X}(\bar{U}) \ : \ \bar{U} \in \mathrm{closure}(\bar{\mathcal{U}})\} \\ &= -\inf\{\mathcal{H}_{\varphi_X}(\bar{U}) \ : \ \bar{U} \in \bar{\mathcal{U}}\} \\ &= -\varphi_X(x_0). \end{aligned} \tag{4.16}$$

Here, the first equality follows from the closedness of $\bar{\mathcal{U}}$. The second equality follows from the following two observations:

- a compact connected set K which contains two points 0 and $x \in A$ has a measure $\mathcal{H}_{\varphi_X}(K) \geq \varphi_X(x) \geq \varphi_X(x_0)$ (see, e.g., [13, Lemma 3.4]); and
- the Hausdorff measure $\mathcal{H}_{\varphi_X}(S)$ of the segment S connecting 0 and x_0 is equal to $\varphi_X(x_0)$.

By combining (4.15) with (4.16), we obtain (4.14) and the proof of Lemma 4.12 is completed. □

Remark 4.13. From Lemma 4.12, we have

$$\mathbb{P}_p(0 \leftrightarrow n\partial C) \approx \exp(-n)$$

for the unit ball C with respect to the norm $\varphi_X(\cdot)$. Therefore, this ball can be interpreted as the region where the cluster C can spread within a certain probability cost.

The second lemma for the proof of Theorem 1.8 is a modified version of Lemma 3.23. Here, we say that a path $\gamma = (e_1, e_2, \ldots, e_r)$ is *open* if e_i is open for any $i = 1, 2, \ldots, r$.

Lemma 4.14. Let $\gamma_1, \ldots, \gamma_n$ be paths in X_1 which start from the origin 0 and let $\widetilde{\gamma}_1, \widetilde{\gamma}_2, \ldots, \widetilde{\gamma}_n$ be their liftings. Then the following inequality holds:

$$\mathbb{P}_p\left(\bigcup_{i=1}^{n}\{\gamma_i \text{ is open}\}\right) \leq \mathbb{P}_p\left(\bigcup_{i=1}^{n}\{\widetilde{\gamma}_i \text{ is open}\}\right).$$

Proof. Recall that an edge $e \in E$ is assumed to be open (resp. closed) if and only if $t_e = 1$ (resp. $t_e = 0$). For a path $\gamma = (e_1, \ldots, e_r)$ in X, we define $T'(\gamma)$ as the number of closed edges in γ:

$$T'(\gamma) = \sum_{i=1}^{r}(1 - t_{e_i}).$$

Lemma 3.23 can be applied to T' and we have

$$\mathbb{P}_p\left(\bigcap_{i=1}^{n}\{T_1'(\gamma_i) \geq 1\}\right) \geq \mathbb{P}_p\left(\bigcap_{i=1}^{n}\{T'(\widetilde{\gamma}_i) \geq 1\}\right).$$

By taking the complement, we obtain Lemma 4.14. □

Theorem 1.8 can be proved as follows.

Proof of Theorem 1.8. Take $x_1 \in \mathcal{D}_1$ with $\varphi_{X_1}(x_1) \leq 1$ arbitrarily. Let $N \in \mathbb{Z}_{>0}$ be the minimum number with $Nx_1 \in \Gamma_1$. We take a number $k \in \mathbb{Z}_{\geq 0}$ arbitrarily. Let Λ_R be the ball with radius R:

$$\Lambda_R := \{y_1 \in X \,:\, d_{X_1}(0, y_1) \leq R\}.$$

We take R sufficiently large that Λ_R includes 0 and kNx_1. Let $\{\gamma_1, \gamma_2, \ldots, \gamma_n\}$ be the set of all self-avoiding paths in Λ_R from 0 to kNx_1. By Lemma 4.14, we have

$$\mathbb{P}_p\left(\bigcup_{i=1}^{n}\{\gamma_i \text{ is open}\}\right) \leq \mathbb{P}_p\left(\bigcup_{i=1}^{n}\{\widetilde{\gamma}_i \text{ is open}\}\right),$$

where $\tilde{\gamma}_i$ is the lifting of γ_i which starts from the origin $0 \in X$. Since the end point of each lifting $\widetilde{\gamma}_i$ is in $\omega^{-1}(kNx_1)$, the right hand side is bounded above by

$$\mathbb{P}_p(0 \leftrightarrow \omega^{-1}(kNx_1) \text{ in } X). \tag{4.17}$$

The projective relation (1.11) implies that vertices in $\omega^{-1}(kNx_1)$ are realized in $P^{-1}(kNx_1)$. Therefore (4.17) is also bounded above by

$$\mathbb{P}_p(0 \leftrightarrow P^{-1}(kNx_1) \text{ in } X) = \mathbb{P}_p(0 \leftrightarrow kNP^{-1}(x_1) \text{ in } X).$$

Letting $R \to \infty$ implies

$$\mathbb{P}_p(0 \leftrightarrow kNx_1 \text{ in } X_1) \leq \mathbb{P}_p(0 \leftrightarrow kNP^{-1}(x_1) \text{ in } X).$$

Therefore we have

$$\begin{aligned} \varphi_{X_1}(x_1) &= \lim_{k\to\infty} -\frac{1}{kN} \log \mathbb{P}_p(0 \leftrightarrow kNx_1 \text{ in } X_1) \\ &\geq \limsup_{k\to\infty} -\frac{1}{kN} \log \mathbb{P}_p(0 \leftrightarrow kNP^{-1}(x_1) \text{ in } X). \end{aligned} \tag{4.18}$$

From Lemma 4.12, the last expression of (4.18) is equal to $\varphi_X(y)$ for some point $y \in P^{-1}(x_1)$. Combining $\varphi_{X_1}(x_1) \leq 1$ with (4.18), we have $\varphi_X(y) \leq 1$. Thus we obtain

$$\mathcal{C}_1 \cap \mathcal{D}_1 \subset P(\mathcal{C}).$$

Since $P(\mathcal{C})$ is closed, the proof of Theorem 1.8 is completed. □

Chapter 5

Conclusion

This thesis first formulated the FPP model on the so-called crystal lattices and established the shape theorem in this context (Theorem 1.5). The monotonicity of the limit shapes under covering maps was also given (Theorem 1.6). This result is derived from the covering monotonicity (1.7) of the critical probabilities, and, in a special case, provides insight into the limit shape of the cubic lattice model. As an analogy of this result, this thesis also presented the covering monotonicity of the unit ball with respect to the inverse correlation length (Theorem 1.8).

Several studies have been conducted on the formulation of periodic lattices in the context of percolation theory, as exemplified by Kesten [19] and Grimmett [16]. This thesis uses the formulation of "crystal lattices", which was introduced by Kotani and Sunada [21]. The crystal lattice studied here comes from discrete geometric analysis and has several good properties regarding the study of percolation theory. In the conclusion of this thesis, we mention these properties and discuss future prospects for percolation models on crystal lattices.

1. Projective relation

The first property of crystal lattices to be considered is the projective relation (1.11) of crystal lattices. In the context of percolation theory, the covering monotonicity (1.7) plays an important role in giving an estimate for the critical probability. Note that the critical probability itself is obtained from the graph structure. The projective relation enables us to derive this result to the covering monotonicity of the "shape" of percolation clusters (Theorems 1.6

and 1.8). Similar to the importance of (1.7) for the evaluation of the critical probabilities, these results are expected to have implications for the shape of percolation clusters.

2. $\mathbb{L}^d$ structures of a d-dimensional crystal lattice

The second property is that a d-dimensional crystal lattice retains the properties of the cubic lattice $\mathbb{L}^d$. As we introduced in Section 2.1.3, a periodic realization (Φ, ρ) of a d-dimensional crystal lattice X induces the lattice group $\Gamma = \rho(L)$, which is a linear transformation of $\mathbb{Z}^d$. By focusing on this Γ, we can obtain the norm $\mu(\cdot)$ in the FPP model (Proposition 3.3) and show that the limit shape depends on only the period of the realization (Proposition 3.10). Moreover, as shown by (2.2) in Section 2.1.3, a crystal lattice can be considered as a d-dimensional array of trees, which plays a key role in the proof of Lemma 3.8.

As a further study, it is expected that these d-dimensional structures can be applied to considerations of the *critical exponents*, values that describe the behavior of physical quantities near phase transitions. For example, in the bond percolation model on the cubic lattice $\mathbb{L}^d$, it is believed that the percolation probability $\theta(p)$ behaves as

$$\theta(p) \approx (p - p_c(\mathbb{L}^d))^\beta$$

as $p \downarrow p_c(\mathbb{L}^d)$ for some $\beta > 0$. This constant β, which depends on the dimension d, is called a critical exponent. In the research community of percolation theory, the critical exponent β is expected to be the same for any d-dimensional graph. The crystal lattice will make a significant contribution to the consideration of such problems.

3. Standard realization

As mentioned in Section 1.1.4, one of the main subjects of study for crystal lattices comes from the concept of standard realization, that is, periodic realization with maximal symmetry. Figure 5.1 shows the standard realizations of the cubic lattice and the triangular lattice.

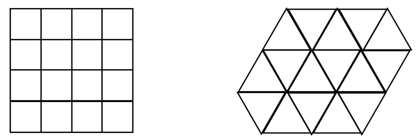

Figure 5.1: Standard realizations of the cubic lattice (left) and the triangular lattice (right).

When we consider the FPP model on the cubic lattice, we usually think of the standard realization, like that on the left. The paper [2] provides a simulation study of the FPP model on the two-dimensional cubic lattice and observes that the larger the variability of the time distribution, the closer to the Euclidean ball the limit shape becomes. Though the paper [9] shows that the limit shape is not the Euclidean ball in the high-dimensional case, it is still interesting that the limit shape obtained from the standard realization is close to a highly symmetric object.

The same can be said for the inverse correlation length. The paper [12] studies the site percolation model on the triangular lattice $\mathbb{T}$, given by the vertices $m + ne^{i\pi/3}$ for $m, n \in \mathbb{Z}$ and edges linking nearest neighbors together. This realization is nothing but the standard realization of $\mathbb{T}$ as shown in Figure 5.1. In this case, the inverse correlation length is proved to approach a constant multiple of the L_2-norm as $p \uparrow p_c^{\mathrm{site}}(\mathbb{T})$.

We thus pose the following question: for the standard realization $\Phi : X \to \mathbb{R}^d$ of a crystal lattice X, does the following hold?

- the limit shape $\mathcal{B}$ in the FPP model approaches the Euclidean ball as the variance of the time distribution increases; and
- the unit ball C with respect to the inverse correlation length φ_X comes closer to the Euclidean ball as $p \uparrow p_c(X)$.

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