A unifying approach to non-minimal quasi-stationary distributions for one-dimensional diffusions

Kosuke Yamato

Sections 1 through 4 of this article have been accepted for publication in a revised form in Journal of Applied Probability. This version is free to view and download for private research and study only. Not for re-distribution or re-use.

Acknowledgements

I would like to thank my supervisor Kouji Yano, who always encourage my study and carefully read drafts of this thesis. Thanks to his valuable comments the present thesis was significantly improved.

I express my gratitude to Servet Martínez, Jaime San Martín and Jorge Littin Curinao who sparing time for online discussions and gave me a lot of suggestions and comments on my draft. They were very insightful and gave me valuable information about the literature on quasi-stationary distributions. Especially, as a co-supervisor, Servet Martínez has carefully read my drafts many times and encouraged me, which helped me improve my thesis. I would like to thank again.

I am grateful to Toshiro Watanabe for the private communication for exponential tail distributions and the suggestion on the counterexample given in Remark 4.3.

This work was supported by JSPS Open Partnership Joint Research Projects grant no. JPJSBP120209921, JSPS KAKENHI Grant Number JP21J11000 and the Top Global University Project of Kyoto University, and was carried out under the ISM Cooperative Research Program (2020-ISMCRP-5013).

Contents

1 Introduction

Let $X = (X_t)_{t\geq 0}$ be a stochastic process on a state space S with a finite lifetime ζ. A probability measure ν on S is called a *quasi-stationary distribution* of X when the distribution of X_t with the initial distribution ν conditioned to survive at time t is time-invariant, that is, the following holds:

$$\mathbb{P}_\nu[X_t \in dx \mid \zeta > t] = \nu(dx) \quad (t > 0), \tag{1.1}$$

where $\mathbb{P}_\nu$ denotes the underlying probability measure of X with its initial distribution ν.

In the present thesis, we study the case where X is an irreducible one-dimensional diffusion on $[0, b)$ or $[0, b]$ $(0 < b \leq \infty)$ stopped at 0 with the finite lifetime $\zeta = T_0$, the first hitting time of the boundary 0. The main objective of the present thesis is to study a domain of attraction of quasi-stationary distributions, that is, for a quasi-stationary distribution ν we study a sufficient condition for an initial distribution μ such that

$$\mu_t(dx) := \mathbb{P}_\mu[X_t \in dx \mid T_0 > t] \xrightarrow[t\to\infty]{} \nu(dx). \tag{1.2}$$

Here and hereafter all the convergence of probability distributions is in the sense of the weak convergence.

Many studies (e.g., Hening and Kolb [11], Kolb and Steinsaltz [15], Littin [19] and Mandl [22]) have dealt with convergence (1.2) in the case when μ is compactly supported and it has been shown that convergence (1.2) holds and the limit distribution ν does not depend on the choice of a compactly supported μ. The limit measure ν is sometimes called *Yaglom limit* or the *minimal quasi-stationary distribution*. On the other hand, for some diffusions there exist infinitely many quasi-stationary distributions. Although it is a natural problem to know what initial distributions we can take so that convergence (1.2) holds for a given quasi-stationary distribution ν, there are very few studies dealing with this problem for non-minimal quasi-stationary distributions. The author only knows two papers: Lladser and San Martín [20] and Martínez, Picco and San Martín [24], whose results we generalize in the present thesis. For this reason, we focus on convergence (1.2) when ν is a non-minimal quasi-stationary distribution.

One of our main results is a general theorem which reduces the convergence (1.2) to the tail behavior of T_0. We denote the set of probability measures on a set I by $\mathcal{P}(I)$ or $\mathcal{P}I$. For a class $\mathcal{P} \subset \mathcal{P}[0, b)$ of initial distributions, we say that the *first hitting uniqueness* holds on $\mathcal{P}$ if

$$\text{the map} \quad \mathcal{P} \ni \mu \longmapsto \mathbb{P}_\mu[T_0 \in dt] \quad \text{is injective.} \tag{1.3}$$

As the class $\mathcal{P}$, we shall take

$$\mathcal{P}_{\mathrm{exp}} = \{\mu \in \mathcal{P}(I) \mid \mathbb{P}_\mu[T_0 \in dt] = \lambda \mathrm{e}^{-\lambda t} dt \quad (\lambda > 0)\}, \tag{1.4}$$

the set of initial distributions with exponential hitting probabilities. For birth and death processes, we can show the first hitting uniqueness on $\mathcal{P}(\mathbb{N})$ by the argument in [8, Proposition 5.6]. We refer to Rogers [25] as a general study of the first hitting uniqueness. He gave a condition for a one-dimensional diffusion to satisfy the first hitting uniqueness on $\mathcal{P}(0,\infty)$. The condition is, however, difficult to check in general and too strong to ensure the first hitting uniqueness. Indeed, he remarked that the Brownian motion with a constant negative drift does not satisfy his condition but the first hitting uniqueness holds on $\mathcal{P}(0,\infty)$. The reason why we consider the class $\mathcal{P}_{\mathrm{exp}}$ is that when ν is a quasi-stationary distribution, $\mathbb{P}_\nu[T_0 \in dt]$ is exponentially distributed. Indeed, from the definition of quasi-stationary distributions, it holds $\mathbb{P}_\nu[X_{t+s} \in dx \mid T_0 > t] = \mathbb{P}_\nu[X_s \in dx]$. Thus it follows $\mathbb{P}_\nu[T_0 > t+s \mid T_0 > t] = \mathbb{P}_\nu[X_{t+s} > 0 \mid T_0 > t] = \mathbb{P}_\nu[X_s > 0] = \mathbb{P}_\nu[T_0 > s]$. The following is one of the main results in the present thesis, which will be proven in Section 4.1.

Theorem 1.1. *Let X be a $\frac{d}{dm}\frac{d}{ds}$-diffusion on $[0,b)$ $(0 < b \leq \infty)$ and set for $\mu \in \mathcal{P}[0,b)$,*

$$\mu_t(dx) = \mathbb{P}_\mu[X_t \in dx \mid T_0 > t]. \tag{1.5}$$

Then for $\lambda > 0$, the following are equivalent:

$$\text{(i)} \quad \lim_{t\to\infty} \frac{\mathbb{P}_\mu[T_0 > t+s]}{\mathbb{P}_\mu[T_0 > t]} = \mathrm{e}^{-\lambda s} \ (s > 0). \tag{1.6}$$

$$\text{(i)'} \quad \lim_{t\to\infty} \frac{1}{\mathbb{P}_\mu[T_0 > t]} \int_t^\infty \mathbb{P}_\mu[T_0 > s] ds = 1/\lambda. \tag{1.7}$$

$$\text{(ii)} \quad \mathbb{P}_{\mu_t}[T_0 \in ds] \xrightarrow[t\to\infty]{} \lambda \mathrm{e}^{-\lambda s} ds. \tag{1.8}$$

Suppose, in addition, that the first hitting uniqueness holds on $\mathcal{P}_{\mathrm{exp}}$ and that

$$\mathbb{P}_\nu[T_0 \in dt] = \lambda \mathrm{e}^{-\lambda t} dt \quad \text{for some } \nu \in \mathcal{P}(0,b). \tag{1.9}$$

Then the following is also equivalent to (i), (i)' and (ii):

$$\text{(iii)} \quad \mu_t \xrightarrow[t\to\infty]{} \nu. \tag{1.10}$$

The other main result is an application of Theorem 1.1 to the class of diffusion called *Kummer diffusions* with negative drifts. A Kummer diffusion $Y^{(\alpha,\beta)}$ $(\alpha > 0,\ \beta \in \mathbb{R})$ is a diffusion on $[0,\infty)$ stopped upon hitting 0 whose local generator $\mathcal{L}^{(\alpha,\beta)}$ on $(0,\infty)$ is

$$\mathcal{L}^{(\alpha,\beta)} = x\frac{d^2}{dx^2} + (-\alpha + 1 - \beta x)\frac{d}{dx}. \tag{1.11}$$

Note that the process $Y^{(\alpha,\beta)}$ is also called a *radial Ornstein-Uhlenbeck process* in some literature (see e.g., [3] and [10]). Write

$$g_\gamma^{(\alpha,\beta)}(x) := \mathbb{E}_x[\mathrm{e}^{-\gamma T_0}] = \int_0^\infty \mathrm{e}^{-\gamma t} \mathbb{P}_x[T_0 \in dt] \quad (\gamma \geq 0), \tag{1.12}$$

which is the Laplace transform of the first hitting time of 0 for $Y^{(\alpha,\beta)}$. Then $g_\gamma^{(\alpha,\beta)}$ is a γ-eigenfunction for $\mathcal{L}^{(\alpha,\beta)}$, i.e., $\mathcal{L}^{(\alpha,\beta)} g_\gamma^{(\alpha,\beta)} = \gamma g_\gamma^{(\alpha,\beta)}$ (see e.g., [26, p.292]). We define a Kummer diffusion with a negative drift $Y^{(\alpha,\beta,\gamma)}$ $(\gamma \geq 0)$ as the h-transform of $Y^{(\alpha,\beta)}$ by the function $g_\gamma^{(\alpha,\beta)}$, that is, the process $Y^{(\alpha,\beta,\gamma)}$ is a diffusion on $[0,\infty)$ stopped at 0 whose local generator on $(0,\infty)$ is

$$\mathcal{L}^{(\alpha,\beta,\gamma)} = \frac{1}{g_\gamma^{(\alpha,\beta)}}(\mathcal{L}^{(\alpha,\beta)} - \gamma) g_\gamma^{(\alpha,\beta)}. \tag{1.13}$$

We sometimes omit (α,β) and write $\mathcal{L}^{[\gamma]} = \mathcal{L}^{(\alpha,\beta,\gamma)}$, $Y^{[\gamma]} = Y^{(\alpha,\beta,\gamma)}$, $g_\gamma = g_\gamma^{(\alpha,\beta)}$, etc. Note that $\mathcal{L}^{[0]} = \mathcal{L}^{(\alpha,\beta,0)} = \mathcal{L}^{(\alpha,\beta)}$ and $Y^{[0]} = Y^{(\alpha,\beta,0)} = Y^{(\alpha,\beta)}$.

If we write

$$\widetilde{Y}^{(\alpha,\beta,\gamma)} := \sqrt{2Y^{(\alpha,\beta,\gamma)}}, \tag{1.14}$$

then the local generator $\widetilde{\mathcal{L}}^{(\alpha,\beta,\gamma)}$ of $\widetilde{Y}^{(\alpha,\beta,\gamma)}$ on $(0,\infty)$ is given as

$$\widetilde{\mathcal{L}}^{(\alpha,\beta,\gamma)} = \frac{1}{2}\frac{d^2}{dx^2} + \left(\frac{1-2\alpha}{2x} - \frac{\beta x}{2} + \frac{(\widetilde{g}_\gamma^{(\alpha,\beta)})'}{\widetilde{g}_\gamma^{(\alpha,\beta)}} \right) \frac{d}{dx}, \tag{1.15}$$

where $\widetilde{g}_\gamma^{(\alpha,\beta)}(x) = \widetilde{\mathbb{E}}_x[\mathrm{e}^{-\gamma \widetilde{T}_0}]$ denotes the Laplace transform of the first hitting time of 0 for $\widetilde{Y}^{(\alpha,\beta,0)}$ starting from x. When $\alpha = 1/2$ and $\gamma = 0$, the process $\widetilde{Y}^{(1/2,\beta,0)}$ is the Ornstein-Uhlenbeck process and, when $\beta = 0$, the process $\widetilde{Y}^{(\alpha,0,\gamma)}$ is the Bessel process with a negative drift (see e.g., [10]). The other main result is to give a concrete sufficient condition for convergence (1.2) for Kummer diffusions with negative drifts. We classify $Y^{[\gamma]} = Y^{(\alpha,\beta,\gamma)}$ $(\alpha > 0,\ \beta \in \mathbb{R},\ \gamma \geq 0)$ into the following five cases by β and γ:

$$\begin{aligned} &\text{Case 1:} && \beta = 0, && \gamma > 0. \\ &\text{Case 2:} && \beta > 0, && \gamma \geq 0. \\ &\text{Case 3:} && \beta < 0, && \gamma > 0. \\ &\text{Case 1':} && \beta = 0, && \gamma = 0. \\ &\text{Case 3':} && \beta < 0, && \gamma = 0. \end{aligned} \tag{1.16}$$

We will show in Proposition 4.8 that non-minimal quasi-stationary distributions exist only in the Case 1-3. The following theorem is another main result in the present thesis. We denote the set of integrable functions on I w.r.t. the measure ν by $L^1(I,\nu)$ and denote $f(x) \sim g(x)$ $(x \to \infty)$ when $\lim_{x\to\infty} f(x)/g(x) = 1$.

Theorem 1.2. *Let $X = Y^{[\gamma]} = Y^{(\alpha,\beta,\gamma)}$ $(\alpha > 0,\ \beta \in \mathbb{R},\ \gamma \geq 0)$ satisfying one of the Case 1-3 in (1.16) and let $\mu \in \mathcal{P}(0,\infty)$. Then the following holds:*

(i) If the Case 1 holds and $\mu(dx) = \rho(x)dx$ for some $\rho \in L^1((0,\infty),dx)$ and

$$\log \rho(x) \sim (\delta - 2\sqrt{\gamma})\sqrt{x} \quad (x \to \infty) \tag{1.17}$$

for some $0 < \delta < 2\sqrt{\gamma}$, then it holds

$$\mu_t \xrightarrow[t\to\infty]{} \nu_\lambda \tag{1.18}$$

with $\lambda = \gamma - \delta^2/4 \in (0, \lambda_0^{[\gamma]})$, where $\lambda_0^{[\gamma]} = \gamma > 0$ is the spectral bottom.

(ii) If the Case 2 holds and

$$\mu(x,\infty) \sim x^{-\alpha-\gamma/\beta+\delta}\ell(x) \quad (x\to\infty) \tag{1.19}$$

for some $0<\delta<\alpha+\gamma/\beta$ *and some slowly varying function* ℓ *at* ∞*, then it holds*

$$\mu_t \xrightarrow[t\to\infty]{} \nu_\lambda \tag{1.20}$$

with $\lambda = \beta(\alpha-\delta)+\gamma \in (0,\lambda_0^{[\gamma]})$*, where* $\lambda_0^{[\gamma]} = \alpha\beta+\gamma>0$ *is the spectral bottom.*

(iii) If the Case 3 holds and

$$\mu(x,\infty) \sim x^{-1+\gamma/\beta+\delta}\ell(x) \quad (x\to\infty) \tag{1.21}$$

for some $0<\delta<1-\gamma/\beta$ *and some slowly varying function* ℓ *at* ∞*. then it holds*

$$\mu_t \xrightarrow[t\to\infty]{} \nu_\lambda \tag{1.22}$$

with $\lambda = -\beta(1-\delta)+\gamma \in (0,\lambda_0^{[\gamma]})$*, where* $\lambda_0^{[\gamma]} = -\beta+\gamma>0$ *is the spectral bottom.*

Renewal dynamical approach for quasi-stationary distributions

To investigate convergence (1.2) for more general diffusions, we obtain several partial results through the renewal dynamical approach, which was introduced in Ferrari, Kesten, Martínez and Picco [9] to show the existence of the minimal quasi-stationary distribution for Markov chains on $\mathbb{N}$. By extending the method to one-dimensional diffusions, we give a characterization of quasi-stationary distributions and a necessary condition for the convergence (1.2) through the renewal dynamical approach. In the approach, we introduce a transform Φ of probability measures.

Let X be an irreducible $\frac{d}{dm}\frac{d}{ds}$-diffusion on $[0,b)$ $(0<b\le\infty)$ stopped at 0 with $\mathbb{P}_x[T_0<\infty]=1$ and assume there exists non-minimal quasi-stationary distributions (note that a necessary and sufficient condition for the existence of non-minimal quasi-stationary distributions is given in Section 3.3). Define

$$\mathcal{P}_m = \{\mu\in\mathcal{P}(0,b) \mid \mathbb{E}_\mu T_0^n<\infty \ (n\ge1)\}. \tag{1.23}$$

For $\mu\in\mathcal{P}(0,b)$ and the transition density $p(t,x,y)$ w.r.t. the speed measure dm of X, we denote

$$p(t,\mu,y) := \int_0^b p(t,x,y)\mu(dx). \tag{1.24}$$

For $\mu\in\mathcal{P}(0,b)$, we define a measure $\Phi\mu$ by

$$\Phi\mu(A) = \frac{1}{\mathbb{E}_\mu T_0}\int_0^\infty \mathbb{P}_\mu[X_t\in A]dt = \frac{1}{\mathbb{E}_\mu T_0}\int_A dm(y)\int_0^\infty p(t,\mu,y)dt. \tag{1.25}$$

When $\mathbb{E}_\mu T_0 < \infty$, the measure $\Phi\mu$ is a probability since $\mathbb{P}_\mu[X_t \in (0,b)] = \mathbb{P}_\mu[T_0 > t]$. The distribution $\Phi\mu$ can be interpreted as the limit distribution of the conservative stochastic process which behaves as X until T_0 and as soon as it hits 0, it jumps into a random point in $(0,b)$ according to the probability μ and starts afresh (see e.g., Ben-Ari and Pinsky [1]).

We will prove in Proposition 5.1 that Φ preserves $\mathcal{P}_m$. Quasi-stationary distributions are characterized as the fixed points of Φ in $\mathcal{P}_m$, which we will show in Proposition 5.9. We also show in Theorem 5.2 that for the convergence (1.2), it is necessary that $\Phi^n\mu$ converges to the same limit as $n \to \infty$. As we have seen above, there is a close relationship between quasi-stationary distributions and the transform Φ. However, the author cannot give any sufficient condition for the convergence (1.2) from the renewal dynamical approach.

Outline of the thesis

The remainder of the present thesis is organized as follows. In Section 2, we will recall previous studies on quasi-stationary distributions for one-dimensional diffusions. In Section 3, we will recall several known results on one-dimensional diffusions, the quasi-stationary distributions and the spectral theory for second-order ordinary differential operators. In Section 4.1, we will show Theorem 1.1, a general result for convergence to quasi-stationary distributions. In Section 4.2, we will give the hitting density of Kummer diffusions with negative drifts. In Section 4.3, we will show Theorem 1.2, which gives a sufficient condition for convergence to non-minimal quasi-stationary distributions for the class of Kummer diffusions with negative drifts. In Section 5, we will consider the renewal dynamical approach of initial distributions and show its relation to convergence to quasi-stationary distributions. In Appendix 6, we will give a sufficient condition for the spectral measure of a diffusion to have its Laplace transform.

2 Previous studies

We briefly review several previous studies on quasi-stationary distributions for one-dimensional diffusions to compare with our main results.

For the case where there is a natural boundary, Mandl [22] gave a sufficient condition for the convergence to the minimal quasi-stationary distributions, which was a first remarkable result for quasi-stationary distributions of one-dimensional diffusions. His condition has been weakened by many authors e.g., Collet, Martínez and San Martín [7], Hening and Kolb [11], Kolb and Steinsaltz [15] and Martínez and San Martín [23]. Under certain weak assumptions it is shown that convergence to the minimal quasi-stationary distribution follows for all compactly supported initial distributions.

The case where there is an entrance boundary has also been widely studied. Cattiaux, Collet, Lambert, Martínez, Méléard and San Martín [4] and Littin [19] showed that there always exists a unique quasi-stationary distribution and convergence to the unique quasi-stationary distribution always holds for all compactly supported initial distributions.

Here we briefly mention Takeda's results [28] of quasi-stationary distributions for general symmetric Markov processes. He proved existence and uniqueness of the quasi-stationary distribution under the *tightness property*, which is equivalent for one-dimensional diffusions to absence of natural boundaries. He also showed that converge to the unique quasi-stationary distribution from all initial distributions holds under the assumption of the tightness property and *intrinsic ultracontractivity*, which is a stronger assumption than the presence of the entrance boundary.

Let us come back to the study for one-dimensional diffusion. In the case where the right boundary is natural, we have non-minimal quasi-stationary distributions. Firstly, Martínez, Picco and San Martín [24] studied Brownian motions with negative drifts and showed convergence to non-minimal quasi-stationary distributions under the assumptions on tail behavior of the initial distribution.

Theorem 2.1 ([24, Theorem 1.1])**.** *Let B_t be a standard Brownian motion and let $\alpha > 0$ and consider the process*

$$X_t = B_t - \alpha t. \tag{2.1}$$

For an initial distribution μ on $(0, \infty)$ assume $\mu(dx) = \rho(x)dx$ for some $\rho \in L^1((0, \infty), dx)$ satisfying

$$\log \rho(x) \sim -(\alpha - \delta)x \quad (x \to \infty) \tag{2.2}$$

for some $\delta \in (0, \alpha)$. Then it holds

$$\mathbb{P}_\mu[X_t \in dx \mid T_0 > t] \xrightarrow[t\to\infty]{} \nu_\lambda(dx), \tag{2.3}$$

with

$$\lambda = (\alpha^2 - \delta^2)/2 \quad \textit{and} \quad \nu_\lambda(dx) = C_\lambda \mathrm{e}^{-\alpha x} \sinh(x\sqrt{\alpha^2 - 2\lambda})dx \tag{2.4}$$

for the normalizing constant C_λ.

Remark 2.2. When $\alpha = 1/2, \beta = 0$ and $\gamma > 0$, the process $\sqrt{2Y^{(1/2,0,\gamma)}}$ is a Brownian motion with a negative drift $-\sqrt{2\gamma}t$. Hence this theorem is generalized by (i) of Theorem 1.2.

Secondly, Lladser and San Martín [20] studied Ornstein-Uhlenbeck processes:

Theorem 2.3 ([20, Theorem 1.1])**.** *Let $\alpha > 0$. Let X be the solution of the following SDE:*

$$dX_t = dB_t - \alpha X_t dt, \tag{2.5}$$

where B is a standard Brownian motion. For an initial distribution μ on $(0,\infty)$ assume $\mu(dx) = \rho(x)dx$ for some $\rho \in L^1((0,\infty), dx)$ satisfying

$$\rho(x) \sim x^{-2+\delta}\ell(x) \quad (x \to \infty) \tag{2.6}$$

for some $\delta \in (0,1)$ and a slowly varying function ℓ at ∞. Then it holds

$$\mathbb{P}_\mu[X_t \in dx \mid T_0 > t] \xrightarrow[t\to\infty]{} \nu_\lambda(dx) \tag{2.7}$$

with

$$\lambda = \alpha(1-\delta) \quad \text{and} \quad \nu_\lambda(dx) = C_\lambda \psi_{-\lambda}(x)\mathrm{e}^{-\alpha x^2}dx \tag{2.8}$$

for the normalizing constant C_λ, where $u = \psi_{-\lambda}$ denotes the unique solution for the following differential equation:

$$\frac{1}{2}\frac{d^2}{dx^2}u - \alpha x \frac{d}{dx}u = -\lambda u, \quad \lim_{x\to+0} u(x) = 0, \quad \lim_{x\to+0}\frac{d}{dx}u(x) = 1 \quad (x \in (0,\infty)). \tag{2.9}$$

Remark 2.4. In Theorem 1.2 (ii), if $\mu(dx) = \rho(x)dx$ for $\rho \in L^1((0,\infty), dx)$ and

$$\rho(x) \sim x^{-\alpha-\gamma/\beta+\delta-1}\ell(x) \quad (x \to \infty), \tag{2.10}$$

for a slowly varying function ℓ, then (1.19) holds from Karamata's theorem [2, Proposition 1.5.8]. Hence (ii) of Theorem 1.2 is an extension of [20, Theorem 1.1].

3 Preliminaries

In this section, we recall several known results on one-dimensional diffusions and their quasi-stationary distributions.

3.1 Feller's canonical form of second-order differential operators

Let $(X, \mathbb{P}_x)_{x\in I}$ be a one-dimensional diffusion on $I = [0,b)$ or $[0,b]$ $\quad (0 < b \leq \infty)$, that is, the process X is a time-homogeneous strong Markov process on I which has a continuous path up to its lifetime. Throughout this thesis, we always assume

$$\mathbb{P}_x[T_y < \infty] > 0 \quad (x \in I \setminus \{0\},\ y \in [0,b)), \tag{3.1}$$

where T_y denotes the first hitting time of y and, assume the point 0 is a trap;

$$X_t = 0 \quad \text{for } t \geq T_0. \tag{3.2}$$

Let us recall Feller's classification of the boundaries (see e.g., Itô [12]). There exist a Radon measure m on $I \setminus \{0\}$ with full support and a strictly increasing continuous function s on $(0,b)$ such that the local generator $\mathcal{L}$ on $(0,b)$ is represented by

$$\mathcal{L} = \frac{d}{dm}\frac{d}{ds}. \tag{3.3}$$

We call m *the speed measure* and s *the scale function* of X and we say X is a $\frac{d}{dm}\frac{d}{ds}$-diffusion. Let $c = 0$ or b and take $d \in (0, b)$. Set

$$I(c) = \int_c^d ds(x) \int_c^x dm(y), \quad J(c) = \int_c^d dm(x) \int_c^x ds(y). \tag{3.4}$$

The boundary c is classified as follows:

$$\text{The boundary } c \text{ is } \begin{cases} \text{regular} & \text{when } I(c) < \infty,\ J(c) < \infty. \\ \text{exit} & \text{when } I(c) = \infty,\ J(c) < \infty. \\ \text{entrance} & \text{when } I(c) < \infty,\ J(c) = \infty. \\ \text{natural} & \text{when } I(c) = \infty,\ J(c) = \infty. \end{cases} \tag{3.5}$$

Since $\mathbb{P}_x[T_0 < \infty] > 0$ for every $x > 0$, the boundary 0 is necessarily regular or exit, equivalently $J(0) < \infty$. Note that in this case $s(0) := \lim_{x \to +0} s(x) > -\infty$ holds. We also assume that the boundary b is not exit and that the boundary b is reflecting when it is regular.

Let us consider a diffusion on I whose local generator $\mathcal{L}$ on $(0, b)$ is

$$\mathcal{L} = a(x)\frac{d^2}{dx^2} + c(x)\frac{d}{dx} \quad (x \in (0, b)) \tag{3.6}$$

for functions a and c. Assume $a(x) > 0$ $(x \in (0, b))$. Then $\mathcal{L} = \frac{d}{dm}\frac{d}{ds}$, where

$$dm(x) = \frac{1}{a(x)} \exp\left(\int_d^x \frac{c(y)}{a(y)} dy\right) dx, \quad ds(x) = \exp\left(-\int_d^x \frac{c(y)}{a(y)} dy\right) dx \tag{3.7}$$

for arbitrary taken $d \in (0, b)$.

3.2 Spectral theory for second-order differential operators

Let us briefly review several results on the spectral theory of second-order differential operators. For the details, see e.g., Coddington and Levinson [6] and Kotani [16].

Set $I = (0, b)$ $(0 < b \leq \infty)$. Let dm be a Radon measure on I with full support and let $s : I \to (-\infty, \infty)$ be a strictly-increasing continuous function. We assume that the boundary 0 is regular or exit, i.e. $\int_0^d dm(x) \int_0^x ds(y) < \infty$ for some $0 < d < b$ and assume the boundary b is natural, i.e., $\int_d^b dm(x) \int_x^b ds(y) = \infty$ and $\int_d^b ds(x) \int_x^b dm(y) = \infty$ for some $0 < d < b$. Let $u = \psi_\lambda$ be defined by (3.13). Set

$$g_\lambda(x) = \psi_\lambda(x) \int_x^b \frac{ds(y)}{\psi_\lambda(y)^2} \quad (\lambda \geq 0). \tag{3.8}$$

Then the function $u = g_\lambda$ is the unique, non-increasing solution for

$$\frac{d}{dm}\frac{d^+}{ds} u = \lambda u, \quad \lim_{x \to +0} u(x) = 1, \tag{3.9}$$

where $\frac{d^+}{ds}$ is the right-differentiation w.r.t. s; $\frac{d^+}{ds}f(x) := \lim_{h\to+0}\frac{f(x+h)-f(x)}{s(x+h)-s(x)}$. Define the Green function

$$G_\lambda(x,y) = G_\lambda(y,x) := \psi_\lambda(x)g_\lambda(y) \quad (0 \le x \le y < b,\ \lambda \ge 0). \tag{3.10}$$

Then there exists a unique Radon measure σ on $[0,\infty)$, which we call the *spectral measure*, such that

$$G_\lambda(x,y) = \int_0^\infty \frac{\psi_{-\xi}(x)\psi_{-\xi}(y)}{\lambda+\xi}\sigma(d\xi) \tag{3.11}$$

and the transition density $p(t,x,y)$ w.r.t. dm of $\frac{d}{dm}\frac{d}{ds}$-diffusion absorbed at 0 is given as

$$p(t,x,y) = \int_0^\infty \mathrm{e}^{-\lambda t}\psi_{-\lambda}(x)\psi_{-\lambda}(y)\sigma(d\lambda) \quad (t>0, x,y \in I) \tag{3.12}$$

(see McKean [13] for the details).

3.3 Quasi-stationary distributions

Let us summarize known results on quasi-stationary distributions for one-dimensional diffusions and give a necessary and sufficient condition for existence of quasi-stationary distributions. Let X be a $\frac{d}{dm}\frac{d}{ds}$-diffusion on $I = [0,b)$ or $[0,b]$ $(0 < b \le \infty)$. We define a function $u = \psi_\lambda$ as the unique solution of the following equation:

$$\frac{d}{dm}\frac{d^+}{ds}u(x) = \lambda u(x), \quad \lim_{x\to+0} u(x) = 0, \quad \lim_{x\to+0}\frac{d}{ds}u(x) = 1 \quad (x \in (0,b), \lambda \in \mathbb{R}). \tag{3.13}$$

Note that from the assumption that the boundary 0 is regular or exit, the function ψ_λ always exists. The operator $L = -\frac{d}{dm}\frac{d}{ds}$ defines a non-negative definite self-adjoint operator on $L^2(I,dm) := \{f : I \to \mathbb{R} \mid \int_I |f|^2 dm < \infty\}$. Here we assume the Dirichlet boundary condition at 0 and the Neumann boundary condition at b if the boundary b is regular. We denote the infimum of the spectrum of L by $\lambda_0 \ge 0$.

Let us consider the case where the boundary b is not natural. It is then known that there is a unique quasi-stationary distribution (noting that Takeda [28] showed the corresponding result for general Markov processes with the tightness property):

Proposition 3.1 (see e.g., [19, Lemma 2.2, Theorem 4.1])**.** *Assume the boundary b is not natural. Then it holds $\lambda_0 > 0$ and the function $\psi_{-\lambda_0}$ is strictly positive and integrable w.r.t. dm and, there is a unique quasi-stationary distribution given as*

$$\nu_{\lambda_0}(dx) = \lambda\psi_{-\lambda_0}(x)dm(x), \quad \mathbb{P}_{\nu_{\lambda_0}}[T_0 \in dt] = \lambda_0\mathrm{e}^{-\lambda_0 t}dt. \tag{3.14}$$

Moreover, for every probability distribution μ on $(0,b)$ with a compact support, it holds

$$\mu_t \xrightarrow[t\to\infty]{} \nu_{\lambda_0}. \tag{3.15}$$

We now assume the boundary b is natural. We check that in this case, $s(b) = \infty$ is equivalent to finiteness of the lifetime T_0. Since it holds

$$\mathbb{P}_x[T_b < \infty] = 0 \quad (x \in (0,b)), \tag{3.16}$$

and

$$\frac{s(x) - s(0)}{s(M) - s(0)} = \mathbb{P}_x[T_M < T_0] \quad (0 < x < M < b) \tag{3.17}$$

(see e.g., Itô [12]), by taking limit $M \to b$ we have from (3.16)

$$\frac{s(x) - s(0)}{s(b) - s(0)} = \mathbb{P}_x[T_0 = \infty]. \tag{3.18}$$

Hence it follows

$$\mathbb{P}_x[T_0 < \infty] = 1 \quad \text{for some / any } x > 0 \quad \Leftrightarrow \quad s(b) = \infty. \tag{3.19}$$

Thus $s(b) = \infty$ is equivalent to finiteness of T_0.

We recall the following good properties for the function ψ_λ:

Proposition 3.2 ([8, Lemma 6.18]). *Suppose the boundary b is natural and $s(b) = \infty$. Then for $\lambda > 0$ the following hold:*

(i) For $0 < \lambda \leq \lambda_0$, the function $\psi_{-\lambda}$ is strictly positive on $I \setminus \{0\}$ and

$$1 = \lambda \int_0^b \psi_{-\lambda}(x) dm(x). \tag{3.20}$$

(ii) For $\lambda > \lambda_0$, the function $\psi_{-\lambda}$ change signs on I.

Now we state a necessary and sufficient condition for existence of non-minimal quasi-stationary distributions. This result has been shown in [8, Theorem 6.34] when the boundary 0 is regular.

Theorem 3.3. *Suppose that the boundary 0 is regular or exit and that the boundary b is natural. Then a non-minimal quasi-stationary distribution exists if and only if*

$$\lambda_0 > 0 \quad \textit{and} \quad s(b) = \infty. \tag{3.21}$$

This condition is equivalent to

$$m(d,b) < \infty \quad \textit{for some } d \in (0,b) \quad \textit{and} \quad \limsup_{x \to b} s(x) m(x,b) < \infty. \tag{3.22}$$

In this case, a probability measure ν is a quasi-stationary distribution if and only if

$$\nu(dx) = \lambda \psi_{-\lambda}(x) dm(x) =: \nu_\lambda(dx), \quad \mathbb{P}_{\nu_\lambda}[T_0 \in dt] = \lambda \mathrm{e}^{-\lambda t} dt \quad \textit{for some } 0 < \lambda \leq \lambda_0. \tag{3.23}$$

Proof. We may assume without loss of generality that $s(0) = 0$. The equivalence between (3.21) and (3.22) follows from [18, Theorem 3 (ii), Appendix I]. The fact that ν_λ ($\lambda \in (0, \lambda_0]$) is a quasi-stationary distribution can be seen by the exactly same argument in [8, Lemma 6.18]. Thus we only show that every quasi-stationary distribution is given by ν_λ for some $\lambda \in (0, \lambda_0]$.

Let μ be a quasi-stationary distribution with

$$\mathbb{P}_\mu[T_0 > t] = \mathrm{e}^{-\beta t} \quad (\beta \in (0, \lambda_0]). \tag{3.24}$$

Since it holds

$$\mathbb{P}_x[X_t \in A, T_0 > t] = \int_0^b p(t, x, y) 1_A(y) dm(y), \tag{3.25}$$

the probability measure μ is absolutely continuous w.r.t. dm, we denote the density by ρ. Then it holds

$$\rho(x) = \mathrm{e}^{\beta t} \int_0^b p(t, x, y) \rho(y) dm(y) \quad (t > 0) \tag{3.26}$$

for dm-a.e. $x \in I$. Define

$$\rho_t(x) := \int_0^b p(t, x, y) \rho(y) dm(y) \ (= \mathrm{e}^{-\beta t} \rho(x)) \quad (t \geq 0). \tag{3.27}$$

Then it follows

$$\int_0^\infty \rho_t(x) dt = (1/\beta) \rho(x). \tag{3.28}$$

On the other hand, we have

$$\int_0^\infty \rho_t(x) dt = \int_0^b \rho(y) dm(y) \int_0^\infty p(t, x, y) dt \tag{3.29}$$

$$= \int_0^b (s(x) \wedge s(y)) \rho(y) dm(y) \tag{3.30}$$

$$= \int_0^x s(y) \rho(y) dm(y) + s(x) \int_x^b \rho(y) dm(y), \tag{3.31}$$

where we used the well-known formula:

$$\mathbb{E}_x \left[\int_0^{T_0} f(X_t) dt \right] = \int_0^b (s(x) \wedge s(y)) f(y) dm(y) \tag{3.32}$$

(see e.g., [26, Theorem 49.1] and [14, Lemma 23.10]). Thus for dm-a.e. $x \in I$, we obtain the equality

$$(1/\beta) \rho(x) = \int_0^x s(y) \rho(y) dm(y) + s(x) \int_x^b \rho(y) dm(y). \tag{3.33}$$

Since the RHS of (3.33) is right-continuous, we have the right-continuous version of ρ. By differentiation, it holds

$$(1/\beta)d\rho(x) = s(x)\rho(x)dm + \left(\int_x^b \rho(y)dm(y)\right) ds - s(x)\rho(x)dm \tag{3.34}$$

$$= \left(\int_x^b \rho(y)dm(y)\right) ds. \tag{3.35}$$

Hence we obtain

$$\frac{d}{dm}\frac{d^+}{ds}\rho = -\beta\rho \quad (x \in I). \tag{3.36}$$

This implies $\rho = C\psi_{-\beta}$ for some constant $C > 0$. Since $\mu = \rho\ dm$ is a probability, from Proposition 3.2 (i), it follows $C = \beta$. □

Remark 3.4. Under the assumptions of Theorem 3.3, the spectral measure has its support on $[\lambda_0, \infty)$.

For probability distributions on $(0, b)$, let us recall the stochastic order. For $\mu_1,\ \mu_2 \in \mathcal{P}(0, b)$, we define $\mu_1 \preceq \mu_2$ by

$$\mu_2(0, x] \leq \mu_1(0, x] \quad (x > 0). \tag{3.37}$$

(We refer to [27] as an extensive reference on stochastic orders.) The following proposition says that the stochastic order is a total one for quasi-stationary distributions, which was shown by Cavender [5] for birth and death processes.

Proposition 3.5. *Suppose the boundary b is natural and* (3.21) *holds. Then it holds*

$$\nu_\lambda \preceq \nu_{\lambda'} \quad (0 < \lambda' \leq \lambda \leq \lambda_0). \tag{3.38}$$

In particular, the distribution ν_{λ_0} is the minimal one in this order. This is why we call ν_{λ_0} the minimal quasi-stationary distribution.

Proof. From (3.13), it holds

$$\psi_{-\lambda}(x) = s(x) - \lambda \int_0^x ds(y) \int_0^y \psi_{-\lambda}(z)dm(z) \quad (x > 0, \lambda \in \mathbb{R}). \tag{3.39}$$

Hence it follows

$$\nu_\lambda(0, x] = \lambda \int_0^x \psi_{-\lambda}(y)dm(y) = 1 - \frac{d^+}{ds}\psi_{-\lambda}(x) \quad (x > 0, 0 < \lambda \leq \lambda_0). \tag{3.40}$$

Let $0 < \lambda' \leq \lambda \leq \lambda_0$. From (3.40) and a similar argument in [8, Lemma 6.11], we have

$$\frac{d^+}{ds}\psi_{-\lambda}(x) \leq \frac{d^+}{ds}\psi_{-\lambda'}(x) \quad (x > 0), \tag{3.41}$$

which yields $\nu_\lambda \preceq \nu_{\lambda'}$. □

4 Proof of main results

In this section, we study the convergence (1.2) from the perspective of the first hitting uniqueness.

4.1 Convergence to quasi-stationary distributions

Here we give a proof of Theorem 1.1.

Proof of Theorem 1.1. At first, we show the equivalence of (i) and (i)'. Let $g(t) := \mathbb{P}_\mu[T_0 > \log t]$. Then it is not difficult to see that (i) is equivalent to

$$\lim_{t\to\infty} \frac{g(st)}{g(t)} = s^{-\lambda} \quad (s > 0), \tag{4.1}$$

the regular variation of the function g at ∞ of order $-\lambda$.

Assume (i) holds. By Potter's theorem [2, Theorem 1.5.6], for $0 < \varepsilon < \lambda$, there exists $C > 0$ such that

$$\frac{\mathbb{P}_\mu[T_0 > t+s]}{\mathbb{P}_\mu[T_0 > t]} \leq C\mathrm{e}^{-(\lambda-\varepsilon)s} \quad (s, t > 0). \tag{4.2}$$

By the dominated convergence theorem, we have

$$\frac{1}{\mathbb{P}_\mu[T_0 > t]} \int_t^\infty \mathbb{P}_\mu[T_0 > s]ds = \int_0^\infty \frac{\mathbb{P}_\mu[T_0 > t+s]}{\mathbb{P}_\mu[T_0 > t]}ds \xrightarrow{t\to\infty} \frac{1}{\lambda}. \tag{4.3}$$

Next we assume (i)'. Since

$$\frac{1}{\mathbb{P}_\mu[T_0 > \log t]} \int_{\log t}^\infty \mathbb{P}_\mu[T_0 > s]ds = \frac{1}{g(t)} \int_t^\infty \frac{g(s)}{s}ds \xrightarrow{t\to\infty} \frac{1}{\lambda}, \tag{4.4}$$

we may see from Karamata's theorem [2, Theorem 1.6.1] that the function $g(t)$ varies regularly at ∞ with exponent $-\lambda$.

Next we show the equivalence of (i) and (ii). From the Markov property, we have

$$\mathbb{P}_{\mu_t}[T_0 > s] = \frac{\mathbb{P}_\mu[T_0 > t+s]}{\mathbb{P}_\mu[T_0 > t]} \quad (t, s \geq 0). \tag{4.5}$$

Now it is obvious that (i) and (ii) are equivalent.

Since it is obvious that (iii) implies (i), we finally show that (ii) implies (iii). Since $\mathcal{P}[0, b]$, the class of probability measures on the compactification $[0, b]$, is compact under the topology of weak convergence, we can take a sequence $\{t_n\}_n$ which diverges to ∞ such that

$$\mu_{t_n} \xrightarrow[n\to\infty]{} \theta \in \mathcal{P}[0,b]. \tag{4.6}$$

From (ii), we have

$$\mathbb{P}_{\mu_{t_n}}[T_0 \in ds] \xrightarrow[n\to\infty]{} \lambda \mathrm{e}^{-\lambda s} ds. \tag{4.7}$$

On the other hand, for fixed $t > 0$ we have

$$\mathbb{P}_{\mu_{t_n}}[T_0 > t] = \int_{[0,b]} \mathbb{P}_x[T_0 > t] \mu_{t_n}(dx), \tag{4.8}$$

where we understand

$$\mathbb{P}_x[T_0 > t] = \begin{cases} 0 & x = 0, \\ 1 & x = b. \end{cases} \tag{4.9}$$

Note that since the boundary b is natural, the function $x \mapsto \mathbb{P}_x[T_0 > t]$ is continuous on $[0,b]$. From (4.6), we obtain

$$\lim_{n\to\infty} \mathbb{P}_{\mu_{t_n}}[T_0 > t] = \int_{[0,b]} \mathbb{P}_x[T_0 > t] \theta(dx). \tag{4.10}$$

Then from (4.7), it follows that

$$\int_{[0,b]} \mathbb{P}_x[T_0 > t] \theta(dx) = \mathrm{e}^{-\lambda t}. \tag{4.11}$$

Since it holds that

$$\lim_{t\to 0} \mathbb{P}_x[T_0 > t] = 1\{x > 0\}, \quad \lim_{t\to\infty} \mathbb{P}_x[T_0 > t] = 1\{x = b\} \quad (x \in [0,b]), \tag{4.12}$$

we have from the dominated convergence theorem and (4.11) that $\theta\{0\} = \theta\{b\} = 0$. Therefore $\theta \in \mathcal{P}(0,b)$ and $\mathbb{P}_\theta[T_0 \in ds] = \lambda \mathrm{e}^{-\lambda s} ds$. Then since the first hitting uniqueness holds on $\mathcal{P}_{\mathrm{exp}}$, we have $\theta = \nu$. The limit distribution θ does not depend on the choice of the sequence $\{t_n\}$ and therefore we obtain (iii). □

Remark 4.1. Provided that the first hitting uniqueness holds on $\mathcal{P}_{\mathrm{exp}}$ and X satisfies the condition of Theorem 3.3, an initial distribution $\mu \in \mathcal{P}[0,b)$ satisfying $\mathbb{P}_\mu[T_0 \in dt] = \lambda \mathrm{e}^{-\lambda t} dt$ for some $0 < \lambda \leq \lambda_0$ must satisfy $\mu = \nu_\lambda$.

We give a sufficient condition for (i) of Theorem 1.1 via the hitting density.

Proposition 4.2. *Assume the hitting densities f_x of 0 exist, i.e., there exists a non-negative jointly measurable function $f_x(t)$ such that*

$$\mathbb{P}_x[T_0 \in dt] = f_x(t) dt \quad (0 < x < b,\ t > 0). \tag{4.13}$$

Let $\mu \in \mathcal{P}(0,b)$ and assume the function

$$f_\mu(t) := \int_0^\infty f_x(t)\mu(dx) \quad (0 < x < b,\ t > 0) \tag{4.14}$$

is differentiable in $t > 0$ and

$$-\lim_{t\to\infty} \frac{d}{dt} \log f_\mu(t) = \lambda \in (0, \lambda_0]. \tag{4.15}$$

Then it holds

$$\lim_{t\to\infty} \frac{\mathbb{P}_\mu[T_0 > t+s]}{\mathbb{P}_\mu[T_0 > t]} = \mathrm{e}^{-\lambda s} \quad (s > 0). \tag{4.16}$$

Proof. Set $h(u) = f_\mu(\log u)$ for $u > 1$. From (4.15), we have

$$\lim_{t\to\infty} \frac{th'(t)}{h(t)} = \lim_{t\to\infty} \frac{\mathrm{e}^t h'(\mathrm{e}^t)}{h(\mathrm{e}^t)} = -\lambda. \tag{4.17}$$

Thus the function h varies regularly at ∞ with exponent $-\lambda$. From L'Hôpital's rule, we have for $u = \mathrm{e}^s > 1$

$$\lim_{t\to\infty} \frac{\mathbb{P}_\mu[T_0 > t + \log u]}{\mathbb{P}_\mu[T_0 > t]} = \lim_{t\to\infty} \frac{f_\mu(t+\log u)}{f_\mu(t)} = \lim_{t\to\infty} \frac{h(\mathrm{e}^t u)}{h(\mathrm{e}^t)} = u^{-\lambda} = \mathrm{e}^{-\lambda s}. \tag{4.18}$$

□

Remark 4.3. We may expect that Proposition 4.2 would be extended with (4.15) being replaced by

$$\log f_\mu(t) \sim -\lambda t \quad (t \to \infty), \tag{4.19}$$

which is weaker than (4.15) by L'Hôpital's rule. In general, however, it does not hold. We give a counterexample which satisfies (4.19) but not the condition (i) in Theorem 1.1. Let us find a positive function f of the form

$$f(t) = \mathrm{e}^{(-\lambda+\varepsilon(t))t} \tag{4.20}$$

with a function $\varepsilon(t)$ varnishing at ∞ but not satisfying

$$\frac{\int_{t+s}^\infty f(u)du}{\int_t^\infty f(u)du} \xrightarrow[t\to\infty]{} \mathrm{e}^{-\lambda s} \quad (s > 0). \tag{4.21}$$

By the change of variables, we can see that (4.21) is equivalent to that the function

$$h(t) := \int_t^\infty u^{-\lambda-1+\varepsilon(\log u)} du \tag{4.22}$$

varies regularly with exponent $-\lambda$ at ∞. If the function ε is non-increasing, by the monotone density theorem [2, Theorem 1.7.2] it is equivalent to the slow variation of

$$k(s) = s^{\varepsilon(\log s)} \quad (s > 1). \tag{4.23}$$

We now set

$$\varepsilon(s) = 2^{-n} \quad (4^n < s \leq 4^{n+1}, n \in \mathbb{N}), \tag{4.24}$$

and then the function ε vanishes at ∞ and

$$\frac{k(\mathrm{e} \cdot \exp(4^n))}{k(\exp(4^n))} = \frac{\exp(2^n + 2^{-n})}{\exp(2^{n+1})} = \exp(-2^n + 2^{-n}) \xrightarrow[n\to\infty]{} 0. \tag{4.25}$$

So the function k does not vary slowly.

We give a sufficient condition for existence of the hitting densities of 0. For this purpose, we need the following condition on decay of the spectral measure σ of $-\frac{d}{dm}\frac{d}{ds}$:

$$\text{(S)} \qquad \int_0^\infty \mathrm{e}^{-\lambda t} \sigma(d\lambda) < \infty \quad (t > 0). \tag{4.26}$$

A sufficient condition for (S) is given in Proposition 6.3. The following result by Yano [30] gives existence and a spectral representation of the hitting densities.

Proposition 4.4 ([30, Proposition 2.1]). *Assume* (S) *holds. Then for any* $0 < x < b$ *the distribution of* T_0 *under* $\mathbb{P}_x$ *has a density* $f_x(t)$ *on* $(0, \infty)$ *w.r.t. the Lebesgue measure, that is, the following hold:*

$$\mathbb{P}_x[T_0 \in dt] = f_x(t)dt \quad (0 < x < b, \ t > 0). \tag{4.27}$$

The hitting densities have a spectral representation:

$$f_x(t) = \int_0^\infty \mathrm{e}^{-\lambda t} \psi_{-\lambda}(x) \sigma(d\lambda) \quad (0 < x < b, \ t > 0). \tag{4.28}$$

and have another representation:

$$f_x(t) = \left. \frac{d}{ds(y)} p(t, x, y) \right|_{y=0} \quad (0 < x < b, \ t > 0). \tag{4.29}$$

4.2 Hitting densities of Kummer diffusions with negative drifts

Let us give the hitting densities of Kummer diffusions with negative drifts. At first we give a speed measure and a scale function for Kummer diffusions with negative drifts. Fix $\alpha > 0$ and $\beta \in \mathbb{R}$. From (1.11), we have

$$\mathcal{L}^{[0]} = \mathcal{L}^{(\alpha,\beta)} = x\frac{d^2}{dx^2} + (-\alpha + 1 - \beta x)\frac{d}{dx} = \frac{d}{dm^{[0]}}\frac{d}{ds^{[0]}} \tag{4.30}$$

with

$$dm^{[0]}(x) := dm^{(\alpha,\beta)}(x) = x^{-\alpha}\mathrm{e}^{-\beta x}dx, \quad ds^{[0]}(x) := ds^{(\alpha,\beta)}(x) = x^{\alpha-1}\mathrm{e}^{\beta x}dx. \tag{4.31}$$

In addition, for $\gamma \geq 0$, we have

$$\mathcal{L}^{[\gamma]} = \mathcal{L}^{(\alpha,\beta,\gamma)} = x\frac{d^2}{dx^2} + \left(-\alpha + 1 - \beta x + \frac{xg_\gamma'(x)}{g_\gamma(x)}\right)\frac{d}{dx} = \frac{d}{dm^{[\gamma]}}\frac{d}{ds^{[\gamma]}} \tag{4.32}$$

with

$$dm^{[\gamma]} = g_\gamma^2 dm^{[0]} \quad ds^{[\gamma]} = g_\gamma^{-2} ds^{[0]}, \tag{4.33}$$

where g_γ is the function given in (1.12). Note that since $g_\gamma(0) = 1$, the classification of the boundary 0 for $\mathcal{L}^{[\gamma]}$ does not depend on $\gamma \geq 0$. The boundary ∞ for $\mathcal{L}^{[\gamma]}$ is always natural, which we will see in Proposition 4.7. We also have

$$\mathcal{L}^{[\gamma]} = \mathcal{L}^{[0]} + \frac{xg_\gamma'}{g_\gamma}\frac{d}{dx} \tag{4.34}$$

and, by the obvious relation

$$\widetilde{g}_\gamma(x) = g_\gamma(x^2/2), \tag{4.35}$$

it follows

$$\widetilde{\mathcal{L}}^{(\alpha,\beta,\gamma)} = \widetilde{\mathcal{L}}^{(\alpha,\beta,0)} + \frac{\widetilde{g}_\gamma'}{\widetilde{g}_\gamma}\frac{d}{dx}, \tag{4.36}$$

which implies (1.15).

We summarize several results on the hitting densities for Kummer diffusions with negative drifts. Note that from (4.31) and Proposition 6.3, the condition (S) holds for $\frac{d}{dm^{[0]}}\frac{d}{ds^{[0]}}$.

Theorem 4.5. *For the process $Y^{(\alpha,\beta,\gamma)}$ ($\alpha > 0,\ \beta \in \mathbb{R},\ \gamma \geq 0$), the hitting densities $f_x^{[\gamma]}$ of* 0 *and the spectral measure $\sigma^{[\gamma]}$ for $\mathcal{L}^{[\gamma]}$ are given as*

$$f_x^{[\gamma]}(t) = \frac{\mathrm{e}^{-\gamma t}}{g_\gamma(x)} f_x^{[0]}(t) \quad (0 < x < \infty,\ t > 0) \tag{4.37}$$

and

$$\sigma^{[\gamma]}(d\lambda) = \sigma^{[0]}(d(\lambda - \gamma)) \tag{4.38}$$

for

$$f_x^{[0]}(t) = \begin{cases} \dfrac{1}{\Gamma(\alpha)} x^\alpha t^{-\alpha-1}\mathrm{e}^{-x/t} & (\beta = 0), \\ \dfrac{x^\alpha \mathrm{e}^{\beta t}}{\Gamma(\alpha)}\left(\dfrac{\beta \mathrm{e}^{-\beta t}}{1 - \mathrm{e}^{-\beta t}}\right)^{1+\alpha} \exp\left(\dfrac{-x\beta\mathrm{e}^{-\beta t}}{1 - \mathrm{e}^{-\beta t}}\right) & (\beta \neq 0), \end{cases} \tag{4.39}$$

and

$$\sigma^{[0]}(d\lambda) = \begin{cases} \beta^{\alpha+1} \sum_{n=0}^{\infty} \frac{(\alpha)_{n+1}}{n!\Gamma(\alpha)} \delta_{\beta(n+\alpha)}(d\lambda) & (\beta > 0), \\ \frac{1}{\Gamma(\alpha)^2} \lambda^{\alpha} d\lambda & (\beta = 0), \\ (-\beta)^{\alpha+1} \sum_{n=0}^{\infty} \frac{(\alpha)_{n+1}}{n!\Gamma(\alpha)} \delta_{(-\beta)(n+1)}(d\lambda) & (\beta < 0), \end{cases} \tag{4.40}$$

where $(a)_k$ $(a \in \mathbb{R},\ k \in \mathbb{N})$ *is a Pochhammer symbol*

$$(a)_k = a(a+1)\cdots(a+k-1). \tag{4.41}$$

In particular, the infimum of the spectrum of $-\mathcal{L}^{[\gamma]}$ *on* $L^2((0,\infty), dm^{[\gamma]})$ *is*

$$\lambda_0^{[\gamma]} = \begin{cases} \alpha\beta + \gamma & (\beta > 0), \\ \gamma & (\beta = 0), \\ -\beta + \gamma & (\beta < 0). \end{cases} \tag{4.42}$$

Remark 4.6. From e.g., [21, Section 3.7], we have

$$g_\gamma(x) = \begin{cases} \frac{1}{2^{\alpha-1}\Gamma(\alpha)} (2\sqrt{\gamma x})^{\alpha} K_\alpha(2\sqrt{\gamma x}) & (\beta = 0), \\ \frac{\Gamma(\alpha + \gamma/\beta)}{\Gamma(\alpha)} (\beta x)^{\alpha} U(\alpha + \gamma/\beta, \alpha+1, \beta x) & (\beta > 0), \\ \frac{\Gamma(1 - \gamma/\beta)}{\Gamma(\alpha)} (-\beta x)^{\alpha} \mathrm{e}^{\beta x} U(1 - \gamma/\beta, \alpha+1; -\beta x) & (\beta < 0), \end{cases} \tag{4.43}$$

where K_α denotes the modified Bessel function of the second kind (see e.g., [21, Section 3.1]) and U denotes the Tricomi confluent hypergeometric function:

$$U(a,b;x) = \frac{1}{\Gamma(a)} \int_0^\infty \mathrm{e}^{-sx} s^{a-1} (1+s)^{b-a-1} ds \quad (a > 0,\ b \in \mathbb{R},\ x > 0). \tag{4.44}$$

Note that

$$K_\alpha(x) \sim 2^{\alpha-1}\Gamma(\alpha) x^{-\alpha}, \quad U(a,b;x) \sim \frac{\Gamma(b-1)}{\Gamma(a)} x^{-b+1} \quad (x \to +0,\ a > 0,\ b > 1) \tag{4.45}$$

and

$$K_\alpha(x) \sim \sqrt{\frac{\pi}{2x}} \mathrm{e}^{-x}, \quad U(a,b,x) \sim x^{-a} \quad (x \to +\infty,\ a > 0) \tag{4.46}$$

(see e.g., [21, Section 3.14.1]).

Although Theorem 4.5 can be easily shown by compiling some known results, we give a proof for completeness.

Proof of Theorem 4.5. At first, we show (4.37) and (4.39). We denote the transition probability of $Y^{[\gamma]} = Y^{(\alpha,\beta,\gamma)}$ by

$$\mathbb{P}_x[Y_t^{[\gamma]} \in dy] = p^{[\gamma]}(t,x,y)dm^{[\gamma]}(y) \tag{4.47}$$

Then it holds

$$p^{[\gamma]}(t,x,y) = \mathrm{e}^{-\gamma t}\frac{p^{[0]}(t,x,y)}{g_\gamma(x)g_\gamma(y)}, \tag{4.48}$$

(see e.g., [29, p.172]), where we write $\mathbb{P}_x$ for the underlying probability measure for $Y^{[\gamma]}$ starting from x. From [3, Appendix 1], the transition density $p^{[0]}(t,x,y)$ is given as

$$p^{[0]}(t,x,y) = \begin{cases} \dfrac{1}{t}(xy)^{\alpha/2}\mathrm{e}^{-(x+y)/t}I_\alpha\left(\dfrac{2\sqrt{xy}}{t}\right) & (\beta = 0), \\ \dfrac{\beta\mathrm{e}^{-\alpha\beta t/2}}{1-\mathrm{e}^{-\beta t}}(xy)^{\alpha/2}\exp\left(-\dfrac{(x+y)\beta\mathrm{e}^{-\beta t}}{1-\mathrm{e}^{-\beta t}}\right)I_\alpha\left(\dfrac{2\sqrt{xy}\beta\mathrm{e}^{-\beta t/2}}{1-\mathrm{e}^{-\beta t}}\right) & (\beta \neq 0), \end{cases} \tag{4.49}$$

where the function I_ν is the modified Bessel function of the first kind:

$$I_\nu(x) = \sum_{n=0}^{\infty}\frac{1}{n!\Gamma(n+\nu+1)}\left(\frac{x}{2}\right)^{\nu+2n} \quad (\nu \in \mathbb{R},\ x \in \mathbb{R}). \tag{4.50}$$

We now have

$$\begin{aligned}
\mathbb{P}_x[T_0^{[\gamma]} > t] &= \int_0^b p^{[\gamma]}(t,x,y)dm^{[\gamma]}(y) && (4.51)\\
&= \frac{\mathrm{e}^{-\gamma t}}{g_\gamma(x)}\int_0^b p^{[0]}(t,x,y)g_\gamma(y)dm^{[0]}(y) && (4.52)\\
&= \frac{\mathrm{e}^{-\gamma t}}{g_\gamma(x)}\int_0^b p^{[0]}(t,x,y)dm^{[0]}(y)\int_0^\infty \mathrm{e}^{-\gamma u}f_y^{[0]}(u)du && (4.53)\\
&= \frac{\mathrm{e}^{-\gamma t}}{g_\gamma(x)}\int_0^\infty \mathrm{e}^{-\gamma u}du\int_0^b p^{[0]}(t,x,y)f_y^{[0]}(u)dm^{[0]}(y) && (4.54)\\
&= \frac{\mathrm{e}^{-\gamma t}}{g_\gamma(x)}\int_0^\infty \mathrm{e}^{-\gamma u}f_x^{[0]}(u+t)du && (4.55)\\
&= \frac{1}{g_\gamma(x)}\int_t^\infty \mathrm{e}^{-\gamma u}f_x^{[0]}(u)du. && (4.56)
\end{aligned}$$

This shows (4.37). Then from Proposition 4.4 we obtain (4.39).

From [29, p.173] we have (4.38). We show (4.40). First we consider the case $\beta > 0$. By some computation, we can check that

$$\psi_\lambda(x) = \frac{1}{\alpha}x^\alpha M(\lambda/\beta+\alpha, 1+\alpha; \beta x) \quad (x > 0,\ \lambda \in \mathbb{R}), \tag{4.57}$$

where the function M is Kummer's confluent hypergeometric function:

$$M(a,b;x) = \sum_{n=0}^{\infty} \frac{(a)_n x^n}{(b)_n n!} \quad (a,b \in \mathbb{R},\ x \in \mathbb{R}). \tag{4.58}$$

We consider the values of λ for which the function ψ_λ is square-integrable. We may assume $\lambda < 0$. Since the asymptotic behavior of the function M is given by

$$M(a,b;x) \sim \frac{\Gamma(b)}{\Gamma(a)} x^{a-b} \mathrm{e}^{x} \quad (x \to \infty) \tag{4.59}$$

for $a \neq 0, -1, -2, \cdots$ (see e.g., [21, p.289]), the function ψ_λ is not square-integrable w.r.t. dm when $\lambda/\beta + \alpha \neq 0, -1, -2, \cdots$. When $\lambda/\beta + \alpha = 0, -1, -2, \cdots$, the function ψ_λ is a polynomial and obviously square-integrable w.r.t. dm. Note that

$$M(-n, 1+\alpha; \beta x) = \frac{n!}{(1+\alpha)_n} L_n^{(\alpha)}(\beta x), \tag{4.60}$$

where $L_n^{(\alpha)}(x)$ is the n-th Laguerre polynomial of parameter α, that is,

$$L_n^{(\alpha)}(x) = \mathrm{e}^{x} \frac{x^{-\alpha}}{n!} \frac{d^n}{dx^n}(\mathrm{e}^{-x} x^{n+\alpha}) \quad (n \in \mathbb{N}) \tag{4.61}$$

(see e.g., [21, p.241]). Since the Laguerre polynomials $\{L_n^{(\alpha)}(x)\}_n$ comprise an orthogonal basis of $L^2((0,\infty), x^{\alpha}\mathrm{e}^{-x}dx)$, the functions $\{\psi_{-\beta(\alpha+n)}(x)\}$ is so on $L^2((0,\infty), x^{-\alpha}\mathrm{e}^{-\beta x}dx)$. Hence the spectral measure only have the point spectrum and the support of σ is $\{\beta(\alpha + n),\ n \geq 0\}$. Since it holds

$$\int_0^\infty L_i^{(\alpha)}(x) L_j^{(\alpha)}(x) x^{\alpha} \mathrm{e}^{-x} dx = \delta_{ij} \frac{\Gamma(i+\alpha+1)}{i!} \quad (i,j \in \mathbb{N}) \tag{4.62}$$

(see e.g., [21, p.241]), it follows

$$\int_0^\infty \psi_{-\beta(\alpha+n)}(x)^2 dm(x) = \frac{(n!)^2}{\alpha^2 \beta^{\alpha+1}\{(1+\alpha)_n\}^2} \int_0^\infty L_n^{(\alpha)}(x)^2 x^{\alpha} \mathrm{e}^{-x} dx \tag{4.63}$$

$$= \frac{n!\Gamma(\alpha)}{\beta^{\alpha+1}(\alpha)_{n+1}}. \tag{4.64}$$

Hence we obtain

$$\sigma\{\beta(n+\alpha)\} = \frac{\beta^{\alpha+1}(\alpha)_{n+1}}{n!\Gamma(\alpha)} \quad (n \geq 0). \tag{4.65}$$

Next we show the case $\beta < 0$. Let us consider the map

$$L^2((0,\infty), dm^{(\alpha,-\beta)}) \ni f \longmapsto \mathrm{e}^{\beta x} f \in L^2((0,\infty), dm^{(\alpha,\beta)}). \tag{4.66}$$

Obviously, this map is unitary. Moreover, since it holds

$$\mathcal{L}^{(\alpha,\beta)}(\mathrm{e}^{\beta x}\psi_\lambda^{(\alpha,-\beta)}(x)) = (\lambda - \beta(\alpha-1))(\mathrm{e}^{\beta x}\psi_\lambda^{(\alpha,-\beta)}(x)) \tag{4.67}$$

and

$$\frac{d}{ds^{(\alpha,\beta)}}(\mathrm{e}^{\beta x}\psi_\lambda^{(\alpha,-\beta)}(x)) = \beta x^{1-\alpha}\psi_\lambda^{(\alpha,-\beta)}(x) + \mathrm{e}^{\beta x}\frac{d}{ds^{(\alpha,-\beta)}}\psi_\lambda^{(\alpha,-\beta)}(x), \tag{4.68}$$

we can see from (4.57) that

$$\psi_\lambda^{(\alpha,\beta)}(x) = \mathrm{e}^{\beta x}\psi_{\lambda+\beta(\alpha-1)}^{(\alpha,-\beta)}, \tag{4.69}$$

where we denote the function defined in (3.13) for $\mathcal{L}^{(\alpha,\beta)}$ by $\psi_\lambda^{(\alpha,\beta)}$. Then from the unitarity of the map (4.66) and the argument for the case $\beta > 0$, the functions $\{\psi_{-\beta(n+1)}^{(\alpha,\beta)}, n \geq 0\}$ comprise the orthogonal basis of $L^2((0,\infty), dm^{(\alpha,\beta)})$ and therefore we obtain (4.40) for $\beta < 0$.

Finally, we show the case $\beta = 0$. Note that we can see from some computation that

$$\psi_\lambda(x) = \Gamma(\alpha)\left(\frac{x}{\lambda}\right)^{\alpha/2} I_\alpha(2\sqrt{\lambda x}) \quad (x > 0,\ \lambda \in \mathbb{R}). \tag{4.70}$$

From (4.28) and (4.39), we have

$$\int_0^\infty \mathrm{e}^{-\lambda t}\psi_{-\lambda}(x)\sigma^{[0]}(d\lambda) = \frac{1}{\Gamma(\alpha)}x^\alpha t^{-\alpha-1}\mathrm{e}^{-x/t}. \tag{4.71}$$

Since it holds that

$$\frac{d}{dx}(x^\nu I_\nu(x)) = x^\nu I_{\nu-1}(x), \quad I_\nu(x) \sim \frac{\mathrm{e}^x}{\sqrt{2\pi x}} \quad (\nu \in \mathbb{R},\ x \to \infty) \tag{4.72}$$

(see e.g., [21, p.67, p.139]), we can see

$$\frac{d}{dx}\int_0^\infty \mathrm{e}^{-\lambda t}\left|\frac{d}{dx}\psi_{-\lambda}(x)\right|\sigma^{[0]}(d\lambda) < \infty \quad (x > 0). \tag{4.73}$$

Thus we have

$$\int_0^\infty \mathrm{e}^{-\lambda t}\sigma^{[0]}(d\lambda) = \frac{d}{ds(x)}\int_0^\infty \mathrm{e}^{-\lambda t}\psi_{-\lambda}(x)\sigma^{[0]}(d\lambda)\bigg|_{x=0} \tag{4.74}$$

$$= \frac{d}{ds(x)}\frac{1}{\Gamma(\alpha)}x^\alpha t^{-\alpha-1}\mathrm{e}^{-x/t}\bigg|_{x=0} \tag{4.75}$$

$$= \frac{\alpha t^{-\alpha-1}}{\Gamma(\alpha)}. \tag{4.76}$$

From the uniqueness of the Laplace transform, we obtain (4.40).

□

We give the classification of the boundary ∞ for $\mathcal{L}^{[\gamma]}$:

Proposition 4.7. *For $\alpha > 0,\ \beta \in \mathbb{R},\ \gamma \geq 0$, the boundary ∞ for $\mathcal{L}^{[\gamma]}$ is natural.*

Proof. Let $\beta > 0$. From (4.43) and (4.46), we have

$$s(x) - s(1) = \int_1^x y^{\alpha-1} \mathrm{e}^{\beta y} \frac{dy}{g_\gamma^2(y)} \tag{4.77}$$

$$\asymp \int_1^x y^{\alpha+2\gamma/\beta-1} \mathrm{e}^{\beta y} dy \xrightarrow[x\to\infty]{} \infty, \tag{4.78}$$

where $f_1 \asymp f_2$ means there exists a constant $c > 0$ such that $(1/c)f_1(x) \leq f_2(x) \leq cf_1(x)$ for large $x > 0$. Note that from L'Hôpital's rule, it holds for $\delta \in \mathbb{R}$

$$\int_x^\infty y^\delta \mathrm{e}^{-\beta y} dy \sim \frac{1}{\beta} x^\delta \mathrm{e}^{-\beta x} \quad (x \to \infty). \tag{4.79}$$

We have

$$\int_1^\infty ds^{[\gamma]}(x) \int_x^\infty dm^{[\gamma]}(y) \asymp \int_1^\infty x^{\alpha+2\gamma/\beta-1} \mathrm{e}^{-\beta x} dx \int_x^\infty y^{-\alpha-2\gamma/\beta} \mathrm{e}^{-\beta y} dy \tag{4.80}$$

$$\asymp \int_1^\infty \frac{dx}{x} = \infty. \tag{4.81}$$

Thus the boundary ∞ is natural. We can show the cases of $\beta = 0$ and $\beta < 0$ by the similar argument and hence we omit them. □

4.3 Convergence to non-minimal quasi-stationary distributions for Kummer diffusions with negative drifts

Let us apply Theorem 1.1 to Kummer diffusions with negative drifts and give a sufficient condition on initial distributions under which the conditional process converges to each non-minimal quasi-stationary distribution specified.

The following proposition gives a necessary and sufficient condition for that of Theorem 3.3:

Proposition 4.8. *For $\mathcal{L}^{(\alpha,\beta,\gamma)}$ ($\alpha > 0, \beta \in \mathbb{R}, \gamma \geq 0$), the condition of Theorem 3.3 holds if and only if one of the Case 1-3 in (1.16) holds.*

Proof. Let $\beta > 0$. Obviously, it holds $m^{[\gamma]}(1,\infty) < \infty$ and $s^{[\gamma]}(\infty) = \infty$. From (4.46), we have

$$m^{[\gamma]}(x,\infty)(s^{[\gamma]}(x) - s^{[\gamma]}(1)) \asymp (x^{-\alpha-2\gamma/\beta} \mathrm{e}^{-\beta x})(x^{\alpha+2\gamma/\beta-1} \mathrm{e}^{\beta x}) \tag{4.82}$$

$$\asymp 1/x \xrightarrow[x\to\infty]{} 0. \tag{4.83}$$

Let $\beta = 0$. We can easily check $s^{[\gamma]}(\infty) = \infty$ for $\gamma \geq 0$ and

$$\lim_{x\to\infty} m^{[0]}(x,\infty)(s^{[0]}(x) - s^{[0]}(1)) = \infty. \tag{4.84}$$

For $\gamma > 0$, from (4.46) it holds

$$m^{[\gamma]}(x, \infty)(s^{[\gamma]}(x) - s^{[\gamma]}(1)) \asymp \mathrm{e}^{-4\sqrt{\gamma x}} \cdot \mathrm{e}^{4\sqrt{\gamma x}} = 1. \tag{4.85}$$

Let $\beta < 0$. It holds $s^{[0]}(\infty) < \infty$ from (4.31). For $\gamma > 0$, we have from (4.43)

$$s^{[\gamma]}(x) - s^{[\gamma]}(1) \asymp \int_1^x y^{-\alpha-\gamma/\beta}\mathrm{e}^{-\beta y}dy \tag{4.86}$$

$$\asymp x^{1-\alpha-2\gamma/\beta}\mathrm{e}^{-\beta x} \xrightarrow[x\to\infty]{} \infty. \tag{4.87}$$

Similarly, we can show $m^{[\gamma]}(1, x) \asymp x^{-2+\alpha+2\gamma/\beta}\mathrm{e}^{\beta x}$ and thus $m^{[\gamma]}(1, \infty) < \infty$. Then it holds

$$m^{[\gamma]}(x, \infty)(s^{[\gamma]}(x) - s^{[\gamma]}(1)) \asymp 1/x \xrightarrow[x\to\infty]{} 0. \tag{4.88}$$

□

We give a proof of Theorem 1.2 after several preparatory results. For the process $Y^{(\alpha,\beta,\gamma)}$, the first hitting uniqueness holds on $\mathcal{P}(0, \infty)$. We show this fact in more general settings as follows:

Theorem 4.9. *Let X be a $\frac{d}{dm}\frac{d}{ds}$-diffusion on $[0, b)$ $(0 < b \le \infty)$ and $s(b) = \infty$. Suppose the hitting densities $f_x(t)$ of 0 have the following form*

$$f_x(t) = u(x)w(t)\mathrm{e}^{-v(x)y(t)} \quad (0 < x < b,\ t > 0) \tag{4.89}$$

for some strictly positive functions $u(x)$ and $v(x)$ on $(0, b)$ and some strictly positive function $w(t)$ and $y(t)$ on $(0, \infty)$. In addition, suppose v is strictly increasing continuous and $y(0, \infty) = (0, \infty)$. Then the first hitting uniqueness holds on $\mathcal{P}(0, \infty)$.

Proof. Suppose μ_1 and $\mu_2 \in \mathcal{P}(I)$ satisfy

$$\mathbb{P}_{\mu_1}[T_0 \in dt] = \mathbb{P}_{\mu_2}[T_0 \in dt] \tag{4.90}$$

and set $\mu = \mu_1 - \mu_2$. It holds

$$\int_0^b f_x(t)\mu(dx) = 0 \quad (t > 0). \tag{4.91}$$

Note that from the continuity of $f_x(t)/w(t)$ w.r.t. t, the equality (4.91) holds for every $t > 0$. From (4.89) and by a change of variables, we have

$$0 = \int_{v(0)}^{v(b)} u(v^{-1}(x))\mathrm{e}^{-xy(t)}\mu(v^{-1}(dx)). \tag{4.92}$$

Since $y(0, \infty) = (0, \infty)$, it holds from the uniqueness of the Laplace transform

$$u(x)\mu(dx) = 0 \quad \text{on } (0, b). \tag{4.93}$$

Since $u(x) > 0$, we obtain the desired result. □

For the proof of (i) of Theorem 1.2, we need the following lemma, which enables us to cut off the integral region for the asymptotic behavior of the Laplace transform:

Lemma 4.10. *Let* $f : (0,\infty) \to [0,\infty)$ *and assume*

$$\log f(x) \sim \delta\sqrt{x} \quad (x \to \infty) \tag{4.94}$$

for $\delta > 0$ *and*

$$\int_0^\infty \mathrm{e}^{-x/t} f(x)dx < \infty \quad (t > 0). \tag{4.95}$$

Then for every $\varepsilon > 0$*, it holds*

$$\log \int_0^\infty \mathrm{e}^{-x/t} f(x)dx \sim \frac{\delta^2}{4}t \tag{4.96}$$

and

$$\int_0^\infty \mathrm{e}^{-x/t} f(x)dx \sim \int_{(\delta^2/4-\varepsilon)t^2}^{(\delta^2/4+\varepsilon)t^2} \mathrm{e}^{-x/t} f(x)dx \quad (t \to \infty). \tag{4.97}$$

Proof. Since it holds

$$\lim_{t\to\infty} \int_0^1 \mathrm{e}^{-x/t} f(x)dx < \infty \quad \text{and} \quad \lim_{t\to\infty} \int_1^\infty \mathrm{e}^{-x/t} f(x)dx = \infty, \tag{4.98}$$

we may assume without loss of generality that $f(x) = 0$ for $0 < x < 1$. It is enough to show

$$\lim_{t\to\infty} \frac{\int_1^{(\delta^2/4-\varepsilon)t} \mathrm{e}^{-x/t} f(x)dx}{\int_1^\infty \mathrm{e}^{-x/t} f(x)dx} = 0 \tag{4.99}$$

and

$$\lim_{t\to\infty} \frac{\int_{(\delta^2/4+\varepsilon)t}^\infty \mathrm{e}^{-x/t} f(x)dx}{\int_1^\infty \mathrm{e}^{-x/t} f(x)dx} = 0. \tag{4.100}$$

Let

$$h(x) = \frac{\log(x^2 f(x))}{\sqrt{x}} - \delta \quad (x > 1). \tag{4.101}$$

Then from (4.94), we have $\lim_{x\to\infty} h(x) = 0$. It follows

$$\int_1^\infty \mathrm{e}^{-x/t} f(x)dx = \int_1^\infty \mathrm{e}^{-\varphi_t(x)} \frac{dx}{x^2} \tag{4.102}$$

where

$$\varphi_t(x) = x/t - (\delta + h(x))\sqrt{x} \tag{4.103}$$

Note that

$$\varphi_t(x) = \frac{1}{t}\left(\sqrt{x} - \frac{\delta + h(x)}{2}t\right)^2 - \frac{(\delta + h(x))^2}{4}t. \tag{4.104}$$

Let $\theta := \delta/2 - \sqrt{\delta^2/4 - \varepsilon} > 0$ and take $R > 1$ so that

$$|h(x)| < \theta \quad \text{and} \quad \frac{2\delta|h(x)| + h(x)^2}{4} < \theta^2/8 \quad (x > R). \tag{4.105}$$

Then for $R < x < (\delta^2/4 - \varepsilon)t^2$, it follows

$$\frac{\delta + h(x)}{2}t - \sqrt{x} > \frac{\delta + h(x)}{2}t - t\sqrt{\delta^2/4 - \varepsilon} > \frac{\theta}{2}t \tag{4.106}$$

and thus

$$\varphi_t(x) \geq (\theta^2/8 - \delta^2/4)t. \tag{4.107}$$

Then it follows

$$\int_R^{(\delta^2/4-\varepsilon)t^2} \mathrm{e}^{-\varphi_t(x)}\frac{dx}{x^2} \leq \mathrm{e}^{(\delta^2/4-\theta^2/8)t}\int_R^{(\delta^2/4-\varepsilon)t^2}\frac{dx}{x^2} \tag{4.108}$$

$$\leq \mathrm{e}^{(\delta^2/4-\theta^2/8)t}. \tag{4.109}$$

For showing (4.99), it is hence enough to show

$$\log\int_1^\infty \mathrm{e}^{-x/t}f(x)dx \sim \frac{\delta^2}{4}t \quad (t \to \infty). \tag{4.110}$$

From [2, Theorem 4.12.10 (ii)], it holds

$$\log\int_0^x f(y)dy \sim \delta\sqrt{x} \quad (x \to \infty). \tag{4.111}$$

From Kohlbecker's Tauberian Theorem [2, Theorem 4.12.1], we obtain therefore (4.110). We can show (4.100) by a similar argument. □

Now we proceed to the proof of Theorem 1.2.

Proof of Theorem 1.2. First we show (i). From Proposition 4.2 and Theorem 4.5, it is enough to show

$$\lim_{t\to\infty}\frac{d}{dt}\log\int_0^\infty \mathrm{e}^{-x/t}\frac{x^{\alpha/2}}{K_\alpha(2\sqrt{\gamma x})}\mu(dx) = \delta^2/4. \tag{4.112}$$

From (4.46), it holds

$$\log\widetilde{\rho}(x) := \log\frac{x^{\alpha/2}\rho(x)}{K_\alpha(2\sqrt{\gamma x})} \sim \delta\sqrt{x} \quad (x \to \infty). \tag{4.113}$$

Take $\varepsilon > 0$. Since it holds

$$\frac{d}{dt} \log \int_0^\infty \mathrm{e}^{-x/t} \frac{x^{\alpha/2}}{K_\alpha(2\sqrt{\gamma x})} \mu(dx) = \frac{\int_0^\infty \mathrm{e}^{-x/t} x \widetilde{\rho}(x) dx}{t^2 \int_0^\infty \mathrm{e}^{-x/t} \widetilde{\rho}(x) dx}, \tag{4.114}$$

we have from (4.113) and Lemma 4.10

$$\frac{\int_0^\infty \mathrm{e}^{-x/t} x \widetilde{\rho}(x) dx}{t^2 \int_0^\infty \mathrm{e}^{-x/t} \widetilde{\rho}(x) dx} \sim \frac{\int_{(\delta^2/4-\varepsilon)t^2}^{(\delta^2/4+\varepsilon)t^2} \mathrm{e}^{-x/t} x \widetilde{\rho}(x) dx}{t^2 \int_{(\delta^2/4-\varepsilon)t^2}^{(\delta^2/4+\varepsilon)t^2} \mathrm{e}^{-x/t} \widetilde{\rho}(x) dx} \tag{4.115}$$

and obviously we have

$$\int_{(\delta^2/4-\varepsilon)t^2}^{(\delta^2/4+\varepsilon)t^2} \mathrm{e}^{-x/t} x \widetilde{\rho}(x) dx \lesseqgtr (\delta^2/4 \pm \varepsilon) t^2 \int_{(\delta^2/4-\varepsilon)t^2}^{(\delta^2/4+\varepsilon)t^2} \mathrm{e}^{-x/t} \widetilde{\rho}(x) dx. \tag{4.116}$$

Since $\varepsilon > 0$ can be taken arbitrary small, we obtain

$$\frac{\int_0^\infty \mathrm{e}^{-x/t} x \widetilde{\rho}(x) dx}{t^2 \int_0^\infty \mathrm{e}^{-x/t} \widetilde{\rho}(x) dx} \xrightarrow{t \to \infty} \delta^2/4. \tag{4.117}$$

Next we show (ii). From the proof of Proposition 4.2, it is enough to show that the function $f_\mu(\log t)$ varies regularly at ∞ with exponent $-\lambda$. From Theorem 4.5, we have

$$f_\mu(\log t) = \frac{1}{\Gamma(\alpha)} t^{\beta-\gamma} h(t)^{1+\alpha} \int_0^\infty \frac{x^\alpha}{g_\gamma(x)} \mathrm{e}^{-h(t)x} \mu(dx), \tag{4.118}$$

where

$$h(t) = \frac{\beta}{t^\beta - 1} \quad (t > 1). \tag{4.119}$$

The inverse function h^{-1} of h is given as

$$h^{-1}(s) = \left(1 + \frac{\beta}{s}\right)^{1/\beta} \quad (s > 0). \tag{4.120}$$

Note that the function $h^{-1}(s)$ varies regularly at $s = 0$ with exponent $-1/\beta$. By considering the function $f(\log h^{-1}(s))$, it follows that the function $f_\mu(\log t)$ varies regularly at $t = \infty$ with exponent $-\lambda$ if and only if the function

$$\int_0^\infty \frac{x^\alpha}{g_\gamma(x)} \mathrm{e}^{-sx} \mu(dx) \tag{4.121}$$

varies regularly at $s = 0$ with exponent $-\alpha - (\gamma - \lambda)/\beta$. From Karamata's Tauberian Theorem [2, Theorem 1.7.1], it is equivalent to that the function

$$\int_0^x \frac{y^\alpha}{g_\gamma(y)} \mu(dy) \tag{4.122}$$

varies regularly at $x = \infty$ with exponent $\alpha + (\gamma - \lambda)/\beta$. Then from (4.46) and [2, Theorem 1.6.4], it is equivalent to the function $\mu(x, \infty)$ varies regularly at $x = \infty$ with exponent $-\lambda/\beta$, and therefore we obtain (ii).

Finally, we show (iii). The proof of this case is quite similar to that of (ii). From Theorem 4.5, we have

$$f_\mu(\log t) = \frac{1}{\Gamma(\alpha)} t^{\beta-\gamma} h(t)^{1+\alpha} \int_0^\infty \frac{x^\alpha}{g_\gamma(x)} \mathrm{e}^{-h(t)x} \mu(dx). \tag{4.123}$$

Note that for $\beta < 0$, it holds $\lim_{t\to\infty} h(t) = -\beta$. Then the function $f_\mu(\log t)$ varies regularly at $t = \infty$ with exponent $-\lambda$ if and only if the function

$$\int_0^\infty \frac{x^\alpha}{g_\gamma(x)} \mathrm{e}^{-h(t)x} \mu(dx) = \frac{(-\beta)^{-\alpha}\Gamma(\alpha)}{\Gamma(1-\gamma/\beta)} \int_0^\infty \frac{\mathrm{e}^{-(h(t)+\beta)x}}{U(1-\gamma/\beta, \alpha+1; -\beta x)} \mu(dx) \tag{4.124}$$

varies regularly at $t = \infty$ with exponent $-\lambda - \beta + \gamma$. Note that the function $h^{-1}(s)$ varies regularly at $s = -\beta + 0$ with exponent $-1/\beta$. Thus, by denoting $u = s + \beta$, the regular variation at $t = \infty$ of (4.124) with exponent $-\lambda - \beta + \gamma$ is equivalent to that at $u = 0$ of

$$\int_0^\infty \frac{\mathrm{e}^{-ux}}{U(1-\gamma/\beta, \alpha+1; -\beta x)} \mu(dx) \tag{4.125}$$

with exponent $1 + (\lambda - \gamma)/\beta$. Using (4.46), the rest of the proof can be made by the same argument in (ii) and hence we omit it. The proof is complete. □

5 Renewal dynamical approach to quasi-stationary distributions

For every $\frac{d}{d\widetilde{m}}\frac{d}{ds}$-diffusion X on $(0, b)$ $(0 < b \leq \infty)$ satisfying (3.21), the diffusion $s(X)$ is $\frac{d}{dm}\frac{d}{dx}$-diffusion on $(0, \infty)$ for $dm(x) = d\widetilde{m}(s^{-1}(x))$. Thus we may assume without loss of generality that the diffusion is under the natural scale: $s(x) = x$. In this section, we only consider such natural scale diffusions on $(0, \infty)$.

Let us recall that the transform Φ was introduced in (1.25). We ensure that Φ preserves $\mathcal{P}_m$.

Proposition 5.1. *For $n \geq 1$, we have*

$$\mathbb{E}_{\Phi\mu} T_0^n = \frac{\mathbb{E}_\mu T_0^{n+1}}{(n+1)\mathbb{E}_\mu T_0}. \tag{5.1}$$

More generally, we have for $0 \leq k \leq m$

$$\mathbb{E}_{\Phi^m\mu} T_0^n = \binom{n+k}{n}^{-1} \cdot \frac{\mathbb{E}_{\Phi^{m-k}\mu} T_0^{n+k}}{\mathbb{E}_{\Phi^{m-k}\mu} T_0^k}. \tag{5.2}$$

Proof. Since (5.2) follows from (5.1) by induction, we only prove (5.1). Note $\mathbb{P}_\mu[T_0 > t] = \int_0^\infty \mu(dx) \int_0^\infty p(t,x,y)dm(y)$. It follows that

$$\mathbb{E}_{\Phi\mu} T_0^n = n \int_0^\infty t^{n-1} \mathbb{P}_{\Phi\mu}[T_0 > t] dt \tag{5.3}$$

$$= n \int_0^\infty t^{n-1} dt \int_0^\infty p(t, \Phi\mu, y) dm(y) \tag{5.4}$$

$$= \frac{n}{\mathbb{E}_\mu T_0} \int_0^\infty ds \int_0^\infty t^{n-1} dt \int_0^\infty dm(y) \int_0^\infty p(s,\mu,x) p(t,x,y) dm(x) \tag{5.5}$$

$$= \frac{n}{\mathbb{E}_\mu T_0} \int_0^\infty t^{n-1} dt \int_0^\infty ds \int_0^\infty p(s+t,\mu,y) dm(y) \tag{5.6}$$

$$= \frac{n}{\mathbb{E}_\mu T_0} \int_0^\infty t^{n-1} dt \int_t^\infty ds \int_0^\infty p(s,\mu,y) dm(y) \tag{5.7}$$

$$= \frac{1}{\mathbb{E}_\mu T_0} \int_0^\infty s^n ds \int_0^\infty p(s,\mu,y) dm(y) \tag{5.8}$$

$$= \frac{\mathbb{E}_\mu T_0^{n+1}}{(n+1)\mathbb{E}_\mu T_0}. \tag{5.9}$$

□

The following theorem shows that for an initial distribution μ, convergence of $\Phi^n \mu$ as $n \to \infty$ is necessary for convergence of μ_t as $t \to \infty$.

Theorem 5.2. *Let $\mu \in \mathcal{P}_m$ and $\nu \in \mathcal{P}(0,\infty)$. Assume*

$$\mu_t \xrightarrow[t\to\infty]{} \nu. \tag{5.10}$$

Then it holds

$$\Phi^n \mu \xrightarrow[n\to\infty]{} \nu. \tag{5.11}$$

For the proof of Theorem 5.2, we need the density of $\Phi^n \mu$.

Proposition 5.3. *For $n \geq 1$, the probability measure $\Phi^n \mu$ has a density w.r.t. dm given by*

$$f_n^\mu(x) = \frac{n}{\mathbb{E}_\mu T_0^n} \int_0^\infty t^{n-1} p(t,\mu,x) dt. \tag{5.12}$$

Proof. We show by induction. The case $n = 1$ is obvious. Assume the assertion holds for

$n = k$. By Proposition 5.1, we have

$$\Phi^{k+1}\mu(A) = \frac{1}{\mathbb{E}_{\Phi^k\mu}T_0}\int_0^\infty dt\int_A p(t,\Phi^k\mu,y)dm(y) \tag{5.13}$$

$$= \frac{(k+1)\mathbb{E}_\mu T_0^k}{\mathbb{E}_\mu T_0^{k+1}}\int_0^\infty dt\int_0^\infty f_k^\mu(x)dm(x)\int_A p(t,x,y)dm(y) \tag{5.14}$$

$$= \frac{k(k+1)}{\mathbb{E}_\mu T_0^{k+1}}\int_0^\infty dt\int_0^\infty dm(x)\int_0^\infty s^{k-1}p(s,\mu,x)ds\int_A p(t,x,y)dm(y) \tag{5.15}$$

$$= \frac{k+1}{\mathbb{E}_\mu T_0^{k+1}}\int_A dm(y)\int_0^\infty t^k p(t,\mu,y)dt. \tag{5.16}$$

□

We give a proof of Theorem 5.2.

Proof of Theorem 5.2. By Proposition 5.3, we have

$$\Phi^n\mu(A) = \frac{n}{\mathbb{E}_\mu T_0^n}\int_A dm(x)\int_0^\infty t^{n-1}p(t,\mu,x)dt \tag{5.17}$$

$$= \frac{n}{\mathbb{E}_\mu T_0^n}\int_0^\infty t^{n-1}\mu_t(A)\mathbb{P}_\mu[T_0 > t]dt. \tag{5.18}$$

By changing variables by $t' = \frac{t^n}{\mathbb{E}_\mu T_0^n}$, it follows for $||T_0||_{p,\mu} = (\mathbb{E}_\mu T_0^p)^{1/p}$ $(p > 0)$ that

$$\Phi^n\mu(A) = \int_0^\infty \mu_{t^{1/n}||T_0||_{n,\mu}}(A)\mathbb{P}_\mu[T_0 > t^{1/n}||T_0||_{n,\mu}]dt \tag{5.19}$$

$$= \int_0^\infty \mu_{t^{1/n}||T_0||_{n,\mu}}(A)\beta_n(dt) \tag{5.20}$$

where $\beta_n(dt) = \mathbb{P}_\mu[T_0 > t^{1/n}||T_0||_{n,\mu}]dt$. Note that β_n is a probability measure on $[0,\infty)$. Since $\lim_{n\to\infty}||T_0||_{n,\mu} = \text{ess.}\sup_\mu T_0 = \infty$, we have from the dominated convergence theorem

$$\beta_n[0,R] = \int_0^R \mathbb{P}_\mu[T_0 > t^{1/n}||T_0||_{n,\mu}]dt \xrightarrow{n\to\infty} 0 \tag{5.21}$$

for every $R > 0$. Hence it follows that $\beta_n \xrightarrow[n\to\infty]{} \delta_\infty$ on $\mathcal{P}[0,\infty]$. Then from (5.10) we have

$$\lim_{n\to\infty}\Phi^n\mu(A) = \nu(A). \tag{5.22}$$

□

For $\mu \in \mathcal{P}_m$ we define the normalized n-th moment of T_0 under μ by

$$m_n^\mu := \frac{1}{n!}\mathbb{E}_\mu T_0^n. \tag{5.23}$$

The following theorem gives a sufficient condition for convergence of $\Phi^n\mu$ as $n \to \infty$.

Theorem 5.4. *Let $\mu \in \mathcal{P}_m$ and assume the following hold:*

$$\lim_{n\to\infty} \frac{m_{n-1}^\mu}{m_n^\mu} = \lambda > 0. \tag{5.24}$$

Then we have

$$\lim_{n\to\infty} f_n^\mu(x) = \lambda \psi_{-\lambda}(x) \quad (x > 0), \tag{5.25}$$

$$\Phi^n \mu \xrightarrow[n\to\infty]{} \nu_\lambda. \tag{5.26}$$

For the proof, we need some preparation. For a function $g : (0, \infty) \to \mathbb{R}$ with

$$\int_0^x dy \int_y^\infty |g(z)| dm(z) < \infty, \tag{5.27}$$

we define

$$Kg(x) := \int_0^x dy \int_y^\infty g(z) dm(z) = \int_0^\infty (x \wedge y) g(y) dm(y) \quad (x > 0). \tag{5.28}$$

Let us recall the formula (3.32). Then for a function g with (5.27), it holds

$$\mathbb{E}_\mu \int_0^{T_0} g(X_t) dt = \int_0^\infty \mu(dx) \int_0^\infty (x \wedge y) g(y) dm(y) = \int_0^\infty Kg(x) \mu(dx). \tag{5.29}$$

Applying (5.29), we obtain another representation of f_n^μ different from (5.12).

Proposition 5.5. *For $\mu \in \mathcal{P}_m$, the density f_n^μ of $\Phi^n \mu$ satisfies*

$$f_n^\mu(x) = \frac{n!}{\mathbb{E}_\mu T_0^n} K^{n-1} G_\mu(x), \tag{5.30}$$

where $G_\mu(x) := \int_0^x \mu(y, \infty) dy$ and we denote $K^\ell g := K(K^{\ell-1} g)$ $(\ell \geq 1)$.

Proof. From (5.29), we have for $g \in L^1((0, \infty), dm)$,

$$\int_0^\infty g(y) \Phi\mu(dy) = \frac{1}{\mathbb{E}_\mu T_0} \int_0^\infty g(y) dm(y) \int_0^\infty p(t, \mu, y) dt \tag{5.31}$$

$$= \frac{1}{\mathbb{E}_\mu T_0} \int_0^{T_0} \mathbb{E}_\mu[g(X_t)] dt \tag{5.32}$$

$$= \frac{1}{\mathbb{E}_\mu T_0} \int_0^\infty \mu(dx) \int_0^\infty (x \wedge y) g(y) dm(y) \tag{5.33}$$

$$= \frac{1}{\mathbb{E}_\mu T_0} \int_0^\infty G_\mu(y) g(y) dm(y). \tag{5.34}$$

Since it holds that

$$G_{\Phi\mu}(x) = \int_0^x \Phi\mu(y,\infty)dy \tag{5.35}$$

$$= \frac{1}{\mathbb{E}_\mu T_0} \int_0^x dy \int_y^\infty G_\mu(z)dm(z) \tag{5.36}$$

$$= \frac{1}{\mathbb{E}_\mu T_0} K G_\mu(x) \tag{5.37}$$

it follows from Proposition 5.1 and (formal) self-adjointness of K under dm that

$$\int_0^\infty g(y)\Phi^n\mu(dy) \tag{5.38}$$

$$= \frac{1}{\mathbb{E}_{\Phi^{n-1}\mu}T_0} \int_0^\infty G_{\Phi^{n-1}\mu}(y)g(y)dm(y) \tag{5.39}$$

$$= \frac{1}{(\mathbb{E}_{\Phi^{n-1}\mu}T_0)(\mathbb{E}_{\Phi^{n-2}\mu}T_0)} \int_0^\infty KG_{\Phi^{n-2}\mu}(y)g(y)dm(y) \tag{5.40}$$

$$= \frac{1}{(\mathbb{E}_{\Phi^{n-1}\mu}T_0)(\mathbb{E}_{\Phi^{n-2}\mu}T_0)} \int_0^\infty G_{\Phi^{n-2}\mu}(y)Kg(y)dm(y) \tag{5.41}$$

$$= \cdots \tag{5.42}$$

$$= \frac{1}{(\mathbb{E}_{\Phi^{n-1}\mu}T_0)(\mathbb{E}_{\Phi^{n-2}\mu}T_0)\cdots(\mathbb{E}_{\Phi\mu}T_0)(\mathbb{E}_\mu T_0)} \int_0^\infty K^{n-1}G_\mu(y)g(y)dm(y) \tag{5.43}$$

$$= \frac{n!}{\mathbb{E}_\mu T_0^n} \int_0^\infty K^{n-1}G_\mu(y)g(y)dm(y). \tag{5.44}$$

The proof is complete. □

Now we prove Theorem 5.4.

Proof of Theorem 5.4. From Proposition 5.5, it holds

$$f_n^\mu(x) = \frac{1}{m_n^\mu} K^{n-1} G_\mu(x). \tag{5.45}$$

For a function g with

$$\int_0^R dy \int_0^y |g(z)|dm(z) < \infty \tag{5.46}$$

for every $R > 0$, define

$$Ig(x) = \int_0^x dy \int_0^y g(z)dm(z) \quad (x > 0). \tag{5.47}$$

Let $g \in L^1((0,\infty), dm)$. We have

$$Kg(x) = \int_0^x dy \int_y^\infty g(z)dm(z) \tag{5.48}$$

$$= x\int_0^\infty g(z)dm(z) - Ig(x). \tag{5.49}$$

Then it follows that

$$K^{n-1}G_\mu(x) = x\int_0^\infty K^{n-2}G_\mu(y)dm(y) - IK^{n-2}G_\mu(x) \tag{5.50}$$
$$= m_{n-1}^\mu x - IK^{n-2}G_\mu(x) \tag{5.51}$$
$$= m_{n-1}^\mu x - I(m_{n-2}^\mu x - IK^{n-3}G_\mu(x)) \tag{5.52}$$
$$= m_{n-1}^\mu x - m_{n-2}^\mu Ix + I^2K^{n-3}G_\mu(x) \tag{5.53}$$
$$= \cdots \tag{5.54}$$
$$= \sum_{k=1}^{n-1}(-1)^{k-1}m_{n-k}^\mu I^{k-1}x + (-1)^{n-1}I^{n-1}G_\mu(x). \tag{5.55}$$

Then we have

$$f_n^\mu(x) = \sum_{k=1}^{n-1}(-1)^{k-1}\frac{m_{n-k}^\mu}{m_n^\mu}I^{k-1}x + (-1)^{n-1}\frac{1}{m_n^\mu}I^{n-1}G_\mu(x). \tag{5.56}$$

From (5.24) we have $M := \sup_{n\geq 0}\frac{m_{n-1}^\mu}{m_n^\mu} < \infty$ where we regard $m_0^\mu = 1$. Hence it follows that

$$\sum_{k=1}^{n-1}\frac{m_{n-k}^\mu}{m_n^\mu}I^{k-1}x \leq \sum_{n=1}^{\infty}M^nI^{n-1}x = M\psi_M(x) < \infty. \tag{5.57}$$

Next we show the second term in the RHS of (5.56) vanishes as $n\to\infty$. It is not difficult to check that

$$I^nG_\mu(x) \leq \frac{1}{n!}x\left(\int_0^x ydm(y)\right)^n. \tag{5.58}$$

and

$$m_n^\mu \geq M^{-n}. \tag{5.59}$$

Therefore we obtain for every $R > 0$

$$\lim_{n\to\infty}\sup_{x\in[0,R]}\frac{1}{m_n^\mu}I^{n-1}|G_\mu(x)| = 0. \tag{5.60}$$

Then from (5.56) and the dominated convergence theorem, we have

$$\lim_{n\to\infty}f_n^\mu(x) = \lambda\psi_{-\lambda}(x). \tag{5.61}$$

From (5.57) and (5.58), we have

$$f_n^\mu(x) \leq M\psi_M(x) + \frac{M^n}{(n-1)!}x\left(\int_0^x ydm(y)\right)^{n-1} \leq M\psi_M(x) + Mx\mathrm{e}^{M\int_0^x ydm(y)} \tag{5.62}$$

and, it is obvious that

$$\int_0^R (M\psi_M(x) + Mx\mathrm{e}^{M\int_0^x ydm(y)})dm(x) < \infty \tag{5.63}$$

for every $R > 0$. Hence from the dominated convergence theorem, it follows that

$$\Phi^n \mu \xrightarrow[n\to\infty]{} \nu_\lambda. \tag{5.64}$$

□

The next proposition shows that condition (i) in Theorem 1.1 is sufficient for (5.24).

Proposition 5.6. *Let $\mu \in \mathcal{P}_m$ and assume the condition (i) in Theorem 1.1 for $\lambda \in (0, \lambda_0]$ is satisfied. Then* (5.24) *holds.*

Proof. By integrating by parts, it holds for $n \geq 2$

$$m_n^\mu = \frac{1}{(n-1)!}\int_0^\infty t^{n-1}\mathbb{P}_\mu[T_0 > t]dt \tag{5.65}$$

$$= \frac{1}{(n-2)!}\int_0^\infty t^{n-2}\left(\int_t^\infty \mathbb{P}_\mu[T_0 > s]ds\right)dt. \tag{5.66}$$

Then for $h(t) = \frac{1}{\mathbb{P}_\mu[T_0>t]}\int_t^\infty \mathbb{P}_\mu[T_0 > s]ds$, it holds

$$m_n^\mu = \frac{1}{(n-2)!}\int_0^\infty t^{n-2}\mathbb{P}_\mu[T_0 > t]h(t)dt \tag{5.67}$$

For $R > 0$ it is not difficult to see

$$\frac{\int_0^R t^{n-2}\mathbb{P}_\mu[T_0 > t]h(t)dt}{\int_R^\infty t^{n-2}\mathbb{P}_\mu[T_0 > t]h(t)dt} \xrightarrow{n\to\infty} 0. \tag{5.68}$$

From Theorem 1.1 (i)', it follows

$$\lim_{n\to\infty}\frac{m_n^\mu}{m_{n-1}^\mu} = \lim_{R\to\infty}\lim_{n\to\infty}\frac{\int_R^\infty t^{n-2}\mathbb{P}_\mu[T_0 > t]h(t)dt}{\int_R^\infty t^{n-2}\mathbb{P}_\mu[T_0 > t]dt} = \frac{1}{\lambda}. \tag{5.69}$$

□

Note that from [8, Theorem 6.26 (iii), (iv)], if the initial distribution μ is compactly supported, the condition (i) in Theorem 1.1 holds for $\lambda = \lambda_0$. By Proposition 5.6 and Theorem 5.4, we obtain the following corollary.

Corollary 5.7. *Let $\mu \in \mathcal{P}(0, \infty)$ with the compact support. Then the convergence* (5.25) *and* (5.26) *holds for $\lambda = \lambda_0$.*

Remark 5.8. In [9, Theorem 4.1], for Markov chains on $\mathbb{N}$ it was shown under some weak assumptions that the condition (5.24) holds for $\lambda = \lambda_0$ if the initial distribution is a point mass.

The set of quasi-stationary distributions coincides with the set of fixed points of Φ on $\mathcal{P}_m$.

Proposition 5.9. *For $\mu \in \mathcal{P}_m$, $\Phi\mu = \mu$ if and only if $\mu = \nu_\lambda$ for $\lambda \in (0, \lambda_0]$.*

Proof. Assume $\Phi\mu = \mu$. From (5.37), it holds

$$G_\mu(x) = \lambda K G_\mu(x), \tag{5.70}$$

where $\lambda = (\mathbb{E}_\mu T_0)^{-1}$. Right-differentiating both sides, we have

$$\mu(x, \infty) = \lambda \int_x^\infty G_\mu(y) dm(y). \tag{5.71}$$

Thus μ has a density w. r. t. dm and we denote it by ρ. Then we have

$$\frac{d}{dm}\frac{d^+}{dx}\rho(x) = -\lambda\rho(x), \quad \rho(0) = 0, \quad \frac{d^+}{dx}\rho(0) = \lambda. \tag{5.72}$$

Hence it follows

$$\rho(x) = \lambda\psi_{-\lambda}(x) \quad m\text{-a.e.} \tag{5.73}$$

Since it holds

$$\inf_{x>0} \psi_{-\lambda}(x) \geq 0 \Leftrightarrow \lambda \leq \lambda_0 \tag{5.74}$$

(see e.g., [8, Lemma 6.7]), it follows $\lambda \in (0, \lambda_0]$.

Let $\lambda \in (0, \lambda_0]$. We show $\Phi\nu_\lambda = \nu_\lambda$. From Proposition 5.5, it is enough to show

$$\psi_{-\lambda}(x) = G_\mu(x). \tag{5.75}$$

It holds

$$\begin{aligned} G_\mu(x) &= \int_0^x dy \int_y^\infty \lambda\psi_{-\lambda}(z) dm(z) && (5.76) \\ &= x - \lambda \int_0^x dy \int_0^y \psi_{-\lambda}(z) dm(z) && (5.77) \\ &= \psi_{-\lambda}(x). && (5.78) \end{aligned}$$

□

6 Appendix: Existence of the Laplace transform of spectral measures

We give a sufficient condition of existence of the Laplace transform of spectral measures by applying the theory of strings with singular boundary developed by Kotani [17].

At first we recall the strings with a left entrance boundary. We say that a function $w : \mathbb{R} \to [0, \infty]$ is a *string* on $\mathbb{R}$ if w is non-decreasing and right-continuous. Set $\ell = \inf\{x \in \mathbb{R} \mid w(x) = \infty\}$. We assume the boundary $-\infty$ is regular or entrance; $\int_{-\infty}^{\delta} w(x)dx < \infty$ for some $\delta < \ell$. By a similar argument for a string on $(0, \infty)$, there exists a spectral measure σ on $(0, \infty)$ such that the transition density $q(t, x, y)$ w.r.t. dw for $\frac{d}{dw}\frac{d}{dx}$-(generalized) diffusion with Neumann boundary condition at $-\infty$ is given by

$$q(t, x, y) = \int_0^{\infty} \mathrm{e}^{-\lambda t} \varphi_{-\lambda}(x) \varphi_{-\lambda}(y) \sigma(d\lambda), \tag{6.1}$$

where the function $u = \varphi_\lambda$ is the solution for the following equation:

$$\frac{d}{dw}\frac{d}{dx}u = \lambda u, \quad u(0) = 1, \quad \frac{d^+}{dx}u(0) = 0 \quad (\lambda \in \mathbb{R}, -\infty < x < \ell) \tag{6.2}$$

For a string m on $(0, \infty)$, its *dual string* $w = m^*$ defined by

$$m^*(x) = \inf\{y > 0 \mid m(y) > x\} \tag{6.3}$$

is a string on $\mathbb{R}$. In addition, if the boundary 0 for m is regular or exit, then the boundary $-\infty$ for m^* is regular or entrance, accordingly. The spectral measure σ of $-\frac{d}{dm^*}\frac{d}{dx}$ is represented by the spectral measure θ of $-\frac{d}{dm}\frac{d}{dx}$ as follows:

Proposition 6.1 (Yano [30, Theorem 2.2])**.** *Let m be a string on $(0, \infty)$ and let m^* denote its dual string. Assume the following holds:*

$$(\mathrm{S}^*) \qquad \int_0^{\infty} \lambda \mathrm{e}^{-\lambda t} \sigma(d\lambda) < \infty \quad \textit{for all } t > 0. \tag{6.4}$$

Then the condition (S) *holds for $\frac{d}{dm}\frac{d}{dx}$ and it holds*

$$\theta(d\lambda) = \lambda \sigma(d\lambda) \quad \textit{on } (0, \infty). \tag{6.5}$$

We recall a result of Kotani [17] of the tail behavior of the spectral measures for strings on $\mathbb{R}$.

Proposition 6.2 (Kotani [17, pp. 803–804])**.** *Let w be a string on $\mathbb{R}$ whose boundary $-\infty$ is regular or entrance. Let $\phi : [0, 1] \to [0, \infty)$ be a function satisfying the following three conditions:*

(i) ϕ is a strictly-increasing, convex function and $\phi(0) = 0$, $\phi'(1-) < \infty$.

(ii) $\limsup_{y\to+0} \phi(xy)/\phi(y) < \infty \quad (x > 0)$.

(iii) $\liminf_{y\to+0} \phi(xy)/\phi(y) > 0 \quad (0 < x \le 1)$.

Then

$$\int_{-\infty}^{\delta} \phi(W(x))dx \in (0,\infty) \quad \textit{for some} \quad \delta < \ell \tag{6.6}$$

if and only if

$$\int_0^1 p(t)\phi(t)dt < \infty, \tag{6.7}$$

where

$$W(x) = \int_{-\infty}^{x} w(y)dy \quad (x \in \mathbb{R}) \quad \textit{and} \quad p(t) = \int_0^\infty \mathrm{e}^{-\lambda t}\sigma(d\lambda) \quad (t > 0). \tag{6.8}$$

Now we can show the result we aimed at.

Proposition 6.3. *Let m be a speed measure and s be a scale function on $(0,b)$ $(0 < b \le \infty)$. Then if $|s(0)| < \infty$ and*

$$m(x,c] \le C(s(x) - s(0))^{-\delta} \quad (0 < x < c) \tag{6.9}$$

for some $C > 0$, $0 < c < b$ and $0 < \delta < 1$, the condition (S) *holds.*

Proof of Proposition 6.3. Define $\widetilde{s}(x) := s(x) - s(0)$ and denote its inverse function by $\widetilde{s}^{-1}$. By considering $dm(\widetilde{s}^{-1}(x))$ instead of dm, we may assume without loss of generality that $s(x) = x$. We may also assume that $b > 1$ and $c = 1$. By abuse of notation, we denote a string defined through the measure m by

$$m(x) = \begin{cases} -\,m(x,1] & (0 < x < 1), \\ m(1,x] & (x \ge 1). \end{cases} \tag{6.10}$$

Since it holds that $|m(x)| \le Cx^{-\delta}$ $(0 < x < 1)$, we have

$$m^*(x) \le C^{1/\delta}|x|^{-1/\delta} \quad (x < -C). \tag{6.11}$$

Then it follows for $M^*(x) = \int_{-\infty}^{x} m^*(y)dy$ that

$$M^*(x) \le C'|x|^{-(1/\delta-1)} \quad (x < -C) \tag{6.12}$$

for some $C' > 0$. When we take $\beta > 1$ so that $\beta(1/\delta - 1) > 1$, we have

$$\int_{-\infty}^{-C} M^*(x)^\beta dx < \infty. \tag{6.13}$$

From Proposition 6.2 for $\phi(t) = t^\beta$, this yields

$$\int_0^1 p(t)t^\beta dt < \infty. \tag{6.14}$$

Note that, since the function p is non-increasing, it follows from (6.7) that $p(t) < \infty$ $(t > 0)$. Therefore, combining with Proposition 6.1, we obtain the desired result. □

References

[1] I. Ben-Ari and R. G. Pinsky. Ergodic behavior of diffusions with random jumps from the boundary. *Stochastic Process. Appl.*, 119(3):864–881, 2009.

[2] N. H. Bingham, C. M. Goldie, and J. L. Teugels. *Regular variation*, volume 27 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1987.

[3] A. N. Borodin and P. Salminen. *Handbook of Brownian motion—facts and formulae*. Probability and its Applications. Birkhäuser Verlag, Basel, second edition, 2002.

[4] P. Cattiaux, P. Collet, A. Lambert, S. Martínez, S. Méléard, and J. San Martín. Quasi-stationary distributions and diffusion models in population dynamics. *Ann. Probab.*, 37(5):1926–1969, 2009.

[5] J. A. Cavender. Quasi-stationary distributions of birth-and-death processes. *Adv. in Appl. Probab.*, 10(3):570–586, 1978.

[6] E. A. Coddington and N. Levinson. *Theory of ordinary differential equations*. McGraw-Hill Book Company, Inc., New York-Toronto-London, 1955.

[7] P. Collet, S. Martínez, and J. San Martín. Asymptotic laws for one-dimensional diffusions conditioned to nonabsorption. *Ann. Probab.*, 23(3):1300–1314, 1995.

[8] P. Collet, S. Martínez, and J. San Martín. *Quasi-stationary distributions*. Probability and its Applications (New York). Springer, Heidelberg, 2013. Markov chains, diffusions and dynamical systems.

[9] P. A. Ferrari, H. Kesten, S. Martinez, and P. Picco. Existence of quasi-stationary distributions. A renewal dynamical approach. *Ann. Probab.*, 23(2):501–521, 1995.

[10] A. Göing-Jaeschke and M. Yor. A survey and some generalizations of Bessel processes. *Bernoulli*, 9(2):313–349, 2003.

[11] A. Hening and M. Kolb. Quasistationary distributions for one-dimensional diffusions with singular boundary points. *Stochastic Process. Appl.*, 129(5):1659–1696, 2019.

[12] K. Itô. *Essentials of stochastic processes*, volume 231 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, RI, 2006. Translated from the 1957 Japanese original by Yuji Ito.

[13] H. P. McKean Jr. Elementary solutions for certain parabolic partial differential equations. *Trans. Amer. Math. Soc.*, 82:519–548, 1956.

[14] O. Kallenberg. *Foundations of modern probability*. Probability and its Applications (New York). Springer-Verlag, New York, second edition, 2002.

[15] M. Kolb and D. Steinsaltz. Quasilimiting behavior for one-dimensional diffusions with killing. *Ann. Probab.*, 40(1):162–212, 2012.

[16] S. Kotani. Krein's strings with singular left boundary. *Rep. Math. Phys.*, 59(3):305–316, 2007.

[17] S. Kotani. Krein's strings whose spectral functions are of polynomial growth. *Kyoto J. Math.*, 53(4):787–814, 2013.

[18] S. Kotani and S. Watanabe. Kreĭn's spectral theory of strings and generalized diffusion processes. In *Functional analysis in Markov processes (Katata/Kyoto, 1981)*, volume 923 of *Lecture Notes in Math.*, pages 235–259. Springer, Berlin-New York, 1982.

[19] J. Littin. Uniqueness of quasistationary distributions and discrete spectra when ∞ is an entrance boundary and 0 is singular. *J. Appl. Probab.*, 49(3):719–730, 2012.

[20] M. Lladser and J. San Martín. Domain of attraction of the quasi-stationary distributions for the Ornstein-Uhlenbeck process. *J. Appl. Probab.*, 37(2):511–520, 2000.

[21] W. Magnus, F. Oberhettinger, and R. P. Soni. *Formulas and theorems for the special functions of mathematical physics.* Third enlarged edition. Die Grundlehren der mathematischen Wissenschaften, Band 52. Springer-Verlag New York, Inc., New York, 1966.

[22] P. Mandl. Spectral theory of semi-groups connected with diffusion processes and its application. *Czechoslovak Math. J.*, 11(86):558–569, 1961.

[23] S. Martínez and J. San Martín. Rates of decay and h-processes for one dimensional diffusions conditioned on non-absorption. *J. Theoret. Probab.*, 14(1):199–212, 2001.

[24] S. Martinez, P. Picco, and J. San Martin. Domain of attraction of quasi-stationary distributions for the Brownian motion with drift. *Adv. in Appl. Probab.*, 30(2):385–408, 1998.

[25] L. C. G. Rogers. A diffusion first passage problem. In *Seminar on stochastic processes, 1983 (Gainesville, Fla., 1983)*, volume 7 of *Progr. Probab. Statist.*, pages 151–160. Birkhäuser Boston, Boston, MA, 1984.

[26] L. C. G. Rogers and D. Williams. *Diffusions, Markov processes, and martingales. Vol. 2.* Cambridge Mathematical Library. Cambridge University Press, Cambridge, 2000. Itô calculus, Reprint of the second (1994) edition.

[27] M. Shaked and J. G. Shanthikumar. *Stochastic orders.* Springer Series in Statistics. Springer, New York, 2007.

[28] M. Takeda. Existence and uniqueness of quasi-stationary distributions for symmetric Markov processes with tightness property. *J. Theoret. Probab.*, 32(4):2006–2019, 2019.

[29] T. Takemura and M. Tomisaki. h transform of one-dimensional generalized diffusion operators. *Kyushu J. Math.*, 66(1):171–191, 2012.

[30] K. Yano. Excursion measure away from an exit boundary of one-dimensional diffusion processes. *Publ. Res. Inst. Math. Sci.*, 42(3):837–878, 2006.