

PBW parametrizations and generalized preprojective algebras

Kota Murakami

PBW parametrizations and generalized preprojective algebras

Kota Murakami

ABSTRACT. Geiß-Leclerc-Schröer [Invent. Math. **209** (2017)] has introduced a notion of generalized preprojective algebras associated with generalized Cartan matrices and their symmetrizers. These algebras realize crystal structures on the set of maximal dimensional irreducible components of the nilpotent varieties [Selecta Math. (N.S.) **24** (2018)]. For general finite types, we give stratifications of these components via partial orders of torsion classes in module categories of generalized preprojective algebras in terms of Weyl groups. In addition, we realize Mirković-Vilonen polytopes from generic modules of these components, and give an identification as crystals between the set of Mirković-Vilonen polytopes and the set of maximal dimensional irreducible components. This generalizes results of Baumann-Kamnitzer [Represent. Theory **16** (2012)] and Baumann-Kamnitzer-Tingley [Publ. Math. Inst. Hautes Études Sci. **120** (2014)].

CONTENTS

1. Introduction	2
2. Preliminary	5
2.1. Cartan matrices, root systems and Weyl groups	5
2.2. Preprojective algebras associated with symmetrizable Cartan matrices	8
2.3. Module varieties for generalized preprojective algebras	11
2.4. Partial orders on module categories	14
2.5. Tilting theory for preprojective algebras	15
2.6. Reflection functors and idempotent ideals	18
3. On stability of module categories	22
3.1. Framed preprojective algebras and stable modules	23
3.2. g -vectors and wall-chamber structures	30
4. Nilpotent varieties and MV polytopes	35
4.1. On crystal structures on irreducible components and PBW parametrizations	35
4.2. MV polytopes from generalized preprojective algebras	46
Acknowledgement	59

2020 *Mathematics Subject Classification*. Primary 17B37, 16T20; Secondary 16S90.

Key words and phrases. Symmetrizable Cartan matrices, Preprojective algebras, Nilpotent varieties, MV-polytopes, Crystal bases.

The original version of this article is an article accepted for publication in *Adv. Math.* as [Mur22]. The original version of this article is available online at <https://doi.org/10.1016/j.aim.2021.108144>. This work was supported by the Kyoto Top Global University project, Grant-in-Aid for JSPS Fellows (JSPS KAKENHI Grant Number JP21J14653) and JSPS bilateral program (Grant Number JPJSBP120213210). This work was also supported by the Research Institute for Mathematical Sciences, an International Joint Usage/Research Center located in Kyoto University.

1. INTRODUCTION

Preprojective algebras are introduced by Gelfand-Ponomarev [GP79] in order to understand representation categories of finite quivers. A preprojective algebra is defined as a double of a finite quiver with preprojective relations, which contains the path algebra of the original quiver as a subalgebra. These algebras have many applications to different contexts, for example tilting theory and McKay correspondences [AIR15, BIRS09], cluster structures of coordinate rings of algebraic groups or flag varieties [GLS11, Lec16], and geometric representation theory [Nak98, Nak01, SV20], *etc.*

Recently, Geiß-Leclerc-Schröer [GLS17] introduced a certain class of 1-Iwanaga-Gorenstein algebras and a notion of *generalized preprojective algebras* for symmetrizable generalized Cartan matrices (=GCMs) and their symmetrizers (see Definition 2.6). These classes of algebras contain path algebras of acyclic quivers and their preprojective algebras as special cases. They generalize a part of links between Kac-Moody Lie algebras and preprojective algebras (or representations of quivers) from simply-laced types to symmetrizable types which contain B, C, F, G types (*e.g.* [GLS16, GLS18b]). On the other hand, in studies of quantum enveloping algebras and algebraic groups, it is important to find bases of an algebra or its irreducible representations which have nice properties. Many bases are defined and are compared in this context, for example (dual) canonical bases [Lus90], (dual) semi-canonical bases [Lus00], Mirković-Vilonen bases [MV07] and theta bases [GHKK18]. The canonical bases are described by some kinds of geometry, that is, affine Grassmannians and Nakajima quiver varieties. Kamnitzer *et al.* [BKK21, KTW⁺19], *etc.* have compared these two kinds of geometry and have developed their relationships from a viewpoint of representation theory about a duality of conical symplectic singularities.

Motivated from their works, we extend relationships between some combinatorics of affine Grassmannians and representation theory of quivers from a symmetric setting to our symmetrizable setting. In particular, we employ the representation theory of generalized preprojective algebras instead of employing Nakajima quiver varieties that are not available for B, C, F, G cases. Namely, we compare Kamnitzer's theory of Mirković-Vilonen polytopes [Kam10] and representation theory of generalized preprojective algebras for symmetrizable Cartan matrices from a viewpoint of tilting theory.

Framed preprojective algebras. In representation theory of quivers, *framing* is a useful technique for understanding some symmetries associated with quivers in terms of quantum groups and cluster algebras [Kel11, Rei08]. A framed quiver has original vertices i and added corresponding vertices i' . Preprojective algebras of framed quivers have similar combinatorial features to Nakajima quiver varieties in the module categories, called “Nakajima's tricks” in the work of Baumann-Kamnitzer [BK12]. Roughly speaking, modules over these algebras describe data of positive roots by dimensions of vector spaces on original vertices, and data of integral weights by those on added vertices. As a by-product of this construction, we obtain some constructible functions on varieties of nilpotent modules of preprojective algebras through a class of modules,

called stable modules. Such modules are known to have a few kinds of constructions other than a use of framed preprojective algebras (e.g. [BIRS09, GLS11]), and these types of constructible functions are often useful for relating preprojective algebras with contexts of canonical bases or cluster algebras. We generalize the definition and develop some algebraic properties of framed preprojective algebras and their module categories from symmetric cases to symmetrizable cases. Namely, we define framed preprojective algebras (see §3.1) for generalized preprojective algebras. In particular, our algebras contain classical ones in [BK12] as special cases. By adopting the technique in the work of Baumann-Kamnitzer [BK12] for classical cases, we prove that there are nice “generalized stable modules” as a generalization of [BK12, Theorem 3.1]. The difference between the original work and our approach is that we must consider symmetrizers and work with suitable subcategories instead of the whole module categories.

Theorem 1.1 (\doteq Theorem 3.7; see Definitions 2.9, 2.36 and §2.2, §2.6 for notation). Let $\Pi := \Pi(C, D)$ be the generalized preprojective algebra associated with a GCM C and a symmetrizer D of C . Let ϖ_i be the fundamental weight with respect to $i \in Q_0$ and let $w \in W$, where W is the Weyl group of C . Then, there is a unique Π -module $N(\gamma)$ for each weight γ in Tits cone, which satisfies the following:

- (1) If $\gamma \in \mathbb{Z}^n$ is an anti-dominant weight, then $N(\gamma) = 0$;
- (2) If $\varpi_i - w\varpi_i \neq 0$, then there exists unique $N(-w\varpi_i) \in \text{rep}_{\text{l.f.}} \Pi$ such that $\text{rank } N(-w\varpi_i) = \varpi_i - w\varpi_i$ and $\text{sub } N(-w\varpi_i) = E_i$. Conversely, if a module $M \in \text{rep}_{\text{l.f.}} \Pi$ satisfies $\text{rank } M = \varpi_i - w\varpi_i$ and $\text{sub } M = E_i$, then $M \cong N(-w\varpi_i)$;
- (3) If γ and δ belong to the same Weyl chamber, then $N(\gamma + \delta) \cong N(\gamma) \oplus N(\delta)$.

This result makes us expect that there is nice geometry like Nakajima quiver varieties for Geiß-Leclerc-Schröer’s theory, via the existence of some nice combinatorial features of framed preprojective algebras. In fact, they enable us to develop a relationship between generalized preprojective algebras and canonical bases in Theorem 1.2 and Theorem 1.3.

Parametrizations of nilpotent varieties. Kashiwara-Saito [KS97] developed combinatorics of irreducible components and constructible functions on varieties of nilpotent modules over preprojective algebras, called *nilpotent varieties*, in terms of Kashiwara’s crystal basis [Kas91]. After their work, Lusztig [Lus00] constructed semi-canonical bases via generic values of constructible functions on irreducible components of nilpotent varieties. Geiß-Leclerc-Schröer [GLS05] compared the multiplication of dual semi-canonical bases with that of dual canonical bases by using the decomposition theory of irreducible components in module varieties of Crawley-Boevey-Schröer [CBS02].

Recently, Geiß-Leclerc-Schröer [GLS18b] defined nilpotent varieties for generalized preprojective algebras. They gave a crystal structure on the set of special irreducible components of nilpotent varieties, called *maximal components*, and gave a conjectural description of dual semi-canonical bases as a generalization of the classical theory of Kashiwara-Saito [KS97] and Lusztig [Lus00]. We give a parametrization of irreducible components of Geiß-Leclerc-Schröer’s varieties by studying a special class of modules, called *crystal modules* by the τ -tilting theory of Adachi-Iyama-Reiten [AIR14]. This kind of results for symmetric GCMs can be found

in the works of Geiss-Leclerc-Schröer [GLS11, Proposition 14.6] and Baumann-Kamnitzer-Tingley [BKT14, paragraph after Proposition 5.23].

Theorem 1.2 (\doteq Theorem 4.13). Let $\Pi := \Pi(C, D)$ be the generalized preprojective algebra associated with a Cartan matrix C and its symmetrizer D , and let $\mathbf{i} := (i_1, \dots, i_r)$ be any reduced expression of an element w of the Weyl group $W(C)$. We define $V_{\mathbf{i},k} := N(-s_{i_1} \cdots s_{i_k} \varpi_{i_k})$ and $M_{\mathbf{i},k} := V_{\mathbf{i},k}/V_{\mathbf{i},k^-}$ for $1 \leq k \leq r$ where $k^- = \max\{0, 1 \leq s \leq k-1 \mid i_s = i_k\}$. Then, there are constructible sets $\Pi_{\mathbf{i}}^{\mathbf{a}}$ of crystal modules M which have a filtration

$$M = T_0 \supseteq T_1 \supseteq \cdots \supseteq T_r = 0$$

such that $T_j/T_{j+1} \cong (M_{\mathbf{i},k}^{\oplus a_j})^*$ for $\mathbf{a} := (a_1, \dots, a_r) \in \mathbb{Z}_{\geq 0}^r$, where $*$ is the natural involution $\text{rep } \Pi \rightarrow \text{rep } \Pi$ induced by taking the K -dual space. The constructible set $\Pi_{\mathbf{i}}^{\mathbf{a}}$ is irreducible. Moreover, $Z_{\mathbf{i}}^{\mathbf{a}} := \overline{\Pi_{\mathbf{i}}^{\mathbf{a}}}$ is a maximal irreducible component of the nilpotent variety.

Note that any maximal irreducible component is given in this way because any maximal irreducible component has a crystal module as a generic point. In particular, each crystal module has a filtration in above Theorem 1.2 for arbitrary reduced expression of w_0 . This theorem gives a geometric interpretation of the PBW type parametrizations of Kashiwara crystal bases by Saito [Sai94].

Mirković-Vilonen polytopes and generalized preprojective algebras. Let G be a complex reductive group and let $\text{Gr}_G = G(\mathbb{C}((t)))/G(\mathbb{C}[[t]])$ be its affine Grassmannian. Gr_G has subvarieties Gr_G^λ whose intersection homologies are isomorphic to the irreducible highest weight representations $V(\lambda)$ of the highest weight λ of the Langlands dual group G^\vee . Mirković-Vilonen [MV07] studied algebraic cycles of affine Grassmannian which form a basis of the intersection homology of Gr_G^λ mentioned above, called the Mirković-Vilonen cycles ($=$ MV cycles).

Kamnitzer [Kam10] defined the *Mirković-Vilonen polytopes* ($=$ MV polytopes) in order to extract combinatorial features of MV cycles. The set of MV polytopes is associated with the root datum of G , and they have much information of the quantum enveloping algebra associated with the Lie algebra of the Langlands dual group G^\vee as a result of the geometric Satake correspondence. In particular, the set of MV polytopes for G bijectively corresponds to Kashiwara's crystal bases, and they have many numerical data about crystal bases which are developed by Lusztig [Lus90] and Berenstein-Zelevinsky [BZ97], including \mathbf{i} -Lusztig data via their edge lengths.

Baumann-Kamnitzer-Tingley [BK12, BKT14] compared combinatorial structures of module categories over preprojective algebras and MV polytopes along with the context of \mathbf{i} -Lusztig data. In particular, they realized MV polytopes in the Grothendieck group of the module category of a preprojective algebra by taking convex hull of the set of submodules of generic modules of irreducible components of the nilpotent variety. We generalize their arguments to our symmetrizable setting. This extends the results of Baumann-Kamnitzer-Tingley [BK12, BKT14] to our setting. Our approach is based on the τ -tilting theory unlike their arguments.

Theorem 1.3 (\doteq Theorem 4.42, Theorem 4.45). Let $\Pi := \Pi(C, D)$ be the generalized preprojective algebra associated with a Cartan matrix C and its symmetrizer D except for type G_2 . Let $\Gamma := \{w\varpi_i \mid w \in W, i \in Q_0\}$ be the set of chamber weights. We consider each maximal irreducible component Z of a variety $\Pi(\mathbf{r})$ of suitable modules over Π and a generic module T of this

component. Then, there is a function D_γ on $\Pi(\mathbf{r})$, which defines a constructible function on each component Z . In addition, $P(T) = \{v \in \mathbb{R}^n \mid \langle \gamma, v \rangle \leq D_\gamma(T), \gamma \in \Gamma\}$ is an MV polytope associated with the Langlands dual root datum of C . In particular, the map $P(-): \mathcal{B} \rightarrow \mathcal{MV}$ from the set of maximal irreducible components of a nilpotent variety coming from C to the set of MV polytopes coming from the root datum of C^Γ is an isomorphism.

Indeed, the function $D_\gamma(-)$ is given by $\dim \text{Hom}_\Pi(N(\gamma), -)$ (up to scalar) from Theorem 1.1, and Theorem 1.2 gives the vertex description of our polytopes. Recently, Baumann-Kamnitzer-Knutson *et al.* [BKK21, HKW20] try to relate the geometry of quiver Grassmannians of modules over preprojective algebras and the geometry of affine Grassmannian slices. In their observation, quiver Grassmannians from generic modules and MV cycles from these modules are related via representation theory of current Lie algebras. Our result suggests existence of some more geometric relationship between the theory of Geiß-Leclerc-Schröer [GLS17, GLS18a, GLS16, GLS18b, GLS18c, GLS20] and the geometry of affine Grassmannians (*e.g.* the mathematical definition of Coulomb branches of quiver gauge theories with symmetrizers by Nakajima-Weekes [NW19]).

Organization. In §2, we prepare our conventions about root systems, representation theory of quivers, tilting theory and reflection functors. In §3, we develop two constructions about stability conditions, namely stable modules and g -vectors as generalizations of results of Baumann-Kamnitzer [BK12] and Mizuno [Miz14]. In §4, we compare the data about stability conditions of module categories of generalized preprojective algebras and crystal basis. In particular, we realize Mirković-Vilonen polytopes for general finite types as generalizations of results of Baumann-Kamnitzer [BK12] and Baumann-Kamnitzer-Tingley [BKT14].

Conventions and notation. Throughout this paper, we refer to a field K as a commutative algebraically closed field unless specified otherwise. However, the results obtained before § 4 are valid for any field. If we refer to a K -algebra Λ , then Λ means a unital associative non-commutative K -algebra. If we refer to a module over a ring Λ , the module means a left Λ -module. The category $\text{Rep } \Lambda$ (*resp.* $\text{rep } \Lambda$) denotes the module category (*resp.* the category of finite dimensional modules) over a K -algebra Λ . For a finite dimensional K -algebra Λ , the duality $\mathbb{D} := \text{Hom}_K(-, K): \text{rep } \Lambda \rightarrow \text{rep } \Lambda^{\text{op}}$ denotes the standard K -duality. For a finite dimensional K -algebra Λ , the covariant functor $\nu := \mathbb{D}(\Lambda) \otimes_\Lambda (-): \text{rep } \Lambda \rightarrow \text{rep } \Lambda$ denotes the Nakayama functor. For a module M over a K -algebra Λ , the category M^\perp denotes the full subcategory $\{N \in \text{rep } \Lambda \mid \text{Hom}_\Lambda(M, N) = 0\}$ of $\text{rep } \Lambda$. The category $\text{proj } \Lambda$ denotes the full subcategory of finitely generated projective modules over a ring Λ . The commutative group $K_0(\mathcal{E})$ denotes the Grothendieck group of an exact category \mathcal{E} . For a commutative ring R , the set $M_n(R)$ denotes the set of $n \times n$ -square matrices. For a matrix A , the matrix A^\top denotes the transpose matrix of A . We always consider a variety as a reduced separated scheme of finite type over an algebraically closed field K , whose topology is the Zariski topology. For an algebraic variety V , the set $\text{Irr}(V)$ denotes the set of irreducible components of V .

2. PRELIMINARY

2.1. Cartan matrices, root systems and Weyl groups. We recall basic concepts about root systems. We refer to Kac [Kac90], Björner-Brenti [BB05] and Humphreys [Hum90].

Definition 2.1. A matrix $C = (c_{ij}) \in M_n(\mathbb{Z})$ is called a *generalized Cartan matrix* (GCM), if it satisfies the following three conditions:

- (C1): For each $1 \leq i \leq n$, we have $c_{ii} = 2$;
- (C2): If $i \neq j$, then $c_{ij} \leq 0$;
- (C3): We have $c_{ij} \neq 0$ if and only if we have $c_{ji} \neq 0$.

In particular, if $C = (c_{ij})$ satisfies the following (C4), then C is called *symmetrizable*.

- (C4): There is a diagonal matrix $D = \text{diag}(c_1, \dots, c_n)$ ($c_i \in \mathbb{Z}, c_i \geq 1$) such that DC is a symmetric matrix.

In the condition (C4), the matrix D is called a *symmetrizer* of C .

The following quadratic form q_{DC} and graph $\Gamma(C)$ give a classification of GCMs.

Definition 2.2. Let $C = (c_{ij}) \in M_n(\mathbb{Z})$ be a symmetrizable GCM and let $D = \text{diag}(c_1, \dots, c_n)$ be a symmetrizer of C .

- (1) The graph $\Gamma(C)$ has vertices $1, \dots, n$. An edge between i and j exists in $\Gamma(C)$ if and only if $c_{ij} < 0$. The edges are labeled by pairs $(|c_{ji}|, |c_{ij}|)$ as:

$$i \xrightarrow{(|c_{ji}|, |c_{ij}|)} j$$

We call $\Gamma(C)$ the *valued graph* of C . If $\Gamma(C)$ is a connected graph, then C is called *connected*.

- (2) We define the form $q_{DC} : \mathbb{Z}^n \rightarrow \mathbb{Z}$ by

$$q_{DC} = \sum_{i=1}^n c_i X_i^2 + \sum_{i < j} c_i c_{ij} X_i X_j.$$

Since we have $c_i c_{ij} = c_j c_{ji}$, the form q_{DC} is symmetric. If q_{DC} is positive definite (*resp.* positive semi-definite), then C is called *finite type* (*resp.* *Euclidean type*).

Remark 2.3. If C is a connected symmetrizable GCM, there exists a unique *minimal* symmetrizer. That is, any symmetrizer of C is equal to mD for the minimal symmetrizer D and some positive integer m .

Next, we define data of roots and weights of complex semisimple Lie algebras. We put $Q_0 := \{1, 2, \dots, n\}$. Let \mathfrak{h} be an n -dimensional \mathbb{C} -vector space, and let $\{\alpha_1, \dots, \alpha_n\} \subset \mathfrak{h}^*$ and $\{\alpha_1^\vee, \dots, \alpha_n^\vee\} \subset \mathfrak{h}$ be the set of simple roots (*resp.* the set of simple co-roots). We define bilinear forms $\langle -, - \rangle_C : \mathfrak{h}^* \times \mathfrak{h}^* \rightarrow \mathbb{C}$ and $(-, -) : \mathfrak{h}^* \times \mathfrak{h}^* \rightarrow \mathbb{C}$ by

$$\langle \alpha_i, \alpha_j \rangle_C := \alpha_i(\alpha_j^\vee) = c_{ji}, \quad (\alpha_i, \alpha_j)_C := c_j c_{ji} = c_i c_{ij}$$

for any $i, j \in Q_0$. We fix a basis $\{\varpi_j \mid 1 \leq j \leq n\}$ of \mathfrak{h}^* such that $\varpi_j(\alpha_i^\vee) = \delta_{ij}$ for $1 \leq i, j \leq n$. We call them fundamental weights. We have $\alpha_i = \sum_{j \in Q_0} c_{ji} \varpi_j$.

We define the *integral weight lattice* and the set of *dominant integral weights* by

$$P := \{\nu \in \mathfrak{h}^* \mid \langle \nu, \alpha_i \rangle \in \mathbb{Z}, i \in Q_0\} = \bigoplus_{i \in Q_0} \mathbb{Z}\varpi_i;$$

$$P^+ := \{\nu \in P \mid \langle \nu, \alpha_i \rangle \geq 0, i \in Q_0\} = \bigoplus_{i \in Q_0} \mathbb{Z}_{\geq 0}\varpi_i.$$

By the above definition, we can identify ϖ_i with the i -th standard basis vector of \mathbb{Z}^n . Furthermore, we define the *root lattice* and the *positive root lattice* by

$$R := \bigoplus_{i \in Q} \mathbb{Z}\alpha_i, \quad R^+ := \bigoplus_{i \in Q} \mathbb{Z}_{\geq 0}\alpha_i.$$

By the above definition, we can also identify α_i with the i -th standard basis vector of \mathbb{Z}^n . Finally, we define the Weyl group and the root system. Define $s_i \in \text{Aut}(\mathbb{Z}^n)$ by $s_i(\alpha_j) := \alpha_j - c_{ij}\alpha_i$ for each $i, j \in Q_0$. $W(C)$ denotes the group generated by s_1, \dots, s_n and we call $W(C)$ the Weyl group of C . In general, the Weyl group $W(C)$ associated with the Kac-Moody Lie algebra $\mathfrak{g}(C)$ is generated by s_i ($i \in Q_0$) and only relations

$$\begin{aligned} s_i^2 &= e \quad (i \in Q_0); \\ s_i s_j &= s_j s_i \quad (c_{ij} c_{ji} = 0); \\ s_i s_j s_i &= s_j s_i s_j \quad (c_{ij} c_{ji} = 1); \\ s_i s_j s_i s_j &= s_j s_i s_j s_i \quad (c_{ij} c_{ji} = 2); \\ s_i s_j s_i s_j s_i s_j &= s_j s_i s_j s_i s_j s_i \quad (c_{ij} c_{ji} = 3). \end{aligned}$$

We define the set of (real) roots by

$$\Delta := \{w(\alpha_i) \mid w \in W(C), i \in Q_0\}.$$

Finally, we recall Weyl chambers. Let $\varpi_1, \dots, \varpi_n \in \mathbb{Z}^n$ be the standard basis of \mathbb{Z}^n . We define a group monomorphism $\sigma^*: W(C) \rightarrow \text{GL}(\mathbb{Z}^n)$ by

$$\sigma^*(s_i)(\varpi_j) := \begin{cases} \varpi_i - \sum_{k \in Q_0} c_{kj} \varpi_k & (i = j) \\ \varpi_j & (i \neq j). \end{cases}$$

For $a, b \in \mathbb{R}^n$, we write $\langle a, b \rangle := a \cdot b^\top$. For each root $x \in \Delta$, we define a hyperplane $H_x := \{y \in \mathbb{R}^n \mid \langle y, x \rangle = 0\}$. We refer to a connected component of $\mathbb{R}^n \setminus \bigcup_{x \in \Delta} H_x$ as a *Weyl chamber*. C_0 denotes the chamber $\{a_1 \varpi_1 + \dots + a_n \varpi_n \mid a_i \in \mathbb{R}_{>0}\}$. A Weyl chamber is given by $\sigma^*(w)(C_0)$ for some $w \in W(C)$ and there is a bijection between W and the set of Weyl chambers by $w \mapsto \sigma^*(w)(C_0)$. We write $\Gamma := \{w\varpi_i \mid w \in W, i \in Q_0\}$.

For $w \in W(C)$, we take an expression $w = s_{i_1} \cdots s_{i_k}$. If k is minimal among such expressions, then we write $l(w) = k$ and refer to this expression as a *reduced expression* of w . If there are no elements $w' \in W(C)$ such that $l(w') > l(w)$ for $w \in W(C)$, then we call w the *longest element* of W and w_0 denotes the longest element of $W(C)$. Finally, we define the right weak Bruhat order \leq_R on $W(C)$. We say that $u \leq_R v$ if there exist simple reflections s_{i_1}, \dots, s_{i_k} such that $v = u s_{i_1} \cdots s_{i_k}$ and $l(u s_{i_1} \cdots s_{i_j}) = l(u) + j$ for $1 \leq j \leq k$.

We write with hats $\widehat{}$ concepts associated with an un-twisted affine type GCM \widehat{C} for a finite type GCM C . (e.g. affine Weyl groups $\widehat{W}(C) := W(\widehat{C})$).

2.2. Preprojective algebras associated with symmetrizable Cartan matrices. In this subsection, we briefly review the representation theory of preprojective algebras with symmetrizable Cartan matrices introduced in Geiß-Leclerc-Schröer [GLS17]. In their work, the representation theory of acyclic quivers and of their preprojective algebras are generalized by using the data of symmetrizable GCMs and their symmetrizers. First, we define a quiver from a symmetrizable GCM.

Definition 2.4. Let $C = (c_{ij}) \in M_n(\mathbb{Z})$ be a symmetrizable GCM and let $D = \text{diag}(c_1, \dots, c_n)$ be its symmetrizer.

- (1) If $c_{ij} < 0$, then we set $g_{ij} := |\gcd(c_{ij}, c_{ji})|$ and $f_{ij} := |c_{ij}|/g_{ij}$.
- (2) If $\Omega \subset \{1, 2, \dots, n\} \times \{1, 2, \dots, n\}$ satisfies the following two conditions, then Ω is called an *orientation* of C .
 - If $\{(i, j), (j, i)\} \cap \Omega \neq \emptyset$ holds, then $c_{ij} < 0$;
 - Every sequence $((i_1, i_2), (i_2, i_3), \dots, (i_t, i_{t+1}))$ ($t \geq 1$) in $\{1, 2, \dots, n\} \times \{1, 2, \dots, n\}$ such that each $(i_s, i_{s+1}) \in \Omega$ ($1 \leq s \leq t$) satisfies $i_1 \neq i_{t+1}$.

We define the opposite orientation $\Omega^* := \{(j, i) \mid (i, j) \in \Omega\}$ and $\overline{\Omega} := \Omega \cup \Omega^*$.

Definition 2.5. Under the setting of Definitions 2.1, 2.4, we define:

- (1) The quiver $Q = Q(C, \Omega) = (Q_0, Q_1, s, t)$ as follows:

$$Q_0 = \{1, 2, \dots, n\},$$

$$Q_1 = \{\alpha_{ij}^{(g)} : j \rightarrow i \mid (i, j) \in \Omega, 1 \leq g \leq g_{ij}\} \cup \{\varepsilon_i : i \rightarrow i \mid 1 \leq i \leq n\},$$

$$s(\alpha_{ij}^{(g)}) = j, t(\alpha_{ij}^{(g)}) = i, s(\varepsilon_i) = t(\varepsilon_i) = i.$$

- (2) The *double quiver* $\overline{Q} = (\overline{Q}_0, \overline{Q}_1, s, t)$ of Q as follows:

$$\overline{Q}_0 = Q_0 = \{1, 2, \dots, n\},$$

$$\overline{Q}_1 = \{\alpha_{ij}^{(g)} : j \rightarrow i \mid (i, j) \in \overline{\Omega}, 1 \leq g \leq g_{ij}\} \cup \{\varepsilon_i : i \rightarrow i \mid 1 \leq i \leq n\},$$

$$s(\alpha_{ij}^{(g)}) = j, t(\alpha_{ij}^{(g)}) = i, s(\varepsilon_i) = t(\varepsilon_i) = i.$$

Finally, we define K -algebras H and Π as quivers with relations.

Definition 2.6. Under the setting of Definition 2.5, we define:

- (1) We define a K -algebra $H = H(C, D, \Omega) := KQ/I$ by the quiver Q with relations I generated by (H1), (H2):
 - (H1) $\varepsilon_i^{c_i} = 0$ ($i \in Q_0$);
 - (H2) For each $(i, j) \in \Omega$, we have $\varepsilon_i^{f_{ji}} \alpha_{ij}^{(g)} = \alpha_{ij}^{(g)} \varepsilon_j^{f_{ij}}$ ($1 \leq g \leq g_{ij}$).
- (2) We define a K -algebra $\Pi = \Pi(C, D, \overline{\Omega}) := K\overline{Q}/\overline{I}$ by the quiver \overline{Q} with relations \overline{I} generated by (P1)-(P3):
 - (P1) $\varepsilon_i^{c_i} = 0$ ($i \in Q_0$);

(P2) For each $(i, j) \in \overline{\Omega}$, we have $\varepsilon_i^{f_{ji}} \alpha_{ij}^{(g)} = \alpha_{ij}^{(g)} \varepsilon_j^{f_{ij}}$ ($1 \leq g \leq g_{ij}$);

(P3) For each $i \in Q_0$, we have

$$\sum_{j \in \overline{\Omega}(-, i)} \sum_{g=1}^{g_{ij}} \sum_{f=0}^{f_{ji}-1} \operatorname{sgn}(i, j) \varepsilon_i^f \alpha_{ij}^{(g)} \alpha_{ji}^{(g)} \varepsilon_i^{f_{ji}-1-f} = 0,$$

where we define

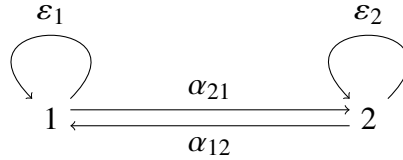
$$\overline{\Omega}(i, -) := \{j \in Q_0 \mid (i, j) \in \overline{\Omega}\}, \quad \overline{\Omega}(-, j) := \{i \in Q_0 \mid (i, j) \in \overline{\Omega}\},$$

and

$$\operatorname{sgn}(i, j) := \begin{cases} 1 & (i, j) \in \Omega, \\ -1 & (i, j) \in \Omega^*. \end{cases}$$

In the definition of Π , the set $\{e_i \mid i \in Q_0\}$ is the complete set of primitive orthogonal idempotent elements corresponding to the set Q_0 of vertices. We note that Π does not depend on the choice of orientation Ω up to isomorphism. We denote the above $\Pi(C, D, \overline{\Omega})$ by $\Pi(C, D)$ in short. We refer to the above $\Pi = \Pi(C, D)$ as the *generalized preprojective algebra* associated with the pair (C, D) .

Example 2.7. Let $C = \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}$, let $D = \operatorname{diag}(2d, d)$ ($d \in \mathbb{Z}_{>0}$) and let $\Omega = \{(1, 2)\}$. We have $c_1 = 2d, c_2 = d, g_{12} = g_{21} = 1, f_{12} = 1, f_{21} = 2$. Then, $\Pi = \Pi(C, D)$ is isomorphic to the K -algebra defined as the quiver



with relations (P1) $\varepsilon_1^{2d} = 0, \varepsilon_2^d = 0$; (P2) $\varepsilon_1^2 \alpha_{12} = \alpha_{12} \varepsilon_2, \varepsilon_2 \alpha_{21} = \alpha_{21} \varepsilon_1^2$; (P3) $\alpha_{12} \alpha_{21} \varepsilon_1 + \varepsilon_1 \alpha_{12} \alpha_{21} = 0, \alpha_{21} \alpha_{12} = 0$.

As a K -algebra, a preprojective algebra of finite type is characterized by the following Proposition.

Proposition 2.8 (Geiß-Leclerc-Schröer [GLS17, Corollary 11.3, 12.7]). Let C be a connected GCM. Then, $\Pi(C, D)$ is a connected finite dimensional self-injective K -algebra if and only if C is of finite type.

In this paper, we mainly deal with a finite type GCM C and its symmetrizer D . We put $D = mD'$ with the minimal symmetrizer D' . Let \widehat{C} denote the corresponding un-twisted affine type GCM, and let $\widehat{D} = m\widehat{D}'$ denote a symmetrizer of \widehat{C} , where \widehat{D}' is the minimal symmetrizer of \widehat{C} . Let $\widehat{\Pi} := \Pi(\widehat{C}, \widehat{D})$ denote the generalized preprojective algebra associated with \widehat{C} and \widehat{D} .

Definition 2.9 (Locally free modules). Under the setting of Definition 2.6, we define:

- (1) $H_i := e_i(K[\varepsilon_i]/(\varepsilon_i^{c_i}))e_i \cong K[\varepsilon_i]/(\varepsilon_i^{c_i})$ for each $i \in Q_0$.
- (2) A module M over H or Π is *locally free*, if $e_i M$ is a free H_i -module for each $i \in Q_0$.

- (3) For a locally free module M , we define $\underline{\text{rank}} M = (a_1, \dots, a_n)$ where a_i ($1 \leq i \leq n$) is the rank of $e_i M$ as a free H_i -module. For a free H_i -module M , we write the rank of M as a free H_i -module by $\text{rank}_i M$.

Let $\text{Rep}_{\text{l.f.}} \Lambda$ (*resp.* $\text{rep}_{\text{l.f.}} \Lambda$) denote the full subcategory of $\text{Rep} \Lambda$ consisting of locally free modules (*resp.* locally free modules such that each free H_i -module $e_i M$ is of finite rank) for $\Lambda = H$ or Π .

Definition 2.10. For each $i \in Q_0$, we say that $E_i \in \text{rep}_{\text{l.f.}} \Lambda$ is a *generalized simple module*, if we have H_i -module isomorphisms

$$e_j E_i \cong \begin{cases} H_i & (j = i) \\ 0 & (j \neq i). \end{cases}$$

By definition, E_i is a uniserial module which has only simple modules S_i as its composition factors. For $M, N \in \text{rep}_{\text{l.f.}} H$, we define

$$\begin{aligned} \langle M, N \rangle_H &:= \dim \text{Hom}_H(M, N) - \dim \text{Ext}_H^1(M, N), \\ (M, N)_H &:= \langle M, N \rangle_H + \langle N, M \rangle_H, \\ q_H(M) &:= \langle M, M \rangle_H. \end{aligned}$$

Then, we have the following proposition:

Proposition 2.11 ([GLS17, Corollary 4.2]). For $M, N \in \text{rep}_{\text{l.f.}} H$ with $\underline{\text{rank}} M = (a_1, \dots, a_n)$ and $\underline{\text{rank}} N = (b_1, \dots, b_n)$, we have

$$\langle M, N \rangle_H = \sum_{i=1}^n c_i a_i b_i + \sum_{(j,i) \in \Omega} c_i c_{ij} a_i b_j.$$

Then, $(M, N) \mapsto \langle M, N \rangle_H$ descends to the Grothendieck group \mathbb{Z}^n of $\text{rep}_{\text{l.f.}} H$ and induces a symmetric bilinear form on $(-, -)_H: \mathbb{Z}^n \times \mathbb{Z}^n \rightarrow \mathbb{Z}$. We have equalities $(-, -)_H = (-, -)_C$ and $q_H = q_{DC}$.

We know the following properties about locally free modules.

Proposition 2.12 ([GLS17, Proof of Lemma 3.8]). The category $\text{Rep}_{\text{l.f.}} \Pi$ is closed under kernel of epimorphisms, cokernel of monomorphisms, and extensions.

Since each projective Π -module is locally free, we have the following:

Proposition 2.13 ([FG19, Corollary 2.7]). Let $M \in \text{Rep} \Pi$. If $\text{proj. dim}_{\Pi} M < \infty$, then $M \in \text{Rep}_{\text{l.f.}} \Pi$.

Proposition 2.14 ([GLS17, Theorem 12.6]). For $M \in \text{Rep}_{\text{l.f.}} \Pi$ and $N \in \text{rep}_{\text{l.f.}} \Pi$, we have the following:

- (1) We have a functorial isomorphism

$$\text{Ext}_{\Pi}^1(M, N) \cong \mathbb{D} \text{Ext}_{\Pi}^1(N, M).$$

- (2) If C does not contain any component of finite type, then we have more general functorial isomorphisms

$$\mathrm{Ext}_{\Pi}^{2-i}(M, N) \cong \mathbb{D} \mathrm{Ext}_{\Pi}^i(N, M) \quad (i = 0, 1, 2).$$

- (3) If M is also of finite rank, we have the following equality for any type of symmetrizable GCM C :

$$\dim \mathrm{Ext}_{\Pi}^1(M, N) = \dim \mathrm{Hom}_{\Pi}(M, N) + \dim \mathrm{Hom}_{\Pi}(N, M) - (M, N)_H.$$

We refer to this formula as the *generalized Crawley-Boevey formula*.

For classical preprojective algebras, a certain class of modules, called nilpotent modules, plays important roles in geometric representation theory about quantum groups. It is known that this class of modules can be characterized as the class of modules such that they have finite Jordan-Hölder filtrations with simple modules $\{S_i\}_{i \in Q_0}$ as composition factors. Geiß-Leclerc-Schröer [GLS18b] adapted this characterization as a first step of a generalization of nilpotent modules.

Definition 2.15. A module $M \in \mathrm{rep}_{\mathrm{l.f.}} \Pi$ is \mathbb{E} -filtered if and only if M has a filtration

$$M = M_n \supseteq M_{n-1} \supseteq \cdots \supseteq M_0 = 0$$

such that each M_{i+1}/M_i is isomorphic to a generalized simple module E_j ($j \in Q_0$). Let $\mathbb{E}\text{-filt}(\Pi)$ denote the category of \mathbb{E} -filtered modules.

Remark 2.16. There is a locally free module which is not \mathbb{E} -filtered already for B_2 -type.

2.3. Module varieties for generalized preprojective algebras. We briefly recall representation varieties for generalized preprojective algebras from [GLS18b, §2.2–§2.5]. Let $\Pi := \Pi(C, D)$. For a dimension vector $d = (d_1, \dots, d_n)$, we define

$$\overline{H}(d) := \prod_{a \in \overline{Q}_1} \mathrm{Hom}_K(K^{d_s(a)}, K^{d_t(a)}),$$

and let $\mathrm{rep}(\Pi, d)$ be the representation variety of Π -modules with dimension vector d . Then, by definition, a point in $\mathrm{rep}(\Pi, d)$ is a tuple $(M(a))_{a \in \overline{Q}_1} \in \overline{H}(d)$ which satisfies equations

$$M(\varepsilon_i)^{c_i} = 0, \quad M(\varepsilon_i)^{f_{ji}} M(\alpha_{ij}^{(g)}) = M(\alpha_{ij}^{(g)}) M(\varepsilon_j)^{f_{ij}},$$

$$\sum_{j \in \overline{\Omega}(-, i)} \sum_{g=1}^{g_{ij}} \sum_{f=0}^{f_{ji}-1} \mathrm{sgn}(i, j) M(\varepsilon_i)^f M(\alpha_{ij}^{(g)}) M(\alpha_{ji}^{(g)}) M(\varepsilon_i)^{f_{ji}-1-f} = 0$$

for all $i \in Q_0$, $(i, j) \in \overline{\Omega}$, and $1 \leq g \leq g_{ij}$. The group $G(d) := \prod_{i \in Q_0} \mathrm{GL}_K(d_i)$ acts on $\mathrm{rep}(\Pi, d)$ by conjugation. For a module $M \in \mathrm{rep}(\Pi, d)$, let $\mathcal{O}(M) := G(d)M$ denote its $G(d)$ -orbit. The set of $G(d)$ -orbits in $\mathrm{rep}(\Pi, d)$ bijectively corresponds to the set of isoclasses of modules in $\mathrm{rep}(\Pi, d)$. Let $\mathrm{rep}_{\mathrm{l.f.}}(\Pi, d) \subseteq \mathrm{rep}(\Pi, d)$ be the subvarieties of locally free modules, and let $\mathrm{nil}_{\mathbb{E}}(\Pi, d) \subseteq \mathrm{rep}_{\mathrm{l.f.}}(\Pi, d)$ be the constructible subset of \mathbb{E} -filtered modules in $\mathrm{rep}_{\mathrm{l.f.}}(\Pi, d)$.

We interpret above module varieties in terms of species (or sometimes called modulated graphs). We define H_i - H_j bimodules as follows:

$$\begin{aligned} {}_iH_j &:= H_i \otimes_K \text{Span}_K \{ \alpha_{ij}^{(g)} \mid 1 \leq g \leq g_{ij} \} \otimes_K H_j \\ &= \text{Span}_K \{ \varepsilon_i^{f_i} \otimes_K \alpha_{ij}^{(g)} \otimes_K \varepsilon_j^{f_j} \mid f_i, f_j \geq 0, 1 \leq g \leq g_{ij} \} \\ &= \bigoplus_{g=1}^{g_{ij}} \bigoplus_{f=0}^{f_{ij}-1} H_i \otimes_K (\alpha_{ij}^{(g)} \otimes_K \varepsilon_j^f) = \bigoplus_{g=1}^{g_{ij}} \bigoplus_{f=0}^{f_{ji}-1} (\varepsilon_i^f \otimes_K \alpha_{ij}^{(g)}) \otimes_K H_j. \end{aligned}$$

In the below, we omit \otimes_K for simplicity. In particular, we have isomorphisms as H_i -modules and as H_j -modules:

$$\begin{aligned} H_i({}_iH_j) &\cong H_i^{|c_{ij}|} \cong ({}_jH_i)_{H_i}; \\ H_j({}_jH_i) &\cong H_j^{|c_{ji}|} \cong ({}_iH_j)_{H_j}. \end{aligned}$$

Let M_i be a finite dimensional H_i -module for each $i \in Q_0$, and let M_{ij} be H_i -linear map

$$M_{ij}: {}_iH_j \otimes_{H_j} M_j \rightarrow M_i$$

for each $(i, j) \in \overline{\Omega}$. Let \otimes_i denote \otimes_{H_i} for simplicity in the below. Let $\mathbf{M} := (M_1, \dots, M_n)$ denote a tuple of H_i -modules M_i and let

$$H(\mathbf{M}) := \prod_{(i,j) \in \overline{\Omega}} \text{Hom}_{H_i}({}_iH_j \otimes_j M_j, M_i), \quad H^*(\mathbf{M}) := \prod_{(j,i) \in \overline{\Omega}^*} \text{Hom}_{H_j}({}_jH_i \otimes_i M_i, M_j).$$

Let $M = (\mathbf{M}, \mathbf{M}_{ij}, \mathbf{M}_{ji})$ denote an object determined by a tuple of a pair of modules \mathbf{M} , elements $\mathbf{M}_{ij} \in H(\mathbf{M})$ and $\mathbf{M}_{ji} \in H^*(\mathbf{M})$. We define a morphism between two tuples $\mathbf{M} = (\mathbf{M}, \mathbf{M}_{ij}, \mathbf{M}_{ji})$ and $\mathbf{N} = (\mathbf{N}, \mathbf{N}_{ij}, \mathbf{N}_{ji})$ by a tuple $(f_i)_i$ of H_i -linear maps $f_i: M_i \rightarrow N_i$ such that there is a commutative diagram

$$\begin{array}{ccc} {}_iH_j \otimes_{H_j} M_j & \xrightarrow{1 \otimes_j f_j} & {}_iH_j \otimes_{H_j} N_j \\ M_{ij} \downarrow & & \downarrow N_{ij} \\ M_i & \xrightarrow{f_i} & N_i \end{array}$$

for each $i \in Q_0$.

Let $\text{rep}(C, D, \overline{\Omega})$ denote the category defined by above data. Now, we have a functorial isomorphism

$$\begin{aligned} \text{ad}_{ji}: \text{Hom}_{H_j}({}_jH_i \otimes_i M_i, M_j) &\rightarrow \text{Hom}_{H_i}(M_i, {}_iH_j \otimes_j M_j) \\ \psi &\mapsto \left(\psi^\vee: m \mapsto \sum_{g=1}^{g_{ij}} \sum_{f=0}^{f_{ji}-1} \varepsilon_i^f \alpha_{ij}^{(g)} \otimes_j \psi(\alpha_{ji}^{(g)} \varepsilon_i^{f_{ji}-1-f} \otimes_i m) \right). \end{aligned}$$

We write $M_{ji}^\vee := \text{ad}_{ji}(M_{ji})$. We define two maps $M_{i,\text{in}}$ and $M_{i,\text{out}}$ by

$$M_{i,\text{in}} := (\text{sgn}(i, j) M_{ij})_j: \bigoplus_{j \in \overline{\Omega}(i, -)} {}_iH_j \otimes_j M_j \rightarrow M_i$$

and

$$M_{i,\text{out}} := (M_{ji}^\vee)_j: M_i \rightarrow \bigoplus_{j \in \overline{\Omega}(i,-)} {}_iH_j \otimes_j M_j.$$

Then, we have the following:

Proposition 2.17 ([GLS17, Proposition 5.2]). *Keep the above notation. The category $\text{rep } \Pi$ is equivalent to the full subcategory of $\text{rep}(C, D, \overline{\Omega})$ with objects $M = (\mathbf{M}, \mathbf{M}_{ij}, \mathbf{M}_{ji})$ such that $M_{i,\text{in}} \circ M_{i,\text{out}} = 0$ for each $i \in Q_0$.*

By Proposition 2.17, we often identify objects in $\text{rep } \Pi$ with those in $\text{rep}(C, D, \overline{\Omega})$. Note that the category $\text{rep } \Pi$ has the natural involution $*$ induced by taking the K -dual space $(M_i)^*$ of each M_i and the K -dual map $(M_{ij})^*$ for each M_{ij} .

We write

$$\overline{H}(\mathbf{M}) := \prod_{(i,j) \in \overline{\Omega}} \text{Hom}_{H_i}({}_iH_j \otimes_j M_j, M_i).$$

From now on, we write the morphism in $\text{Hom}_{H_i}({}_iH_j \otimes_j M_j, M_i)$ by $M_{ij}: {}_iH_j \otimes_j M_j \rightarrow M_i$ for $M \in \overline{H}(\mathbf{M})$. The group $G(\mathbf{M}) := \prod_{i \in Q_0} \text{GL}_{H_i}(M_i)$ acts by conjugation on $\overline{H}(\mathbf{M})$, where $\text{GL}_{H_i}(M_i)$ is the group of H_i -automorphisms of M_i . Now, we can think of $\text{rep}(\Pi, \mathbf{M}) := \{M \in \overline{H}(\mathbf{M}) \mid M_{i,\text{in}} \circ M_{i,\text{out}} = 0\}$ as an affine variety of Π -modules M with equalities $e_i M = M_i$ for $i \in Q_0$. This $\text{rep}(\Pi, \mathbf{M}) \subseteq \overline{H}(\mathbf{M})$ is $G(\mathbf{M})$ -stable. The set of $G(\mathbf{M})$ -orbits in $\text{rep}(\Pi, \mathbf{M})$ bijectively corresponds to the set of isoclasses of Π -modules M with equalities $e_i M = M_i$ for $i \in Q_0$. We call \mathbf{M} *locally free* if each M_i is a free H_i -module. In this case, we write $\underline{\text{rank}} \mathbf{M} := (\text{rank}_1(M_1), \dots, \text{rank}_n(M_n))$ and call it the *rank vector* of \mathbf{M} . For a locally free \mathbf{M} , we define $\text{nil}_{\mathbb{E}}(\Pi, \mathbf{M}) = \Pi(\mathbf{M})$ be the constructible subset of \mathbb{E} -filtered modules in $\text{rep}(\Pi, \mathbf{M})$. For a rank vector $\mathbf{r} = (r_1, \dots, r_n)$, we define

$$\begin{aligned} \mathbf{M}(\mathbf{r}) &:= (H_1^{\oplus r_1}, \dots, H_n^{\oplus r_n}); \\ H(\mathbf{r}) &:= H(\mathbf{M}(\mathbf{r})); \\ \overline{H}(\mathbf{r}) &:= \overline{H}(\mathbf{M}(\mathbf{r})); \\ \text{rep}(\Pi, \mathbf{r}) &:= \text{rep}(\Pi, \mathbf{M}(\mathbf{r})); \\ \text{nil}_{\mathbb{E}}(\Pi, \mathbf{r}) &:= \Pi(\mathbf{r}) := \Pi(\mathbf{M}(\mathbf{r})); \\ G(\mathbf{r}) &:= G(\mathbf{M}(\mathbf{r})). \end{aligned}$$

Then, each variety $\Pi(\mathbf{M})$ is isomorphic to $\Pi(\mathbf{r})$ for each $\mathbf{r} = \underline{\text{rank}} \mathbf{M}$ and we identify these varieties $\Pi(\mathbf{r})$ and $\Pi(\mathbf{M})$.

Finally, we compare these two constructions. We have the natural projection $\text{rep}(\Pi, d) \xrightarrow{p_\Pi} \prod_{i \in Q_0} \text{rep}(H_i, d_i)$. Then, we have $p_\Pi^{-1}(\mathbf{M}) \cong \text{rep}(\Pi, \mathbf{M})$ for $\mathbf{M} = (M_1, \dots, M_n)$. In this setting, we find that the $G(\mathbf{M})$ -orbit of M on this fiber can be written as $G(\mathbf{M})M = G(d)M \cap \text{rep}(\Pi, \mathbf{M})$. Now, a closed $G(\mathbf{M})$ -stable subset Z of $\text{rep}(\Pi, \mathbf{M})$ has $\dim Z = \dim G(\mathbf{M}) + m$ for some $m \in \mathbb{Z}$ if and only if the corresponding subset $G(d)Z$ of $\text{rep}(\Pi, d)$ has $\dim G(d)Z = \dim G(d) + m$.

2.4. Partial orders on module categories. In this subsection, we briefly review a notion of τ -rigid modules introduced in Auslander-Smalø [AS81] and later developed as the τ -tilting theory due to Adachi-Iyama-Reiten [AIR14]. Let Λ be a basic finite dimensional K -algebra and τ be the Auslander-Reiten translation for $\text{rep } \Lambda$. Let $|M|$ denote the number of non-isomorphic indecomposable direct summands of M , and let $\langle e \rangle$ denote the two-sided ideal generated by an element $e \in \Lambda$.

Definition 2.18. Let $M \in \text{rep } \Lambda$ and let $P \in \text{proj } \Lambda$. We define:

- (1) M is a τ -rigid Λ -module, if $\text{Hom}_\Lambda(M, \tau M) = 0$;
- (2) M is a τ -tilting Λ -module, if M is τ -rigid and $|M| = |\Lambda|$;
- (3) M is an *almost complete* τ -tilting Λ -module, if M is τ -rigid and $|M| = |\Lambda| - 1$;
- (4) M is a *support* τ -tilting module, if there exists an idempotent $e \in \Lambda$ such that M is a τ -tilting $(\Lambda/\langle e \rangle)$ -module;
- (5) (M, P) is a τ -rigid pair, if M is τ -rigid and $\text{Hom}_\Lambda(P, M) = 0$;
- (6) (M, P) is a *support* τ -tilting pair (*resp. almost complete* τ -tilting pair), if (M, P) is a τ -rigid pair and $|M| + |P| = |\Lambda|$ (*resp. $|M| + |P| = |\Lambda| - 1$*);
- (7) (M, P) is a *direct summand of* (M', P') , if (M, P) and (M', P') are τ -rigid pairs and M (*resp. P*) is a direct summand of M' (*resp. a direct summand of P'*);
- (8) (M, P) is *basic*, if M and P are basic (*i.e.* each direct summand of $M \oplus P$ is multiplicity free).

Let $\text{s}\tau\text{-tilt } \Lambda$ denote the full subcategory of basic support τ -tilting Λ -modules. We can think of τ -rigid modules as a generalization of classical partial tilting modules in the sense of the classical Bongartz's lemma (see [AIR14, Theorem 2.10]). Let $\text{add } M$ denote the full subcategory of finite direct summands of finite direct sums of M . A characterization of τ -rigid pairs and support τ -tilting pairs is given as the following theorem:

Theorem 2.19 ([AIR14, Proposition 2.3]). Let $M \in \text{rep } \Lambda$, let $P \in \text{proj } \Lambda$, and let $e \in \Lambda$ be an idempotent element such that $\text{add } P = \text{add } \Lambda e$.

- (1) (M, P) is a τ -rigid pair, if and only if M is a τ -rigid $(\Lambda/\langle e \rangle)$ -module;
- (2) (M, P) is a support τ -tilting pair if and only if M is a τ -tilting $(\Lambda/\langle e \rangle)$ -module;
- (3) (M, P) is an almost complete support τ -tilting pair, if and only if M is an almost complete τ -tilting $(\Lambda/\langle e \rangle)$ module;
- (4) If (M, P) and (M, Q) are support τ -tilting pairs in $\text{rep } \Lambda$, then $\text{add } P = \text{add } Q$.

By Theorem 2.19, we can identify basic support τ -tilting modules with basic support τ -tilting pairs.

Definition 2.20 (*cf.* [AIR14, Proposition 1.1]). (1) A full subcategory \mathcal{T} in $\text{rep } \Lambda$ (*resp. \mathcal{F} in $\text{rep } \Lambda$*) is a *torsion class* (*resp. a torsion-free class*), if \mathcal{T} (*resp. \mathcal{F}*) is closed under extensions and taking a factor module of objects (*resp. taking a submodule of objects*).

(2) $\text{Fac } M$ (*resp. $\text{Sub } M$*) denotes the full subcategory of factor modules (*resp. submodules*) of finite direct sums of $M \in \text{rep } \Lambda$.

Let $\text{tors } \Lambda$ (*resp. $\text{torf } \Lambda$*) denote the set of torsion classes in $\text{rep } \Lambda$ (*resp. torsion-free classes*). In the τ -tilting theory, one of the most important classes of algebras is τ -tilting finite algebras:

Definition 2.21 ([DIJ19]). An algebra Λ is called τ -tilting finite, if $\text{s}\tau\text{-tilt } \Lambda$ is a finite set.

Theorem 2.22 ([DIJ19], [AIR14, Theorem 2.7]). For a τ -tilting finite algebra, we have a bijection between $\text{s}\tau\text{-tilt } \Lambda$ and $\text{tors } \Lambda$:

$$\begin{aligned} \text{s}\tau\text{-tilt } \Lambda &\longrightarrow \text{tors } \Lambda \\ M &\longmapsto \text{Fac } M. \end{aligned}$$

We can define a partial order of $\text{s}\tau\text{-tilt } \Lambda$ by Theorem 2.22 as follows:

Definition 2.23. For $T, T' \in \text{s}\tau\text{-tilt } \Lambda$, we define a partial order \leq on $\text{s}\tau\text{-tilt } \Lambda$ by $T \leq T'$ if $\text{Fac } T \subseteq \text{Fac } T'$.

The above partial order is understood in terms of mutations:

Definition-Proposition 2.24 ([AIR14, Theorem 2.18]). Any basic almost complete τ -tilting pair (U, Q) is a direct summand of precisely two different basic support τ -tilting pairs (T, P) and (T', P') . In addition, these $T, T' \in \text{s}\tau\text{-tilt } \Lambda$ satisfy $T' < T$ or $T' > T$. In this setting, if $T' < T$ (resp. $T' > T$), we say that (T', P') is a *left* (resp. *right*) mutation of (T, P) . For this $T \in \text{s}\tau\text{-tilt } \Lambda$ and the indecomposable summand X of T such that $T = X \oplus U$, we say that T is the *left* (resp. *right*) mutation of T' at X , if $T' < T$ (resp. $T' > T$).

Proposition 2.25 ([AIR14, Definition-Proposition 2.28]). Under the setting of Definition-Proposition 2.24, T' is the left mutation of T at X if and only if $X \notin \text{Fac } U$.

We know that for a τ -tilting finite algebra Λ , the finite set $\text{s}\tau\text{-tilt } \Lambda$ forms a finite complete lattice by Definition-Proposition 2.21 and Definition 2.33. In particular, we find that $M, N \in \text{s}\tau\text{-tilt } \Lambda$ are related by a mutation if and only if one is next to the other in the finite complete lattice of $\text{s}\tau\text{-tilt } \Lambda$ by the following Theorem 2.26:

Theorem 2.26 ([AIR14, Theorem 2.33]). Let $T, U \in \text{s}\tau\text{-tilt } \Lambda$. The following conditions are equivalent:

- (1) U is a left mutation of T ;
- (2) T is a right mutation of U ;
- (3) T and U satisfy $T > U$, and there does not exist $V \in \text{s}\tau\text{-tilt } \Lambda$ such that $T > V > U$.

2.5. Tilting theory for preprojective algebras. In this subsection, we briefly review the work of Fu-Geng [FG19] about a relationship between the generalized preprojective algebras and the Weyl groups of the Kac-Moody Lie algebras. Let $\Pi = \Pi(C, D)$ be the generalized preprojective algebra associated with a symmetrizable GCM C and its symmetrizer D . We define an *idempotent ideal* I_i for each $i \in Q_0$ by $I_i := \Pi(1 - e_i)\Pi$. Note that we have an isomorphism $\Pi/I_i \cong E_i$ for each $i \in Q_0$. In [FG19], some results of Buan-Iyama-Reiten-Scott [BIRS09] and Mizuno [Miz14] are generalized for our situation.

Definition 2.27. (1) We say that a two-sided ideal T of Π is a *tilting ideal* if T is a left tilting Π -module and a right tilting Π -module in the sense of Miyasita [Miy86].
 (2) We assume Π is finite dimensional over K . Then, we say that a two-sided ideal T of Π is a *support τ -tilting ideal* if T is a left support τ -tilting Π -module and a right support τ -tilting Π -module.

Theorem 2.28 ([FG19, Lemma 3.2, 3.9, Theorem 3.12, 4.7, 5.14, 5.17]). Let $C \in M_n(\mathbb{Z})$ be a symmetrizable GCM, and let D be any symmetrizer of C .

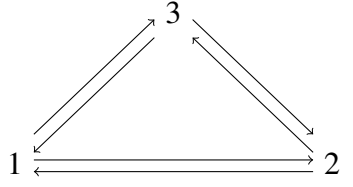
- (1) There is a bijection ψ from $W(C)$ to the monoid

$$\langle I_1, I_2, \dots, I_n \rangle := \{I_{i_1} I_{i_2} \cdots I_{i_k} \mid i_1, i_2, \dots, i_k \in Q_0, k \geq 0\}$$

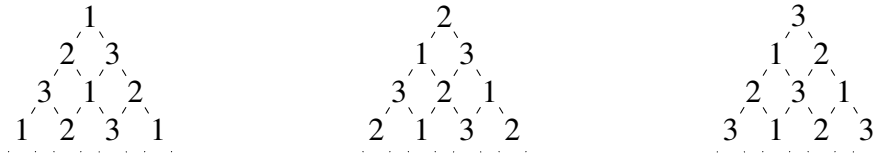
given by $\psi(w) = I_w = I_{i_1} I_{i_2} \cdots I_{i_k}$, where $w = s_{i_1} s_{i_2} \cdots s_{i_k}$ is a reduced expression of $w \in W(C)$. Here, ψ does not depend on the choice of reduced expression of w .

- (2) Assume that C has no components of finite type. Then, $T \in \langle I_1, I_2, \dots, I_n \rangle$ if and only if T is a tilting ideal of Π such that Π/T has finite length. Any object in $\langle I_1, I_2, \dots, I_n \rangle \subseteq \text{Rep } \Pi$ has projective dimension at most 1.
- (3) Assume that C is of finite type. Then, $T \in \langle I_1, I_2, \dots, I_n \rangle$ if and only if T is a basic support τ -tilting ideal. In particular, $\psi: w \mapsto I_w$ gives a bijection between $W(C)$ and $s\tau\text{-tilt } \Pi$.

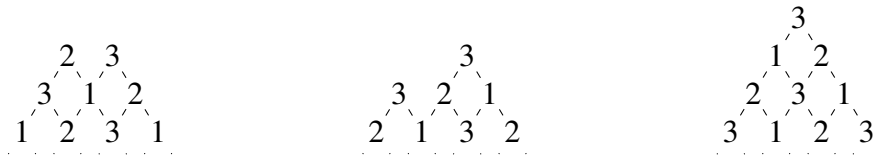
Example 2.29. Let $C = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$, let $D = \text{diag}(1, 1, 1)$ and let $\Pi := \Pi(C, D)$. Namely, we consider the following double quiver with preprojective relation:



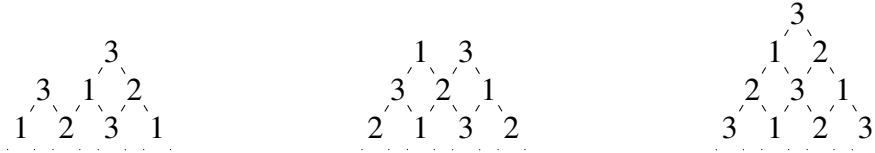
Then, the Loewy series of indecomposable projective modules Πe_1 , Πe_2 and Πe_3 are respectively given by



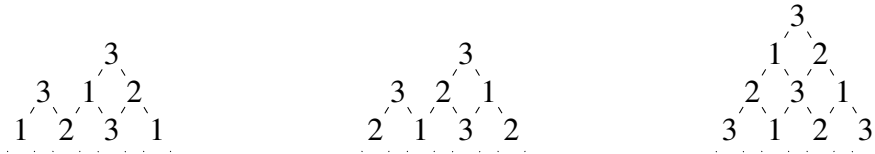
The Loewy series of $I_{s_1 s_2}$ is given by the direct sum of the following three modules



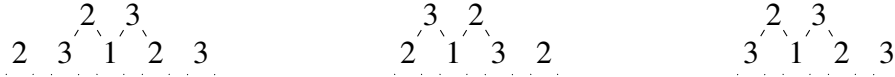
The Loewy series of $I_{s_2s_1}$ is given by the direct sum of



The Loewy series of $I_{s_1s_2s_1} = I_{s_2s_1s_2}$ is given by the direct sum of



The Loewy series of $I_{s_1s_2s_3s_2s_1}$ is given by the direct sum of



Note that the multiplication gives an isomorphism $I_{i_1} \otimes_{\Pi} \cdots \otimes_{\Pi} I_{i_n} \cong I_{i_1} \cdots I_{i_n}$ if $w = s_{i_1} \cdots s_{i_n}$ is a reduced expression by the above Theorem 2.28. We note that we can obtain a similar result in $\text{Rep } \Pi^{\text{op}}$ to Theorem 2.28. From now till the end of this subsection, let $C \in M_n(\mathbb{Z})$ be a symmetrizable GCM of finite type, and let D be any symmetrizer D of C .

Proposition 2.30 ([Mur19, Lemma 3.10, Theorem 3.11]). For each $w \in W$, the two sided ideal I_w of $\Pi(C, D)$ is an \mathbb{E} -filtered module. In particular, any basic support τ -tilting Π -module is \mathbb{E} -filtered. We have an isomorphism $I_w^* \cong \Pi/I_{ww_0}$ in $\text{rep}_{\text{l.f.}} \Pi$.

Since C is of finite type, we have a relationship between the right weak Bruhat order \leq_R on $W(C)$ and the mutation in $\text{s}\tau$ -tilt Π .

Theorem 2.31 ([FG19, Lemma 5.11, Proposition 5.13 and its proof]). Let $T \in \langle I_1, \dots, I_n \rangle$. If $TI_i \neq T$, then T has the left mutation TI_i at Te_i in $\text{s}\tau$ -tilt Π .

Theorem 2.32 ([FG19, Theorem 5.16]). For $i \in Q_0$ and $w \in W$, we write the corresponding τ -tilting pair (I_w, P_w) .

- (1) $I_w, I_{ws_i} \in \text{s}\tau$ -tilt Π are related by a right or left mutation. In particular, if $l(ws_i) > l(w)$, then we have

$$I_{ws_i} = \begin{cases} I_w(1 - e_i) & (I_w I_i = 0) \\ I_w I_i e_i \oplus I_w(1 - e_i) & (I_w I_i \neq 0). \end{cases}$$

(2) We have the following isomorphism:

$$P_w \cong \bigoplus_{i \in Q_0, I_w e_i = 0} \Pi e_{\sigma(i)},$$

where σ is the Nakayama permutation determined by the self-injectivity of Π , *i.e.* $\Pi e_i \cong \mathbb{D}(e_{\sigma(i)} \Pi)$.

Note that the Nakayama permutation σ of Π in the above Theorem 2.32 has a description in terms of the Weyl group $W(C)$, as we can understand by using g -matrices in the §3.2. We have the following generalization of a result of Mizuno [Miz14]:

Theorem 2.33 ([Mur19, Theorem 2.31]). Let $w \in W$ and $i \in Q_0$. The following are equivalent:

- $l(w) < l(ws_i) = l(w) + 1$;
- $I_w I_i \neq I_w$;
- I_w has a left mutation I_{ws_i} at $I_w e_i$.

Theorem 2.34 ([Mur19, Theorem 2.32]). Let $w \in W$ and $i \in Q_0$. The following are equivalent:

- $l(w) > l(ws_i) = l(w) - 1$;
- $I_w I_i = I_w$;
- I_{ws_i} has a left mutation I_w at $I_{ws_i} e_i$.

In particular, we find that $u \leq_R v$ in W if and only if $I_u \geq I_v$ in $\mathfrak{s}\tau$ -tilt Π . That is, (W, \leq_R) can be identified with $(\mathfrak{s}\tau\text{-tilt } \Pi, \leq)^{\text{op}}$ as a poset. We note that we can consider the similar poset structure of $\mathfrak{s}\tau\text{-tilt } \Pi^{\text{op}}$, if we consider the left weak order instead of the right weak Bruhat order. Finally, we recall homological properties about I_w .

Proposition 2.35 ([Mur19, Proposition 3.8, 3.9]). We have the following equalities:

- (1) If $l(ws_i) > l(w)$, then we have an equality $\text{Tor}_1^{\Pi}(I_w, E_i) = 0$.
- (2) If $l(s_i w) > l(w)$, then we have an equality $\text{Ext}_{\Pi}^1(I_w, E_i) = 0$.

2.6. Reflection functors and idempotent ideals. In the work of Baumann-Kamnitzer [BK12] and Geiß-Leclerc-Schröer [GLS17], reflection functors on module categories of generalized preprojective algebras are developed in terms of species. We briefly review the definitions of reflection functors and their descriptions through idempotent ideals.

Definition 2.36. Keep the notation of Proposition 2.17. Let $M \in \text{rep } \Pi$. We identify M with $(\mathbf{M}, \mathbf{M}_{ij}, \mathbf{M}_{ji})$ such that $M_{i,\text{in}} \circ M_{i,\text{out}} = 0$ for each $i \in Q_0$. We construct a new Π -module $\Sigma_i(M)$ by replacing the diagram

$$\widetilde{M}_i \xrightarrow{M_{i,\text{in}}} M_i \xrightarrow{M_{i,\text{out}}} \widetilde{M}_i$$

with

$$\widetilde{M}_i \xrightarrow{\overline{M}_{i,\text{out}} M_{i,\text{in}}} \text{Ker } M_{i,\text{in}} \xrightarrow{\text{can}} \widetilde{M}_i,$$

where $\widetilde{M}_i := \bigoplus_{\overline{\Omega}(i,j)} H_j \otimes_j M_j$, and $\overline{M}_{i,\text{out}}: M_i \rightarrow \text{Ker } M_{i,\text{in}}$ is induced by $M_{i,\text{out}}$, and can is the canonical injection. Similarly, we construct a new module $\Sigma_i^-(M)$ by replacing the diagram

$$\widetilde{M}_i \xrightarrow{M_{i,\text{in}}} M_i \xrightarrow{M_{i,\text{out}}} \widetilde{M}_i$$

with

$$\widetilde{M}_i \xrightarrow{\text{can}} \text{Cok } M_{i,\text{out}} \xrightarrow{M_{i,\text{out}}\overline{M}_{i,\text{in}}} \widetilde{M}_i,$$

where $\overline{M}_{i,\text{in}}: \text{Cok } M_{i,\text{out}} \rightarrow M_i$ is induced by $M_{i,\text{in}}$, and can is the canonical projection. Then, $(\Sigma_i(M))^* \cong \Sigma_i^-(M^*)$ and $(\Sigma_i^-(M))^* \cong \Sigma_i(M^*)$ for each $i \in Q_0$.

Remark 2.37. By Definition 2.36, we have a diagram

$$\begin{array}{ccccc} \widetilde{M}_i & \xrightarrow{\text{can}} & \text{Cok } M_{i,\text{out}} & \xrightarrow{M_{i,\text{out}}\overline{M}_{i,\text{in}}} & \widetilde{M}_i \\ \parallel & & \downarrow \overline{M}_{i,\text{in}} & & \parallel \\ \widetilde{M}_i & \xrightarrow{M_{i,\text{in}}} & M_i & \xrightarrow{M_{i,\text{out}}} & \widetilde{M}_i \\ \parallel & & \downarrow \overline{M}_{i,\text{out}} & & \parallel \\ \widetilde{M}_i & \xrightarrow{\overline{M}_{i,\text{out}}M_{i,\text{in}}} & \text{Ker } M_{i,\text{in}} & \xrightarrow{\text{can}} & \widetilde{M}_i \end{array}$$

By this diagram, we have canonical morphisms $\Sigma_i^- M \rightarrow M \rightarrow \Sigma_i M$.

For $M \in \text{rep } \Pi$ and $i \in Q_0$, let $\text{sub}_i M$ (*resp.* $\text{fac}_i M$) be the largest submodule (*resp.* factor module) of M such that each composition factor of $\text{sub}_i M$ (*resp.* $\text{fac}_i M$) is isomorphic to S_i . We say that M has a trivial i -sub (*resp.* i -factor) if $\text{sub}_i M = 0$ (*resp.* $\text{fac}_i M = 0$). We define $\text{sub } M := \bigoplus_{i \in Q_0} \text{sub}_i M$ and $\text{fac } M := \bigoplus_{i \in Q_0} \text{fac}_i M$. The following properties about reflection functors are important.

Theorem 2.38 ([GLS17, Proposition 9.1], Külshammer [Kül17, Lemma 5.2]). For each $i \in Q_0$, the following statements (1)–(5) hold:

- (1) The pair (Σ_i^-, Σ_i) is a pair of adjoint functors, *i.e.* there is a functorial isomorphism

$$\text{Hom}_\Pi(\Sigma_i^- M, N) \cong \text{Hom}_\Pi(M, \Sigma_i N).$$

- (2) The adjunction morphisms $\text{id} \rightarrow \Sigma_i \Sigma_i^-$ and $\Sigma_i^- \Sigma_i \rightarrow \text{id}$ can be inserted in functorial short exact sequences

$$0 \rightarrow \text{sub}_i \rightarrow \text{id} \rightarrow \Sigma_i \Sigma_i^- \rightarrow 0$$

and

$$0 \rightarrow \Sigma_i^- \Sigma_i \rightarrow \text{id} \rightarrow \text{fac}_i \rightarrow 0.$$

- (3) We define subcategories $\mathcal{S}_i := \{M \in \text{rep } \Pi \mid \text{sub}_i M = 0\}$ and $\mathcal{T}_i := \{M \in \text{rep } \Pi \mid \text{fac}_i M = 0\}$. Then, we have mutual inverse equivalences of categories $\Sigma_i: \mathcal{T}_i \rightarrow \mathcal{S}_i$ and $\Sigma_i^-: \mathcal{S}_i \rightarrow \mathcal{T}_i$.
- (4) If M is a locally free module, then we have an equality $\underline{\text{rank}} \Sigma_i M = s_i(\underline{\text{rank}} M)$ (*resp.* $\underline{\text{rank}} \Sigma_i^- M = s_i(\underline{\text{rank}} M)$) if and only if $\text{fac}_i M = 0$ (*resp.* $\text{sub}_i M = 0$).
- (5) For any $i \in Q_0$, we have $\Sigma_i \cong \text{Hom}_\Pi(I_i, \bullet)$ and $\Sigma_i^- \cong I_i \otimes_\Pi (\bullet)$.

The following formulas proved in the work of Geiß-Leclerc-Schröer [GLS18b] are useful:

Proposition 2.39 ([GLS18b]). Let $M \in \text{rep } \Pi$ and $i \in Q_0$. Then, we have the following:

- (1) $\dim \text{Hom}_\Pi(E_i, M) = \dim \text{Ker } M_{i,\text{out}} = \dim \text{sub}_i(M)$;
- (2) $\dim \text{Hom}_\Pi(M, E_i) = \dim \text{Cok } M_{i,\text{in}} = \dim \text{fac}_i(M)$;
- (3) If M is locally free, then we have

$$\begin{aligned} \dim \text{Ext}_\Pi^1(M, E_i) &= \dim \widetilde{M}_i - \dim \text{Im } M_{i,\text{in}} - \dim \text{Im } M_{i,\text{out}} \\ &= \dim(\text{Ker } M_{i,\text{in}}/\text{Im } M_{i,\text{out}}). \end{aligned}$$

Finally, we recall a notion of torsion pairs, which play important roles in this paper.

Definition 2.40. Let \mathcal{A} be an Abelian category. Then, a pair $(\mathcal{T}, \mathcal{F})$ of extension closed subcategories is a torsion pair if and only if we have the following two equalities:

- $\mathcal{T} = \{M \in \mathcal{A} \mid \text{Hom}_{\mathcal{A}}(M, N) = 0 \text{ for any } N \in \mathcal{F}\}$;
- $\mathcal{F} = \{N \in \mathcal{A} \mid \text{Hom}_{\mathcal{A}}(M, N) = 0 \text{ for any } M \in \mathcal{T}\}$.

For a torsion pair $(\mathcal{T}, \mathcal{F})$ and $M \in \text{rep } \Pi$, we have the *canonical short exact sequence* $0 \rightarrow tM \rightarrow M \rightarrow M/tM \rightarrow 0$ with torsion submodule tM of M . Any short exact sequence of the form $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$ with $L \in \mathcal{T}$ and $N \in \mathcal{F}$ is isomorphic to the canonical sequence (see [ASS06, §VI Proposition 1.5]).

For the module category over Π , we describe torsion pairs in terms of the Weyl group $W = W(C)$.

Definition 2.41. Let $w \in W$. We define subcategories of $\text{rep } \Pi$ as follows:

$$\mathcal{T}^w = \{T \in \text{rep } \Pi \mid \text{Ext}_\Pi^1(I_w, T) = 0\}, \quad \mathcal{F}^w = \{T \in \text{rep } \Pi \mid \text{Hom}_\Pi(I_w, T) = 0\}; \quad (2.1)$$

$$\mathcal{T}_w = \{T \in \text{rep } \Pi \mid I_w \otimes_\Pi T = 0\}, \quad \mathcal{F}_w = \{T \in \text{rep } \Pi \mid \text{Tor}_1^\Pi(I_w, T) = 0\}. \quad (2.2)$$

Proposition 2.42 ([Mur19, Corollary 3.12]). Let C be a symmetrizable GCM of finite type.

- (1) The natural K -duality $*$ on $\text{rep } \Pi$ gives a bijection $*$: $\text{tors } \Pi \rightarrow \text{torf } \Pi$ by $\text{Fac } I_w \mapsto \text{Sub } \Pi / I_{ww_0}$.
- (2) We have a bijection between W and the set of torsion pairs given by $w \mapsto (\text{Fac } I_w, \text{Sub } \Pi / I_w)$ and we can write

$$\text{Fac } I_w = \{I_w \otimes_\Pi T \mid T \in \text{rep } \Pi\}, \quad \text{Sub } \Pi / I_w = \{T \in \text{rep } \Pi \mid \text{Hom}_\Pi(I_w, T) = 0\}.$$

Proof. For the reader's convenience, we give a sketch of the proof of (2). The assertion $\text{Fac } I_w = \{I_w \otimes_\Pi T \mid T \in \text{rep } \Pi\}$ is a consequence of the Brenner-Butler theory. Namely, we have an algebra epimorphism $\widehat{\Pi} \rightarrow \widehat{\Pi}/\langle e_0 \rangle = \Pi$ for the generalized preprojective algebra of the un-twisted affine type associated with Π , so that we have an exact inclusion of categories $\text{rep } \Pi \rightarrow \text{rep } \widehat{\Pi}$. Now, the $\widehat{\Pi}$ -modules \widehat{I}_w are tilting modules for $w \in W(C)$ by Theorem 2.28. So, we have functorial isomorphisms $\text{Ext}_{\widehat{\Pi}}^1(\widehat{I}_w, M) \cong \text{Ext}_\Pi^1(I_w, M)$ and $\text{Hom}_{\widehat{\Pi}}(\widehat{I}_w, N) \cong \text{Hom}_\Pi(I_w, N)$ for $M, N \in \text{rep } \Pi$ etc. In particular, we have two forms of torsion pairs $(\mathcal{T}^w, \mathcal{F}^w)$ and $(\mathcal{T}_w, \mathcal{F}_w)$. Note that the well-known fact in the τ -tilting theory [AIR14, Proposition 2.16] tells us that $(\text{Fac } I_w, I_w^\perp)$ is a torsion pair in $\text{rep } \Pi$ for any support τ -tilting module I_w , so that we have $\text{Fac } I_w = \mathcal{T}^w$. Any module T has an epimorphism $\Pi^{\oplus m} \rightarrow T$ from a free Π -module $\Pi^{\oplus m}$, so that we have an epimorphism

$I_w^{\oplus m} \rightarrow I_w \otimes_{\Pi} T$ by applying the functor $I_w \otimes_{\Pi} (-)$. So, any module of the form $I_w \otimes_{\Pi} T$ belongs to $\text{Fac } I_w$. Now, we have a pair of exact functors whose compositions are identities

$$\mathcal{F}_w \begin{array}{c} \xrightarrow{I_w \otimes_{\Pi} (-)} \\ \xleftarrow{\text{Hom}_{\Pi}(I_w, -)} \end{array} \mathcal{T}^w$$

by our exact inclusion of categories $\text{rep } \Pi \rightarrow \text{rep } \widehat{\Pi}$. So, \mathcal{T}^w is the essential image of $I_w \otimes_{\Pi} (-)$. Next, by the self-injectivity of Π , any ideal of Π is basic if Π is basic. So, the torsion submodule of Π with respect to $\text{Fac } I_w$ is isomorphic to I_w . Thus, the canonical short exact sequence of Π associated with a torsion pair $(\mathcal{T}^w, \mathcal{F}^w)$ is isomorphic to

$$0 \rightarrow I_w \rightarrow \Pi \rightarrow \Pi/I_w \rightarrow 0.$$

Since the bijection $W(C) \ni w \mapsto \text{Sub } \Pi/I_w \in \text{torf } \Pi$ in Theorem 2.22 and (1) implies that $(\text{Fac } I_w, \text{Sub } \Pi/I_w)$ is a torsion pair by the finiteness of Weyl group $W(C)$. Thus, we obtain our assertion. \square

Remark 2.43. By a dual argument of Proposition 2.42, we find that the torsion pair $(\mathcal{T}_w, \mathcal{F}_w)$ in the Proof of Proposition 2.42 coincides with the torsion pair $(\text{Fac } I_{w^{-1}w_0}, \text{Sub } \Pi/I_{w^{-1}w_0})$ for each $w \in W$ and

$$\mathcal{F}_w = \{\text{Hom}_{\Pi}(I_w, T) \mid T \in \text{rep } \Pi\}.$$

Proposition 2.44. Let $u, v \in W$. Suppose that u and v satisfy $l(uv) = l(u) + l(v)$. Then, we have mutually inverse equivalences

$$\mathcal{F}^v \begin{array}{c} \xrightarrow{I_u \otimes_{\Pi} (-)} \\ \xleftarrow{\text{Hom}_{\Pi}(I_u, -)} \end{array} \mathcal{F}^{uv} \cap \mathcal{T}^u$$

Proof. If $T \in \mathcal{F}^v$, then we have $T \in \mathcal{F}_u$ by our choice of u and v by Remark 2.43. In fact, we have $\mathcal{F}^v = \text{Sub } \Pi/I_v \subset \text{Sub } \Pi/I_{uv} \subset \text{Sub } \Pi/I_{uv_0} = \mathcal{F}_u$. Thus, we obtain

$$T \cong \text{Hom}_{\Pi}(I_u, I_u \otimes_{\Pi} T)$$

by applying an equivalence $\text{Hom}_{\Pi}(I_u, -): \mathcal{T}^u \rightarrow \mathcal{F}_u$ for $I_u \otimes_{\Pi} T \in \mathcal{T}^u$ by Proposition 2.42. We obtain

$$\begin{aligned} \text{Hom}_{\Pi}(I_{uv}, I_u \otimes_{\Pi} T) &\cong \text{Hom}_{\Pi}(I_v, \text{Hom}_{\Pi}(I_u, I_u \otimes_{\Pi} T)) \\ &\cong \text{Hom}_{\Pi}(I_v, T) = 0. \end{aligned}$$

In particular, we obtain $I_u \otimes_{\Pi} T \in \mathcal{F}^{uv}$. Conversely, if $T \in \mathcal{F}^{uv}$, then

$$\text{Hom}_{\Pi}(I_v, \text{Hom}_{\Pi}(I_u, T)) = \text{Hom}_{\Pi}(I_{uv}, T) = 0.$$

Thus, we obtain $\text{Hom}_{\Pi}(I_u, T) \in \mathcal{F}^u$. Thus, we obtain our assertion by the equivalence between \mathcal{F}_u and \mathcal{T}^u induced by $\text{Hom}_{\Pi}(I_u, -)$ and $I_u \otimes_{\Pi} (-)$. \square

The evaluation map $I_w \otimes_{\Pi} (\text{Hom}_{\Pi}(I_w, M)) \rightarrow M$ is injective and the image is the torsion submodule of M with respect to the torsion pair $(\text{Fac } I_w, \text{Sub } \Pi/I_w)$. Let $T_w := I_w \otimes_{\Pi} (\text{Hom}_{\Pi}(I_w, M))$ denote the torsion submodule of T with respect to a torsion pair $(\text{Fac } I_w, \text{Sub } \Pi/I_w)$. Finally, we refer to a relationship between rank vectors and reflection functors.

Proposition 2.45. Let $w \in W$ and $M \in \text{rep}_{\text{l.f.}} \Pi$. If $M \in \mathcal{F}_w$ (resp. $M \in \mathcal{T}^w$), then $I_w \otimes_{\Pi} M$ (resp. $\text{Hom}_{\Pi}(I_w, M)$) is a locally free module with $\underline{\text{rank}}(I_w \otimes_{\Pi} M) = w(\underline{\text{rank}} M)$ (resp. $\underline{\text{rank}}(\text{Hom}_{\Pi}(I_w, M)) = w(\underline{\text{rank}} M)$), where we identify $\underline{\text{rank}} M$ with some element of the root lattice.

Proof. Since we can prove the second assertion about $M \in \mathcal{T}^w$ dually as the first assertion about $M \in \mathcal{F}_w$, we only prove the first assertion. By Remark 2.43, we put $M = \text{Hom}_{\Pi}(I_w, T)$ with some $T \in \text{rep } \Pi$. We prove by induction on $l(w)$. If $l(w) = 1$ and we put $w = s_i$, then we have an equality

$$\dim \text{sub}_i M = \dim \text{Hom}_{\Pi}(E_i, \text{Hom}_{\Pi}(I_{s_i}, T)) = \dim \text{Hom}_{\Pi}(I_{s_i} \otimes_{\Pi} E_i, T) = 0.$$

Thus, the assertion is proved by Proposition 2.39 and [GLS17, Proposition 9.4].

We choose a reduced expression $w = s_{i_1} \cdots s_{i_k}$. Then, we have an isomorphism $\text{Hom}_{\Pi}(I_w, (-)) \cong \text{Hom}_{\Pi}(I_{s_{i_k}}, (\cdots (\text{Hom}_{\Pi}(I_{s_{i_1}}, (-))))))$ by $I_w \cong I_{s_{i_1}} \otimes_{\Pi} \cdots \otimes_{\Pi} I_{s_{i_k}}$. Now, the module

$$I_{s_{i_1} w} \otimes_{\Pi} \text{Hom}_{\Pi}(I_w, T) \cong I_{s_{i_1} w} \otimes_{\Pi} \text{Hom}_{\Pi}(I_{s_{i_1} w}, \text{Hom}_{\Pi}(I_{s_{i_1}}, T))$$

is a torsion submodule of $\text{Hom}_{\Pi}(I_{s_{i_1}}, T)$. We take a locally free module $N := \text{Hom}_{\Pi}(I_w, T) \cong \text{Hom}_{\Pi}(I_{s_{i_1} w}, \text{Hom}_{\Pi}(I_{s_{i_1}}, T))$, so that $I_{s_{i_1} w} \otimes_{\Pi} N$ belongs to $\mathcal{T}^{s_{i_1} w}$ and this module is locally free by induction hypothesis. Then, $I_{s_{i_1} w} \otimes_{\Pi} N$ is a submodule of an object in $\mathcal{F}_{s_{i_1}}$. Since $\mathcal{F}_{s_{i_1}}$ is a torsion-free class, any submodule of an object in $\mathcal{F}_{s_{i_1}}$ belongs to $\mathcal{F}_{s_{i_1}}$. Thus, we deduce our assertion inductively from the $l(w) = 1$ case because $I_w \otimes_{\Pi} N \cong I_w \otimes_{\Pi} \text{Hom}_{\Pi}(I_w, T) \cong I_{s_{i_1}} \otimes_{\Pi} I_{s_{i_1} w} \otimes_{\Pi} N$. \square

Theorem 2.38 ensures that \mathbb{E} -filtered property is stable under taking reflection functors in some subcategories of $\text{rep}_{\text{l.f.}} \Pi$ in the following sense:

Proposition 2.46. Let $M \in \mathbb{E}\text{-filt}(\Pi)$ such that $\text{fac}_i M = 0$ (resp. $\text{sub}_i M = 0$). Then, the locally free module $\Sigma_i M$ (resp. $\Sigma_i^- M$) is an \mathbb{E} -filtered module.

Proof. By Definition 2.36, we find that $\Sigma_i^- E_j \neq 0$ if and only if $i \neq j$. Now, we take an \mathbb{E} -filtered module M and a filtration $0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_n = M$ such that $M_i/M_{i-1} \cong E_{k_i}$ for each $1 \leq i \leq n$ and some $k_i \in Q_0$. Let $m(M) := n$ the length of this filtration. We prove our assertion by induction on n . If $m(M) = 1$, then $M = E_j$ for some $j \in Q_0$. By the definition of Σ_i^- , we deduce $\Sigma_i^- E_j$ is \mathbb{E} -filtered by inspection. Next, we assume that our assertion is true for each $m(M) \leq l$ and take $m(M) = l + 1$. Now, we write $M_{l+1}/M_l \cong E_{k_{l+1}}$. If $k_{l+1} \neq i$, then we have an exact sequence

$$0 \rightarrow \Sigma_i^- M_l \rightarrow \Sigma_i^- M_{l+1} \rightarrow \Sigma_i^- E_{k_{l+1}} \rightarrow 0$$

by Proposition 2.35, Theorem 2.38 and our choice of k_{l+1} . Then, $\Sigma_i^- M_l$ is \mathbb{E} -filtered by our induction hypothesis. So, $\Sigma_i^- M_{l+1}$ is also \mathbb{E} -filtered by the first step of our induction and the above exact sequence. If $k = i$, then we have an epimorphism $f: \Sigma_i^- M_l \rightarrow \Sigma_i^- M_{l+1}$. Since $\text{rep}_{\text{l.f.}} \Pi$ is closed under taking kernels of epimorphisms, $\text{Ker } f$ is a locally free module. Furthermore, $\text{Ker } f$ is a direct sum of E_i by the definition of Σ_i^- . So, we can deduce the assertion by the exact sequence

$$0 \rightarrow \text{Ker } f \rightarrow \Sigma_i^- M_l \rightarrow \Sigma_i^- M_{l+1} \rightarrow 0$$

and the induction hypothesis. Thus, we obtain our assertion by induction on $m(M)$. \square

3. ON STABILITY OF MODULE CATEGORIES

Keep the setting of the previous section.

3.1. Framed preprojective algebras and stable modules. In this subsection, we develop representation theory of *framed preprojective algebras* as an analogue of the work of Baumann-Kamnitzer [BK12]. Namely, we attach a symmetrizable GCM \tilde{C} and its symmetrizer \tilde{D}

$$\tilde{C} = \begin{pmatrix} C & -I_n \\ -I_n & 2I_n \end{pmatrix} \quad \tilde{D} = \begin{pmatrix} D & O \\ O & D \end{pmatrix}$$

to any symmetrizable GCM $C = (c_{ij})$ and its symmetrizer D , and consider a new generalized preprojective algebra $\tilde{\Pi} := \Pi(\tilde{C}, \tilde{D})$. A $\tilde{\Pi}$ -module M is a direct sum of Q_0 -graded vector spaces $\bigoplus_{i \in Q_0} M_i$ and $\bigoplus_{i \in Q_0} M_{i'}$ where we set $\tilde{Q} = (Q_0 \amalg Q'_0, \tilde{Q}_1)$ as the quiver associated with (\tilde{C}, \tilde{D}) . The rank-vector of a locally free module M is defined as a pair $\underline{\text{rank}} M = (\nu, \lambda) \in \mathbb{N}^{Q_0} \times \mathbb{N}^{Q_0}$, where $\nu_i = \text{rank}_i M_i$ and $\lambda_{i'} = \text{rank}_{i'} M_{i'}$. We will identify $\nu = (\nu_i)_{i \in Q_0}$ with a positive root $\sum_{i \in Q_0} \nu_i \alpha_i \in R$ and identify $\lambda = (\lambda_{i'})_{i \in Q_0}$ with an anti-dominant weight $-\sum_{i \in Q_0} \lambda_{i'} \varpi_i \in P$ (see §2.1). Under this identification and our choice of \tilde{C} , we have an equality

$$\langle (\nu, \lambda), (\alpha_i, 0) \rangle_{\tilde{C}} = \langle \nu + \lambda, \alpha_i \rangle$$

where the right hand side is the standard duality pairing between P and R . We develop properties of stable modules defined as follows:

Definition 3.1. Let $M \in \mathbb{E}\text{-filt}(\tilde{\Pi})$. We say that M is a stable module if $\text{sub}_i M = 0$ for each $i \in Q_0$.

Lemma 3.2. Let $M \in \text{rep}_{\text{l.f.}} \tilde{\Pi}$ be a stable module with $\underline{\text{rank}} M = (\nu, \lambda)$ and $c := (\dim \text{fac}_i M) / c_i$. Then, we have equalities $\dim \text{Hom}_{\tilde{\Pi}}(M, E_i) = cc_i$ and $\dim \text{Ext}_{\tilde{\Pi}}^1(M, E_i) = cc_i - c_i \langle \nu + \lambda, \alpha_i \rangle$. In particular, we have an inequality $c \geq \max\{0, \langle \nu + \lambda, \alpha_i \rangle\}$.

Proof. By Proposition 2.14, we have the following equality:

$$\dim \text{Ext}_{\tilde{\Pi}}^1(M, E_i) = \dim \text{Hom}_{\tilde{\Pi}}(M, E_i) + \dim \text{Hom}_{\tilde{\Pi}}(E_i, M) - c_i \langle \underline{\text{rank}} M, \alpha_i \rangle_{\tilde{C}}.$$

Since M is a stable module, Proposition 2.39 yields our statement by identifying $\langle \underline{\text{rank}} M, \alpha_i \rangle_{\tilde{C}} = \langle \nu + \lambda, \alpha_i \rangle$. \square

By Lemma 3.2, we have the inequality $c \geq \max\{0, \langle \nu + \lambda, \alpha_i \rangle\}$. We define the following:

Definition 3.3 (Small i -factor). Let $M \in \text{rep}_{\text{l.f.}} \tilde{\Pi}$ be stable. We say that M has a *small i -factor* if and only if $\text{rank}_i M = \max\{0, \langle \nu + \lambda, \alpha_i \rangle\}$.

Lemma 3.4. Let $M \in \text{rep}_{\text{l.f.}} \tilde{\Pi}$ be stable. Then, we have the following:

- (1) If M has $\underline{\text{rank}} M = (\nu, \lambda)$ and $\dim \text{fac}_i M = cc_i$, then there exists a stable $\tilde{\Pi}$ -module M' with $\underline{\text{rank}} M' = (\nu + k\alpha_i, \lambda)$ for any $k \in \mathbb{Z} \cap [-c, c - \langle \nu + \lambda, \alpha_i \rangle]$.
- (2) If $\langle \nu + \lambda, \alpha_i \rangle \leq 0$, then we have the following equivalence of categories:

$$\left\{ \begin{array}{l} \text{stable } \tilde{\Pi}\text{-modules with} \\ \text{rank-vector } (\nu, \lambda) \\ \text{with small } i\text{-factor} \end{array} \right\} \begin{array}{c} \xrightarrow{\Sigma_i} \\ \xleftarrow{\Sigma_i^-} \end{array} \left\{ \begin{array}{l} \text{stable } \tilde{\Pi}\text{-modules with} \\ \text{rank-vector } (\nu - \langle \nu + \lambda, \alpha_i \rangle \alpha_i, \lambda) \\ \text{with small } i\text{-factor} \end{array} \right\}.$$

- (3) If there exists some stable module M with $\underline{\text{rank}} M = (\nu, \lambda)$, then $\nu + \lambda \in \bigcap_{w \in W} w(\lambda + Q_+)$.

Proof. For $M \in \text{rep } \widetilde{\Pi}$, we consider this module in terms of species. Namely, we consider the following data based on Proposition 2.17:

$$\begin{aligned} & \left(\bigoplus_{j \in \widetilde{\Omega}(i, -)} {}_i H_j \otimes_j M_j \right) \oplus ({}_i H_{i'} \otimes_{i'} M_{i'}) \xrightarrow{\sum \text{sign}(i, j) M_{ij} - M_{i'}} M_i \\ & \xrightarrow{(M_{ji}^\vee, M_{i'i}^\vee)^\top} \left(\bigoplus_{j \in \widetilde{\Omega}(i, -)} {}_i H_j \otimes_j M_j \right) \oplus ({}_i H_{i'} \otimes_{i'} M_{i'}). \end{aligned}$$

For simplicity, we write this diagram by $\widetilde{M}_i \xrightarrow{M_{i, \text{in}}} M_i \xrightarrow{M_{i, \text{out}}} \widetilde{M}_i$. We remark that M is stable if and only if $M_{i, \text{out}}$ is injective for any i by Proposition 2.39. If we put $u := M_{i, \text{out}} \circ M_{i, \text{in}} \in \text{End}_{\widetilde{\Pi}}(\widetilde{M}_i)$, then we have

$$\text{Im } u \subseteq \text{Im } M_{\text{out}(i)} \subseteq \text{Ker } u$$

and $u(\text{Im } M_{i, \text{out}}) = 0$ by a preprojective relation. Thus, we obtain the following equalities because M is stable:

$$\dim \text{Im } M_{i, \text{out}} - \dim \text{Im } u = \dim \text{Cok } M_{i, \text{in}} = cc_i$$

and

$$\begin{aligned} & \dim \text{Ker } u - \dim \text{Im } M_{i, \text{out}} \\ &= \dim \widetilde{M}_i - \dim \text{Im } u - \dim \text{Im } M_{i, \text{out}} \\ &= \dim \widetilde{M}_i + (\dim \text{Im } M_{i, \text{out}} - \dim \text{Im } u) - 2 \dim \text{Im } M_{i, \text{out}} \\ &= cc_i + \dim \widetilde{M}_i - 2 \dim M_i \\ &= cc_i - c_i \langle \nu + \lambda, \alpha_i \rangle. \end{aligned}$$

By these equalities, we obtain a submodule $V \subseteq \widetilde{M}_i$ of $\text{rank}_i \widetilde{M}_i = k + \text{rank}_i M_i$ such that $\text{Im } u \subseteq V \subseteq \text{Ker } u$. Thus, we can construct a module required in (1) by taking $\widetilde{M}_i \xrightarrow{u} V \xrightarrow{M_{i, \text{out}}} \widetilde{M}_i$ instead of $\widetilde{M}_i \xrightarrow{M_{i, \text{in}}} M_i \xrightarrow{M_{i, \text{out}}} \widetilde{M}_i$.

We prove (2). We find that M has a small i -factor if and only if $c = 0$ or $c - \langle \nu + \lambda, \alpha_i \rangle = 0$ in the Proof of (1). This is equivalent to $\text{Im } M_{i, \text{out}} = \text{Im } u$ or $\text{Ker } u$.

If we assume $c = 0$, then we have $\text{Im } M_{i, \text{out}} = \text{Im } u$ and $\text{Ker } M_{i, \text{in}} = \text{Ker } u$ by injectivity of $M_{i, \text{out}}$. Then, by definition, the functor $\Sigma_i M$ is given by replacing the diagram

$$\widetilde{M}_i \xrightarrow{M_{i, \text{in}}} M_i \xrightarrow{M_{i, \text{out}}} \widetilde{M}_i$$

with

$$\widetilde{M}_i \xrightarrow{\overline{M}_{i, \text{out}} M_{i, \text{in}}} \text{Ker } u \xrightarrow{\text{can}} \widetilde{M}_i.$$

On the other hand, if we assume $c - \langle \nu + \lambda, \alpha_i \rangle = 0$, then we have $\text{Im } M_{i, \text{out}} = \text{Ker } u$ and $\text{Cok } \overline{M}_{i, \text{out}} = \widetilde{M}_i / \text{Ker } u \cong \text{Im } u$. Then, $\Sigma_i^- M$ is given by replacing the diagram

$$\widetilde{M}_i \xrightarrow{M_{i, \text{in}}} M_i \xrightarrow{M_{i, \text{out}}} \widetilde{M}_i$$

with

$$\widetilde{M}_i \xrightarrow{\text{can}} \text{Im } u \xrightarrow{M_{i,\text{out}} \widetilde{M}_{i,\text{in}}} \widetilde{M}_i.$$

This yields the assertion (2) from our equalities of dimensions.

Finally, if we fix λ and allow to vary ν , then (1) implies that the set

$$\{\nu + \lambda \in P \mid \text{there exists a stable module with rank-vector } (\nu, \lambda)\} \subseteq \lambda + Q_+$$

is W -invariant. Therefore, the set is contained in $\bigcap_{w \in W} w(\lambda + Q_+)$. \square

By these lemmas, we have the following theorem as a generalization of [BK12, Theorem 3.4]:

Theorem 3.5. Let $\lambda \in \mathbb{Z}^n$ be an anti-dominant weight and $w \in W$. If M is a stable $\widetilde{\Pi}$ -module with $\underline{\text{rank}} M = (w\lambda - \lambda, \lambda)$, then M has a small i -factor for any $i \in Q_0$. Moreover, there exists a unique stable $\widetilde{\Pi}$ -module $\widetilde{N}(w\lambda)$ with $\underline{\text{rank}} \widetilde{N}(w\lambda) = (w\lambda - \lambda, \lambda)$.

Proof. We prove the existence and the uniqueness later. Let M be a stable module with $\underline{\text{rank}} M = (w\lambda - \lambda, \lambda)$ and $i \in Q_0$. First, we prove that M has a small i -factor for any $i \in Q_0$.

If $l(s_i w) > l(w)$, then there exists a stable $\widetilde{\Pi}$ -module M' with $\underline{\text{rank}} M' = (w\lambda - \lambda - c\alpha_i, \lambda)$ by Lemma 3.4 (1). Now, we have $(w\lambda - \lambda - c\alpha_i) + \lambda = w\lambda - c\alpha_i \in w(\lambda + R^+)$ by Lemma 3.4 (3). Thus, $-cw^{-1}(\alpha_i) \in R^+$. By $l(s_i w) > l(w)$, it is found that $w^{-1}(\alpha_i)$ is a positive root. So, c must be equal to 0. Since λ is anti-dominant, we have $\langle w\lambda, \alpha_i \rangle \leq 0$.

If $l(s_i w) < l(w)$, then there exists a stable module M'' with $\underline{\text{rank}} M'' = (w\lambda - \lambda + (c - \langle w\lambda, \alpha_i \rangle)\alpha_i, \lambda)$ by Lemma 3.4 (1). Thus, $(w\lambda - \lambda + (c - \langle w\lambda, \alpha_i \rangle)\alpha_i) + \lambda = w\lambda + (c - \langle w\lambda, \alpha_i \rangle)\alpha_i \in w(\lambda + R^+)$ by Lemma 3.4 (3). Since $w^{-1}(\alpha_i)$ is a negative root, we have $c - \langle w\lambda, \alpha_i \rangle \leq 0$. On the other hand, we have an inequality $c - \langle w\lambda, \alpha_i \rangle \geq 0$ by the existence of M and Lemma 3.2. Thus, we obtain $c = \langle w\lambda, \alpha_i \rangle$. So, M has a small i -factor.

We prove the existence and the uniqueness by induction on $l(w)$. If $l(w) = 0$, the assertion follows from the fact that any two vertices in Q'_0 are not connected by arrows. Namely, a stable module M with $\underline{\text{rank}} M = (0, \lambda)$ has only generalized simple modules for some vertices in Q'_0 as its direct summands. If $l(w) > 0$, then we can find $i \in Q_0$ such that $l(s_i w) < l(w)$. Then, the following

$$\left\{ \begin{array}{l} \text{stable } \widetilde{\Pi}\text{-modules with} \\ \text{rank-vector } (s_i w\lambda - \lambda, \lambda) \end{array} \right\} \begin{array}{c} \xrightarrow{\Sigma_i} \\ \xleftarrow{\Sigma_i^-} \end{array} \left\{ \begin{array}{l} \text{stable } \widetilde{\Pi}\text{-modules with} \\ \text{rank-vector } (w\lambda - \lambda, \lambda) \end{array} \right\}$$

gives an equivalence of categories by our first half argument and Lemma 3.4 (2). Thus, the assertion for w is equivalent to that for $s_i w$. So, we obtain the assertion for any w by induction. \square

By the above lemmas, we have a description of these $\widetilde{N}(\lambda)$ under the actions of reflection functors and dimension equalities as follows:

Corollary 3.6. Let γ be a weight and $i \in Q_0$.

(1) If $\langle \gamma, \alpha_i \rangle \leq 0$, then we have

$$\Sigma_i \widetilde{N}(\gamma) \cong \widetilde{N}(s_i \gamma); \quad \Sigma_i \widetilde{N}(s_i \gamma) \cong \widetilde{N}(s_i \gamma); \quad \Sigma_i^- \widetilde{N}(s_i \gamma) \cong \widetilde{N}(\gamma).$$

(2) We have the following equalities:

$$\begin{aligned}\dim \operatorname{Hom}_{\tilde{\Pi}}(\tilde{N}(\gamma), E_i) &= \max\{0, c_i\langle \gamma, \alpha_i \rangle\}; \\ \dim \operatorname{Ext}_{\tilde{\Pi}}^1(\tilde{N}(\gamma), E_i) &= \max\{0, -c_i\langle \gamma, \alpha_i \rangle\}.\end{aligned}$$

Proof. (1) Let $w \in W$ such that $l(s_i w) > l(w)$ and let $\gamma = w\lambda$ with λ anti-dominant. The first isomorphism follows from Theorem 3.5. The second isomorphism follows from

$$\Sigma_i \tilde{N}(s_i \gamma) \cong \Sigma_i^2 \tilde{N}(\gamma) \cong \Sigma_i \tilde{N}(\gamma) \cong \tilde{N}(s_i \gamma) \quad (\text{cf. Proposition 2.38}).$$

Since $\tilde{N}(\gamma)$ has a trivial i -factor, Theorem 2.38 yields

$$\Sigma_i^- \tilde{N}(s_i \gamma) \cong \Sigma_i^- \Sigma_i \tilde{N}(\gamma) \cong \tilde{N}(\gamma).$$

(2) These equalities follow from Lemma 3.2 and Theorem 3.5. \square

By examining the rank-vectors, we find that $\tilde{N}(\lambda)$ is contained in $\tilde{N}(w\lambda)$ as a submodule, and the canonical inclusion $\tilde{N}(\lambda) \hookrightarrow \tilde{N}(w\lambda)$ is an isomorphism at vertices in Q'_0 . We define a new $\tilde{\Pi}$ -module $N(w\lambda)$ from $\tilde{N}(w\lambda)$ by $N(w\lambda) := \tilde{N}(w\lambda)/\tilde{N}(\lambda)$. Since $\operatorname{rep}_{\text{l.f.}} \tilde{\Pi}$ is closed under taking cokernels of epimorphisms, $N(w\lambda)$ is a locally free $\tilde{\Pi}$ -module. This $N(w\lambda)$ has the natural Π -module structure because $N(w\lambda) \in \operatorname{rep} \tilde{\Pi}$ has dimension 0 on any Q'_0 component. Thus, $N(w\lambda)$ is an \mathbb{E} -filtered module by our construction. The following theorem which is a generalization of [BK12, Theorem 3.1] plays an important role in order to realize MV polytopes:

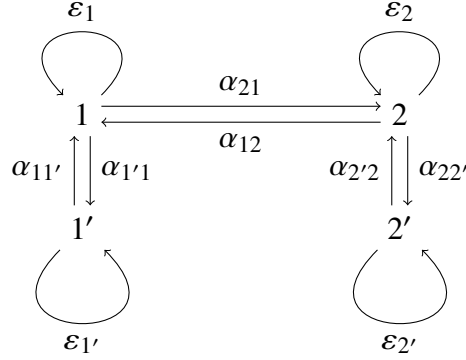
Theorem 3.7. Let ϖ_i be the fundamental weight with respect to $i \in Q_0$ and $w \in W$.

- (1) If $\gamma \in \mathbb{Z}^n$ is an anti-dominant weight, then we have $N(\gamma) = 0$.
- (2) If $\varpi_i - w\varpi_i \neq 0$, then there exists a unique Π -module $N(-w\varpi_i) \in \mathbb{E}\text{-filt}(\Pi)$ such that $\underline{\operatorname{rank}} N(-w\varpi_i) = \varpi_i - w\varpi_i$ and $\operatorname{sub} N(-w\varpi_i) = E_i$. Conversely, if a module $M \in \operatorname{rep} \Pi$ satisfies $\underline{\operatorname{rank}} M = \varpi_i - w\varpi_i$ and $\operatorname{sub} M = E_i$, then we have $M \cong N(-w\varpi_i)$.
- (3) If γ and δ belong to the same Weyl chamber, then we have $N(\gamma + \delta) \cong N(\gamma) \oplus N(\delta)$.

Proof. (1) Since $\underline{\operatorname{rank}} \tilde{N}(\gamma) = (0, \gamma)$, the assertion is clear.

- (2) Let $w \in W$ and $i \in Q_0$. The module $\tilde{N}(-w\varpi_i)$ is stable with $\underline{\operatorname{rank}} \tilde{N}(-w\varpi_i) = (\varpi_i - w\varpi_i, -\varpi_i)$. Now, a graded vector space $\bigoplus_{j \in Q'_0} \tilde{N}(-w\varpi_i)_j$ coincides with $\tilde{N}(-\varpi_i) \cong E_i$. On the other hand, $N(-w\varpi_i)$ has $\underline{\operatorname{rank}} N(-w\varpi_i) = \varpi_i - w\varpi_i$. For each $j \in Q_0$, we have $\operatorname{sub}_j \tilde{N}(-w\varpi_i) = \operatorname{Ker} M_{j', j} \cap (\operatorname{sub}_j N(-w\varpi_i))$. Since $\tilde{N}(-w\varpi_i)$ is stable, this equality yields that the inequality $\operatorname{rank}_i \operatorname{sub}_i N(-w\varpi_i) \leq 1$ and $\operatorname{sub}_j N(-w\varpi_i) = 0$ for $i \neq j$. Since $N(-w\varpi_i) \neq 0$, we obtain $\operatorname{sub} N(-w\varpi_i) \cong E_i$.

Conversely, we assume that $M \in \operatorname{rep} \Pi$ satisfies $\underline{\operatorname{rank}} M = \varpi_i - w\varpi_i$ and $\operatorname{sub} M = E_i$. Now, we can construct a stable module $N \in \operatorname{rep} \tilde{\Pi}$ such that $\underline{\operatorname{rank}} N = (\varpi_i - w\varpi_i, -\varpi_i)$.

FIGURE 1. The framed quiver of type B_2

In fact, we take

$$\begin{aligned}
 N_j &= M_j & j \in Q_0 \\
 N_h &= M_h & h \in \overline{Q}_1 \\
 N_{i'} &= H_{i'}^{\oplus r} & (r := \text{rank}_i(\text{sub}_i M)) \\
 N_{i'} &: {}_{i'}H_i \otimes_i M_i \rightarrow N_{i'}
 \end{aligned}$$

where $N_{i'}$ is any $H_{i'}$ -isomorphism and the other arrows and spaces are zero. By the uniqueness of stable modules, we deduce $N \cong \tilde{N}(-w\varpi_i)$. Thus, we obtain $M \cong N(-w\varpi_i)$.

- (3) By the uniqueness of stable modules, we have an isomorphism $\tilde{N}(w\lambda) \oplus \tilde{N}(w\mu) \cong \tilde{N}(w(\lambda + \mu))$. Then, we take quotients to obtain $N(w\lambda) \oplus N(w\mu) \cong N(w(\lambda + \mu))$. \square

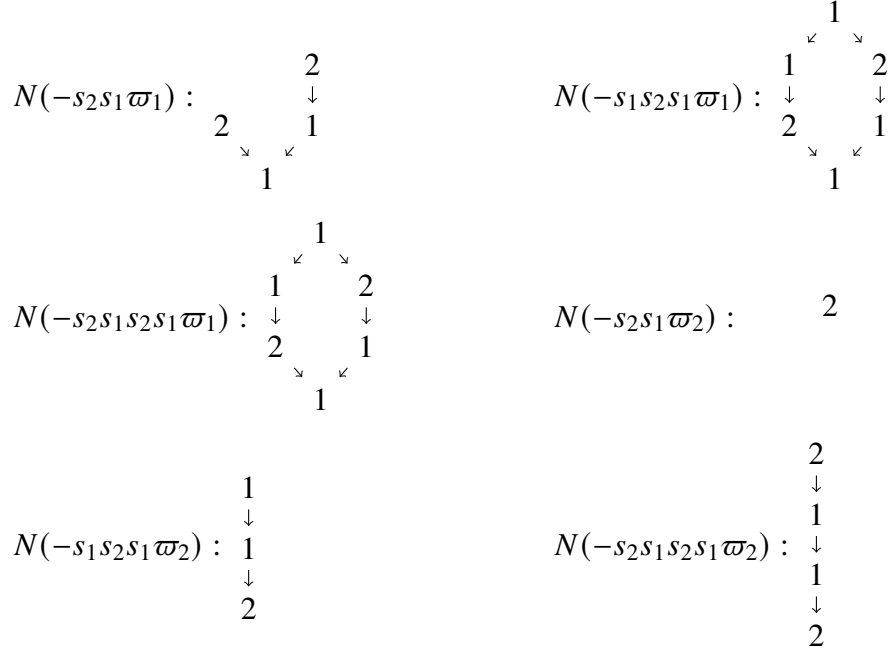
Example 3.8. Let $C = \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}$, $D = \text{diag}(2, 1)$. The double of the framed quiver \tilde{Q} is illustrated in Figure 1. The $\tilde{\Pi}$ -module $\tilde{N}(-\varpi_1)$ (*resp.* $\tilde{N}(-\varpi_2)$) is isomorphic to a generalized simple module $E_{1'}$ (*resp.* $E_{2'}$). The Π -module $N(-s_1\varpi_1)$ is isomorphic to a generalized simple module E_1 . The Π -modules $N(-s_2s_1\varpi_1)$, $N(-s_1s_2s_1\varpi_1)$ and $N(-s_2s_1s_2s_1\varpi_1)$ are respectively illustrated as Figure 2. In particular, we have an isomorphism $N(-s_1s_2s_1\varpi_1) \cong N(-s_2s_1s_2s_1\varpi_1)$. The Π -modules $N(-s_2s_1\varpi_2)$, $N(-s_1s_2s_1\varpi_2)$ and $N(-s_2s_1s_2s_1\varpi_2)$ are also illustrated in Figure 2.

If C is of finite type, we have the following:

Theorem 3.9. The Π -module $N(\varpi_i)$ is isomorphic to the projective cover of a simple module S_i for each $i \in Q_0$.

Proof. Note that the weight $w_0\varpi_i$ is anti-dominant. We have the canonical inclusion $\tilde{N}(w_0\varpi_i) \hookrightarrow \tilde{N}(\varpi_i)$ in $\text{rep } \tilde{\Pi}$. In particular, we have an exact sequence

$$0 \rightarrow \tilde{N}(w_0\varpi_i) \rightarrow \tilde{N}(\varpi_i) \rightarrow N(\varpi_i) \rightarrow 0.$$

FIGURE 2. Stable modules for type B_2

Applying the functor $\text{Hom}_{\tilde{\Pi}}(-, E_j)$, we obtain

$$\text{Hom}_{\tilde{\Pi}}(\tilde{N}(w_0\varpi_i), E_j) \rightarrow \text{Ext}_{\tilde{\Pi}}^1(N(\varpi_i), E_j) \rightarrow \text{Ext}_{\tilde{\Pi}}^1(\tilde{N}(\varpi_i), E_j). \quad (3.1)$$

Here, both of two extremal terms in (3.1) are equal to 0 by Corollary 3.6. So, we obtain $\text{Ext}_{\tilde{\Pi}}^1(N(\varpi_i), E_j) = \text{Ext}_{\tilde{\Pi}}^1(\tilde{N}(\varpi_i), E_j) = 0$. We remind that E_j is a uniserial module. Namely, E_j has a unique filtration $0 = S_j^{(0)} \subsetneq S_j^{(1)} \subsetneq \cdots \subsetneq S_j^{(c_j)} = E_j$ of Π -modules. We have the canonical short exact sequence

$$0 \rightarrow S_j^{(c_j-1)} \hookrightarrow E_j \rightarrow S_j \rightarrow 0.$$

Then, we apply the functor $\text{Hom}_{\tilde{\Pi}}(N(\varpi_i), -)$ and obtain an exact sequence

$$\text{Hom}_{\tilde{\Pi}}(N(\varpi_i), S_j) \rightarrow \text{Ext}_{\tilde{\Pi}}^1(N(\varpi_i), S_j^{(c_j-1)}) \rightarrow \text{Ext}_{\tilde{\Pi}}^1(N(\varpi_i), E_j). \quad (3.2)$$

In the case of $i \neq j$, both of two extremal terms of (3.2) are equal to 0 by the above argument and the uniseriality of E_j . In fact, we have exact sequences for $X \in \text{rep } \Pi$:

$$0 \rightarrow \text{Hom}_{\Pi}(X, S_j^{(c_j-1)}) \rightarrow \text{Hom}_{\Pi}(X, E_j) \rightarrow \text{Hom}_{\Pi}(X, S_j)$$

and

$$0 \rightarrow \text{Hom}_{\Pi}(X, S_j^{(k)}) \rightarrow \text{Hom}_{\Pi}(X, S_j^{(k+1)}) \rightarrow \text{Hom}_{\Pi}(X, S_j)$$

for $1 \leq k \leq c_j - 1$. Thus, an induction on k and $S_j^{(1)} \cong S_j$ gives $\text{Hom}_{\Pi}(X, S_j) = 0$ if $\text{Hom}_{\Pi}(X, E_j) = 0$. By a similar argument, we obtain $\text{Ext}_{\tilde{\Pi}}^1(N(\varpi_i), S_j) = 0$.

In the case of $i = j$, the middle morphism of

$$\begin{aligned} \mathrm{Hom}_{\Pi}(N(\varpi_i), E_i) &\rightarrow \mathrm{Hom}_{\Pi}(N(\varpi_i), S_i) \\ &\rightarrow \mathrm{Ext}_{\Pi}^1(N(\varpi_i), S_i^{(c_i-1)}) \rightarrow \mathrm{Ext}_{\Pi}^1(N(\varpi_i), E_i) = 0 \end{aligned}$$

is surjective. Since we have $\dim \mathrm{Hom}_{\Pi}(N(\varpi_i), S_i) = 1$, it is found that $\mathrm{Ext}_{\Pi}^1(N(\varpi_i), S_i^{(c_i-1)}) = 0$. So, we obtain $\mathrm{Ext}_{\Pi}^1(N(\varpi_i), S_i) = 0$ by induction. Thus, $N(\varpi_i)$ is a projective cover of a simple module. On the other hand, we have an equality

$$\begin{aligned} \dim \mathrm{Hom}_{\Pi}(N(\varpi_i), S_j) &= \dim \mathrm{Hom}_{\tilde{\Pi}}(N(\varpi_i), S_j) \\ &= \left(\dim \mathrm{Hom}_{\tilde{\Pi}}(\tilde{N}(\varpi_i), E_j) \right) \Big/ c_i = \langle \alpha_j, \varpi_i \rangle. \end{aligned}$$

So, we obtain our assertion. \square

We prepare the following modules and develop properties of them:

Definition 3.10. Let $w \in W$ and $w = s_{i_1} \cdots s_{i_l}$ be a reduced expression. We record this reduced expression by $\mathbf{i} := (i_1, \dots, i_l)$. We define $V_{\mathbf{i},k} := N(-s_{i_1} \cdots s_{i_k} \varpi_{i_k})$ and $M_{\mathbf{i},k} := V_{\mathbf{i},k}/V_{\mathbf{i},k^-}$ for $1 \leq k \leq l$ where $k^- = \max\{0, 1 \leq s \leq k-1 \mid i_s = i_k\}$.

These kinds of modules have been studied in the works of Geiss-Leclerc-Schröer [GLS11] and Buan-Iyama-Reiten-Scott [BIRS09] for classical cases. We adopt the argument in the work of Baumann-Kamnitzer-Tingley [BKT14, Example 5.12, 5.14] and obtain the following theorem about these modules:

Theorem 3.11. In the setting of Definition 3.10, we have isomorphisms

$$V_{\mathbf{i},k} \cong ((\Pi/I_{s_{i_1} \cdots s_{i_k}}) \otimes_{\Pi} \Pi e_{i_k})^*$$

and

$$M_{\mathbf{i},k} \cong (I_{s_{i_1} \cdots s_{i_{k-1}}} \otimes_{\Pi} E_{i_k})^* \cong (I_{s_{i_1} \cdots s_{i_{k-1}}}/I_{s_{i_1} \cdots s_{i_k}})^*.$$

Proof. First, we prove that $M_{\mathbf{i},k} \cong (I_{s_{i_1} \cdots s_{i_{k-1}}} \otimes_{\Pi} E_{i_k})^*$. Since $\tilde{N}(-s_{i_k} \varpi_{i_k}) \cong \Sigma_{i_k} \tilde{N}(-\varpi_{i_k})$ and $\tilde{N}(-\varpi_{i_k}) = E_{i_k}'$, we have the following exact sequence (3.3) in $\mathrm{rep} \tilde{\Pi}$:

$$0 \rightarrow \tilde{N}(-\varpi_{i_k}) \rightarrow \tilde{N}(-s_{i_k} \varpi_{i_k}) \rightarrow E_{i_k} \rightarrow 0. \quad (3.3)$$

Since these three modules appearing in the sequence (3.3) have trivial i_{k-1} -factors, the functor $\Sigma_{i_{k-1}}$ is exact on the above short exact sequence by Theorem 2.38:

$$0 \rightarrow \tilde{N}(-s_{i_{k-1}} \varpi_{i_k}) \rightarrow \tilde{N}(-s_{i_{k-1}} s_{i_k} \varpi_{i_k}) \rightarrow \Sigma_{i_{k-1}} E_{i_k} \rightarrow 0.$$

Since $\tilde{N}(-s_{i_{k-1}} \varpi_{i_k})$ and $\tilde{N}(-s_{i_{k-1}} s_{i_k} \varpi_{i_k})$ have trivial i_{k-2} -factors by Corollary 3.6 (2), so does $\Sigma_{i_{k-1}} E_{i_k}$. Since \mathbf{i} is a reduced expression, we can repeat these procedures by Corollary 3.6 (2). We obtain

$$0 \rightarrow \tilde{N}(-s_{i_1} \cdots s_{i_{k-1}} \varpi_{i_k}) \rightarrow \tilde{N}(-s_{i_1} \cdots s_{i_k} \varpi_{i_k}) \rightarrow \Sigma_{i_1} \cdots \Sigma_{i_{k-1}} E_{i_k} \rightarrow 0.$$

Now, $\tilde{N}(-s_{i_r} \varpi_{i_k}) = \Sigma_{i_r} \tilde{N}(-\varpi_{i_k}) \cong \tilde{N}(-\varpi_{i_k})$ for any $k^- < r < k$. So, we obtain

$$0 \rightarrow \tilde{N}(-s_{i_1} \cdots s_{i_{k^-}} \varpi_{i_k}) \rightarrow \tilde{N}(-s_{i_1} \cdots s_{i_k} \varpi_{i_k}) \rightarrow \Sigma_{i_1} \cdots \Sigma_{i_{k-1}} E_{i_k} \rightarrow 0.$$

The map $\widetilde{N}(-s_{i_1} \cdots s_{i_k} \varpi_{i_k}) \rightarrow \Sigma_{i_1} \cdots \Sigma_{i_{k-1}} E_{i_k}$ sends the submodule $\widetilde{N}(-\varpi_{i_k})$ to 0. So, we have the following short exact sequence in $\text{rep } \Pi$:

$$0 \rightarrow V_{\mathbf{i},k^-} \rightarrow V_{\mathbf{i},k} \rightarrow \Sigma_{i_1} \cdots \Sigma_{i_{k-1}} E_{i_k} \rightarrow 0.$$

So, we obtain

$$\begin{aligned} M_{\mathbf{i},k} &= V_{\mathbf{i},k} / V_{\mathbf{i},k^-} \\ &\cong \Sigma_{i_1} \cdots \Sigma_{i_{k-1}} E_{i_k} \\ &\cong (\Sigma_{i_1}^- \cdots \Sigma_{i_{k-1}}^- E_{i_k})^* \quad (E_i^* \cong E_i) \\ &\cong (I_{s_{i_1} \cdots s_{i_{k-1}}} \otimes_{\Pi} E_{i_k})^* \quad (\text{Theorem 2.38}). \end{aligned}$$

Then, we prove that $V_{\mathbf{i},k} \cong N(-s_{i_1} \cdots s_{i_k} \varpi_{i_k}) \cong ((\Pi / I_{s_{i_1} \cdots s_{i_k}}) \otimes_{\Pi} \Pi e_{i_k})^*$. We pay attention to that $\mathbb{B}\text{-filt}(\Pi)$ is closed under taking quotients by submodules in $\mathbb{B}\text{-filt}(\Pi)$. For a reduced expression $w = s_{i_1} \cdots s_{i_l}$, we have a descending chain of ideals $\Pi \supseteq I_{s_{i_1}} \supseteq \cdots \supseteq I_{s_{i_1} \cdots s_{i_l}}$. This filtration induces the filtration $\Pi / I_w \supseteq I_{s_{i_1}} / I_w \supseteq \cdots \supseteq I_{s_{i_1} \cdots s_{i_l}} / I_w = 0$. The k -th successive quotient in this filtration is isomorphic to $I_{s_{i_1} \cdots s_{i_{k-1}}} / I_{s_{i_1} \cdots s_{i_k}} \cong I_{s_{i_1} \cdots s_{i_{k-1}}} \otimes_{\Pi} (\Pi / I_{i_k}) \cong I_{s_{i_1} \cdots s_{i_{k-1}}} \otimes_{\Pi} E_{i_k}$. By applying the exact functor $-\otimes_{\Pi} \Pi e_i$, all of the successive quotients $I_{s_{i_1} \cdots s_{i_{k-1}}} / I_{s_{i_1} \cdots s_{i_k}}$ in this filtration such that $i_k \neq i$ are killed by Theorem 2.32. Thus, we obtain

$$\begin{aligned} \underline{\text{rank}}((\Pi / I_w) \otimes_{\Pi} \Pi e_i) &= \sum_{i_k=i} I_{s_{i_1} \cdots s_{i_{k-1}}} \otimes_{\Pi} E_{i_k} \\ &= \sum_{i_k=i} s_{i_1} \cdots s_{i_{k-1}} \alpha_{i_k} \quad (\text{Proposition 2.45}) \\ &= \varpi_i - w \varpi_i \end{aligned}$$

by a bit of calculation. Now, we have $\text{fac } \Pi e_i \cong E_i$, so that $\text{fac}((\Pi / I_w) \otimes_{\Pi} \Pi e_i)$ is isomorphic to 0 or E_i . So, the module $((\Pi / I_{s_{i_1} \cdots s_{i_k}}) \otimes_{\Pi} \Pi e_{i_k})^*$ must coincide with $N(-s_{i_1} \cdots s_{i_k} \varpi_{i_k})$ by Theorem 3.7. \square

3.2. g -vectors and wall-chamber structures. In this subsection, we develop g -vectors for finite dimensional generalized preprojective algebras in order to realize MV polytopes. Our discussion is an analogue of the work of Mizuno [Miz14]. In this subsection, Π means a finite dimensional generalized preprojective algebra, and $\widehat{\Pi}$ means the completed generalized preprojective algebra (cf. [GLS17, Proposition 12.3]) of un-twisted affine type associated with Π . Here, this completion is with respect to the two-sided ideal generated by $\bigoplus_{(i,j) \in \overline{Q}} i H_j$. In particular, $\widehat{\Pi}$ is semi-perfect, and $\text{proj } \widehat{\Pi}$ is a Krull-Schmidt category. In this subsection, we identify the standard basis of $K_0(\text{proj } \Pi) \cong \mathbb{Z}^n$ with $\{[\Pi e_i] \mid i \in Q_0\}$.

Definition 3.12. Let $M \in \text{rep } \Pi$ and $P_1(M) \rightarrow P_0(M) \rightarrow M \rightarrow 0$ be the minimal projective presentation of M . Then, we define the g -vector of M by

$$g(M) := (g_1(M), \dots, g_n(M))^{\top} = [P_0(M)] - [P_1(M)] \in K_0(\text{proj } \Pi).$$

A significance of g -vectors and τ -tilting pairs is characterized by the following theorem:

Proposition 3.13 ([AIR14, Theorem 5.1]). Let (M, P) be a support τ -tilting pair for a finite dimensional K -algebra Λ with $M = \bigoplus_{i=1}^l M_i$ and $P = \bigoplus_{i=1}^n P_i$ with M_i and P_i indecomposable. Then, $g(M_1), \dots, g(M_l), g(P_{l+1}), \dots, g(P_n)$ form a basis of the Grothendieck group $K_0(\text{proj } \Lambda)$.

Let $\widehat{I}_w \in \text{Rep } \widehat{\Pi}$ denote the tilting ideal for $w \in \widehat{W}$ in Theorem 2.28. This module induces an automorphism on the bounded homotopy category $\mathcal{K}^b(\text{proj } \widehat{\Pi})$. In the work of Fu-Geng [FG19], they considered the action of affine Weyl group $\widehat{W} = W(\widehat{C})$ on the Grothendieck group of the bounded homotopy category $K_0(\mathcal{K}^b(\text{proj } \widehat{\Pi})) \cong K_0(\text{proj } \widehat{\Pi}) \cong \mathbb{Z}^{n+1}$ by using tilting complexes.

Proposition 3.14 ([FG19, Proof of Theorem 4.7]). Let $\widehat{C} = (c_{ij})$. We have a faithful group action $\sigma^*: \widehat{W} \rightarrow \text{GL}_{n+1}(\mathbb{Z}^{n+1})$ defined by

$$s_i \mapsto [\widehat{I}_i \otimes_{\widehat{\Pi}}^{\mathbb{L}} -] \simeq K_0(\text{proj } \widehat{\Pi})$$

$$[\widehat{\Pi}e_j] \mapsto \begin{cases} [\widehat{\Pi}e_i] - \sum_{k \in \widehat{Q}_0} c_{kj} [\widehat{\Pi}e_k] & (i = j) \\ [\widehat{\Pi}e_j] & (i \neq j). \end{cases}$$

We introduce g -matrices for generalized preprojective algebras in view of above Proposition 3.13 and Proposition 3.14.

Definition 3.15. Let $\widehat{I}_w \in \widehat{\Pi}$ be the tilting ideal corresponding to $w \in \widehat{W}$ and let σ be the Nakayama permutation of Π from the self-injectivity of Π .

- (1) We define the g -matrix attached to $w \in \widehat{W}$ by $\hat{g}(w) = (g(\widehat{I}_w e_0), \dots, g(\widehat{I}_w e_n)) \in M_{n+1}(\mathbb{Z})$.
- (2) We define the g -matrix attached to $w \in W$ by $g(w) = (g_1, \dots, g_n) \in M_n(\mathbb{Z})$, where $g_i := g(I_w e_i) - (0, \dots, \delta_{I_w e_i, 0}, \dots, 0)$. (Here, $\delta_{I_w e_i, 0}$ is on the $\sigma(i)$ -th coordinate.)

Namely, $g(w)$ is the matrix whose column vectors correspond to g -vectors of indecomposable direct summands of I_w and P_w for the τ -tilting pair (I_w, P_w) . The aim of this subsection is to compare $\hat{g}(w)$ with $g(w)$ for $w \in W \subset \widehat{W}$, and calculate $g(w)$. Recall that $\text{proj.dim}_{\widehat{\Pi}} \widehat{I}_w \leq 1$.

Theorem 3.16. Let $w \in W$ and $0 \rightarrow P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} \widehat{I}_w e_i \rightarrow 0$ be the minimal projective resolution for $\widehat{I}_w e_i$. Applying $\overline{(-)} := \Pi \otimes_{\widehat{\Pi}} -$ to this sequence, we obtain an exact sequence

$$0 \rightarrow \nu^{-1}(\Pi e_i / I_w e_i) \rightarrow \overline{P_1} \xrightarrow{\overline{f_1}} \overline{P_0} \xrightarrow{\overline{f_0}} I_w e_i \rightarrow 0,$$

where ν is the Nakayama functor for $\text{rep } \Pi$, and the sequence

$$\overline{P_1} \xrightarrow{\overline{f_1}} \overline{P_0} \xrightarrow{\overline{f_0}} I_w e_i \rightarrow 0$$

is the minimal projective presentation of $I_w e_i \in \text{rep } \Pi$ if $I_w e_i \neq 0$.

Proof. First, we remind that $\Pi \cong \widehat{\Pi} / \langle e_0 \rangle$ and $\Pi \otimes_{\widehat{\Pi}} \widehat{I}_w e_i \cong (\widehat{I}_w e_i) / (\langle e_0 \rangle \widehat{I}_w e_i) \cong (\widehat{I}_w e_i) / (\langle e_0 \rangle e_i) \cong I_w e_i$, where we use $\widehat{I}_w \supset \langle e_0 \rangle$. We apply the tensor functor $\Pi \otimes_{\widehat{\Pi}} (-)$ to the minimal projective resolution of $\widehat{I}_w e_i$. Then, we obtain the following exact sequence in $\text{rep } \widehat{\Pi}$:

$$0 \rightarrow \text{Tor}_1^{\widehat{\Pi}}(\Pi, I_w e_i) \rightarrow \overline{P_1} \xrightarrow{\overline{f_1}} \overline{P_0} \xrightarrow{\overline{f_0}} I_w e_i \rightarrow 0.$$

Since \bar{f}_0 is minimal by construction, we have enough to show that $\text{Tor}_1^{\widehat{\Pi}}(\Pi, I_w e_i)$ is not projective. Since any $\widehat{\Pi}$ -module with finite projective dimension is locally free and $\text{proj.dim}_{\widehat{\Pi}} \widehat{I}_w \leq 1$ by Proposition 2.13 and Theorem 2.28, we have the following isomorphisms:

$$\begin{aligned}
\text{Tor}_1^{\widehat{\Pi}}(\Pi, I_w e_i) &\cong \mathbb{D} \text{Ext}_{\widehat{\Pi}}^1(I_w e_i, \mathbb{D}\Pi) && \text{([ASS06, Appendix Proposition 4.11])} \\
&= \mathbb{D} \text{Ext}_{\widehat{\Pi}}^2(\widehat{\Pi} e_i / \widehat{I}_w e_i, \mathbb{D}\Pi) \\
&= \text{Hom}_{\widehat{\Pi}}(\mathbb{D}\Pi, \widehat{\Pi} e_i / \widehat{I}_w e_i) && \text{(Proposition 2.14 and locally-freeness)} \\
&= \text{Hom}_{\Pi}(\mathbb{D}\Pi, \Pi e_i / I_w e_i) \\
&= \nu^{-1}(\Pi e_i / I_w e_i). && \text{(Definition of Nakayama functor)}
\end{aligned}$$

Note that we have an exact sequence

$$0 \rightarrow I_w e_i \rightarrow \Pi e_i \rightarrow \Pi e_i / I_w e_i \rightarrow 0,$$

and we apply the functor $\text{Hom}_{\widehat{\Pi}}(-, \mathbb{D}\Pi)$ to obtain

$$\text{Ext}_{\widehat{\Pi}}^1(\widehat{\Pi} e_i, \mathbb{D}\Pi) \rightarrow \text{Ext}_{\widehat{\Pi}}^1(\widehat{I}_w e_i, \mathbb{D}\Pi) \rightarrow \text{Ext}_{\widehat{\Pi}}^2(\widehat{\Pi} e_i / \widehat{I}_w e_i, \mathbb{D}\Pi) \rightarrow \text{Ext}_{\widehat{\Pi}}^2(\widehat{\Pi} e_i, \mathbb{D}\Pi). \quad (3.4)$$

Since both of two extremal terms in the sequence (3.4) are equal to 0, we have $\text{Ext}_{\widehat{\Pi}}^1(\widehat{I}_w e_i, \mathbb{D}\Pi) \cong \text{Ext}_{\widehat{\Pi}}^2(\widehat{\Pi} e_i / \widehat{I}_w e_i, \mathbb{D}\Pi)$. Since Π is a self-injective algebra and $I_w e_i \neq 0$, it follows that $\Pi e_i / I_w e_i$ is not a projective indecomposable object. In particular, $\nu^{-1}(\Pi e_i / I_w e_i)$ is not projective. So, we obtain our assertion. \square

We obtain the following Corollary 3.17 by the similar argument as in [Miz14, Proposition 3.4].

Corollary 3.17. Let $w \in W$. The matrix $g(w)$ is obtained from $\hat{g}(w)$ by forgetting the 0-th row and the 0-th column.

Proof. We take the minimal projective presentation of $\widehat{I}_w e_i \in \text{Rep } \widehat{\Pi}$:

$$0 \xrightarrow{f_1} P_1 \xrightarrow{f_0} P_0 \rightarrow \widehat{I}_w e_i \rightarrow 0.$$

We decompose $P_0 = P'_0 \oplus (\widehat{\Pi} e_0)^{\oplus l}$ and $P_1 = P'_1 \oplus (\widehat{\Pi} e_0)^{\oplus m}$, where P'_0 and P'_1 do not belong to $\text{add } \widehat{\Pi} e_0$. For the i -th column $\hat{g}^i(w) = \left(g_0(\widehat{I}_w e_i), \dots, g_n(\widehat{I}_w e_i) \right)^{\top}$ of the g -matrix $\hat{g}(w) = \left(g(\widehat{I}_w e_0), \dots, g(\widehat{I}_w e_n) \right)$, we find that $\left(g_1(\widehat{I}_w e_i), \dots, g_n(\widehat{I}_w e_i) \right)$ depends only upon P'_0 and P'_1 by the definition of g -vectors. So, we have enough to show that it coincides with the i -th column of the g -matrix $g(w) = (g(I_w e_1), \dots, g(I_w e_n))$. Now, we have the following exact sequence:

$$0 \rightarrow \nu^{-1}(\Pi e_i / I_w e_i) \rightarrow \overline{P}_1 \xrightarrow{\bar{f}_1} \overline{P}_0 \xrightarrow{\bar{f}_0} I_w e_i \rightarrow 0.$$

If we have the equality $P'_0 \neq 0$, then our assertion follows from Theorem 3.16. On the other hand, if we have $P'_0 = 0$, then it holds that $\overline{P}_1 \cong \nu^{-1}(\Pi e_i) \cong \Pi e_{\sigma(i)}$, and so we have $P'_1 \cong \widehat{\Pi} e_{\sigma(i)}$. So, we obtain our assertion by the definition of $g(w)$. \square

Under these preparations, we can realize Tits cones by using g -matrices.

Theorem 3.18. We have an equality $g(w) = \sigma^*(w)$ for each $w \in W$.

Proof. We can obtain our assertion by combining our propositions and the fact [FG19, Proposition 3.5] that the minimal projective resolution of $\widehat{I}_i e_i$ is

$$0 \rightarrow \widehat{\Pi} e_i \rightarrow \bigoplus_{j \in \overline{\Omega}(i, -)} (\widehat{\Pi} e_j)^{|c_{ji}|} \rightarrow \widehat{I}_i e_i \rightarrow 0.$$

Note that $\widehat{I}_i e_j = \widehat{\Pi} e_j$ if $i \neq j$ by Corollary 3.17. Thus, we can prove our assertion by induction on length of $w \in W$. \square

Note that the Nakayama permutation for Π is characterized by

$$w_0(\varpi_i) = -\varpi_{\sigma(i)}$$

as a direct consequence of Theorem 3.18 ¹. In particular, the Nakayama permutation for the generalized preprojective algebra coincides with the trivial permutation if we consider one of cases of B, C, D_{2n} , E_7 , E_8 , F_4 , G_2 . Thus, we realize Weyl chambers in terms of g -matrices. In particular, we obtain the following by the similar argument as in [Miz14, Theorem 3.9].

Theorem 3.19. We define the cone $C(w)$ associated with $w \in W$ by $\{a_1 g_1(w) + \cdots + a_n g_n(w) \mid a_i \in \mathbb{R}_{>0}\}$. Then, this cone is identified with a Weyl chamber in the §2.1 by our choice of basis of \mathbb{Z}^n .

Proof. We obtain the assertion by computing

$$\begin{aligned} C(w) &= \{a_1 g_1(w) + \cdots + a_n g_n(w) \mid a_i \in \mathbb{R}_{>0}\} \\ &= \{a_1 \sigma^*(w)(\varpi_1) + \cdots + a_n \sigma^*(w)(\varpi_n) \mid a_i \in \mathbb{R}_{>0}\} \\ &= \sigma^*(w)(C_0). \end{aligned}$$

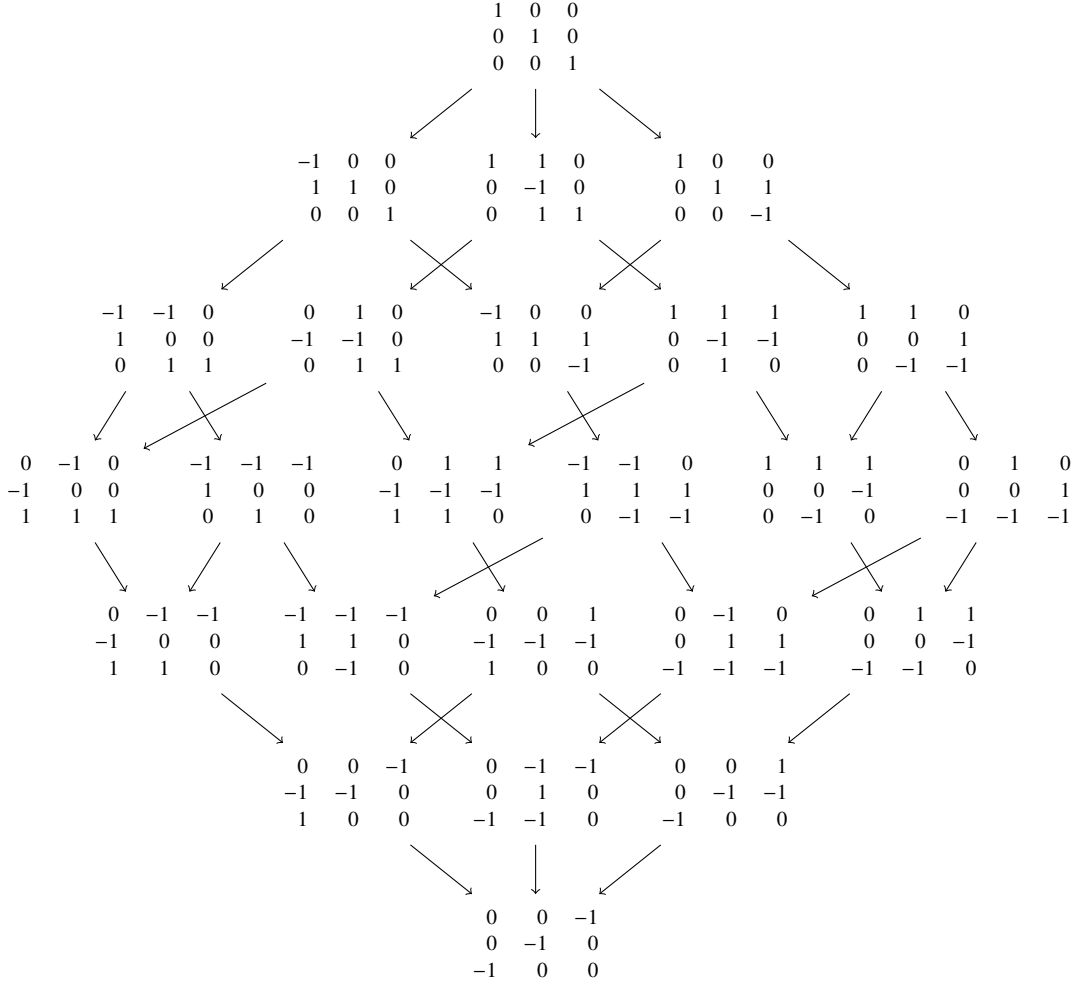
Thus, we obtain the realization. \square

Example 3.20. (1) Let Π be the generalized preprojective algebra of type A_3 with any symmetrizer. g -matrices for Π are given by the lattice corresponding to mutations of support τ -tilting modules as Figure 3. (The same example for the classical case has already appeared in the work of Mizuno [Miz14, Example 3.10 (b)]):

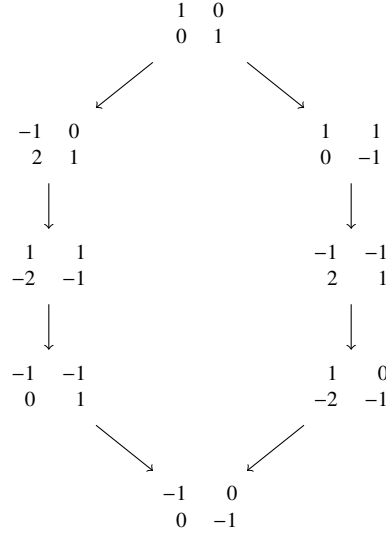
(2) Let Π be the generalized preprojective algebra of type B_2 with any symmetrizer. g -matrices for Π are given by the lattice corresponding to mutations of support τ -tilting modules as Figure 4.

Remark 3.21. By definition, any $V_{i,k}$ is an indecomposable τ -rigid module. For A, D, E and minimal symmetrizer cases, the set of isoclasses of layer modules coincides with the set of isoclasses of bricks. In general, there exists a set $\{B_1, \dots, B_n\}$ of isoclasses of bricks whose dimension vectors coincide with the set of column vectors $\{c_1(w), \dots, c_n(w)\}$ of a c -matrix $c(w) := (g(w)^{-1})^\top$ defined by Fu [Fu17] up to sign (cf. Treffinger [Tre19]). Note that these column vectors of the c -matrix are naturally identified with some elements of $\Delta(C^\top)$ by Theorem

¹The author was motivated by some questions about this from Y. Mizuno and S. Kato to use g -vectors in study of MV polytopes. B. Leclerc was kind to point out the author's error about a description of the Nakayama permutation. He thanks for their comments.

FIGURE 3. g -matrices for type A_3

3.18. Let \mathcal{B}_w be the set of isoclasses of bricks such that $\text{Hom}_\Pi(B, B') = 0$ for any non-isomorphic $B, B' \in \mathcal{B}_w$. Then, the smallest torsion class which contains \mathcal{B}_w coincides with $\text{Fac } I_w$. In addition, \mathcal{B}_w is given by $\text{ind}(I_w/\text{rad}_\Gamma I_w)$, where let $\Gamma := \text{End}_\Pi(I_w)^{\text{op}}$ and let $\text{ind}(I_w/\text{rad}_\Gamma I_w)$ denote the set of isoclasses of indecomposable summands of $I_w/\text{rad}_\Gamma I_w$ (cf. Asai [Asa20]). Then, \mathcal{B}_w bijectively corresponds to the set $\{c_i(w) \in \Delta(C^\top) \mid i \in Q_0\} \cap \Delta^+(C^\top)$ by taking dimension vectors. A similar kind of correspondence between bricks and positive Schur roots for Langlands dual datum has been considered in the work of Geiß-Leclerc-Schröer [GLS20] for $H(C, D, \Omega)$ (see Definition 2.6). So, the combinatorial structures about torsion-free classes are expected to be described in terms of the root system of $\mathfrak{g}(C^\top)$ as in Asai [Asa22] or Enomoto [Eno21].

FIGURE 4. g -matrices for type B_2

4. NILPOTENT VARIETIES AND MV POLYTOPES

Keep the setting of the previous sections.

4.1. On crystal structures on irreducible components and PBW parametrizations. In this subsection, we realize Lusztig's PBW parametrizations via nilpotent varieties of Geiß-Leclerc-Schröer. First, we recall definitions and basic properties about crystals.

Definition 4.1. Let C be a symmetrizable GCM of finite type and let $P := \bigoplus_{i \in Q_0} \mathbb{Z}\varpi_i$ be the associated integral weight lattice. A crystal is a tuple $(B, \text{wt}(-), \tilde{e}_i, \tilde{f}_i, \varepsilon_i, \varphi_i)$ of a set B , maps $\text{wt}(-): B \rightarrow P$, $\tilde{e}_i, \tilde{f}_i: B \rightarrow B \cup \{\emptyset\}$ and $\varepsilon_i, \varphi_i: B \rightarrow \mathbb{Z}$ which satisfy the following conditions for any $i \in Q_0$ and for any $b \in B$:

- (cr1) $\varphi_i(b) = \varepsilon_i(b) + \langle \text{wt}(b), \alpha_i \rangle$;
- (cr2) $\varphi_i(\tilde{e}_i(b)) = \varphi_i(b) + 1$, $\varepsilon_i(\tilde{e}_i(b)) = \varepsilon_i(b) - 1$, $\text{wt}(\tilde{e}_i(b)) = \text{wt}(b) + \alpha_i$ ($\tilde{e}_i(b) \neq \emptyset$);
- (cr3) Two equalities $\tilde{f}_i(b) = b'$ and $\tilde{e}_i(b') = b$ are equivalent.

A lowest weight crystal is a crystal with an element $b_- \in B$ such that

- (cr4) there exists a sequence (i_1, \dots, i_t) with $i_k \in Q_0$ for any $1 \leq k \leq t$ such that $b_- = \tilde{f}_{i_1} \cdots \tilde{f}_{i_t}(b)$ for each $b \in B$;
- (cr5) We have $\varphi_i(b) = \max\{m \mid \tilde{f}_i^m(b) \neq \emptyset\}$.

For a lowest weight crystal B , the functions $\text{wt}(-)$, \tilde{f}_i , ε_i and φ_i are determined by the \tilde{e}_i and $\text{wt}(b_-)$. We mainly deal with infinity crystals $B(-\infty)$. Tingley-Webster [TW16, Proposition 1.4] gave a criterion whether a lowest weight crystal is isomorphic to the infinity crystal $B(-\infty)$ as a reformulation of the well-known criterion by Kashiwara-Saito [KS97, Proposition 3.2.3]. We think of the Tingley-Webster criterion as a definition of $B(-\infty)$ as in the work of [GLS18b].

Definition-Proposition 4.2 ([TW16, Proposition 3.2.3]). Fix a set B with operators $\tilde{e}_i, \tilde{f}_i, \tilde{e}_i^*, \tilde{f}_i^* : B \rightarrow B \cup \{\emptyset\}$, and assume $(B, \tilde{e}_i, \tilde{f}_i)$ and $(B, \tilde{e}_i^*, \tilde{f}_i^*)$ are both lowest weight crystals with the same lowest weight element b_- . Now, we assume that the other data are determined by setting $\text{wt}(b_-) = 0$. We have $(B, \tilde{e}_i, \tilde{f}_i) \cong (B, \tilde{e}_i^*, \tilde{f}_i^*) \cong B(-\infty)$ if they satisfy the following conditions for each $i, j \in Q_0$ and any $b \in B$:

- $\tilde{e}_i(b), \tilde{e}_i^*(b) \neq \emptyset$;
- If $i \neq j$, then we have $\tilde{e}_i^* \tilde{e}_j(b) = \tilde{e}_j \tilde{e}_i^*(b)$;
- We have $\varphi_i(b) + \varphi_i^*(b) - \langle \text{wt}(b), \alpha_i \rangle \geq 0$;
- If we have $\varphi_i(b) + \varphi_i^*(b) - \langle \text{wt}(b), \alpha_i \rangle = 0$, then $\tilde{e}_i(b) = \tilde{e}_i^*(b)$;
- If we have $\varphi_i(b) + \varphi_i^*(b) - \langle \text{wt}(b), \alpha_i \rangle \geq 1$, then $\varphi_i(\tilde{e}_i^*(b)) = \varphi_i(b)$ and $\varphi_i^*(\tilde{e}_i(b)) = \varphi_i^*(b)$;
- If we have $\varphi_i(b) + \varphi_i^*(b) - \langle \text{wt}(b), \alpha_i \rangle \geq 2$, then $\tilde{e}_i^* \tilde{e}_i(b) = \tilde{e}_i \tilde{e}_i^*(b)$.

Geiß-Leclerc-Schröer [GLS18b] generalized a geometric realization of infinity crystals $B(-\infty)$ by Kashiwara-Saito [KS97]. They define a special class of \mathbb{E} -filtered modules, called crystal modules, and they construct a crystal structure on the set of special irreducible components of varieties of \mathbb{E} -filtered modules with specific rank vectors.

Definition 4.3. Let $M \in \mathbb{E}\text{-filt}(\Pi)$, and $0 \rightarrow K_i(M) \rightarrow M \rightarrow \text{fac}_i(M) \rightarrow 0$ and $0 \rightarrow \text{sub}_i(M) \rightarrow M \rightarrow C_i(M) \rightarrow 0$ be canonical short exact sequences. We say that M is a crystal module if and only if:

- (1) $\{0\}$ is a crystal module.
- (2) $K_i(M)$ and $C_i(M)$ are crystal modules for any $i \in Q_0$, and $\text{fac}_i M$ and $\text{sub}_i M$ are locally free modules for any $i \in Q_0$.

Let $\mathbf{i} := (i_1, \dots, i_l)$ be a reduced expression of the longest element of W . Note that any series of inclusions of torsion classes, which has length $l(w_0)$ (called *maximal green sequence*) corresponds to a reduced expression of the longest element of W by the τ -tilting theory. Namely, we can associate the maximal green sequence $\text{rep } \Pi = \text{Fac } I_e \supseteq \text{Fac } I_{i_1} \supseteq \dots \supseteq \text{Fac } I_{s_{i_1} \dots s_{i_l}} = \{0\}$ to \mathbf{i} . We characterize objects in $\text{Fac}(I_w)$ and $\text{Sub } \Pi/I_w$.

Theorem 4.4. Let $M \in \text{rep } \Pi$ be a crystal module and $w \in W$ such that $l(ws_i) = l(w) + 1$. Then, $T_w/T_{ws_i} \cong (I_w/I_{ws_i})^{\oplus a}$ for some $a \in \mathbb{Z}_{\geq 0}$, where T_w is the torsion submodule of M associated to the torsion class $\text{Fac } I_w$ for $w \in W$.

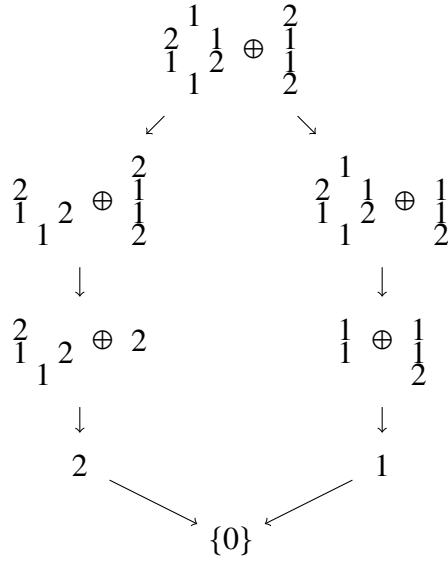
Proof. We put $X = \text{Hom}_\Pi(I_w, M)$. Then, we have the canonical exact sequence from Theorem 2.38:

$$0 \rightarrow \Sigma_i^- \Sigma_i X \rightarrow X \rightarrow \text{fac}_i X \rightarrow 0.$$

By applying $I_w \otimes_\Pi (-)$, we obtain an exact sequence:

$$0 \rightarrow I_{ws_i} \otimes_\Pi \text{Hom}_\Pi(I_{ws_i}, M) \rightarrow I_w \otimes_\Pi \text{Hom}_\Pi(I_w, M) \rightarrow I_w \otimes_\Pi E_i^{\oplus m} \rightarrow 0,$$

where $c_i m = \dim \text{fac}_i(X)$. Here, we use $\text{Tor}_\Pi^1(I_w, E_i) = 0$ for $l(ws_i) > l(w)$ by Proposition 2.35. So, we obtain our assertion from Theorem 3.11. \square

FIGURE 5. $s\tau$ -tilt Π of type B_2

Corollary 4.5. Let $T_0 := M \in \text{rep } \Pi$ and T_j be the torsion submodule of T_{j-1} associated to the torsion class $\text{Fac } I_{s_{i_1} \dots s_{i_j}}$ ($1 \leq j \leq l$). Then, we have a filtration $M = T_0 \supseteq T_1 \supseteq \dots \supseteq T_l = 0$ such that $T_{j-1}/T_j \cong (I_{s_{i_1} \dots s_{i_{j-1}}}/I_{s_{i_1} \dots s_{i_j}})^{\oplus a_j}$ for some $a_j \in \mathbb{Z}_{\geq 0}$.

We refer to the filtration $M = T_0 \supseteq T_1 \supseteq \dots \supseteq T_l = 0$ in Corollary 4.5 as a *Harder-Narasimhan filtration* for $\mathbf{i} = (i_1, \dots, i_l)$ and refer to the modules $I_{s_{i_1} \dots s_{i_{j-1}}}/I_{s_{i_1} \dots s_{i_j}}$ ($1 \leq j \leq l$) as *layer modules*. The aim of this section is to prove that it gives rise to a stratification of irreducible components of nilpotent varieties from these data.

Remark 4.6. For non-Dynkin cases, idempotent ideals corresponding to elements of affine Weyl group never coincide with $\{0\}$. However, the similar short exact sequences as in the Proof of Theorem 4.4 can be obtained also for non-Dynkin cases, though generalized preprojective algebras of non-Dynkin type are always infinite dimensional. In addition, we have Harder-Narasimhan filtrations determined by stabilities (cf. [BKT14, §3.2]) also for non-Dynkin cases, though the τ -tilting theory does not work.

Example 4.7. Let $C = \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}$ and $D = \text{diag}(2, 1)$. In this B_2 case, the lattice of mutations of $s\tau$ -tilt Π is illustrated as Figure 5. All layer modules are illustrated as the following Loewy

series:

$$\begin{aligned} \Pi/I_{s_1} &= \frac{1}{1}; I_{s_1}/I_{s_1s_2} = \frac{2}{1}; & I_{s_1s_2}/I_{s_1s_2s_1} &= \frac{2}{1} \frac{2}{1}; I_{s_1s_2s_1}/I_{s_1s_2s_1s_2} = 2; \\ \Pi/I_{s_2} &= 2; I_{s_2}/I_{s_2s_1} = 2 \frac{1}{2}; & I_{s_2s_1}/I_{s_2s_1s_2} &= \frac{1}{2}; & I_{s_2s_1s_2}/I_{s_2s_1s_2s_1} &= \frac{1}{1}. \end{aligned}$$

Definition 4.8. A module $M \in Z$ for $Z \in \text{lrr}(\Pi(\mathbf{r}))$ is called *generic* if and only if M is contained in a sufficiently small open dense subset of Z defined by a finite set of suitable open conditions which are suitable for contexts. In particular, all the upper semi-continuous constructible functions on Z defined below take its minimal value at M . For example, we can take an open dense subset U_Z for each $Z \in \text{lrr}(\Pi(\mathbf{r}))$ such that $\text{sub}_k(M) \cong \text{sub}_k(N)$ and $\text{fac}_k(M) \cong \text{fac}_k(N)$ for any $M, N \in U_Z$ by upper semi-continuity. So, we can define $\text{fac}_k(Z) := \text{fac}_k(M)$ and $\text{sub}_k(Z) := \text{sub}_k(M)$ up to isomorphism. We demand $M \in Z$ with $\text{sub}_k(M) \cong \text{sub}_k(Z)$ and $\text{fac}_k(M) \cong \text{fac}_k(Z)$ in this context. We say that $Z \in \text{lrr}(\Pi(\mathbf{r}))$ is crystal if Z contains an open dense subset of crystal modules.

Theorem 4.9 ([GLS18b, Theorem 4.1]). For each $Z \in \text{lrr}(\Pi(\mathbf{r}))$, we have

$$\dim Z \leq \dim H(\mathbf{r}).$$

We use the following criterion given by [GLS18b].

Proposition 4.10 ([GLS18b, Proposition 4.4]). For $Z \in \text{lrr}(\Pi(\mathbf{r}))$, the irreducible component Z is crystal if and only if $\dim Z = \dim H(\mathbf{r}) = \dim G(\mathbf{r}) - q_{DC}(\mathbf{r})$.

Let $\text{lrr}(\Pi(\mathbf{r}))^{\max}$ denote the set of crystal irreducible components of $\Pi(\mathbf{r})$. Geiß-Leclerc-Schröer [GLS18b] realized $B(-\infty)$ for general symmetrizable GCMs and for general symmetrizers as follows:

Definition-Theorem 4.11 ([GLS18b]). We give the following:

- (1) Let $\mathcal{B} := \coprod_{\mathbf{r} \in \mathbb{N}^n} \text{lrr}(\Pi(\mathbf{r}))^{\max}$. We write

$$\begin{aligned} \Pi(\mathbf{r})^{k,p} &:= \{M \in \Pi(\mathbf{r}) \mid \text{fac}_k(M) \cong E_k^p\}; \\ \Pi(\mathbf{r})_{k,p} &:= \{M \in \Pi(\mathbf{r}) \mid \text{sub}_k(M) \cong E_k^p\}. \end{aligned}$$

Let $\varphi_i, \varphi_i^* : \mathbb{E}\text{-filt}(\Pi) \rightarrow \mathbb{Z}$ denote functions

$$\varphi_i : M \mapsto [\text{sub}_i(M) : E_i], \quad \varphi_i^* : M \mapsto [\text{fac}_i(M) : E_i],$$

where $[M : E_i]$ denotes the maximal number $p \geq 0$ such that there exists a direct sum decomposition $M = U \oplus V$ with $U \cong E_i^p$ for $M \in \text{rep } H_i$.

- (2) Let $\text{wt}(Z) := \mathbf{r}$ for $Z \in \text{lrr}(\mathbf{r})^{\max}$. We write

$$\varphi_i(Z) := \min\{\varphi_i(M) \mid M \in Z\}, \quad \varepsilon_i(Z) := \varphi_i(Z) - \langle \text{wt}(Z), \alpha_i \rangle.$$

Dually, we write

$$\varphi_i^*(Z) := \min\{\varphi_i^*(M) \mid M \in Z\}, \quad \varepsilon_i^*(Z) := \varphi_i^*(Z) - \langle \text{wt}(Z), \alpha_i \rangle.$$

(3) Let $r := p - q \geq 1$. Then, we have a bijection

$$f_{k,r}: \coprod_{\mathbf{r} \in \mathbb{N}^n} \text{lrr}(\Pi(\mathbf{r})_{k,p})^{\max} \rightarrow \coprod_{\mathbf{r} \in \mathbb{N}^n} \text{lrr}(\Pi(\mathbf{r} - r\alpha_k)_{k,q})^{\max}.$$

Let $e_{k,r} := (f_{k,r})^{-1}$ denote the inverse map of $f_{k,r}$. Dually, we have a bijection

$$f_{k,r}^*: \coprod_{\mathbf{r} \in \mathbb{N}^n} \text{lrr}(\Pi(\mathbf{r})^{k,p})^{\max} \rightarrow \coprod_{\mathbf{r} \in \mathbb{N}^n} \text{lrr}(\Pi(\mathbf{r} - r\alpha_k)^{k,q})^{\max}.$$

Let $e_{k,r}^* := (f_{k,r}^*)^{-1}$ denote the inverse map of $f_{k,r}^*$.

(4) Under above notation, we have a crystal isomorphism

$$(\mathcal{B}, \text{wt}(-), \tilde{e}_i, \tilde{f}_i, \varepsilon_i, \varphi_i) \cong (\mathcal{B}, \text{wt}(-), \tilde{e}_i^*, \tilde{f}_i^*, \varepsilon_i^*, \varphi_i^*) \cong B(-\infty)$$

by assigning the unique irreducible component Z_- of $\Pi(0)$ to the lowest weight element b_- , where we write

$$\tilde{f}_i(Z) := \begin{cases} \overline{f_{i,1}(Z)} & (\varphi_i(Z) \geq 1) \\ \emptyset & (\varphi_i(Z) = 0) \end{cases} \quad \tilde{e}_i(Z) := \overline{e_{i,1}(Z)},$$

and

$$\tilde{f}_i^*(Z) := \begin{cases} \overline{f_{i,1}^*(Z)} & (\varphi_i^*(Z) \geq 1) \\ \emptyset & (\varphi_i^*(Z) = 0) \end{cases} \quad \tilde{e}_i^*(Z) := \overline{e_{i,1}^*(Z)}.$$

In this setting, we define the Kashiwara $*$ -operator as follows:

Definition 4.12 ([GLS18b, §5.4]). Let $M = (M(\varepsilon_i), M(\alpha_{ij})) \in \text{nil}_{\mathbb{B}}(\Pi, d)$ and let $M^* := (M(\varepsilon_i)^*, M(\alpha_{ij})^*) \in \text{nil}_{\mathbb{B}}(\Pi, d)$, where $M(\varepsilon_i)^* := M(\varepsilon_i)^\top$ and $M(\alpha_{ij})^* := M(\alpha_{ji})^\top$. For each dimension vector d , we obtain an automorphism $*_d$ of the variety $\text{nil}_{\mathbb{B}}(\Pi, d)$ defined as $*_d(M) := M^*$. This yields an automorphism $*_{\mathbf{r}}$ of $\Pi(\mathbf{r})$. This $*_{\mathbf{r}}$ yields an involution $*_{\mathbf{r}}: \mathcal{B}_{\mathbf{r}} \rightarrow \mathcal{B}_{\mathbf{r}}$. This induces an involution $*: \mathcal{B} \rightarrow \mathcal{B}$. In this setting, we have

$$* \tilde{e}_i^* = \tilde{e}_i^* \quad * \tilde{e}_i^* = \tilde{e}_i, \quad * \tilde{f}_i^* = \tilde{f}_i^*, \quad * \tilde{f}_i^* = \tilde{f}_i.$$

The aim of this subsection is to prove the following theorem by the help of the argument in the work of Baumann-Kamnitzer-Tingley [BKT14, §4]. This kind of theorem for a classical situation has appeared in the work of Geiss-Leclerc-Schröer [GLS11, §14.2].

Theorem 4.13. We choose a reduced expression $\mathbf{i} = (i_1, \dots, i_r)$ of $w \in W$. Let $\Pi_{\mathbf{i}}^{\mathbf{a}}$ be the set of crystal modules which have a Harder-Narasimhan filtration $M = T_0 \supseteq T_1 \supseteq \dots \supseteq T_r = 0$ such that $T_{j-1}/T_j \cong (I_{s_{i_1} \dots s_{i_{j-1}}}/I_{s_{i_1} \dots s_{i_j}})^{\oplus a_j}$ for $\mathbf{a} := (a_1, \dots, a_r) \in \mathbb{Z}^r$. Then, $\Pi_{\mathbf{i}}^{\mathbf{a}}$ is an irreducible constructible set and $Z_{\mathbf{i}}^{\mathbf{a}} := \overline{\Pi_{\mathbf{i}}^{\mathbf{a}}}$ is a maximal irreducible component.

We give a proof of this theorem after Proposition 4.16. We define $S := \prod_{i \in Q_0} H_i$. We fix a torsion pair $(\mathcal{T}, \mathcal{F})$ in $\text{rep } \Pi$. We consider the set $\mathfrak{Z}(\mathbf{r})$ (*resp.* $\mathfrak{F}(\mathbf{r})$) of all maximal irreducible components of $\Pi(\mathbf{r})$ whose generic points belong to \mathcal{T} (*resp.* \mathcal{F}). We define $\mathfrak{Z} := \coprod_{\mathbf{r} \in \mathbb{Z}_{\geq 0}^n} \mathfrak{Z}(\mathbf{r})$ (*resp.* $\mathfrak{F} := \coprod_{\mathbf{r} \in \mathbb{Z}_{\geq 0}^n} \mathfrak{F}(\mathbf{r})$). Let $(\mathbf{r}_t, \mathbf{r}_f)$ be a tuple of \mathbb{N}^n , and let $\mathbf{r} = \mathbf{r}_t + \mathbf{r}_f$. We define

$$\Pi^{\mathcal{T}}(\mathbf{r}_t) := \{M_t \in \Pi(\mathbf{r}_t) \mid M_t \in \mathcal{T}\}, \quad \Pi^{\mathcal{F}}(\mathbf{r}_f) := \{M_f \in \Pi(\mathbf{r}_f) \mid M_f \in \mathcal{F}\}.$$

Since $\mathcal{F} = \{M \in \text{rep } \Pi \mid \text{Hom}_{\Pi}(I_w, M) = 0\}$ for some $w \in W$, it follows that $\Pi^{\mathcal{F}}(\mathbf{r})$ is an open subset of $\Pi(\mathbf{r})$ as $\dim \text{Hom}_{\Pi}(I_w, -)$ is an upper semi-continuous function. Since the set of torsion pairs bijectively corresponds to the Weyl group $W(C)$, any torsion class has a description of (2.2), so that $\Pi^{\mathcal{T}}(\mathbf{r})$ is also open similarly. Finally, we define a quasi-affine algebraic variety $E(\mathbf{r}_t, \mathbf{r}_f)$ as the set of tuples (M, M_t, M_f, f, g) such that $M \in \Pi(\mathbf{r})$ and take $(M_t, M_f) \in \Pi^{\mathcal{T}}(\mathbf{r}_t) \times \Pi^{\mathcal{F}}(\mathbf{r}_f)$ with an exact sequence

$$0 \rightarrow M_t \xrightarrow{f} M \xrightarrow{g} M_f \rightarrow 0$$

in $\text{rep } \Pi$. Then, we consider the diagram:

$$\Pi^{\mathcal{T}}(\mathbf{r}_t) \times \Pi^{\mathcal{F}}(\mathbf{r}_f) \xleftarrow{p} E(\mathbf{r}_t, \mathbf{r}_f) \xrightarrow{q} \Pi(\mathbf{r}),$$

where p and q are canonical projections.

Lemma 4.14. In the above setting, the projection p is a locally trivial fibration with a smooth and connected fiber of dimension $\dim G(\mathbf{M}) - (\mathbf{r}_t, \mathbf{r}_f)$. The image of q is equal to the set of $M \in \Pi(\mathbf{r})$ such that $\text{rank } tM = \mathbf{r}_t$. The non-empty fibers of q are isomorphic to $G(\mathbf{r}_t) \times G(\mathbf{r}_f)$.

Proof. Our assertion for q follows from the uniqueness of the canonical short exact sequence for a Π -module M and the torsion pair $(\mathcal{T}, \mathcal{F})$ (cf. [ASS06, VI. Proposition 1.5]). In fact, q is a principal $G(\mathbf{r}_t) \times G(\mathbf{r}_f)$ -bundle by considering the ordinary action on $E(\mathbf{r}_t, \mathbf{r}_f)$. We take points $(M_a), (M_{t,a}), (M_{f,a})$ in $\Pi(\mathbf{r}), \Pi^{\mathcal{T}}(\mathbf{r}_t)$ and $\Pi^{\mathcal{F}}(\mathbf{r}_f)$ respectively, and define locally free Π -module structures on $\bigoplus_{i \in Q_0} (H_i)^{r_i}$, $\bigoplus_{i \in Q_0} (H_i)^{r_{ti}}$ and $\bigoplus_{i \in Q_0} (H_i)^{r_{fi}}$, where $H_i := K[\varepsilon_i]/(\varepsilon_i^{c_i})$. We consider the following complex from [GLS17, §12.2] which can be seen as a fiberwise description of tautological vector bundles on $\Pi^{\mathcal{T}}(\mathbf{r}_t) \times \Pi^{\mathcal{F}}(\mathbf{r}_f)$ over a point $((M_{t,a}), (M_{f,a}))$:

$$\begin{aligned} 0 &\rightarrow \bigoplus_{i \in Q_0} \text{Hom}_{H_i}((H_i)^{r_{fi}}, (H_i)^{r_{ti}}) \\ &\xrightarrow{d_{M_f, M_t}^0} \bigoplus_{(i,j) \in \bar{\Omega}} \text{Hom}_{H_i}({}_i H_j \otimes_{H_j} (H_j)^{r_{fj}}, (H_i)^{r_{ti}}) \\ &\xrightarrow{d_{M_f, M_t}^1} \bigoplus_{i \in Q_0} \text{Hom}_{H_i}((H_i)^{r_{fi}}, (H_i)^{r_{ti}}), \end{aligned}$$

where

$$\begin{aligned} (d_{M_f, M_t}^0(\phi_k)_{k \in Q_0})_{(i,j)} &= (M_t)_{ij} \circ (\text{id}_{H_j} \otimes_j \phi_j) - \phi_i \circ (M_f)_{ij}, \\ (d_{M_f, M_t}^1((\psi_{ij})_{(i,j) \in \bar{\Omega}}))_k &= \sum_{j \in \bar{\Omega}(-,k)} \text{sgn}(j, k) (M_t)_{kj} \circ \text{ad}_{jk}(\psi_{jk}) - \psi_{kj} \circ \text{ad}_{jk}((M_f)_{jk}). \end{aligned}$$

Note that the map d_{M_f, M_t}^0 has rank $\dim \text{Hom}_{\mathcal{S}}(M_f, M_t) - \dim \text{Hom}_{\Pi}(M_f, M_t)$, and that $\text{Ext}_{\Pi}^1(M_f, M_t) \cong \text{Ker } d_{M_f, M_t}^1 / \text{Im } d_{M_f, M_t}^0$. By the generalized Crawley-Boevey formula in Proposition 2.14 and the

definition of torsion pairs, we compute:

$$\begin{aligned}
\dim \operatorname{Ker} d_{M_f, M_t}^1 &= \dim \operatorname{Ext}_{\Pi}^1(M_f, M_t) + \operatorname{rank} d_{M_f, M_t}^0 \\
&= \dim \operatorname{Hom}_{\Pi}(M_f, M_t) + \dim \operatorname{Hom}_{\Pi}(M_t, M_f) \\
&\quad - (\mathbf{r}_f, \mathbf{r}_t) + \dim \operatorname{Hom}_S(M_f, M_t) - \dim \operatorname{Hom}_{\Pi}(M_f, M_t) \\
&= \dim \operatorname{Hom}_S(M_f, M_t) - (\mathbf{r}_f, \mathbf{r}_t).
\end{aligned}$$

In particular, $\dim \operatorname{Ker} d_{M_f, M_t}^1$ only depends on \mathbf{r}_f and \mathbf{r}_t . Let E be the set of all S -submodules of $E(\mathbf{r}_t, \mathbf{r}_f)$ that yields exact sequences isomorphic to $0 \rightarrow M_f \xrightarrow{f} M \xrightarrow{g} M_t \rightarrow 0$ of S -modules. This is a homogeneous space for the action of $G(\mathbf{M})$ and the stabilizer of a point of $(f, g) \in E$ is equal to $\{\operatorname{id}_M + fhg \mid h \in \operatorname{Hom}_S(M_f, M_t)\}$, and it is a smooth connected variety of dimension $\dim G(\mathbf{M}) - \dim \operatorname{Hom}_S(M_f, M_t)$. Now, we consider the fiber of p over $(M_{t,a}, M_{f,a}) \in \Pi^{\mathcal{T}}(\mathbf{r}_t) \times \Pi^{\mathcal{F}}(\mathbf{r}_f)$. The fiber consists of (f, g) in E and $M_a \in \Pi(\mathbf{r})$ which satisfy compatibilities in the definition of morphisms of Π -modules. We determine (M_a) from given $((M_{t,a}), (M_{f,a}))$ and (f, g) . Then, the space of possible choices for (M_a) is isomorphic to $\operatorname{Ker} d_{M_f, M_t}^1$, and in fact defines a subsheaf of a tautological vector bundle on $\Pi^{\mathcal{T}}(\mathbf{r}_t) \times \Pi^{\mathcal{F}}(\mathbf{r}_f)$. The linear map d_{M_f, M_t}^1 depends smoothly on $((M_{a,t}), (M_{a,f}))$, and has constant rank by the above equality. In particular, the fiber of the sheaf defined by $\operatorname{Ker} d_{M_f, M_t}^1$ has constant rank. Therefore, our p is a vector bundle by Nakayama's lemma. In particular, p is a locally trivial fibration. Finally, we conclude that the dimension of fibers of p is equal to $\dim E + \dim \operatorname{Ker} d_{M_f, M_t}^1 = \dim G(\mathbf{M}) - (\mathbf{r}_f, \mathbf{r}_t)$. \square

Now, we take a tuple of maximal irreducible components $(Z_t, Z_f) \in \mathfrak{Z}(\mathbf{r}_t) \times \mathfrak{F}(\mathbf{r}_f)$. Then, $p^{-1}(Z_t \times Z_f)$ is an irreducible component of $E(\mathbf{r}_t, \mathbf{r}_f)$ by Lemma 4.14. We consider the irreducible subset $q(p^{-1}(Z_t \times Z_f))$ of $\Pi(\mathbf{r})$, which has dimension

$$\dim(Z_t \times Z_f) + (\dim G(\mathbf{M}) - (\mathbf{r}_t, \mathbf{r}_f)) - \dim(G(\mathbf{r}_t) \times G(\mathbf{r}_f)).$$

Since we take Z_t and Z_f as maximal components, this dimension is equal to $G(\mathbf{M}) - q_{DC}(\mathbf{r})$ by Proposition 4.10. Namely, it is equal to the common dimension of maximal irreducible components of $\Pi(\mathbf{r})$. Thus, $q(p^{-1}(Z_t \times Z_f))$ is a maximal irreducible component of $\Pi(\mathbf{r})$. Since the canonical short exact sequence for M associated with a torsion pair is unique up to isomorphism, this ensures that crystal modules are closed under taking torsion parts or torsion-free parts. In particular, we obtain a map $\Xi: \mathfrak{Z} \times \mathfrak{F} \rightarrow \mathfrak{B}$ by running $(\mathbf{r}_t, \mathbf{r}_f)$.

Theorem 4.15. The map $\Xi: \mathfrak{Z} \times \mathfrak{F} \rightarrow \mathfrak{B}$ is a bijection.

Proof. For any $(\mathbf{r}_t, \mathbf{r}_f)$, the map Ξ induces an injection from $\mathfrak{Z}(\mathbf{r}_t) \times \mathfrak{F}(\mathbf{r}_f)$ to the set of all irreducible components of $\overline{q(E(\mathbf{r}_t, \mathbf{r}_f))}$. Now, we decompose $\Pi(\mathbf{r})$ as a disjoint union of constructible subsets:

$$\Pi(\mathbf{r}) = \bigsqcup_{\mathbf{r}_t + \mathbf{r}_f = \mathbf{r}} q(E(\mathbf{r}_t, \mathbf{r}_f)).$$

Thus, any crystal irreducible component is contained in one and only one of maximal dimensional $\overline{q(E(\mathbf{r}_t, \mathbf{r}_f))}$. In particular, we obtain a bijection $\Xi: \bigsqcup_{\mathbf{r}_t + \mathbf{r}_f = \mathbf{r}} \mathfrak{Z}(\mathbf{r}_t) \times \mathfrak{F}(\mathbf{r}_f) \rightarrow \mathfrak{B}(\mathbf{r})$. \square

Let $Z := \overline{q(p^{-1}(Z_t \times Z_f))}$, and M be a generic point of Z . Then, our construction shows that the torsion submodule tM of M has the rank-vector \mathbf{r}_t and $(tM, M/tM)$ is a generic point in $Z_t \times Z_f$. In fact, $p^{-1}(U_t, U_f)$ is an open dense subset of $p^{-1}(Z_t \times Z_f)$ for a $G(\mathbf{r}_t)$ -invariant open dense subset $U_t \subseteq Z_t$ and a $G(\mathbf{r}_f)$ -invariant open dense subset $U_f \subseteq Z_f$. So, the subset $U := q(p^{-1}(U_t, U_f))$ is open dense in Z , and this subset is constructible by Chevalley's theorem. If we have $M \in U$, then $\underline{\text{rank}} tM = \mathbf{r}_t$ and $(tM, M/tM)$ belongs to $U_t \times U_f$.

Now, we consider the torsion pair $(\text{Fac } I_w, \text{Sub } \Pi/I_w)$ for $w \in W$. Let $\mathfrak{T}^w \times \mathfrak{F}^w$ denote the set of pairs (Z_1, Z_2) of irreducible components Z_1, Z_2 , where a generic point of Z_1 (*resp.* a generic point of Z_2) belongs to $\text{Fac } I_w$ (*resp.* $\text{Sub } \Pi/I_w$). Then, we have the bijection Ξ_w from $\mathfrak{T}^w \times \mathfrak{F}^w$ to \mathfrak{B} .

Proposition 4.16. Let $w \in W$ and $i \in Q_0$ such that $l(ws_i) > l(w)$. Then, the map Ξ_{ws_i} restricts to a bijection $\mathfrak{T}^{ws_i} \times (\mathfrak{F}^{ws_i} \cap \mathfrak{T}^w) \rightarrow \mathfrak{T}^w$, and the map Ξ_w restricts to a bijection $(\mathfrak{F}^{ws_i} \cap \mathfrak{T}^w) \times \mathfrak{F}^w \rightarrow \mathfrak{F}^{ws_i}$. In particular, we have the following commutative diagram:

$$\begin{array}{ccc} \mathfrak{T}^{ws_i} \times (\mathfrak{F}^{ws_i} \cap \mathfrak{T}^w) \times \mathfrak{F}^w & \xrightarrow{\Xi_{ws_i} \times \text{id}} & \mathfrak{T}^w \times \mathfrak{F}^w \\ \text{id} \times \Xi_w \downarrow & & \downarrow \Xi_w \\ \mathfrak{T}^{ws_i} \times \mathfrak{F}^{ws_i} & \xrightarrow{\Xi_{ws_i}} & \mathfrak{B} \end{array}$$

Proof. Let $(Z_1, Z_2) \in \mathfrak{T}^{ws_i} \times \mathfrak{F}^{ws_i}$ and $Z := \Xi_{ws_i}(Z_1, Z_2)$. We take a generic point M of Z and let T_{ws_i} be the torsion submodule of M with respect to $(\text{Fac } I_{ws_i}, \text{Sub } \Pi/I_{ws_i})$ as in Theorem 4.4. Then, T_{ws_i} belongs to $\text{Fac } I_w$ and $(T_{ws_i}, M/T_{ws_i})$ is generic in (Z_1, Z_2) . Since any torsion class is closed under taking quotients and extensions, M belongs to $\text{Fac } I_w$ if and only if M/T_{ws_i} belongs to $\text{Fac } I_w$. Namely, Z belongs to \mathfrak{T}^w if and only if Z_2 belongs to \mathfrak{T}^w . Thus, Ξ_{ws_i} restricts to a bijection $\mathfrak{T}^{ws_i} \times (\mathfrak{F}^{ws_i} \cap \mathfrak{T}^w) \rightarrow \mathfrak{T}^w$. Our assertion about Ξ_w can be proved similarly to Ξ_{ws_i} . Let $(Z_1, Z_2, Z_3) \in \mathfrak{T}^{ws_i} \times (\mathfrak{F}^{ws_i} \cap \mathfrak{T}^w) \times \mathfrak{F}^w$, and let $Z_4 := \Xi_{ws_i}(Z_1, Z_2)$ and $Z := \Xi_w(Z_4, Z_3)$. Then, the point $(T_w, M/T_w)$ is generic in $Z_4 \times Z_3$. In particular, $(T_{ws_i}, T_w/T_{ws_i}, M/T_w)$ is generic in $Z_1 \times Z_2 \times Z_3$ because T_{ws_i} is the torsion submodule of T_w with respect to the torsion pair $(\text{Fac } I_{ws_i}, \text{Sub } \Pi/I_{ws_i})$. A similar argument for $(Z'_1, Z'_2, Z'_3) := (\Xi_{ws_i} \circ (\text{id} \times \Xi_w))^{-1}(Z)$ shows that $(T_{ws_i}, T_w/T_{ws_i}, M/T_w)$ is generic also in (Z'_1, Z'_2, Z'_3) . This implies the commutativity of the diagram. \square

Proof of Theorem 4.13. In view of Proposition 4.16 and its proof, we deal with any maximal green sequence. Namely, $\mathfrak{T}^{s_{i_1} \cdots s_{i_{l-1}}} \times (\mathfrak{F}^{s_{i_1} \cdots s_{i_{l-1}}} \cap \mathfrak{T}^{s_{i_1} \cdots s_{i_{l-2}}}) \times \cdots \times (\mathfrak{F}^{s_{i_1} s_{i_2}} \cap \mathfrak{T}^{s_{i_1}}) \times \mathfrak{F}^{s_{i_1}}$ bijectively corresponds to \mathfrak{B} . In particular, the bijection is given by the correspondence of generic points

$$(T_{s_1 \cdots s_{l-1}}, T_{s_1 \cdots s_{l-2}}/T_{s_1 \cdots s_{l-1}}, \dots, M/T_{s_1})$$

and M . This shows Theorem 4.13 under consideration of Corollary 4.5. \square

As an application of Theorem 4.13, we construct Saito reflections in our setting. In particular, we describe the PBW parametrizations of canonical bases in our setting. For classical cases, a dual argument to ours has appeared in the work of Jiang [Jia14].

Theorem 4.17. Choose a reduced expression $\mathbf{i} := (i_1, \dots, i_l)$ of $w \in W$. Let $\Pi_{\mathbf{i}}^{\mathbf{a}}$ be the constructible set of modules M which have the HN filtration $M = M_0 \supseteq M_1 \supseteq \dots \supseteq M_l = 0$ such that $M_{j-1}/M_j \cong (I_{s_{i_1} \dots s_{i_{j-1}}}/I_{s_{i_1} \dots s_{i_j}})^{\oplus a_j}$ for $\mathbf{a} := (a_1, \dots, a_l) \in \mathbb{Z}_{\geq 0}^r$. Then $\Sigma_{i_1} M \in \Pi_{\mathbf{i}'}^{\mathbf{a}'}$, for $\mathbf{i}' := (i_2, \dots, i_l)$ and $\mathbf{a}' := (a_2, \dots, a_l)$.

Proof. Let $M \in \Pi_{\mathbf{i}}^{\mathbf{a}}$. We have a HN filtration $M = M_0 \supseteq M_1 \supseteq \dots \supseteq M_l = 0$ such that $M_{j-1}/M_j \cong (I_{s_{i_1} \dots s_{i_{j-1}}}/I_{s_{i_1} \dots s_{i_j}})^{\oplus a_j}$ for some $a_j \in \mathbb{Z}_{\geq 0}$. Let $N := M_1$ be the torsion submodule of M with respect to a torsion pair $(\text{Fac } I_{s_{i_1}}, \text{Sub } \Pi/I_{s_{i_1}})$. Since we have an exact sequence $0 \rightarrow N \rightarrow M \rightarrow E_{i_1}^{\oplus a_1} \rightarrow 0$ by Theorem 2.38, it is found that $\Sigma_{i_1} M \cong \Sigma_{i_1} N$ by $\Sigma_{i_1} E_{i_1} = 0$. Then, we have a filtration $N = N_0 \supseteq N_1 \supseteq \dots \supseteq N_{l-1} = 0$ such that $N_{j-1}/N_j \cong M_j/M_{j+1} \cong (I_{s_{i_1} \dots s_{i_j}}/I_{s_{i_1} \dots s_{i_{j+1}}})^{\oplus a_{j+1}} \cong (I_{s_{i_1} \dots s_{i_j}} \otimes_{\Pi} E_{i_{j+1}})^{\oplus a_{j+1}}$ for $j = 1, \dots, l-1$. Thus, N_j/N_{j+1} has a trivial i_1 -factor for $j > 1$. Thus, we have a filtration $N' = N'_0 \supseteq N'_1 \supseteq \dots \supseteq N'_{l-1} = 0$ where $N'_j := \Sigma_{i_1} N_j$ and $N'_j/N'_{j+1} \cong \Sigma_{i_1} (I_{s_{i_1} \dots s_{i_j}} \otimes_{\Pi} E_{i_{j+1}})^{\oplus a_{j+1}} \cong (I_{s_{i_2} \dots s_{i_j}} \otimes_{\Pi} E_{i_{j+1}})^{\oplus a_{j+1}}$ because $I_{s_{i_2} \dots s_{i_j}} \otimes_{\Pi} E_{i_{j+1}}$ has a trivial i_1 -sub. Thus, we obtain that $\Sigma_{i_1} M \cong \Sigma_{i_1} N \in \Pi_{\mathbf{i}'}^{\mathbf{a}'}$. \square

Saito [Sai94, Corollary 3.4.8] has defined mutually bijections between certain subsets of crystals $B(-\infty)$, called Saito reflections. Namely, we define $\tilde{f}_i^{\max}(b) := \tilde{f}_i^{\varphi_i(b)}(b)$ (resp. $\tilde{f}_i^{*\max}(b) := \tilde{f}_i^{*\varphi_i^*(b)}(b)$) and $S_i(b) := \tilde{e}_i^{\varepsilon_i^*(b)} \tilde{f}_i^{*\max}(b)$ (resp. $S_i^*(b) := (\tilde{e}_i^*)^{\varepsilon_i(b)} \tilde{f}_i^{\max}(b)$). Then, these S_i and S_i^* give mutual bijections

$$\{b \in B(-\infty) \mid \varphi_i(b) = 0\} \xrightleftharpoons[S_i^*]{S_i} \{b \in B(-\infty) \mid \varphi_i^*(b) = 0\}.$$

We compare the Saito reflection S_i and our reflection functor Σ_i based on the idea of Baumann-Kamnitzer [BK12] for a classical setting. We prepare the following Propositions as analogies of the work of [BK12]:

Proposition 4.18. Let $Z \in \text{Irr}(\Pi(\mathbf{r}))^{\max}$ and set $c = \varphi_i(Z)$ and $\mathbf{r}' = \mathbf{r} - c\alpha_i$. Let V be an open dense $G(\mathbf{r})$ -invariant subset of $\tilde{f}_i^{\max} Z$. Then, $\{M \in Z \mid [\Sigma_i^- \Sigma_i M] \subseteq V\}$ contains an open dense subset in Z .

Proof. Note that $\Sigma_i^- \Sigma_i M$ is the torsion submodule of M with respect to a torsion pair $(\text{Fac } I_{s_i}, \text{Sub } \Pi/I_{s_i})$. If $M \in \mathbb{E}\text{-filt}(\Pi)$ with $\text{rank } M = \mathbf{r}$ and $\text{rank}_i \text{fac}_i M = c$, then $\Sigma_i^- \Sigma_i M$ is the unique submodule with the rank vector \mathbf{r}' of M up to isomorphism by Proposition 2.38. Let $Y(\mathbf{r}', i, c)$ be the set of tuples $((N_a), (M_a), g)$ of $(N_a) \in \Pi(\mathbf{r}')_{i,0}$, $(M_a) \in \Pi(\mathbf{r})_{i,c}$ and an isomorphism $g: N \rightarrow \Sigma_i^- \Sigma_i M$. We have a diagram

$$\Pi(\mathbf{r}')_{i,0} \xleftarrow{p} Y(\mathbf{r}', i, c) \xrightarrow{q} \Pi(\mathbf{r})_{i,c},$$

where p and q are the first and the second projections respectively. By a similar argument as in the Proof of Lemma 4.14, we find that p and q are principal bundles, and that the set $p(q^{-1}(M))$ is the orbit $[\Sigma_i^- \Sigma_i M]$ for each $(M_a) \in \Pi(\mathbf{r})_{i,c}$.

Now, $Z \cap \Pi(\mathbf{r})_{i,c}$ is an open dense subset of Z . By definition of \tilde{f}_i^{\max} , we obtain a restriction

$$(\tilde{f}_i^{\max} Z) \cap \Pi(\mathbf{r}')_{i,0} \xleftarrow{p} Y' \xrightarrow{q} Z \cap \Pi(\mathbf{r})_{i,c},$$

where Y' is a maximal dimensional irreducible component of $Y(\mathbf{r}', i, c)$. We put $U := Z \cap \Pi(\mathbf{r})_{i,c}$ and $U' := (\tilde{f}_i^{\max} Z) \cap \Pi(\mathbf{r}')_{i,0}$. Since U' is open dense in $\tilde{f}_i^{\max} Z$, it is found that $U' \cap V$ is open dense in U' . Thus, $q(p^{-1}(U' \cap V))$ is open dense in U . We observe that

$$\begin{aligned} \{M \in Z \mid [\Sigma_i^- \Sigma_i M] \subseteq V\} &= \{M \in Z \mid [\Sigma_i^- \Sigma M] \text{ meets } V\} \\ &\subseteq \{M \in U \mid p(q^{-1}(M)) \text{ meets } U' \cap V\} \\ &= q(p^{-1}(U' \cap V)) \end{aligned}$$

Since we take V as $G(\mathbf{r}')$ -invariant, we obtain our assertion. \square

Let $b \in B(-\infty)$ and let Z_b be the corresponding maximal irreducible component under Definition-Theorem 4.11. We take $\mathbf{r} \in R_+$. We define the set $\Theta(\mathbf{r}, i)$ of triples $((M_a), (N_a), h)$ such that $(M_a) \in \Pi(\mathbf{r})_{i,0}$, $(N_a) \in \Pi(s_i \mathbf{r})_{i,0}^*$ and $h: N \rightarrow \Sigma_i M$ is an isomorphism between \mathbb{E} -filtered modules. We have a diagram

$$\Pi(\mathbf{r})_{i,0} \xleftarrow{r} \Theta(\mathbf{r}, i) \xrightarrow{s} (\Pi(s_i \mathbf{r})_{i,0})^*, \quad (4.1)$$

where r and s are principal bundles over the images with structural groups $G(s_i \mathbf{r})$ and $G(\mathbf{r})$ respectively. In fact, for $(M_a) \in \Pi(\mathbf{r})_{i,0}$, the H_i -linear map $M_{i,\text{in}}$ is full rank and the image is locally free. Thus, $(M_a) \mapsto \text{Ker } M_{i,\text{in}}$ is a continuous map from $\Pi(\mathbf{r})_{i,0}$ to a Grassmannian of the \tilde{M}_i in a similar manner to [GLS18b, §3.4]. Thus, by taking an open covering of $\Pi(\mathbf{r})_{i,0}$, we can prove r is a principal $G(s_i \mathbf{r})$ -bundle. By a similar argument, s is a principal $G(\mathbf{r})$ -bundle.

In the crystal side, let $b \in B(-\infty)$ of weight \mathbf{r} and $\varphi_i(b) = 0$, and let $c = \varphi_i(b^*)$; $d = \varepsilon_i(b^*)$ and $\mathbf{r}' = \text{wt}((\tilde{f}_i^*)^{\max} b)$. Then, $\text{wt}(b) = \mathbf{r} = \mathbf{r}' + c\alpha_i$. A bit of calculation yields that $d = c - \langle \alpha_i, \mathbf{r} \rangle \geq 0$ and that $\varphi_i((\tilde{f}_i^*)^{\max} b) = 0$, which imply

$$d = \varphi_i(S_i b); \quad c + d = 2c - \langle \alpha_i, \mathbf{r}' + c\alpha_i \rangle = -\langle \alpha_i, \mathbf{r}' \rangle.$$

In view of this observation, we obtain the following proposition as an analogue of [BK12, Theorem 5.3].

Proposition 4.19. Let $b \in B(-\infty)$ of weight \mathbf{r} and such that $\varphi_i(b) = 0$. We have

$$r^{-1}(Z_b \cap \Pi(\mathbf{r})_{i,0}) = s^{-1}(Z_{S_i b} \cap (\Pi(s_i \mathbf{r})_{i,0})^*).$$

Proof. Let M (resp. N) be an \mathbb{E} -filtered module such that $\text{fac}_i(M) = 0$ (resp. $\text{sub}_i(N) = 0$) and $N \cong \Sigma_i M$. We have an equality

$$\text{rank}_i M_i - \text{rank}_i \text{sub}_i M = \text{rank}_i N_i - \text{rank}_i \text{fac}_i N.$$

Let $\mathbf{r}' \in R^+$ be such that $\mathbf{r} = \mathbf{r}' + c\alpha_i$ and $s_i \mathbf{r} = \mathbf{r}' + d\alpha_i$ for $c, d \in \mathbb{Z}$. Note that $c + d = -\langle \alpha_i, \mathbf{r}' \rangle$. We take two \mathbb{E} -filtered modules M, N such that

$$\begin{aligned} \underline{\text{rank}} M &= \mathbf{r}, & \text{fac}_i M &= 0; \\ \underline{\text{rank}} N &= s_i \mathbf{r}, & \text{sub}_i N &= 0; & \Sigma_i M &\cong N \end{aligned}$$

Note that we have $\text{rank}_i \text{sub}_i M = c$ if and only if $\text{rank}_i \text{fac}_i N = d$. We restrict the diagram (4.1) to a suitable locally closed subset $\Theta(\mathbf{r}, i)_{c,d} \subseteq \Theta(\mathbf{r}, i)$. Namely, we obtain a diagram

$$\Pi(\mathbf{r})_{i,0} \cap (\Pi(\mathbf{r})_{i,c})^* \xleftarrow{r} \Theta(\mathbf{r}, i)_{c,d} \xrightarrow{s} (\Pi(s_i \mathbf{r})_{i,0})^* \cap \Pi(s_i \mathbf{r})_{i,d}. \quad (4.2)$$

For $\mathbf{r} \in R^+$, a vertex $i \in Q_0$ and $c \in \mathbb{Z}_{\geq 0}$, we set $\Pi(\mathbf{r})_{i,c}^\times := \Pi(\mathbf{r})_{i,c} \cap (\Pi(\mathbf{r})_{i,0})^*$.

Let Ψ be the set of tuples $((L_a), (M_a), (N_a)), f, g, h$ such that

$$(L_a) \in \Pi(\mathbf{r}')_{i,0}^\times; \quad (M_a) \in (\Pi(\mathbf{r})_{i,c}^\times)^*; \quad (N_a) \in \Pi(s_i \mathbf{r})_{i,d}^\times \quad (4.3)$$

and that $f: M \rightarrow L, g: L \hookrightarrow N$ and $h: N \xrightarrow{\cong} \Sigma_i M$ in $\mathbb{E}\text{-filt}(\Pi)$ which satisfy that $hgf: M \rightarrow \Sigma_i M$ is the canonical map in Remark 2.37.

$$\begin{array}{ccccc} (\Pi(\mathbf{r}')_{i,0}^\times)^* & \xleftarrow{*p^{\times*}} & (Y(\mathbf{r}', i, c)^\times)^* & \xrightarrow{*q^{\times*}} & (\Pi(\mathbf{r})_{i,c}^\times)^* \\ \parallel & & \uparrow & & \uparrow r \\ & & \Psi & \xrightarrow{\quad} & \Theta(\mathbf{r}, i)_{c,d} \\ & & \downarrow & & \downarrow s \\ \Pi(\mathbf{r}')_{i,0}^\times & \xleftarrow{p^\times} & Y(\mathbf{r}', i, d)^\times & \xrightarrow{q^\times} & \Pi(s_i \mathbf{r})_{i,d}^\times \end{array}$$

Note that Ψ defines principal bundles, with structural groups $G(s_i \mathbf{r}), G(\mathbf{r}')$ and $G(\mathbf{r})$ in the above diagram as an analogue of [BK12, §5.2]. In particular, all the maps are locally trivial fibrations with a smooth connected fiber in the diagram, so that they induce bijections between the sets of irreducible components.

By definition, we have $\varphi_i((S_i b)^*) = 0$. The commutativity of the above commutative diagram implies that $Z_b \cap (\Pi(\mathbf{r})_{i,c}^\times)$ and $Z_{S_i b} \cap \Pi(s_i \mathbf{r})_{i,d}^\times$ correspond in the bijection defined by (4.2). Since we take Z_b and $Z_{S_i b}$ as crystal components, Theorem 4.9 yields our assertion by taking closures in $\Pi(\mathbf{r})_{i,0}$ and $(\Pi(s_i \mathbf{r})_{i,0})^*$. \square

We have the following analogue of [BK12, Proposition 5.5] to our setting.

Proposition 4.20. Let $b \in B(-\infty)$ and set

$$\mathbf{r} = \text{wt}(b); \quad c = \varphi_i(b); \quad U = Z_b \cap \Pi(\mathbf{r})_{i,c}; \quad b' = S_i(\tilde{f}_i^{\max} b); \quad \mathbf{r}' = \text{wt}(b')$$

Let V be an open dense $G(\mathbf{r}')$ -invariant subset of $Z_{b'}$. Then, $\{(M_a) \in U \mid [\Sigma_i M] \subseteq V\}$ contains an open dense subset of U .

Proof. If $c = 0$, we have $b' = S_i b$ and $\mathbf{r}' = s_i \mathbf{r}$. Then, we use the diagram (4.1) and Proposition 4.19 to obtain our assertion by an analogue of Proposition 4.18. In general case, we use $\Sigma_i M \cong \Sigma_i(\Sigma_i^- \Sigma_i M)$ and Proposition 4.18 to obtain our assertion from $c = 0$ case. \square

As a conclusion of this subsection, we have the following Theorem about Saito reflections:

Theorem 4.21 (Saito reflections). Let $\overline{\Pi}_i^{\mathbf{a}} = Z_i^{\mathbf{a}}$. We put $\mathbf{i}' = (i_2, \dots, i_l)$ and $\mathbf{a}' = (a_2, \dots, a_l)$. Then, we have $S_{i_1} \tilde{f}_{i_1}^{\max}(Z_i^{\mathbf{a}}) = Z_{\mathbf{i}'}^{\mathbf{a}'}$.

Proof. Keep with the notation in Theorem 4.13. For any $M \in \overline{\Pi}_i^{\mathbf{a}}$, it is found that $T_{j-1}/T_j \cong (I_{s_{i_1} \dots s_{i_{j-1}}} / I_{s_{i_1} \dots s_{i_j}})^{\oplus a_j}$ has a trivial i_1 -factor for $j = 2, \dots, l-1$. Thus, we have $\text{fac}_{i_1} M \cong E_{i_1}^{\oplus a_1}$.

Now, by our construction, the irreducible component $Z_{\mathbf{i}}^{\mathbf{a}}$ has an open dense subset contained in $\Pi_{\mathbf{i}}^{\mathbf{a}}$, so that we obtain $\varphi_{i_1}^* Z_{\mathbf{i}}^{\mathbf{a}} = a_1$. Now, we take a generic point $M \in \Pi_{\mathbf{i}}^{\mathbf{a}}$ and assume M belongs to an open dense subset U . We find that the set of $M_a \in U$ such that $\Sigma_{i_1} M_a$ is generic in $S_{i_1} \tilde{f}_{i_1}^{\max}(Z_{\mathbf{i}}^{\mathbf{a}})$, contains an open dense subset in U by Proposition 4.20. Thus, we obtain our assertion by Theorem 4.17. \square

In particular, we obtain the PBW parametrization as a corollary of Theorem 4.21.

Corollary 4.22. $\tilde{f}_{i_r}^{\max} S_{i_{r-1}} \tilde{f}_{i_{r-1}}^{\max} \cdots S_{i_1} \tilde{f}_{i_1}^{\max} Z_{\mathbf{i}}^{\mathbf{a}} = Z_0$. \square

Remark 4.23. In Theorem 4.13, if we take an element $w \in W$, we can obtain an injection $\psi_{\mathbf{i}}: \mathbb{Z}_{\geq 0}^r \rightarrow \mathcal{B}$ given by $\mathbf{a} \mapsto Z_{\mathbf{i}}^{\mathbf{a}}$ and the image of $\psi_{\mathbf{i}}$ does not depend on the choice of \mathbf{i} . In particular, we obtain the PBW parametrization of canonical bases of the quantum unipotent subalgebra $U_q(\mathfrak{n}(w))$ with respect to w of any symmetrizable finite Dynkin type.

4.2. MV polytopes from generalized preprojective algebras. In this subsection, we give a description of a version of Harder-Narasimhan polytopes for Π . Namely, we give a description of *generalized Harder-Narasimhan polytopes* (see Definition 4.38) by using g -matrices developed in §3.2. We prove that they realize Mirković-Vilonen polytopes. First, we recall the Auslander-Reiten inner product for the Grothendieck group:

Definition-Theorem 4.24 (Auslander-Reiten [AR85, Theorem 1.4]). Let M and N be Π -modules. Then, we have

$$\langle g(M), \underline{\dim} N \rangle = \dim \operatorname{Hom}_{\Pi}(M, N) - \dim \operatorname{Hom}_{\Pi}(N, \tau M),$$

where the left hand side is the standard inner product on $\mathbb{Z}^n \times \mathbb{Z}^n$.

This pairing is extended to a group homomorphism $\theta: K_0(\operatorname{rep} \Pi) \rightarrow \mathbb{R}$ for $\theta \in \mathbb{R}^n \cong K_0(\operatorname{proj} \Pi) \otimes_{\mathbb{Z}} \mathbb{R}$. We have a realization of Weyl chambers via g -matrices for τ -tilting pairs by Theorem 3.19. In our setting, we want to consider rank vectors instead of dimension vectors, so that we introduce the following inner product:

Definition-Proposition 4.25. Let M be a locally free Π -module with $\alpha := \underline{\operatorname{rank}} M$ and let (I_w, P_w) be the τ -tilting pair with respect to $w \in W$ and $g(w) = (g_1, \dots, g_n)$ be the g -matrix for this pair. Let $\gamma = \sum_{i \in Q_0} a_i g_i \in \mathbb{Z}^n$ with $a_i \in \mathbb{Z}_{\geq 0}$. If we regard α as a positive root and γ as an integral weight $\sum_{i \in Q_0} a_i w \varpi_i$ by Theorem 3.19, then the pairing $\langle \gamma, \alpha \rangle := \langle \gamma, \alpha \rangle_C$ (see §2.1) coincides with the natural standard inner product given by

$$\begin{aligned} \langle \gamma, \alpha \rangle &= \left\langle \sum_{i \in Q_0} a_i g_i, \alpha \right\rangle \\ &= \sum_{I_w e_i \neq 0} \frac{a_i}{c_i} (\dim \operatorname{Hom}_{\Pi}(I_w e_i, M) - \dim \operatorname{Hom}_{\Pi}(M, \tau(I_w e_i))) \\ &\quad - \sum_{I_w e_j = 0} \frac{a_j}{c_j} (\dim \operatorname{Hom}_{\Pi}(P_w e_j, M)), \end{aligned}$$

where $\operatorname{diag}(c_1, \dots, c_n)$ is a symmetrizer of C .

Proof. This is a direct consequence of the fact that

$$\underline{\text{rank}} M = (\text{rank}_1(e_1 M), \dots, \text{rank}_n(e_n M)) = ((\dim e_1 M)/c_1, \dots, (\dim e_n M)/c_n),$$

Theorem 3.19, and Definition-Proposition 4.24. Note that we have $\tau P = 0$ for any projective module P . We obtain an equality

$$\begin{aligned} \langle \sum_{i \in Q_0} a_i g_i, \alpha \rangle &= \sum_{I_w e_i \neq 0} \frac{a_i}{c_i} (\dim \text{Hom}_\Pi(I_w e_i, M) - \dim \text{Hom}_\Pi(M, \tau(I_w e_i))) \\ &\quad - \sum_{I_w e_j = 0} \frac{a_j}{c_j} (\dim \text{Hom}_\Pi(P_w e_j, M) - \dim \text{Hom}_\Pi(M, \tau(P_w e_j))) \\ &= \sum_{I_w e_i \neq 0} \frac{a_i}{c_i} (\dim \text{Hom}_\Pi(I_w e_i, M) - \dim \text{Hom}_\Pi(M, \tau(I_w e_i))) \\ &\quad - \sum_{I_w e_j = 0} \frac{a_j}{c_j} (\dim \text{Hom}_\Pi(P_w e_j, M)). \quad \square \end{aligned}$$

Definition 4.26. Let $w \in W$ and let $\gamma \in P^+$. Let $M \in \text{rep } \Pi$. We write $\pm w\gamma := \pm \sum_{i \in Q_0} \langle \gamma, \alpha_i \rangle w\varpi_i \in \mathbb{Z}^n$.

$$D_{\pm w\gamma}(M) := \sum_{i \in Q_0} \frac{\langle \gamma, \alpha_i \rangle}{c_i} \dim \text{Hom}_\Pi(N(\pm w\varpi_i), M) (\geq 0).$$

We have the following propositions by similar arguments as in [BK12, Proposition 4.1, 4.2, 4.3]:

Proposition 4.27. Let γ be a weight such that $\langle \gamma, \alpha_i \rangle \leq 0$ for $i \in Q_0$. Then, we have the following equalities for a crystal module $T \in \mathbb{E}\text{-filt}(\Pi)$:

$$D_\gamma(T) = D_{s_i\gamma}(\Sigma_i T); \quad (4.4)$$

$$D_{s_i\gamma}(T) = D_\gamma(\Sigma_i^- T) - \langle \gamma, \underline{\text{rank}}(\text{sub}_i T) \rangle; \quad (4.5)$$

$$D_\gamma(T) = D_\gamma(\Sigma_i^- \Sigma_i T). \quad (4.6)$$

Proof. Note that we can regard T as a module in $\text{rep } \Pi$ or $\text{rep } \widetilde{\Pi}$. By Definition 2.36, this difference does not matter when we consider the module $\Sigma_i T$. The first equality (4.4) follows from the following isomorphisms:

$$\begin{aligned} \text{Hom}_\Pi(N(\gamma), T) &\cong \text{Hom}_{\widetilde{\Pi}}(\widetilde{N}(\gamma), T) \\ &\cong \text{Hom}_{\widetilde{\Pi}}(\Sigma_i^- \widetilde{N}(s_i\gamma), T) \\ &\cong \text{Hom}_{\widetilde{\Pi}}(\widetilde{N}(s_i\gamma), \Sigma_i T). \end{aligned}$$

Next, we consider the second equality (4.5). In the above isomorphisms, we obtain, by substituting $T = \Sigma_i^- T$,

$$D_\gamma(\Sigma_i^- T) = D_{s_i\gamma}(\Sigma_i \Sigma_i^- T).$$

Now, Corollary 3.6 says that

$$\frac{1}{c_i} \dim \text{Hom}_{\widetilde{\Pi}}(\widetilde{N}(s_i\gamma), E_i) = -\langle \gamma, \alpha_i \rangle; \quad \frac{1}{c_i} \dim \text{Ext}_{\widetilde{\Pi}}^1(\widetilde{N}(s_i\gamma), E_i) = 0.$$

Since T is a crystal module, the second equality (4.6) is obtained by applying the functor $\mathrm{Hom}_{\tilde{\Pi}}(\tilde{N}(s_i\gamma), -)$ to the short exact sequence

$$0 \rightarrow \mathrm{sub}_i T \rightarrow T \rightarrow \Sigma_i \Sigma_i^- T \rightarrow 0.$$

Finally, we consider the third equality (4.6). We have an exact sequence

$$0 \rightarrow \Sigma_i^- \Sigma_i T \rightarrow T \rightarrow \mathrm{fac}_i T \rightarrow 0 \quad (4.7)$$

and an equality

$$\frac{1}{c_i} \dim \mathrm{Hom}_{\tilde{\Pi}}(\tilde{N}(\gamma), T) = 0.$$

Thus, we obtain the third equality by applying the functor $\mathrm{Hom}_{\tilde{\Pi}}(\tilde{N}(\gamma), -)$ to (4.7). \square

Proposition 4.28 (edge relations). Let $T \in \mathbb{E}\text{-filt}(\Pi)$ be a crystal module, and let $i \in Q_0$ and $w \in W$. We have an inequality

$$D_{-w\varpi_i}(T) + D_{-ws_i\varpi_i}(T) + \sum_{j \neq i} c_{ji} D_{-w\varpi_j}(T) \geq 0. \quad (4.8)$$

Proof. Note that an equality $s_i\varpi_i = \varpi_i - \alpha_i$ is equivalent to

$$\varpi_i + s_i\varpi_i + \sum_{j \neq i} c_{ji}\varpi_j = 0. \quad (4.9)$$

Let $L_w(T)$ be the left hand side of (4.8). Since $L_{s_i w}(T) = L_w(T)$, we may assume $l(ws_i) > l(w)$ without loss of generality. We take w with $l(w) > 0$, and write $w = s_k v$ with $k \in Q_0$ and $l(v) = l(w) - 1$. Note that we have an inequality of length

$$l(s_k v) > l(v), \quad l(s_k v s_i) > l(w) = l(v) + 1 \geq l(v s_i). \quad (4.10)$$

We calculate as follows:

$$\begin{aligned} L_w(T) &= D_{-w\varpi_i}(T) + D_{-ws_i\varpi_i}(T) + \sum_{j \neq i} c_{ji} D_{-w\varpi_j}(T) \\ &= D_{-s_k v s_i \varpi_i}(T) + D_{-s_k v \varpi_i}(T) + \sum_{j \neq i} c_{ji} D_{-s_k v \varpi_j}(T) \\ &= D_{-v s_i \varpi_i}(\Sigma_k^- T) + D_{-v \varpi_i}(\Sigma_k^- T) + \sum_{j \neq i} c_{ji} D_{-v \varpi_j}(\Sigma_k^- T) \\ &= L_v(\Sigma_k^- T). \end{aligned}$$

By calculating inductively, for a reduced expression $w = s_{i_1} \cdots s_{i_r}$, we have

$$L_w(T) = L_1(\Sigma_{i_r}^- \cdots \Sigma_{i_1}^- T).$$

Since $D_\gamma(M) = 0$ for any anti-dominant weight γ and for any $M \in \mathrm{rep} \Pi$, we obtain

$$L_w(T) = D_{-s_i \varpi_i}(\Sigma_{i_r}^- \cdots \Sigma_{i_1}^- T) \geq 0. \quad \square$$

Proposition 4.29. Let $M \in \mathrm{rep} \Pi$ be an \mathbb{E} -filtered module. We have

$$D_\gamma(M) - D_{-\gamma}(M^*) = \langle \gamma, \underline{\mathrm{rank}} M \rangle.$$

Proof. For each $i \in Q_0$ and each $M \in \mathbb{E}\text{-filt}(\Pi)$, the number $D_{\varpi_i}(M)$ is equal to the multiplicity of E_i in a filtration in Definition 2.15. By definition, this is equal to $\langle \varpi_i, \underline{\text{rank}} M \rangle$. Since we have $D_{-\varpi_i}(M) = 0$, we have a desired equality for $\gamma = \varpi_i$. We fix an anti-dominant weight λ and let $\gamma := w\lambda$. Then, we put $F_w(M) = D_{w\lambda}(M) - D_{-w\lambda}(M^*) - \langle \gamma, \underline{\text{rank}} M \rangle$. We assume that $l(s_i w) > l(w)$. Now, we have $\Sigma_i^- \tilde{N}(-w\lambda) \cong \tilde{N}(-s_i w\lambda)$. Then, we have

$$\begin{aligned} \text{Hom}_{\tilde{\Pi}}(\tilde{N}(-s_i w\lambda), M^*) &\cong \text{Hom}_{\tilde{\Pi}}(\tilde{N}(-w\lambda), \Sigma_i^- M^*) \\ &\cong \text{Hom}_{\tilde{\Pi}}(\tilde{N}(-w\lambda), (\Sigma_i^- M)^*), \end{aligned}$$

so that $D_{-s_i w\lambda}(M^*) = D_{-w\lambda}((\Sigma_i^- M)^*)$. Now, we use (4.5) and Proposition 2.45 to obtain

$$\underline{\text{rank}} M = \underline{\text{rank}} \Sigma_i \Sigma_i^- M + \underline{\text{rank}} \text{sub}_i M = s_i (\underline{\text{rank}} \Sigma_i^- M) + \underline{\text{rank}} \text{sub}_i M.$$

This shows that the equality $F_{s_i w}(M) = 0$ is equivalent to $F_w(\Sigma_i^- M) = 0$. Thus, we obtain our assertion by induction on $l(w)$. \square

Then, we review the definition and basic properties of Harder-Narasimhan polytopes of objects in Abelian categories developed in the work of Baumann-Kamnitzer-Tingley [BKT14].

Definition 4.30. Let $K_0(\mathcal{A})$ be the Grothendieck group of an Abelian category \mathcal{A} . For $T \in \mathcal{A}$, we consider the convex hull $\text{Pol}(T)$ in $K_0(\mathcal{A}) \otimes_{\mathbb{Z}} \mathbb{R}$ of all of points $[X]$ defined by subobjects $X \subseteq T$. We refer to this $\text{Pol}(T)$ as the *HN (=Harder-Narasimhan) polytope* of T .

Theorem 4.31 ([BKT14, §3.2]). Under the setting of the above definition, we have the following:

- (1) The HN polytope $\text{Pol}(T)$ has $P_\theta = \{x \in \text{Pol}(T) \mid \langle \theta, x \rangle = \psi_{\text{Pol}(T)}(\theta)\}$ as faces, where $\theta \in K_0(\mathcal{A} \otimes_{\mathbb{Z}} \mathbb{R})^*$ and $\psi_{\text{Pol}(T)}$ is a linear function $\mathbb{R}^n \rightarrow \mathbb{R}$ which θ maps to its maximum on $\text{Pol}(T)$.
- (2) If x is a vertex of the HN polytope $\text{Pol}(T)$, then there is a unique subobject X of T such that $[X] = x$ up to isomorphism.
- (3) Let $T = T_0 \supseteq T_1 \supseteq \cdots \supseteq T_l = 0$ be a HN filtration in \mathcal{A} . Then, each $[T_i]$ is a vertex of $\text{Pol}(T)$.

We define torsion classes and torsion-free classes determined by stability conditions as follows:

Definition 4.32. Let $\theta: K_0(\mathcal{A}) \rightarrow \mathbb{R}$ for $\theta \in \mathbb{R}^n$ be a group homomorphism. Then, we define

$$\begin{aligned} \mathcal{T}_\theta &:= \{M \in \mathcal{A} \mid \langle \theta, [N] \rangle > 0 \text{ for any quotient } N \text{ of } M\} \\ \overline{\mathcal{T}}_\theta &:= \{M \in \mathcal{A} \mid \langle \theta, [N] \rangle \geq 0 \text{ for any quotient } N \text{ of } M\} \\ \mathcal{F}_\theta &:= \{M \in \mathcal{A} \mid \langle \theta, [L] \rangle < 0 \text{ for any submodule } L \text{ of } M\} \\ \overline{\mathcal{F}}_\theta &:= \{M \in \mathcal{A} \mid \langle \theta, [L] \rangle \leq 0 \text{ for any submodule } L \text{ of } M\} \end{aligned}$$

We refer to objects in $\mathcal{R}_\theta := \overline{\mathcal{T}}_\theta \cap \overline{\mathcal{F}}_\theta$ as θ -semistable objects.

It is well-known that $(\overline{\mathcal{T}}_\theta, \mathcal{F}_\theta)$ and $(\mathcal{T}_\theta, \overline{\mathcal{F}}_\theta)$ are torsion pairs with $\mathcal{T}_\theta \subseteq \overline{\mathcal{T}}_\theta$ in \mathcal{A} for any $\theta \in \mathbb{R}^n$. Here, we review a connection between stability conditions and g -vectors. By the work of Brüstle-Smith-Treffinger [BST19, Proposition 3.27, Remark 3.28] (see also Yurikusa [Yur18]) which developed stability conditions for general finite dimensional algebras by the τ -tilting theory, we

know the following proposition by applying their theorem to our setting by Definition-Proposition 4.25:

Proposition 4.33. Let $w \in W$ and let $\theta \in C(w)$. We define

$$\overline{\mathcal{T}}_\theta := \{M \in \text{rep } \Pi \mid \langle \theta, [N] \rangle \geq 0 \text{ for any quotient } N \text{ of } M\}$$

and

$$\mathcal{F}_\theta := \{M \in \text{rep } \Pi \mid \langle \theta, [L] \rangle < 0 \text{ for any submodule } L \text{ of } M\}.$$

Then, $\overline{\mathcal{T}}_\theta = \text{Fac } I_w$ and $\mathcal{F}_\theta = \text{Sub } \Pi/I_w$ for $\theta \in C(w)$. \square

We can consider a three-step filtration $0 \subseteq T_\theta^{\min} \subseteq T_\theta^{\max} \subseteq M$ of torsion submodules $T_\theta^{\min} \in \overline{\mathcal{T}}_\theta$ and $T_\theta^{\max} \in \mathcal{F}_\theta$ for $M \in \text{rep } \Pi$. We compare this operation and Weyl group combinatorics in the case of $\mathcal{A} = \text{rep } \Pi$. Now, we relate these torsion pairs with the Weyl group.

Definition 4.34. For $J \subseteq Q_0$, we define

$$F_J = \{\theta \in \mathbb{R}^n \mid \langle \theta, \alpha_j \rangle = 0 \text{ for any } j \in J \text{ and } \langle \theta, \alpha_i \rangle > 0 \text{ for any } i \in Q_0 \setminus J\}.$$

We say that $w \in W$ is J -reduced on the right if $l(ws_j) > l(w)$ for any $j \in J$. We define a parabolic subgroup $W_J := \langle s_j \mid j \in J \rangle$ of W . Let w_J be the longest element of W_J .

The following Lemma is an analogue of a result of Buan-Iyama-Reiten-Scott [BIRS09, Proof of Theorem III.3.5].

Lemma 4.35. Let $J \subseteq Q_0$. Let C' and D' respectively be the GCM and its symmetrizer associated with the quiver determined by $Q_0 \setminus J$. Let $\Pi' := \Pi(C', D')$. Then, we have an algebra isomorphism $\Pi' \cong \Pi/I_{w_J}$. In particular, $\text{Sub } \Pi/I_{w_J} = \text{rep } \Pi'$.

Proof. Let $I_{Q'} := \Pi(\sum_{i \in Q_0 \setminus J} e_i)\Pi$. Namely, we have $\Pi/I_{Q'} \cong \Pi'$. We enough to prove $I_{w_J} = I_{Q'}$. By definition of $I_{Q'}$, the two-sided ideal $I_{Q'}$ contains all two-sided ideals I such that any composition factor of Π/I is S_i for $i \in J$. Since w_J is the longest element of W_J , we have $l(w_J s_i) < l(w_J)$. We have $I_{w_J} I_i = I_{w_J}$ for any $i \in J$ by Theorem 2.34. On the other hand, any composition factor of Π/I_{w_J} is S_i for $i \in J$ by our choice of w_J and definition of I_{w_J} . Thus, we obtain our assertion. \square

The following result is an analogue of the work of Baumann-Kamnitzer-Tingley [BKT14, Theorem 5.18 and Remark 5.19] to our setting:

Proposition 4.36. Keep the notation in Theorem 4.4. Let $\theta \in F_J$ and let $w \in W$ such that w is J -reduced on the right.

- (1) For any crystal module M , we have $T_{w\theta}^{\min} = T_{ww_J}$. In particular, we have an equality

$$(\mathcal{T}_{w\theta}, \overline{\mathcal{F}}_{w\theta}) = (\text{Fac } I_{ww_J}, \text{Sub } \Pi/I_{ww_J}).$$

- (2) For any $w \in W$, we have $\mathcal{R}_{w\theta} \cong \text{rep } \Pi/I_{w_J}$. In particular, $\mathcal{R}_{w\theta}$ is in common for all possible w .

Proof. For any crystal module M , we have enough to show that $T_{ww_J} \in \mathcal{T}_{w\theta}$ and $M/T_{ww_J} \in \overline{\mathcal{F}}_{w\theta}$ by Theorem 4.16. We have $T_{ww_J} = I_w \otimes_\Pi X$ for some locally free module $X \in \mathcal{F}_w \cap \text{Fac } I_{w_J}$ by arguments in §2.6. If we assume $T_{ww_J} \neq 0$, then $X \notin \text{Sub } \Pi/I_{w_J}$. So, rank X cannot be written

as a linear combination of elements in $\{\alpha_j \mid j \in J\}$, and $\langle w\theta, [T_{ww_J}] \rangle = (w\theta, \underline{\text{rank}} T_{ww_J})_C = (\theta, \underline{\text{rank}} X)_C > 0$ by $\theta \in F_J$ and Proposition 2.45. Thus, we obtain $T_{ww_J} \in \mathcal{T}_{w\theta}$. On the other hand, M/T_{ww_J} has a rank vector in \mathbb{N} -linear combination of roots in $N_{ww_J} = \{s_{i_1} \cdots s_{i_{k-1}} \alpha_{i_k} \mid 1 \leq k \leq r\}$ for a reduced expression $ww_J = s_{i_1} \cdots s_{i_r}$ by Theorem 4.5 and Theorem 3.11. We have $\langle w\theta, [M/T_{ww_J}] \rangle = (w\theta, \underline{\text{rank}} M/T_{ww_J})_C = (ww_J\theta, \underline{\text{rank}} M/T_{ww_J})_C \leq 0$. So, we have our assertion. The second assertion follows from the equivalence $\text{Sub } \Pi/I_{w_J} \cong \text{Sub } \Pi/I_{ww_J} \cap \text{Fac } I_w$ for $l(u) + l(v) = l(uv)$ by Proposition 2.44. Thus, we obtain $\mathcal{R}_\theta \cong \text{Sub } \Pi/I_{w_J} \cong \text{rep } \Pi/I_{w_J}$ by the first assertion, Proposition 4.33 and Lemma 4.35. \square

Remark 4.37. By the K -duality, we obtain an involution of HN polytopes determined by

$$\overline{\mathcal{T}}_{w\theta} = \text{Fac } I_w \mapsto \text{Sub } \Pi/I_{ww_0} = \overline{\mathcal{F}}_{-w\theta},$$

$$\mathcal{F}_{w\theta} = \text{Sub } \Pi/I_w \mapsto \text{Fac } I_{ww_0} = \mathcal{T}_{-w\theta},$$

and

$$\mathcal{R}_{w\theta} \mapsto \mathcal{R}_{-w\theta}.$$

We define a *generalized Harder-Narasimhan polytope* as follows:

Definition 4.38. For each maximal irreducible component Z of $\text{lrr}(\Pi(\mathbf{r}))^{\max}$ and a generic module T of this component, we define

$$P(T) := \{(x_1/c_1, \dots, x_n/c_n) \mid x = (x_1, \dots, x_n) \in \text{Pol}(T)\},$$

where $\text{diag}(c_1, \dots, c_n)$ is a symmetrizer of C . We call this polytope the *generalized HN polytope* of T . Let $\psi_{P(T)}$ be a linear function $\mathbb{R}^n \rightarrow \mathbb{R}$ which θ maps to its maximum on $P(T)$.

By definition, generalized HN polytopes can be thought as a kind of HN polytopes for the category of locally free modules. By our construction, $P(T)$ for a crystal generic module T is GGMS (=Gelfand-Goresky-MacPherson-Serganova) in the sense of [BKT14, §2.6]. In particular, we have for any $\theta \in \mathbb{R}^n$

$$\psi_{P(T)}(\theta) = \langle \theta, \underline{\text{rank}} T_\theta^{\min} \rangle = \langle \theta, \underline{\text{rank}} T_\theta^{\max} \rangle.$$

We have the following analogue of [BKT14, Corollary 5.20].

Theorem 4.39. Let $J \subseteq Q_0$, let $\theta \in F_J$, and let $w \in W$ be J -reduced. We assume $\theta := \sum_{i \in Q_0} \langle \theta, \alpha_i \rangle \varpi_i$ is a dominant integral weight. Then, we have an equality

$$D_{\pm w\theta}(M) = \psi_{P(M)}(\pm w\theta).$$

Proof. We put $\psi'_{P(M)}(w\theta) := (\theta, \underline{\text{rank}} T_\theta^{\max})_C$ and

$$D'_{w\theta}(M) := \sum_{i \in Q_0} \langle \theta, \alpha_i \rangle \dim \text{Hom}_\Pi(N(w\varpi_i), M).$$

We calculate the $D'_{w\theta}(M)$:

$$\begin{aligned}
& \sum_{i \in Q_0} \langle \theta, \alpha_i \rangle \dim \operatorname{Hom}_{\Pi}(N(w\varpi_i), M) \\
&= \sum_{i \in Q_0} \langle \theta, \alpha_i \rangle \dim \operatorname{Hom}_{\Pi}(I_w \otimes_{\Pi} \Pi e_i, M) \\
&= \sum_{i \in Q_0} \langle \theta, \alpha_i \rangle \dim \operatorname{Hom}_{\Pi}(\Pi e_i, \operatorname{Hom}_{\Pi}(I_w, M)) \\
&= \sum_{i \in Q_0} \frac{1}{c_i} (\theta, \alpha_i)_C \dim e_i \operatorname{Hom}_{\Pi}(I_w, M) \\
&= (\theta, \underline{\operatorname{rank}} \operatorname{Hom}_{\Pi}(I_w, M))_C \\
&= (w\theta, \underline{\operatorname{rank}} I_w \otimes_{\Pi} \operatorname{Hom}_{\Pi}(I_w, M))_C \quad (\text{Proposition 2.45}) \\
&= (w\theta, \underline{\operatorname{rank}} M_w)_C \quad (\text{Remark 2.43}) \\
&= (w\theta, \underline{\operatorname{rank}} M_{w\theta}^{\max})_C \quad (\text{Proposition 4.33}) \\
&= \psi'_{P(M)}(w\theta) \quad (\text{cf. [BKT14, Proposition 2.16]}),
\end{aligned}$$

where M_w is the torsion submodule of M with respect to the torsion class $\operatorname{Fac} I_w$. We obtain our assertion $D_{w\theta}(M) = \psi_{P(M)}(w\theta)$ by Definition 4.26 and Definition-Proposition 4.25. Note that the fifth equality in the above equalities is derived as follows: Since M is a crystal module, $I_w \otimes_{\Pi} \operatorname{Hom}_{\Pi}(I_w, M)$ is an \mathbb{E} -filtered module in $\operatorname{Fac} I_w$. We have applied the functor $\operatorname{Hom}_{\Pi}(I_w, -)$ and have used Proposition 2.45 and the equivalence in the Proof of Proposition 2.42.

Now, we apply the functor $\operatorname{Hom}_{\Pi}(-, M^*)$ for an exact sequence

$$0 \rightarrow I_w \otimes_{\Pi} \Pi e_i \rightarrow \Pi e_i \rightarrow (\Pi/I_w) \otimes_{\Pi} \Pi e_i \rightarrow 0,$$

so that we have:

$$\begin{aligned}
0 &\rightarrow \operatorname{Hom}_{\Pi}((\Pi/I_w) \otimes_{\Pi} \Pi e_i, M^*) \rightarrow \operatorname{Hom}_{\Pi}(\Pi e_i, M^*) \\
&\rightarrow \operatorname{Hom}_{\Pi}(I_w \otimes_{\Pi} \Pi e_i, M^*) \rightarrow \operatorname{Ext}_{\Pi}^1((\Pi/I_w) \otimes_{\Pi} \Pi e_i, M^*) \rightarrow 0.
\end{aligned}$$

By using the generalized Crawley-Boevey formula and the above sequence, we have:

$$\begin{aligned}
& \dim \operatorname{Hom}_{\Pi}(M^*, (\Pi/I_w) \otimes_{\Pi} \Pi e_i) \\
&= -\dim \operatorname{Hom}_{\Pi}((\Pi/I_w) \otimes_{\Pi} \Pi e_i, M^*) + \operatorname{Ext}_{\Pi}^1((\Pi/I_w) \otimes_{\Pi} \Pi e_i, M^*) + (\underline{\operatorname{rank}}(\Pi/I_w) \otimes_{\Pi} \Pi e_i, \underline{\operatorname{rank}} M^*)_C \\
&= \dim \operatorname{Hom}_{\Pi}(I_w \otimes_{\Pi} \Pi e_i, M^*) - \dim \operatorname{Hom}_{\Pi}(\Pi e_i, M^*) + (\underline{\operatorname{rank}}(\Pi/I_w) \otimes_{\Pi} \Pi e_i, \underline{\operatorname{rank}} M^*)_C \\
&= \dim \operatorname{Hom}_{\Pi}(I_w \otimes_{\Pi} \Pi e_i, M^*) - \dim e_i M^* + (\varpi_i - w\varpi_i, \underline{\operatorname{rank}} M^*)_C.
\end{aligned}$$

Here, we used an equality $\underline{\text{rank}}((\Pi/I_w) \otimes_{\Pi} \Pi e_i) = \varpi_i - w\varpi_i$ obtained in the Proof of Theorem 3.11 for the third equality. Now, by multiplying $\frac{1}{c_i} \langle \theta, \alpha_i \rangle$ and summing up for $i \in Q_0$, we obtain

$$\begin{aligned} & \sum_{i \in Q_0} \frac{1}{c_i} \langle \theta, \alpha_i \rangle \dim \text{Hom}_{\Pi}(N(w\varpi_i), M) \\ &= \sum_{i \in Q_0} \frac{1}{c_i} \langle \theta, \alpha_i \rangle \dim \text{Hom}_{\Pi}(M^*, (\Pi/I_w) \otimes_{\Pi} \Pi e_i) \quad (\text{Theorem 3.11}) \\ &= \left(\sum_{i \in Q_0} \frac{1}{c_i} \langle \theta, \alpha_i \rangle \dim \text{Hom}_{\Pi}(N(w\varpi_i), M^*) \right) - \langle \theta, \underline{\text{rank}} M^* \rangle + \langle \theta - w\theta, \underline{\text{rank}} M^* \rangle \\ &= \psi_{P(M^*)}(w\theta) - \langle w\theta, \underline{\text{rank}} M \rangle. \end{aligned}$$

Thus, we obtain the assertion because $P(M^*)$ is the image of $P(M)$ by an involution $\mathbf{r} \mapsto \underline{\text{rank}} M - \mathbf{r}$ derived from Remark 4.37. \square

By our argument Proposition 4.36 and Theorem 4.39, we obtain the following corollary:

Corollary 4.40. Let $J \subseteq Q_0$, let $\theta \in F_J$, and let $w \in W$ be J -reduced. Let T be a crystal module and let $X := \text{Hom}_{\Pi}(I_w, T_{w\theta}^{\max}/T_{w\theta}^{\min}) \in \text{rep } \Pi/I_{wJ}$. Then, we have an equality

$$\{x \in P(T) \mid \langle w\theta, x \rangle = D_{w\theta}(T)\} = \underline{\text{rank}} T_{w\theta}^{\min} + wP(X).$$

Now, we compare our generalized HN polytopes and Kamnitzer's Mirković-Vilonen polytopes [Kam10, Kam07]. We recall the definition of MV polytopes:

Definition 4.41. Let $V := R \otimes_{\mathbb{Z}} \mathbb{R}$ for root lattice R associated with the GCM C , and let V^* be its \mathbb{R} -dual space. By Hahn-Banach theorem, a non-empty convex subset P of V can be written as $\{v \in V \mid \langle v, \alpha \rangle \leq \psi_P(\alpha) \text{ for any } \alpha \in V^*\}$ where $\psi_P: V^* \rightarrow \mathbb{R}$ maps α to the maximal value such that α takes on P . We say that a polytope P is a pseudo-Weyl polytope if ψ_P is linear on each Weyl chamber. We write $A_{\gamma} := \psi_P(\gamma)$ for $\gamma \in \Gamma$. We say that a pseudo-Weyl polytope P is an MV polytope if the tuple of data $(A_{\gamma})_{\gamma \in \Gamma}$ satisfies the following Berenstein-Zelevinsky data (BZ1)-(BZ3):

- (BZ1): Each A_{γ} is an integer and each $A_{-\varpi_i} = 0$.
- (BZ2): We have the edge inequalities $A_{-w\varpi_i} + A_{-ws_i\varpi_i} + \sum_{j \in I, j \neq i} c_{ji} A_{-w\varpi_j} \geq 0$.
- (BZ3): All of A_{γ} satisfy tropical Plücker relations.

Here, we say that $(A_{\gamma})_{\gamma \in \Gamma}$ satisfies tropical Plücker relations at (w, i, j) for $w \in W$ and $i, j \in Q_0$ if $c_{ij} = 0$,

- (1) if $c_{ij} = c_{ji} = -1$, then

$$A_{-ws_i\varpi_i} + A_{-ws_j\varpi_j} = \min(A_{-w\varpi_i} + A_{-ws_i s_j \varpi_j}, A_{-ws_j s_i \varpi_i} + A_{-w\varpi_j});$$

(2) if $c_{ij} = -1, c_{ji} = -2$, then

$$\begin{aligned} & A_{-ws_j\varpi_j} + A_{-ws_i s_j \varpi_j} + A_{-ws_i \varpi_i} \\ = & \min(2A_{-ws_i s_j \varpi_j} + A_{-w\varpi_i}, 2A_{-w\varpi_j} + A_{-ws_i s_j s_i \varpi_i}, A_{-w\varpi_j} + A_{-ws_j s_i s_j \varpi_j} + A_{-ws_i \varpi_i}), \\ & A_{-ws_j s_i \varpi_i} + 2A_{-ws_i s_j \varpi_j} + A_{-ws_i \varpi_i} + A_{-ws_i \varpi_i} \\ = & \min(2A_{-w\varpi_j} + 2A_{-ws_i s_j s_i \varpi_i}, 2A_{-ws_j s_i s_j \varpi_j} + 2A_{-ws_i \varpi_i}, A_{-ws_i s_j s_i \varpi_i} + 2A_{-ws_i s_j \varpi_j} + A_{-w\varpi_i}); \end{aligned}$$

(3) if $c_{ij} = -2, c_{ji} = -1$, then

$$\begin{aligned} & A_{-ws_j s_i \varpi_i} + A_{-ws_i \varpi_i} + A_{-ws_i s_j \varpi_j} \\ = & \min(2A_{-ws_i \varpi_i} + A_{-ws_j s_i s_j \varpi_j}, 2A_{-ws_i s_j s_i \varpi_i} + A_{-w\varpi_j}, A_{-ws_i s_j s_i \varpi_i} + 2A_{-ws_i s_j \varpi_j} + A_{-w\varpi_i}), \\ & A_{-ws_j \varpi_j} + 2A_{-ws_i \varpi_i} + A_{-ws_i s_j \varpi_j} \\ = & \min(2A_{-ws_i s_j s_i \varpi_i} + 2A_{-w\varpi_j}, 2A_{-w\varpi_i} + 2A_{-ws_i s_j \varpi_j}, A_{-w\varpi_j} + 2A_{-ws_i \varpi_i} + A_{-ws_j s_i s_j \varpi_j}). \end{aligned}$$

Note that any maximal irreducible component $Z \in \text{lrr}(\Pi(\mathbf{r}))^{\max}$ contains an open dense subset on which D_γ takes a constant value, where $\gamma \in \Gamma$. Let $D_\gamma(Z)$ denote this constant value.

Theorem 4.42. For each maximal irreducible component $Z \in \text{lrr}(\Pi(\mathbf{r}))^{\max}$ and a generic module T of this component, $(D_\gamma(Z))_{\gamma \in \Gamma}$ describes BZ data. In particular,

$$P(T) = \{v \in \mathbb{R}^n \mid \langle \gamma, v \rangle \leq D_\gamma(T), \gamma \in \Gamma\}$$

is an MV polytope associated with the Langlands dual root datum of C except for type G_2 .

Proof. If we take D as the minimal symmetrizer of C , then any generalized preprojective algebra associated with C is isomorphic to $\Pi(C, kD)$ for some $k \in \mathbb{Z}_{>0}$. Then, we have the reduction functor $\text{rep}(\Pi(C, kD)) \ni M \mapsto M/\varepsilon M \in \text{rep}(\Pi(C, D))$, where $\varepsilon := \sum_{i \in Q_0} \varepsilon_i^{c_i}$ (cf. Geiss-Leclerc-Schröer [GLS18a]). Through this reduction functor, it is enough to consider minimal symmetrizer cases. The condition (BZ1) is clear from Theorem 3.7, and (BZ2) is a consequence of Proposition 4.28. By Corollary 4.40, we have enough to show that each 2-face of a generalized HN polytope describes tropical Plücker relations. Namely, we enough to consider HN filtrations of generic modules in the category of Π/I_{w_J} -modules, where we put $J = \{i, j\}$ with $i \neq j$ by Lemma 4.35 and Proposition 4.36. Since the case of $c_{ij} = 0$ is clear, we consider cases of $c_{ij} \neq 0$. We check that the multiplicities of layer modules in our HN filtrations satisfy the following relations case by case by using classification theorems of generic modules:

- (1) If $c_{ij} = c_{ji} = -1$, then any generic module has a form $X_1^{\oplus a} \oplus X_2^{\oplus b} \oplus X_3^{\oplus c}$. Let (a_1, a_2, a_3) (resp. (a'_1, a'_2, a'_3)) be the multiplicities of layer modules in the HN filtration with respect to a reduced expression $\mathbf{i} = (i, j, i)$ (resp. $\mathbf{i}' = (j, i, j)$). Then, we can prove they satisfy the following list (a)–(b) of equalities by calculation, where P_i (resp. E_i) is the indecomposable projective (resp. generalized simple) module with respect to a vertex i . In particular, they satisfy relations (4.11):

$$a'_1 = a_2 + a_3 - p, a'_2 = p, a'_3 = a_1 + a_2 - p, \quad (4.11)$$

where $p = \min(a_1, a_3)$.

$$(a) (X_1, X_2, X_3) = (P_1, P_2, E_1)$$

$$(a_1, a_2, a_3) = (a + c, b, a)$$

$$(a'_1, a'_2, a'_3) = (b, a, b + c)$$

$$p = a$$

$$(b) (X_1, X_2, X_3) = (P_1, P_2, E_2)$$

$$(a_1, a_2, a_3) = (a, b, a + c)$$

$$(a'_1, a'_2, a'_3) = (b + c, a, b)$$

$$p = a$$

- (2) If $c_{ij} = -2, c_{ji} = -1$, then any generic module has a form $X_1^{\oplus a} \oplus X_2^{\oplus b} \oplus X_3^{\oplus c} \oplus X_4^{\oplus d}$ where X_i is one of the 8 Loewy series of modules below ($i = 2, j = 1$) and (X_1, X_2, X_3, X_4) is one of the below list (cf. [GLS18b, §8.2.2]).

$$P_1 = \begin{matrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{matrix}; P_2 = \begin{matrix} & & & 2 \\ & & & 1 \\ & & 1 & \\ & 1 & & \end{matrix}; E_1 = \begin{matrix} 1 \\ 1 \end{matrix}; E_2 = 2;$$

$$T_1 = \begin{matrix} 1 \\ 2 \end{matrix}; T_2 = \begin{matrix} 2 \\ 1 \end{matrix}; T_3 = 2 \begin{matrix} 1 \\ 2 \end{matrix}; T_4 = \begin{matrix} 2 & & & \\ & 1 & & \\ & & 1 & \\ & & & 2 \end{matrix}.$$

Now, let (a_1, a_2, a_3, a_4) (resp. (a'_1, a'_2, a'_3, a'_4)) be the multiplicities of layer modules in the HN filtration from a reduced expression $\mathbf{i} = (j, i, j, i)$ (resp. $\mathbf{i}' = (i, j, i, j)$). Then, they satisfy the following list (a)–(f) of equalities by calculating case by case (cf. Example 4.7). In particular, they satisfy relations (4.12):

$$a'_1 = a_2 + a_3 + a_4 - p_1; a'_2 = 2p_1 - p_2; a'_3 = p_2 - p_1; a'_4 = a_1 + 2a_2 + a_3 - p_2, \quad (4.12)$$

where

$$p_1 = \min(a_1 + a_2, a_1 + a_4, a_3 + a_4),$$

$$p_2 = \min(a_1 + 2a_2, a_1 + 2a_4, a_3 + 2a_4).$$

$$(a) (X_1, X_2, X_3, X_4) = (P_1, P_2, E_1, T_1)$$

$$(a_1, a_2, a_3, a_4) = (b, a, b + d, a + c)$$

$$(a'_1, a'_2, a'_3, a'_4) = (a + c + d, b, a, b + d)$$

$$(p_1, p_2) = (a + b, 2a + b)$$

$$(b) (X_1, X_2, X_3, X_4) = (P_1, P_2, E_1, T_2)$$

$$(a_1, a_2, a_3, a_4) = (b + d, a, b, a + c + d)$$

$$(a'_1, a'_2, a'_3, a'_4) = (a + c, b + d, a, b)$$

$$(p_1, p_2) = (a + b + d, 2a + b + d)$$

$$(c) (X_1, X_2, X_3, X_4) = (P_1, P_2, E_2, T_3)$$

$$(a_1, a_2, a_3, a_4) = (b + c, a + d, b, a)$$

$$(a'_1, a'_2, a'_3, a'_4) = (a + d, b, a, b + c + 2d)$$

$$(p_1, p_2) = (a + b, 2a + b)$$

$$(d) (X_1, X_2, X_3, X_4) = (P_1, P_2, E_2, T_4)$$

$$(a_1, a_2, a_3, a_4) = (b + c + 2d, a, b, a + d)$$

$$(a'_1, a'_2, a'_3, a'_4) = (a, b, a + d, b + c)$$

$$(p_1, p_2) = (a + b + d, 2a + b + 2d)$$

$$(e) (X_1, X_2, X_3, X_4) = (P_1, P_2, T_1, T_3)$$

$$(a_1, a_2, a_3, a_4) = (b, a + d, b + c, a)$$

$$(a'_1, a'_2, a'_3, a'_4) = (a + c + d, b, a, b + c + 2d)$$

$$(p_1, p_2) = (a + b, 2a + b)$$

$$(f) (X_1, X_2, X_3, X_4) = (P_1, P_2, T_2, T_4)$$

$$(a_1, a_2, a_3, a_4) = (b + c + 2d, a, b, a + c + d)$$

$$(a'_1, a'_2, a'_3, a'_4) = (a, b + c, a + d, b)$$

$$(p_1, p_2) = (a + b + c + d, 2a + b + c + 2d)$$

- (3) If $c_{ij} = -1, c_{ji} = -2$, then any generic module has a form $X_1^{\oplus a} \oplus X_2^{\oplus b} \oplus X_3^{\oplus c} \oplus X_4^{\oplus d}$ where X_i is one of the 8 Loewy series of modules below ($i = 1, j = 2$) and (X_1, X_2, X_3, X_4) is one of the below list (cf. [GLS18b, §8.2.2]).

$$P_1 = \begin{array}{c} 1 \\ 2 \\ 1 \end{array} \begin{array}{c} 1 \\ 2 \\ 1 \end{array}; P_2 = \begin{array}{c} 2 \\ 1 \\ 2 \end{array}; E_1 = \begin{array}{c} 1 \\ 1 \end{array}; E_2 = \begin{array}{c} 2 \\ 2 \end{array};$$

$$T_1 = \begin{array}{c} 1 \\ 1 \\ 2 \end{array}; T_2 = \begin{array}{c} 2 \\ 1 \\ 1 \end{array}; T_3 = \begin{array}{c} 1 \\ 2 \\ 1 \end{array}; T_4 = \begin{array}{c} 2 \\ 1 \\ 1 \end{array} \begin{array}{c} 2 \\ 2 \\ 2 \end{array}.$$

Now, let (a_1, a_2, a_3, a_4) (resp. (a'_1, a'_2, a'_3, a'_4)) be the multiplicities of layer modules in the HN filtration from a reduced expression $\mathbf{i} = (j, i, j, i)$ (resp. $\mathbf{i}' = (i, j, i, j)$). Then, they satisfy the following list (a)–(f) of equalities by calculating case by case (cf. Example 4.7). In particular, they satisfy relations (4.13):

$$a'_1 = a_2 + 2a_3 + a_4 - p_2, a'_2 = p_2 - p_1, a'_3 = 2p_1 - p_2, a'_4 = a_1 + a_2 + a_3 - p_1, \quad (4.13)$$

where

$$p_1 = \min(a_1 + a_2, a_1 + a_4, a_3 + a_4),$$

$$p_2 = \min(2a_1 + a_2, 2a_1 + a_4, 2a_3 + a_4).$$

- (a) $(X_1, X_2, X_3, X_4) = (P_1, P_2, E_1, T_1)$
 $(a_1, a_2, a_3, a_4) = (b, a, b + d, a + c)$
 $(a'_1, a'_2, a'_3, a'_4) = (a + c + d, b, a, b + d)$
 $(p_1, p_2) = (a + b + d, 2a + b + d)$
- (b) $(X_1, X_2, X_3, X_4) = (P_1, P_2, E_1, T_2)$
 $(a_1, a_2, a_3, a_4) = (b + d, a, b, a + c + d)$
 $(a'_1, a'_2, a'_3, a'_4) = (a + c, b + d, a, b)$
 $(p_1, p_2) = (a + b, 2a + b)$
- (c) $(X_1, X_2, X_3, X_4) = (P_1, P_2, E_2, T_3)$
 $(a_1, a_2, a_3, a_4) = (b + c, a + d, b, a)$
 $(a'_1, a'_2, a'_3, a'_4) = (a + d, b, a, b + c + 2d)$
 $(p_1, p_2) = (a + b + d, 2a + b + 2d)$
- (d) $(X_1, X_2, X_3, X_4) = (P_1, P_2, E_2, T_4)$
 $(a_1, a_2, a_3, a_4) = (a, b, a + d, b + c)$
 $(a'_1, a'_2, a'_3, a'_4) = (b + c + 2d, a, b, a + d)$
 $(p_1, p_2) = (a + b, 2a + b)$
- (e) $(X_1, X_2, X_3, X_4) = (P_1, P_2, T_1, T_3)$
 $(a_1, a_2, a_3, a_4) = (a + c + d, b, a, b + c + 2d)$
 $(a'_1, a'_2, a'_3, a'_4) = (b, a + d, b + c, a)$
 $(p_1, p_2) = (a + b + c + d, 2a + b + c + 2d)$
- (f) $(X_1, X_2, X_3, X_4) = (P_1, P_2, T_2, T_4)$
 $(a_1, a_2, a_3, a_4) = (a, b + c, a + d, b)$
 $(a'_1, a'_2, a'_3, a'_4) = (b + c + 2d, a, b, a + c + d)$
 $(p_1, p_2) = (a + b, 2a + b)$

In fact, these are equivalent to the tropical Plücker relations [Kam10, Proposition 5.2]. \square

Remark 4.43. Since we have Theorem 4.21 and Theorem 4.31 also for type G_2 , two Harder-Narasimhan filtrations of generic modules of maximal irreducible components with respect to two reduced expressions of the longest element have Lusztig data as multiplicities of layer modules in general. In particular, generalized Harder-Narasimhan polytopes of generic modules of maximal irreducible components satisfies BZ data also for type G_2 (cf. McNamara [McN11, §7]).

Since A_2 and B_2 type generalized preprojective algebras with minimal symmetrizers are special biserial algebras, their module categories are understood by Auslander-Reiten theory [BR87]. However, for type G_2 case, this approach for explicit calculations for 2-faces based is invalid.

Especially, there exist infinitely many maximal rigid objects ². But, also in this case, we expect that maximal rigid objects determined by some combinatorics of two-sided ideals I_w describe BZ data for type G_2 . A general theory about this direction will appear in the author's future work.

Finally, we relate the crystal structure on nilpotent varieties in [GLS18b] and that of the set \mathcal{MV} of MV polytopes as a generalization of [BK12, Theorem 6.3]. We recall the crystal structure on \mathcal{MV} , which has been developed by Kamnitzer [Kam07, §3.6].

Definition-Proposition 4.44. Let $P \in \mathcal{MV}$ with vertex data $(\mu_w)_{w \in W}$ and BZ data $(A_\gamma)_{\gamma \in \Gamma}$ and let

$$c := A_{w\varpi_i} + A_{ws_i\varpi_i} + \sum_{j \in I, j \neq i} c_{ji} A_{w\varpi_j}.$$

Then, we define

$$\begin{aligned} \text{wt}(P) &:= \mu_{w_0}; & \varphi_i(P) &= c; & \varepsilon_i(P) &= c - \langle \alpha_i, \text{wt}(P) \rangle; \\ \text{wt}(\tilde{e}_i P) &= \text{wt}(P) + \alpha_i; & \text{wt}(\tilde{f}_i P) &= \text{wt}(P) - \alpha_i \quad (\varphi_i(P) > 0). \end{aligned}$$

Then, $(\mathcal{MV}, \text{wt}((-)), \tilde{e}_i, \tilde{f}_i, \varepsilon_i, \varphi_i)$ is isomorphic to $B(-\infty)$ as crystals.

Theorem 4.45. The map $P(-): \mathcal{B} \rightarrow \mathcal{MV}$ is a crystal bijection.

Proof. Let $Z \in \mathcal{B}$ and let T be a generic point of Z . Then, we have

$$\begin{aligned} \text{rank}_i \text{fac}_i T &= \frac{1}{c_i} \dim \text{Hom}_\Pi(T, E_i) \\ &= \frac{1}{c_i} \dim \text{Hom}_\Pi(E_i, T^*) \\ &= D_{s_i\varpi_i}(T) - \langle s_i\varpi_i, \underline{\text{rank}} T \rangle \\ &= D_{s_i\varpi_i}(T) + D_{\varpi_i}(T) + \sum_{j \in J, j \neq i} c_{ji} D_{\varpi_j}(T) \end{aligned}$$

Hence, $T \mapsto \text{rank}_i \text{fac}_i T$ takes a constant value

$$D_{s_i\varpi_i}(P(Z)) + D_{\varpi_i}(P(Z)) + \sum_{j \in J, j \neq i} c_{ji} D_{\varpi_j}(P(Z)) = \varphi_i(P(T))$$

on an open dense subset in Z . Let $i \in Q_0$ and let $Z' := \tilde{f}_i^{\max} Z$. Now, we have

$$\text{wt}(P(Z')) = \text{wt}(P(Z)) - \varphi_i(P(Z))\alpha_i = \text{wt}(\tilde{f}_i^{\max} P(Z)).$$

Now, $D_\gamma(T) = D_\gamma(Z)$ and $D_\gamma(T_{s_i}) = D_\gamma(Z')$ for any chamber weight γ . Since $D_\gamma(T) = D_\gamma(T_{s_i})$ for $\langle \gamma, \alpha_i \rangle \leq 0$, we obtain $D_\gamma(Z) = D_\gamma(Z')$. Thus, $P(-)$ preserves \tilde{f}_i^{\max} . Similarly, $P(-)$ preserves \tilde{e}_i^{\max} . The assertions about φ_i and ε_i are directly proved from definitions. \square

²The author was informed this result by C. Geiß and B. Leclerc while his stayed at Caen. He thanks for their kindness.

Acknowledgement. The author thanks Ryo Fujita, Christof Geiß, Yuya Ikeda, Syu Kato, Yoshiyuki Kimura, Bernard Leclerc and Yuya Mizuno for their valuable discussions and comments on this work. Especially, the author expresses deep gratitude to his supervisor Syu Kato, and appreciates for his guidance on geometric representation theory and his kind encouragement for many years. The author was indebted to Bernard Leclerc because he carefully read a draft of this paper and was kind to give many constructive suggestions. A part of this work about Harder-Narasimhan polytopes is motivated from a conversation with Yuya Mizuno and Yoshiyuki Kimura. He also thanks all of his colleagues for their special effort in difficult situations of the COVID-19 pandemic. Finally, the author was indebted to anonymous referees for their constructive suggestions.

REFERENCES

- [AIR14] T. Adachi, O. Iyama, and I. Reiten, τ -tilting theory, *Compos. Math.* **150** (2014), no. 3, 415–452. [MR3187626](#)
- [AIR15] C. Amiot, O. Iyama, and I. Reiten, *Stable categories of Cohen-Macaulay modules and cluster categories*, *Amer. J. Math.* **137** (2015), no. 3, 813–857. [MR3357123](#)
- [AR85] M. Auslander and I. Reiten, *Modules determined by their composition factors*, *Illinois J. Math.* **29** (1985), no. 2, 280–301. [MR784524](#)
- [AS81] M. Auslander and S. O. Smalø, *Almost split sequences in subcategories*, *J. Algebra* **69** (1981), no. 2, 426–454. [MR0617088](#)
- [Asa20] S. Asai, *Semibricks*, *Int. Math. Res. Not. IMRN* **16** (2020), 4993–5054. [MR4139031](#)
- [Asa22] ———, *Bricks over preprojective algebras and join-irreducible elements in Coxeter groups*, *J. Pure Appl. Algebra* **226** (2022), no. 1, 106812. [MR4273083](#)
- [ASS06] I. Assem, D. Simson, and A. Skowroński, *Elements of the representation theory of associative algebras. Vol. 1*, London Mathematical Society Student Texts, vol. 65, Cambridge University Press, Cambridge, 2006. [MR2197389](#)
- [BB05] A. Björner and F. Brenti, *Combinatorics of Coxeter groups*, Graduate Texts in Mathematics, vol. 231, Springer, New York, 2005. [MR2133266](#)
- [BIRS09] A B Buan, O Iyama, I Reiten, and J Scott, *Cluster structures for 2-Calabi-Yau categories and unipotent groups*, *Compos. Math.* **145** (2009), no. 4, 1035–1079. [MR2521253](#)
- [BK12] P. Baumann and J. Kamnitzer, *Preprojective algebras and MV polytopes*, *Represent. Theory* **16** (2012), 152–188. [MR2892443](#)
- [BKK21] P. Baumann, J. Kamnitzer, and A. Knutson, *The Mirković-Vilonen basis and Duistermaat-Heckman measures*, *Acta Math.* **227** (2021), no. 1, 1–101. [MR4346265](#)
- [BKT14] P. Baumann, J. Kamnitzer, and P. Tingley, *Affine Mirković-Vilonen polytopes*, *Publ. Math. Inst. Hautes Études Sci.* **120** (2014), 113–205. [MR3270589](#)
- [BR87] M. C. R. Butler and C. M. Ringel, *Auslander-Reiten sequences with few middle terms and applications to string algebras*, *Comm. Algebra* **15** (1987), no. 1-2, 145–179. [MR876976](#)
- [BST19] T. Brüstle, D. Smith, and H. Treffinger, *Wall and chamber structure for finite-dimensional algebras*, *Adv. Math.* **354** (2019), 106746, 31. [MR3989130](#)
- [BZ97] A. Berenstein and A. Zelevinsky, *Total positivity in Schubert varieties*, *Comment. Math. Helv.* **72** (1997), no. 1, 128–166. [MR1456321](#)
- [CBS02] W. Crawley-Boevey and J. Schröer, *Irreducible components of varieties of modules*, *J. Reine Angew. Math.* **553** (2002), 201–220. [MR1944812](#)
- [DIJ19] L. Demonet, O. Iyama, and G. Jasso, τ -tilting finite algebras, bricks, and g -vectors, *Int. Math. Res. Not. IMRN* **3** (2019), 852–892. [MR3910476](#)
- [Eno21] H. Enomoto, *Bruhat inversions in Weyl groups and torsion-free classes over preprojective algebras*, *Comm. Algebra* **49** (2021), no. 5, 2156–2189. [MR4232491](#)

- [FG19] C. Fu and S. Geng, *Tilting modules and support τ -tilting modules over preprojective algebras associated with symmetrizable Cartan matrices*, *Algebr. Represent. Theory* **22** (2019), no. 5, 1239–1260. MR4026632
- [Fu17] C. Fu, *c-vectors via τ -tilting theory*, *J. Algebra* **473** (2017), 194–220. MR3591148
- [GHKK18] M. Gross, P. Hacking, S. Keel, and M. Kontsevich, *Canonical bases for cluster algebras*, *J. Amer. Math. Soc.* **31** (2018), no. 2, 497–608. MR3758151
- [GLS05] C. Geiss, B. Leclerc, and J. Schröer, *Semicanonical bases and preprojective algebras*, *Ann. Sci. École Norm. Sup. (4)* **38** (2005), no. 2, 193–253. MR2144987
- [GLS11] C. Geiß, B. Leclerc, and J. Schröer, *Kac-Moody groups and cluster algebras*, *Adv. Math.* **228** (2011), no. 1, 329–433. MR2822235
- [GLS16] C. Geiß, B. Leclerc, and J. Schröer, *Quivers with relations for symmetrizable Cartan matrices III: Convolution algebras*, *Represent. Theory* **20** (2016), 375–413. MR3555157
- [GLS17] C. Geiss, B. Leclerc, and J. Schröer, *Quivers with relations for symmetrizable Cartan matrices I: Foundations*, *Invent. Math.* **209** (2017), no. 1, 61–158. MR3660306
- [GLS18a] C. Geiß, B. Leclerc, and J. Schröer, *Quivers with relations for symmetrizable Cartan matrices II: change of symmetrizers*, *Int. Math. Res. Not. IMRN* **2018** (2018), no. 9, 2866–2898. MR3801499
- [GLS18b] C. Geiss, B. Leclerc, and J. Schröer, *Quivers with relations for symmetrizable Cartan matrices IV: crystal graphs and semicanonical functions*, *Selecta Math. (N.S.)* **24** (2018), no. 4, 3283–3348. MR3848021
- [GLS18c] C. Geiß, B. Leclerc, and J. Schröer, *Quivers with relations for symmetrizable Cartan matrices V: Caldero-Chapoton formulas*, *Proc. Lond. Math. Soc. (3)* **117** (2018), no. 1, 125–148. MR3830892
- [GLS20] ———, *Rigid modules and Schur roots*, *Math. Z.* **295** (2020), no. 3–4, 1245–1277. MR4125687
- [GP79] I. M. Gel'fand and V. A. Ponomarev, *Model algebras and representations of graphs*, *Funktional. Anal. i Prilozhen.* **13** (1979), no. 3, 1–12. MR545362
- [HKW20] J. Hilburn, J. Kamnitzer, and A. Weekes, *BFN Springer Theory*, 2020. preprint, [arXiv:2004.14998v2](https://arxiv.org/abs/2004.14998v2).
- [Hum90] J. E. Humphreys, *Reflection groups and Coxeter groups*, *Cambridge Studies in Advanced Mathematics*, vol. 29, Cambridge University Press, Cambridge, 1990. MR1066460
- [Jia14] Y. Jiang, *Parametrizations of canonical bases and irreducible components of nilpotent varieties*, *Int. Math. Res. Not. IMRN* **2014** (2014), no. 12, 3263–3278. MR3217661
- [Kül17] J. Külshammer, *Pro-species of algebras I: Basic properties*, *Algebr. Represent. Theory* **20** (2017), no. 5, 1215–1238. MR3707912
- [Kac90] V. G. Kac, *Infinite-dimensional Lie algebras*, Third, Cambridge University Press, Cambridge, 1990. MR1104219
- [Kam07] J. Kamnitzer, *The crystal structure on the set of Mirković-Vilonen polytopes*, *Adv. Math.* **215** (2007), no. 1, 66–93. MR2354986
- [Kam10] ———, *Mirković-Vilonen cycles and polytopes*, *Ann. of Math. (2)* **171** (2010), no. 1, 245–294. MR2630039
- [Kas91] M. Kashiwara, *On crystal bases of the Q -analogue of universal enveloping algebras*, *Duke Math. J.* **63** (1991), no. 2, 465–516. MR1115118
- [Kel11] B. Keller, *On cluster theory and quantum dilogarithm identities*, *Representations of algebras and related topics*, 2011, pp. 85–116. MR2931896
- [KS97] M. Kashiwara and Y. Saito, *Geometric construction of crystal bases*, *Duke Math. J.* **89** (1997), no. 1, 9–36. MR1458969
- [KTW⁺19] J. Kamnitzer, P. Tingley, B. Webster, A. Weekes, and O. Yacobi, *Highest weights for truncated shifted Yangians and product monomial crystals*, *J. Comb. Algebra* **3** (2019), no. 3, 237–303. MR4011667
- [Lec16] B. Leclerc, *Cluster structures on strata of flag varieties*, *Adv. Math.* **300** (2016), 190–228. MR3534832
- [Lus00] G. Lusztig, *Semicanonical bases arising from enveloping algebras*, *Adv. Math.* **151** (2000), no. 2, 129–139. MR1758244
- [Lus90] ———, *Canonical bases arising from quantized enveloping algebras*, *J. Amer. Math. Soc.* **3** (1990), no. 2, 447–498. MR1035415
- [McN11] P. J. McNamara, *Metaplectic Whittaker functions and crystal bases*, *Duke Math. J.* **156** (2011), no. 1, 1–31. MR2746386

- [Miy86] Y. Miyashita, *Tilting modules of finite projective dimension*, Math. Z. **193** (1986), no. 1, 113–146. MR852914
- [Miz14] Y. Mizuno, *Classifying τ -tilting modules over preprojective algebras of Dynkin type*, Math. Z. **277** (2014), no. 3-4, 665–690. MR3229959
- [Mur19] K. Murakami, *On the module category of generalized preprojective algebras of Dynkin types*, 2019. preprint, [arXiv:1906.08739v2](https://arxiv.org/abs/1906.08739v2), to appear in Osaka J. Math.
- [Mur22] _____, *PBW parametrizations and generalized preprojective algebras*, Adv. Math. **395** (2022), Paper No. 108144. MR4355737
- [MV07] I. Mirković and K. Vilonen, *Geometric Langlands duality and representations of algebraic groups over commutative rings*, Ann. of Math. (2) **166** (2007), no. 1, 95–143. MR2342692
- [Nak01] H. Nakajima, *Quiver varieties and finite-dimensional representations of quantum affine algebras*, J. Amer. Math. Soc. **14** (2001), no. 1, 145–238. MR1808477
- [Nak98] _____, *Quiver varieties and Kac-Moody algebras*, Duke Math. J. **91** (1998), no. 3, 515–560. MR1604167
- [NW19] H. Nakajima and A. Weekes, *Coulomb branches of quiver gauge theories with symmetrizers*, 2019. preprint, [arXiv:1907.06552v2](https://arxiv.org/abs/1907.06552v2).
- [Rei08] M. Reineke, *Framed quiver moduli, cohomology, and quantum groups*, J. Algebra **320** (2008), no. 1, 94–115. MR2417980
- [Sai94] Y. Saito, *PBW basis of quantized universal enveloping algebras*, Publ. Res. Inst. Math. Sci. **30** (1994), no. 2, 209–232. MR1265471
- [SV20] O. Schiffmann and E. Vasserot, *On cohomological Hall algebras of quivers: generators*, J. Reine Angew. Math. **760** (2020), 59–132. MR4069884
- [Tre19] H. Treffinger, *On sign-coherence of c -vectors*, J. Pure Appl. Algebra **223** (2019), no. 6, 2382–2400. MR3906554
- [TW16] P. Tingley and B. Webster, *Mirković-Vilonen polytopes and Khovanov-Lauda-Rouquier algebras*, Compos. Math. **152** (2016), no. 8, 1648–1696. MR3542489
- [Yur18] T. Yurikusa, *Wide subcategories are semistable*, Doc. Math. **23** (2018), 35–47. MR3846064

DEPARTMENT OF MATHEMATICS, KYOTO UNIVERSITY, KITASHIRAKAWA OIWAKE-CHO, SAKYO-KU, KYOTO 606-8502, JAPAN

Email address: k-murakami@math.kyoto-u.ac.jp