Affine super Yangians and rectangular W-superalgebras

Mamoru Ueda

Abstract

We define the affine super Yangian $Y_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m|n))$ with a coproduct structure. We also obtain an evaluation homomorphism, that is, a surjective algebra homomorphism from $Y_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m|n))$ to the completion of the universal enveloping algebra of $\widehat{\mathfrak{gl}}(m|n)$. Motivated by the AGT conjecture, we also construct a homomorphism Φ from the affine super Yangian $Y_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m|n))$ to the universal enveloping algebra of the rectangular W-superalgebra $\mathcal{W}^k(\mathfrak{gl}(ml|nl),(l^{(m|n)}))$ for all $m\neq n, m,n\geq 2$ or $m\geq 3, n=0$. Furthermore, we show that the image of this homomorphism is dense provided that $k+(m-n)(l-1)\neq 0$. We also give Φ by using the evaluation map and coproduct for the affine super Yangian. Moreover, we define the twisted affine Yangian as a coideal of the affine Yangian and construct a homomorphism from the twisted affine Yangian to the universal enveloping algebra of the rectangular W-algebra of type D.

1 Introduction

Drinfeld ([11], [12]) defined the Yangian of a finite dimensional simple Lie algebra $\mathfrak{g}$ in order to obtain a solution of the Yang-Baxter equation. The Yangian is a quantum group which is the deformation of the current algebra $\mathfrak{g}[z]$. He defined it by three different presentations. One of those presentations is called the Drinfeld presentation whose generators are $\{h_{i,r}, x^{\pm}_{i,r}\mid r\in\mathbb{Z}_{\geq 0}\}$, where $\{h_i, x^{\pm}_i\}$ are Chevalley generators of $\mathfrak{g}$. The definition of Yangian as an associative algebra naturally extends to the case that $\mathfrak{g}$ is a symmetrizable Kac-Moody Lie algebra in the Drinfeld presentation. Defining its quasi-Hopf algebra structure is more involved, but this problem has been settled for affine Kac-Moody Lie algebras in [21], [5] and [47].

It is known that the Yangians are closely related to W-algebras. It was shown in [40] that there exist surjective homomorphisms from Yangians of type A to rectangular finite W-algebras of type A. More generally, Brundan and Kleshchev ([9]) constructed a surjective homomorphism from a shifted Yangian, a subalgebra of the Yangian of type A, to a finite W-algebra of type A. Using a geometric realization of the Yangian, Schiffmann and Vasserot ([43]) have constructed a surjective homomorphism from the Yangian of $\widehat{\mathfrak{gl}}(1)$ to the universal enveloping algebra of the principal W-algebra of type A, and proved the celebrated AGT conjecture ([16], [6]).

In the case of the Lie superalgebra $\mathfrak{sl}(m|n)$, the corresponding Yangian in the Drinfeld presentation was first introduced by Stukopin ([44], see also [17]). The relationship between Yangians and W-algebras were also studied in the case of finite Lie superalgbras; by Briot and Ragoucy [7] for $\mathfrak{sl}(m|n)$ and by Peng [39] for $\mathfrak{gl}(1|n)$. In the recent paper [15], Gaberdiel, Li, Peng and H. Zhang defined the Yangian $\widehat{\mathfrak{gl}}(1|1)$ for the affine Lie superalgebra $\widehat{\mathfrak{gl}}(1|1)$ and obtained the similar result as [43] in the super setting.

In sections 2-4, we define the affine super Yangian $Y_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m|n))$ as a quantum group (=an associative algebra equipped with a coproduct satisfying compatibility conditions) in the Drinfeld presentation. We upgrade the definition of the Yangian associated with $\mathfrak{sl}(m|n)$ of Gow [17] to define the affine super Yangian $Y_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m|n))$ as an associative algebra, see Definition 3.1. However, to define the coproduct for $Y_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m|n))$, we need to obtain yet another presentation, that is, a *minimalistic presentation.*

Theorem 1.1. *The affine super Yangian $Y_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m|n))$ is isomorphic to the associative superalgebra over $\mathbb{C}$ generated by $x^+_{i,r}, x^-_{i,r}, h_{i,r}$ $(0 \leq i \leq m+n-1, r = 0, 1)$ subject to the defining relations* (3.17)-(3.25).

By Theorem 1.1, the following assertion gives a coproduct Δ for $Y_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m|n))$ that is compatible with the defining relations (3.17)-(3.25).

Theorem 1.2. *We can define an algebra homomorphism*

$$\Delta\colon Y_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m|n)) \to Y_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m|n))\widehat{\otimes} Y_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m|n))$$

that satisfies the coassociativity. Here, $Y_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m|n))\widehat{\otimes} Y_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m|n))$ is the degreewise completion of $Y_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m|n)) \otimes Y_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m|n))$ in the sense of [33].

When $\mathfrak{g}$ is $\mathfrak{sl}(n)$, $Y_h(\mathfrak{sl}(n))$ has an evaluation map $\mathrm{ev}\colon Y_h(\mathfrak{sl}(n)) \twoheadrightarrow U(\mathfrak{sl}(n))$, which enables us to define actions of $Y_h(\mathfrak{sl}(n))$ on any highest weight representation of $\mathfrak{sl}(n)$. In [20], Guay showed that the affine Yangian $Y_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(n))$ has the evaluation map $\mathrm{ev}\colon Y_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(n)) \to \widetilde{U}(\widehat{\mathfrak{gl}}(n))$, where $\widetilde{U}(\widehat{\mathfrak{gl}}(n))$ is a completion of the universal enveloping algebra of $\widehat{\mathfrak{gl}}(n)$. The surjectivity of the Guay's evaluation map is not trivial and was recently shown in [29]. In section 5, we construct an evaluation map of the affine super Yangian $Y_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m|n))$ (see Theorem 5.1).

Theorem 1.3. *Assume $c\hbar = (-m+n)\varepsilon_1$. Then, for all $a \in \mathbb{C}$, there exists a non-trivial algebra homomorphism $\mathrm{ev}_a\colon Y_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m|n)) \to U(\widehat{\mathfrak{gl}}(m|n))_{\mathrm{comp},+}$ determined by* (5.2)-(5.5)*, where $U(\widehat{\mathfrak{gl}}(m|n))_{\mathrm{comp},+}$ is a completion of the universal enveloping algebra of $\widehat{\mathfrak{gl}}(m|n)$.*

We know only a little about irreducible representations of the affine super Yangian. In the case when $\mathfrak{g}$ is $\widehat{\mathfrak{sl}}(n)$, the easiest irreducible representations of the affine Yangian are obtained by the pullback of irreducible highest weight representations of $\widehat{\mathfrak{gl}}(n)$ since there exists a surjective homomorphism from the affine Yangian to the completion of the universal enveloping algebra of $\widehat{\mathfrak{gl}}(n)$ ([20], [30], and [29]). In [29], Kodera showed that the image of this homomorphism topologically generates the completed universal enveloping algebra by using a braid group action on the affine Yangian. It is natural to try to obtain irreducible representations of the affine super Yangian in the similar way. In [45], we have constructed a homomorphism from the affine super Yangian to the completion of the universal enveloping algebra of $\widehat{\mathfrak{gl}}(m|n)$. However, we cannot prove that he image of this homomorphism is dense in the similar way to the one in [29] since we have no braid group actions on the affine super Yangian. In section 6, we show the statement in the more primitive way. Owing to this result, we obtain irreducible representations of the affine super Yangian via this homomorphism.

In sections 7-10 and appendix A, we give a result similar to the work of Ragoucy-Sorba [40] in the affine super setting. We construct a homomorphism from the affne super Yangian $Y_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m|n))$ to the universal enveloping algebra (see [14] and [33]) of $\mathcal{W}^k(\mathfrak{gl}(ml|nl),(l^{(m|n)}))$, the rectangular W-algebra associated with $\mathfrak{g} = \mathfrak{gl}(nl)$ and a nilpotent element f whose Jordan form corresponds to the partition (l^n). The following theorem is the main result of this paper.

Theorem 1.4. *Suppose that $m, n \geq 2, m \neq n$ or $m \geq 3, n = 0$, and assume that $l \geq 2$ and*

$$\varepsilon_1 = \frac{\alpha}{m-n}, \quad \varepsilon_2 = -1 - \frac{\alpha}{m-n}.$$

Then, there exists an algebra homomorphism

$$\Phi\colon Y_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m|n)) \to \mathcal{U}(\mathcal{W}^k(\mathfrak{gl}(ml|nl),(l^{(m|n)}))),$$

where $\mathcal{U}(\mathcal{W}^k(\mathfrak{gl}(ml|nl),(l^{(m|n)})))$ is the universal enveloping algebra of $\mathcal{W}^k(\mathfrak{gl}(ml|nl),(l^{(m|n)}))$. Moreover, the image of Φ is dense in $\mathcal{U}(\mathcal{W}^k(\mathfrak{gl}(ml|nl),(l^{(m|n)})))$ provided that $\alpha \neq 0$.

By Theorem 1.4, provided that $\alpha \neq 0$, any irreducible representation of $\mathcal{W}^k(\mathfrak{gl}(ml|nl),(l^{(m|n)}))$ can be seen as an irreducible representation of $Y_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m|n))$. In the case that $l=1$, the corresponding homomorphism is the evaluation map. Thus, the corresponding theorem was previously shown in [20], [30], [29], [45] and [48].

We expect that the above result will be useful for studying the AGT correspondence for parabolic sheaves. See [37] for the corresponding result in the quantum toroidal setting.

In order to prove Theorem 1.4, we give explicit generators of the rectangular W-superalgebra $\mathcal{W}^k(\mathfrak{gl}(ml|nl),(l^{(m|n)}))$. We define the homomorphism Φ concretely by using these generators of $\mathcal{W}^k(\mathfrak{gl}(ml|nl),(l^{(m|n)}))$ and check that Φ is compatible with the defining relations of the minimalistic presentation of the affine super Yangian by a direct computation.

For type CD cases, Brown [8] constructed surjective homomorphisms from twisted Yangians to rectangular finite W-algebras of type CD by using twisted Yangians instead of Yangians. Twisted Yangians were introduced by Olshanskii ([38]) and were further studied in [22, 34, 35] etc. The twisted Yangian $T_h(\mathfrak{g},\mathfrak{k})$ is an associative algebra associated with one parameter h, a finite dimensional simple Lie algebra $\mathfrak{g}$, subspaces $\mathfrak{k},\mathfrak{m}\subset\mathfrak{g}$, and a symmetric involution $\theta\colon\mathfrak{g}\to\mathfrak{g}$ such that $\mathfrak{g}^\theta=\mathfrak{k}$ and $\mathfrak{m}=\{x\in\mathfrak{g}\mid\theta(x)=-x\}$. The twisted Yangian $T_h(\mathfrak{g},\mathfrak{k})$ can be realized as a coideal of the finite Yangian $Y_h(\mathfrak{g})$.

In sections12-13 and appendix B, we construct the similar result to Theorem 1.4 in type D setting. The corresponding Yangian is the twisted affine Yangian $TY_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{so}}(n))$ which is defined by using the Drinfeld J presentation of the Guay's affine Yangian in the sense of [21]. The Drinfeld J presentation of the finite Yangian is Drinfeld's original definition of the finite Yangian ([11]) whose generators are $\{x,J(x)\mid x\in\mathfrak{g}\}$, where $J(x)$ is corresponding to $x\otimes z\in\mathfrak{g}\otimes\mathbb{C}[z]$. Reffering to the Drinfeld J presentation of $Y_h(\mathfrak{g})$, Belliard and Regelskis ([4]) constructed the Drinfeld J presentation of the twisted Yangian whose generators are $\{x,B(y)\mid x\in\mathfrak{k},y\in\mathfrak{m}\}$, where $B(y)$ is corresponding to $y\otimes z\in\mathfrak{m}\otimes\mathbb{C}[z]$ when we set $h=0$. In [21], Guay-Nakajima-Wendland constructed the terms $J(h_i),J(x_i^\pm)\in\widetilde{Y}_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(n))$ in the analogy of the Drinfeld J presentation of the finite Yangians. We define $TY_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{so}}(n))$ as a subalgebra of $\widetilde{Y}_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(n))$ in terms of $J(h_i)$. We note that $TY_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{so}}(n))$ becomes a coideal of $\widetilde{Y}_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(n))$.

We construct a surjective homomorphism from the twisted affine Yangian $TY_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{so}}(n))$ to the universal enveloping algebra of $\mathcal{W}^k(\mathfrak{so}(nl),(l^n))$, the rectangular W-algebra associated with $\mathfrak{g}=\mathfrak{so}(nl)$ and a nilpotent element f whose Jordan form corresponds to the partition (l^n).

Theorem 1.5. *Let $n\geq 4$ and l be positive even. For any $k\in\mathbb{C}$, we set*

$$\varepsilon_1=-\frac{(k+(l-1)n-2)\hbar}{n},\quad \varepsilon_2=\hbar+\frac{(k+(l-1)n-2)\hbar}{n}.$$

There exists an algebra homomorphism

$$\Phi\colon TY_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{so}}(n))\to\mathcal{U}(\mathcal{W}^k(\mathfrak{so}(nl),(l^n))).$$

Moreover, the homomorphism Φ is surjective provided that $k+(l-1)n-2\neq 0$.

By Theorem 1.5, any (irreducible) representation of $\mathcal{W}^k(\mathfrak{so}(nl),(l^n))$ can be pulled back as that of $TY_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{so}}(n))$. We note that the homomorphism Φ can be written by using the coproduct and the evaluation map for the Guay's affine Yangian as in [31].

We note that section 2-5 are derived from [45], section 6 is derived from [48], section 7-10 and appendix A are derived from [46], and section 11-13 and appendix B are derived from [49].

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2 Preliminaries

In this section, we recall the definition and presentation of the Lie superalgebra $\widehat{\mathfrak{sl}}(m|n)$ (see [25]). First, we recall the definition of $\mathfrak{sl}(m|n)$ and $\mathfrak{gl}(m|n)$.

Definition 2.1. Let us set $M_{k,l}(\mathbb{C})$ as the set of $k \times l$ matrices over $\mathbb{C}$. We define the Lie superalgebras $\mathfrak{sl}(m|n)$ and $\mathfrak{gl}(m|n)$ as follows;

$$\mathfrak{gl}(m|n) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \,\middle|\, A \in M_{m,m}(\mathbb{C}), B \in M_{m,n}(\mathbb{C}), C \in M_{n,m}(\mathbb{C}), \text{ and } D \in M_{n,n}(\mathbb{C}) \right\},$$

$$\mathfrak{sl}(m|n) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathfrak{gl}(m|n) \,\middle|\, tr(A) - tr(D) = 0 \right\},$$

where we define $\left[\begin{pmatrix} A & B \\ C & D \end{pmatrix}, \begin{pmatrix} E & F \\ G & H \end{pmatrix} \right]$ as

$$\left[\begin{pmatrix} A & B \\ C & D \end{pmatrix}, \begin{pmatrix} E & F \\ G & H \end{pmatrix} \right] = \begin{pmatrix} AE - EA + (BG + FC) & AF + BH - (EB + FD) \\ CE + DG - (GA + HC) & DH - HD + (CF + GB) \end{pmatrix}.$$

As with $\mathfrak{sl}(m)$, $\mathfrak{sl}(m|n)$ has a presentation whose generators are Chevalley generators (see [42] and [18]).

Proposition 2.2. *We set* $p\colon \{1, \cdots, m+n\} \to \{0, 1\}$ *as*

$$p(i) = \begin{cases} 0 & (1 \le i \le m), \\ 1 & (m+1 \le i \le m+n). \end{cases}$$

Suppose that $m, n \ge 2, m \ne n$ *and* $A = (a_{i,j})_{1 \le i,j \le m+n-1}$ *is an* $(m+n-1) \times (m+n-1)$ *matrix whose components are*

$$a_{i,j} = \begin{cases} (-1)^{p(i)} + (-1)^{p(i+1)} & \text{if } i = j, \\ -(-1)^{p(i+1)} & \text{if } j = i+1, \\ -(-1)^{p(i)} & \text{if } j = i-1, \\ 0 & \text{otherwise.} \end{cases}$$

Then, $\mathfrak{sl}(m|n)$ *is isomorphic to the Lie superalgebra over* $\mathbb{C}$ *defined by the generators* $\{x_i^\pm, h_i \mid 1 \le i \le m+n-1\}$ *and by the relations*

$$[h_i, h_j] = 0, \qquad [h_i, x_j^\pm] = \pm a_{i,j} x_j^\pm, \qquad [x_i^+, x_j^-] = \delta_{i,j} h_i, \qquad \mathrm{ad}(x_i^\pm)^{1+|a_{i,j}|} x_j^\pm = 0,$$
$$[x_m^\pm, x_m^\pm] = 0, \qquad [[x_{m-1}^\pm, x_m^\pm], [x_{m+1}^\pm, x_m^\pm]] = 0,$$

where the generators $x_m^\pm$ *are odd and all other generators are even.*

The isomorphism Ψ *is given by*

$$\Psi(h_i) = (-1)^{p(i)} E_{ii} - (-1)^{p(i+1)} E_{i+1,i+1}, \quad \Psi(x_i^+) = E_{i,i+1}, \quad \Psi(x_i^-) = (-1)^{p(i)} E_{i+1,i}.$$

Next, we recall the definition of the affinization of $\mathfrak{sl}(m|n)$ and $\mathfrak{gl}(m|n)$ (see [36]). Lie superalgebra $\mathfrak{sl}(m|n)$ has a non-degenerate invariant bilinear form $\kappa:\mathfrak{g}\otimes\mathfrak{g}\to\mathbb{C}$. The bilinear form is uniquely determined up to the scalar multiple, so we fix it.

Definition 2.3. Suppose that $\mathfrak{g}$ is $\mathfrak{sl}(m|n)$ or $\mathfrak{gl}(m|n)$. Then, we set a Lie superalgebra $\widetilde{\mathfrak{g}}$ as $\mathfrak{g}\otimes\mathbb{C}[t^{\pm 1}]\oplus\mathbb{C}c\oplus\mathbb{C}d$ whose commutator relations are following;

$$[a\otimes t^s, b\otimes t^u]=[a,b]\otimes t^{s+u}+s\delta_{s+u,0}\kappa(a,b)c,$$
$$c \text{ is a central element of } \widetilde{\mathfrak{g}},$$
$$[d,a\otimes t^s]=sa\otimes t^s.$$

We also set a subalgebra $\widehat{\mathfrak{g}}\subset\widetilde{\mathfrak{g}}$ as $\mathfrak{g}\otimes\mathbb{C}[t^{\pm 1}]\oplus\mathbb{C}c$.

We have another presentation of $\widehat{\mathfrak{sl}}(m|n)$ (see [50]).

Proposition 2.4. *Suppose that $m,n\geq 2, m\neq n$ and $A=(a_{i,j})_{0\leq i,j\leq m+n-1}$ is a $(m+n)\times(m+n)$ matrix whose components are*

$$a_{i,j}=\begin{cases}(-1)^{p(i)}+(-1)^{p(i+1)} & \text{if } i=j,\\ -(-1)^{p(i+1)} & \text{if } j=i+1,\\ -(-1)^{p(i)} & \text{if } j=i-1,\\ 1 & \text{if } (i,j)=(0,m+n-1),(m+n-1,0),\\ 0 & \text{otherwise,}\end{cases}$$

Then, $\widetilde{\mathfrak{sl}}(m|n)$ is isomorphic to the Lie superalgebra over $\mathbb{C}$ defined by the generators $\{x_i^{\pm},h_i,d\mid 0\leq i\leq m+n-1\}$ and by the relations

$$[d,h_i]=0,\quad [d,x_i^+]=\begin{cases}x_i^+ & (i=0),\\ 0 & (\text{otherwise}),\end{cases}\quad [d,x_i^-]=\begin{cases}-x_i^- & (i=0),\\ 0 & (\text{otherwise}),\end{cases}\tag{2.5}$$

$$[h_i,h_j]=0,\quad [h_i,x_j^{\pm}]=\pm a_{i,j}x_j^{\pm},\quad [x_i^+,x_j^-]=\delta_{i,j}h_i,\quad \mathrm{ad}(x_i^{\pm})^{1+|a_{i,j}|}x_j^{\pm}=0,\tag{2.6}$$

$$[x_0^{\pm},x_0^{\pm}]=0,\quad [x_m^{\pm},x_m^{\pm}]=0,\tag{2.7}$$

$$[[x_{m-1}^{\pm},x_m^{\pm}],[x_{m+1}^{\pm},x_m^{\pm}]]=0,\quad [[x_{m+n-1}^{\pm},x_0^{\pm}],[x_1^{\pm},x_0^{\pm}]]=0,\tag{2.8}$$

where the generators $x_m^{\pm}$ and $x_0^{\pm}$ are odd and all other generators are even.

The isomorphism Ξ is given by

$$\Xi(h_i)=\begin{cases}-E_{1,1}-E_{m+n,m+n}+c & (i=0),\\ (-1)^{p(i)}E_{ii}-(-1)^{p(i+1)}E_{i+1,i+1} & (1\leq i\leq m+n-1),\end{cases}$$

$$\Xi(x_i^+)=\begin{cases}E_{m+n,1}\otimes t & (i=0),\\ E_{i,i+1} & (\text{otherwise}),\end{cases}\quad \Xi(x_i^-)=\begin{cases}-E_{1,m+n}\otimes t^{-1} & (i=0),\\ (-1)^{p(i)}E_{i+1,i} & (\text{otherwise}).\end{cases}$$

Moreover, $\widehat{\mathfrak{sl}}(m|n)$ is isomorphic to the Lie superalgebra over $\mathbb{C}$ defined by the generators $\{x_i^{\pm},h_i\mid 0\leq i\leq m+n-1\}$ and by the relations (2.6)-(2.8).

Finally, we set some notations. Let us set $\{\alpha_i\}_{0\leq i\leq m+n-1}$ as a set of simple roots of $\widetilde{\mathfrak{sl}}(m|n)$ and δ as a positive root $\sum\limits_{0\leq i\leq m+n-1}\alpha_i$. Moreover, we set Δ (resp. Δ_+) as a set of roots (resp. positive roots) of $\widetilde{\mathfrak{sl}}(m|n)$. We denote the parity of $E_{i,j}$ as $p(E_{i,j})$. Obviously, $p(E_{i,j})$ is equal to $p(i)+p(j)$. We also set Δ_+^{re} and Δ^{re} as $\Delta_+\setminus\mathbb{Z}_{>0}\delta$ and $\Delta\setminus\mathbb{Z}\delta$. We also take an inner product on $\bigoplus\limits_{0\leq i\leq m+n-1}\mathbb{C}\alpha_i$ determined by $(\alpha_i,\alpha_j)=a_{i,j}$. Assume that $\mathfrak{g}=\widetilde{\mathfrak{sl}}(m|n)$ and let $\mathfrak{g}_\alpha$ be the root α space of $\mathfrak{g}$. We set $\{x_\alpha^{k_\alpha}\}_{1\leq k_\alpha\leq\dim\mathfrak{g}_\alpha}$ as a basis of $\mathfrak{g}_\alpha$ which satisfies $\kappa(x_\alpha^p,x_{-\alpha}^q)=\delta_{p,q}$ for all $\alpha\in\Delta_+$. We also denote the parity of $x_\alpha^{k_\alpha}$ by $p(\alpha)$. Moreover, we sometimes identify $\{0,\cdots,m+n-1\}$ with $\mathbb{Z}/(m+n)\mathbb{Z}$ and denote it by I.

3 The minimalistic presentation of the Affine Super Yangian

First, we define the affine super Yangian $Y_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m|n))$. This definition is a generalization of Stukopin's super Yangian ([44]). Let us set $\{x,y\}$ as $xy+yx$.

Definition 3.1. Suppose that $m,n\geq 2$ and $m\neq n$. The affine super Yangian $Y_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m|n))$ is the associative super algebra over $\mathbb{C}$ generated by $x^+_{i,r},x^-_{i,r},h_{i,r}$ ($i\in\{0,1,\cdots,m+n-1\},r\in\mathbb{Z}_{\geq 0}$) with parameters $\varepsilon_1,\varepsilon_2\in\mathbb{C}$ subject to the defining relations:

$$[h_{i,r},h_{j,s}]=0, \tag{3.2}$$

$$[x^+_{i,r},x^-_{j,s}]=\delta_{i,j}h_{i,r+s}, \tag{3.3}$$

$$[h_{i,0},x^\pm_{j,r}]=\pm a_{i,j}x^\pm_{j,r}, \tag{3.4}$$

$$[h_{i,r+1},x^\pm_{j,s}]-[h_{i,r},x^\pm_{j,s+1}]=\pm a_{i,j}\frac{\varepsilon_1+\varepsilon_2}{2}\{h_{i,r},x^\pm_{j,s}\}-b_{i,j}\frac{\varepsilon_1-\varepsilon_2}{2}[h_{i,r},x^\pm_{j,s}], \tag{3.5}$$

$$[x^\pm_{i,r+1},x^\pm_{j,s}]-[x^\pm_{i,r},x^\pm_{j,s+1}]=\pm a_{i,j}\frac{\varepsilon_1+\varepsilon_2}{2}\{x^\pm_{i,r},x^\pm_{j,s}\}-b_{i,j}\frac{\varepsilon_1-\varepsilon_2}{2}[x^\pm_{i,r},x^\pm_{j,s}], \tag{3.6}$$

$$\sum_{w\in\mathfrak{S}_{1+|a_{i,j}|}}[x^\pm_{i,r_{w(1)}},[x^\pm_{i,r_{w(2)}},\dots,[x^\pm_{i,r_{w(1+|a_{i,j}|)}},x^\pm_{j,s}]\dots]]=0\ (i\neq j), \tag{3.7}$$

$$[x^\pm_{i,r},x^\pm_{i,s}]=0\ (i=0,m), \tag{3.8}$$

$$[[x^\pm_{i-1,r},x^\pm_{i,0}],[x^\pm_{i,0},x^\pm_{i+1,s}]]=0\ (i=0,m), \tag{3.9}$$

where

$$a_{i,j}=\begin{cases}(-1)^{p(i)}+(-1)^{p(i+1)} & \text{if } i=j,\\ -(-1)^{p(i+1)} & \text{if } j=i+1,\\ -(-1)^{p(i)} & \text{if } j=i-1,\\ 1 & \text{if } (i,j)=(0,m+n-1),(m+n-1,0),\\ 0 & \text{otherwise,}\end{cases}$$

$$b_{i,j}=\begin{cases}-(-1)^{p(i+1)} & \text{if } i=j+1,\\ (-1)^{p(i)} & \text{if } i=j-1,\\ -1 & \text{if } (i,j)=(0,m+n-1),\\ 1 & \text{if } (i,j)=(m+n-1,0),\\ 0 & \text{otherwise,}\end{cases}$$

and the generators $x^\pm_{m,r}$ and $x^\pm_{0,r}$ are odd and all other generators are even.

Remark 3.10. In this paper, we set $[x,y]$ as $xy-(-1)^{p(x)p(y)}yx$ for all homogeneous elements x,y. Thus, (3.8) is non-trivial.

We also define the affine super Yangian associated with $\widetilde{\mathfrak{sl}}(m|n)$.

Definition 3.11. Suppose that $m,n\geq 2$ and $m\neq n$. We define $Y_{\varepsilon_1,\varepsilon_2}(\widetilde{\mathfrak{sl}}(m|n))$ is the associative super algebra over $\mathbb{C}$ generated by $\{x^\pm_{i,r},h_{i,r},d\mid i\in\{0,1,\cdots,m+n-1\},r\in\mathbb{Z}_{\geq 0}\}$ with parameters $\varepsilon_1,\varepsilon_2\in\mathbb{C}$ subject to the defining relations (3.2)-(3.9) and

$$[d,h_{i,r}]=0,\quad [d,x^+_{i,r}]=\begin{cases}1 & \text{if } i=0,\\ 0 & \text{if } i\neq 0,\end{cases}\quad [d,x^-_{i,r}]=\begin{cases}-1 & \text{if } i=0,\\ 0 & \text{if } i\neq 0,\end{cases} \tag{3.12}$$

where the generators $x^\pm_{m,r}$ and $x^\pm_{0,r}$ are odd and all other generators are even.

One of the difficulty of Definition 3.1 is that the number of generators is infinite. The rest of this section, we construct a new presentation of the affine super Yangian such that the number of generators are finite.

Let us set $\widetilde{h}_{i,1}=h_{i,1}-\dfrac{\varepsilon_1+\varepsilon_2}{2}h_{i,0}^2$. By the definition of $\widetilde{h}_{i,1}$, we can rewrite (3.5) as

$$[\widetilde{h}_{i,1},x_{j,r}^{\pm}]=\pm a_{i,j}\left(x_{j,r+1}^{\pm}-b_{i,j}\frac{\varepsilon_1-\varepsilon_2}{2}x_{j,r}^{\pm}\right). \tag{3.13}$$

By (3.13), we find that $Y_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m|n))$ is generated by $x_{i,r}^{+},x_{i,r}^{-},h_{i,r}$ $(i\in\{0,1,\cdots,m+n-1\},r=0,1)$. In fact, by (3.13) and (3.3), we have the following relations;

$$x_{i,r+1}^{\pm}=\pm\frac{1}{a_{i,i}}[\widetilde{h}_{i,1},x_{i,r}^{\pm}],\quad h_{i,r+1}=[x_{i,r+1}^{+},x_{i,0}^{-}]\text{ if }i\neq m,0, \tag{3.14}$$

$$x_{i,r+1}^{\pm}=\pm\frac{1}{a_{i+1,i}}[\widetilde{h}_{i+1,1},x_{i,r}^{\pm}]+b_{i+1,i}\frac{\varepsilon_1-\varepsilon_2}{2}x_{i,r}^{\pm},\quad h_{i,r+1}=[x_{i,r+1}^{+},x_{i,0}^{-}]\text{ if }i=m,0 \tag{3.15}$$

for all $r\geq 1$. In the following theorem, we construct the minimalistic presentation of the affine super Yangian $Y_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m|n))$ whose generators are $x_{i,r}^{+},x_{i,r}^{-},h_{i,r}$ $(i\in\{0,1,\cdots,m+n-1\},r=0,1)$. We remark that we have not checked that the presentation is minimalistic yet. However, we call this presentation "minimalistic presentation" since, in the non-super case, the corresponding presentation is called "minimalistic presentation".

Theorem 3.16. *Suppose that $m,n\geq 2$ and $m\neq n$. The affine super Yangian $Y_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m|n))$ is isomorphic to the associative super algebra generated by $x_{i,r}^{+},x_{i,r}^{-},h_{i,r}$ $(i\in\{0,1,\cdots,m+n-1\},r=0,1)$ subject to the defining relations:*

$$[h_{i,r},h_{j,s}]=0, \tag{3.17}$$

$$[x_{i,0}^{+},x_{j,0}^{-}]=\delta_{i,j}h_{i,0}, \tag{3.18}$$

$$[x_{i,1}^{+},x_{j,0}^{-}]=\delta_{i,j}h_{i,1}=[x_{i,0}^{+},x_{j,1}^{-}], \tag{3.19}$$

$$[h_{i,0},x_{j,r}^{\pm}]=\pm a_{i,j}x_{j,r}^{\pm}, \tag{3.20}$$

$$[\widetilde{h}_{i,1},x_{j,0}^{\pm}]=\pm a_{i,j}\left(x_{j,1}^{\pm}-b_{i,j}\frac{\varepsilon_1-\varepsilon_2}{2}x_{j,0}^{\pm}\right), \tag{3.21}$$

$$[x_{i,1}^{\pm},x_{j,0}^{\pm}]-[x_{i,0}^{\pm},x_{j,1}^{\pm}]=\pm a_{i,j}\frac{\varepsilon_1+\varepsilon_2}{2}\{x_{i,0}^{\pm},x_{j,0}^{\pm}\}-b_{i,j}\frac{\varepsilon_1-\varepsilon_2}{2}[x_{i,0}^{\pm},x_{j,0}^{\pm}], \tag{3.22}$$

$$(\mathrm{ad}\,x_{i,0}^{\pm})^{1+|a_{i,j}|}(x_{j,0}^{\pm})=0\ (i\neq j), \tag{3.23}$$

$$[x_{i,0}^{\pm},x_{i,0}^{\pm}]=0\ (i=0,m), \tag{3.24}$$

$$[[x_{i-1,0}^{\pm},x_{i,0}^{\pm}],[x_{i,0}^{\pm},x_{i+1,0}^{\pm}]]=0\ (i=0,m), \tag{3.25}$$

where the generators $x_{m,r}^{\pm}$ and $x_{0,r}^{\pm}$ are odd and all other generators are even.

The outline of the proof of Theorem 3.16 is similar to that of Theorem 2.13 of [21]. To simplify the notation, we denote the assocfitive super algebra defined in Theorem 3.16 as $\widetilde{Y}_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m|n))$. We construct $x_{i,r}^{\pm}$ and $h_{i,r}$ as the elements of $\widetilde{Y}_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m|n))$ inductively by (3.14) and (3.15). Since (3.17)-(3.25) are contained in the defining relations of the affine super Yangian, it is enough to check that the defining relations of the affine super Yangians (3.2)-(3.9) are deduced from (3.17)-(3.25) in $\widetilde{Y}_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m|n))$. The proof of Theorem 3.16 is divided into eight lemmas, that is, Lemma 3.26, Lemma 3.31, Lemma 3.35, Lemma 3.36, Lemma 3.37, Lemma 3.38, Lemma 3.57, and Lemma 3.58.

Most of the defining relations (3.2)-(3.9) are obtained in the same way as that of [32] or [21]. For example, we have the following lemma in a similar way as that of Lemma 2.22 of [21].

Lemma 3.26. (1) *The defining relation* (3.4) *holds for all $i,j\in I$ in $\widetilde{Y}_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m|n))$.*

(2) *For all $i,j\in I$, we obtain*

$$[\widetilde{h}_{i,1},x^{\pm}_{j,r}]=\pm a_{i,j}\left(x^{\pm}_{j,r+1}-b_{i,j}\frac{\varepsilon_1-\varepsilon_2}{2}x^{\pm}_{j,r}\right) \tag{3.27}$$

in $\widetilde{Y}_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m|n))$.

Proof. We only show the case that $j=0,m$. The other case is proven in the same way as that of Lemma 2.22 of [21]. We prove (1), (2) by the induction on r. When $r=0$, they are nothing but (3.20) and (3.21). Suppose that (3.4) and (3.27) hold when $r=k$. First, let us show that (3.4) holds when $r=k+1$. By (3.15), we obtain

$$[h_{i,0},x^{\pm}_{j,k+1}]=\pm\frac{1}{a_{j,j+1}}[h_{i,0},[\widetilde{h}_{j+1,1},x^{\pm}_{j,k}]]+b_{j,j+1}\frac{\varepsilon_1-\varepsilon_2}{2}[h_{i,0},x^{\pm}_{j,k}]. \tag{3.28}$$

By $[h_{i,0},h_{j,1}]=0$, we find that the first term of the right hand side of (3.28) is equal to

$$\pm\frac{1}{a_{j,j+1}}[h_{i,0},[\widetilde{h}_{j+1,1},x^{\pm}_{j,k}]]=\pm\frac{1}{a_{j,j+1}}[\widetilde{h}_{j+1,1},[h_{i,0},x^{\pm}_{j,k}]].$$

By the induction hypothesis on r, we can rewrite the right hand side of (3.28) as

$$\begin{aligned}
&\pm\frac{1}{a_{j,j+1}}[\widetilde{h}_{j+1,1},[h_{i,0},x^{\pm}_{j,k}]]+b_{j,j+1}\frac{\varepsilon_1-\varepsilon_2}{2}[h_{i,0},x^{\pm}_{j,k}]\\
&=\frac{a_{i,j}}{a_{j,j+1}}[\widetilde{h}_{j+1,1},x^{\pm}_{j,k}]\pm a_{i,j}b_{j,j+1}\frac{\varepsilon_1-\varepsilon_2}{2}x^{\pm}_{j,k}\\
&=\frac{a_{i,j}}{a_{j,j+1}}\big(\pm a_{j,j+1}(x^{\pm}_{j,k+1}-b_{j,j+1}\frac{\varepsilon_1-\varepsilon_2}{2}x^{\pm}_{j,k})\big)\pm a_{i,j}b_{j,j+1}\frac{\varepsilon_1-\varepsilon_2}{2}x^{\pm}_{j,k}\\
&=\pm a_{i,j}x^{\pm}_{j,k+1}.
\end{aligned}$$

Thus, we have shown that $[h_{i,0},x^{\pm}_{j,k+1}]=\pm a_{i,j}x^{\pm}_{j,k+1}$.

Next, we show that (3.4) holds when $r=k+1$. Since we have already proved that (3.4) holds when $r=k+1$, it is enough to check the relation

$$[\widetilde{h}_{i,1},x^{\pm}_{j,k+1}]=\pm a_{i,j}\left(x^{\pm}_{j,k+2}-b_{i,j}\frac{\varepsilon_1-\varepsilon_2}{2}x^{\pm}_{j,k+1}\right).$$

By (3.15), we obtain

$$[\widetilde{h}_{i,1},x^{\pm}_{j,k+1}]=\pm\frac{1}{a_{j,j+1}}[\widetilde{h}_{i,1},[\widetilde{h}_{j+1,1},x^{\pm}_{j,k}]]+b_{j,j+1}\frac{\varepsilon_1-\varepsilon_2}{2}[\widetilde{h}_{i,1},x^{\pm}_{j,k}]. \tag{3.29}$$

By $[h_{i,1},h_{j,1}]=0$, we find that the right hand side of (3.29) is equal to

$$\pm\frac{1}{a_{j,j+1}}[\widetilde{h}_{j+1,1},[\widetilde{h}_{i,1},x^{\pm}_{j,k}]]+b_{j,j+1}\frac{\varepsilon_1-\varepsilon_2}{2}[\widetilde{h}_{i,1},x^{\pm}_{j,k}].$$

By the induction hypothesis on r, we can rewrite the right hand side of (3.29) as

$$\begin{aligned}
&\pm\frac{1}{a_{j,j+1}}[\widetilde{h}_{j+1,1},[\widetilde{h}_{i,1},x^{\pm}_{j,k}]]+b_{j,j+1}\frac{\varepsilon_1-\varepsilon_2}{2}[\widetilde{h}_{i,1},x^{\pm}_{j,k}]\\
&=\frac{a_{i,j}}{a_{j,j+1}}\left([\widetilde{h}_{j+1,1},x^{\pm}_{j,k+1}]-b_{i,j}\frac{\varepsilon_1-\varepsilon_2}{2}[\widetilde{h}_{j+1,1},x^{\pm}_{j,k}]\right)\\
&\qquad\pm a_{i,j}b_{j,j+1}\frac{\varepsilon_1-\varepsilon_2}{2}\left(x^{\pm}_{j,k+1}-b_{i,j}\frac{\varepsilon_1-\varepsilon_2}{2}x^{\pm}_{j,k}\right).
\end{aligned} \tag{3.30}$$

Since $x^{\pm}_{j,k+2}$ is defined by (3.15), we find that the right hand side of (3.30) is equal to

$$\pm a_{i,j}\left(x^{\pm}_{j,k+2}-b_{j,j+1}\frac{\varepsilon_1-\varepsilon_2}{2}x^{\pm}_{j,k+1}\right)\mp a_{i,j}b_{i,j}\frac{\varepsilon_1-\varepsilon_2}{2}\left(x^{\pm}_{j,k+1}-b_{j,j+1}\frac{\varepsilon_1-\varepsilon_2}{2}x^{\pm}_{j,k}\right)$$
$$\pm a_{i,j}b_{j,j+1}\frac{\varepsilon_1-\varepsilon_2}{2}\left(x^{\pm}_{j,k+1}-b_{i,j}\frac{\varepsilon_1-\varepsilon_2}{2}x^{\pm}_{j,k}\right).$$

By direct computation, it is equal to

$$\pm a_{i,j}\left(x^{\pm}_{j,k+2}-b_{i,j}\frac{\varepsilon_1-\varepsilon_2}{2}x^{\pm}_{j,k+1}\right).$$

This completes the proof. □

Similarly, we also obtain the following lemma in a similar way to the one of [21].

Lemma 3.31. (1) *The relation* (3.3) *holds in* $\widetilde{Y}_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m|n))$ *when* $i=j$ *and* $r+s\leq 2$.

(2) *Suppose that* $i,j\in I$ *and* $i\neq j$. *Then, the relations* (3.3) *and* (3.6) *hold for any* r *and* s *in* $\widetilde{Y}_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m|n))$.

(3) *The relation* (3.6) *holds in* $\widetilde{Y}_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m|n))$ *when* $i=j$, $(r,s)=(1,0)$.

(4) *The relation* (3.5) *holds in* $\widetilde{Y}_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m|n))$ *when* $i=j$, $(r,s)=(1,0)$.

(5) *For all* $i,j\in I$, *the relation* (3.5) *holds in* $\widetilde{Y}_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m|n))$ *when* $(r,s)=(1,0)$.

(6) *Set* $\widetilde{h}_{i,2}$ *as* $h_{i,2}-h_{i,0}h_{i,1}+\dfrac{1}{3}h^3_{i,0}$. *Then, the following equation holds for all* $i,j\in I$ *in* $\widetilde{Y}_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m|n))$;

$$[\widetilde{h}_{i,2},x^{\pm}_{j,0}]=\pm a_{i,j}x^{\pm}_{j,2}\pm\frac{1}{12}a^3_{i,j}x^{\pm}_{j,0}\mp a_{i,j}b_{i,j}\frac{\varepsilon_1-\varepsilon_2}{2}(x^{\pm}_{j,1}-\frac{1}{2}x^{\pm}_{j}h_i-b_{i,j}\frac{\varepsilon_1-\varepsilon_2}{2}x^{\pm}_{j}).$$

(7) *For all* $i,j\in I$, *the relation* (3.7) *holds in* $\widetilde{Y}_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m|n))$ *when*

1. $r_1=\cdots=r_b=0$, $s\in\mathbb{Z}_{\geq 0}$,
2. $r_1=1$, $r_2=\cdots=r_b=0$, $s\in\mathbb{Z}_{\geq 0}$,
3. $r_1=2$, $r_2=\cdots=r_b=0$, $s\in\mathbb{Z}_{\geq 0}$,
4. ($b\geq 2$ *and*) $r_1=r_2=1$, $r_3=\cdots=r_b=0$, $s\in\mathbb{Z}_{\geq 0}$.

(8) *In* $\widetilde{Y}_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m|n))$, *we have*

$$[h_{j,1},x^{\pm}_{i,1}]=\frac{a_{i,j}}{a_{i,i}}[h_{i,1},x^{\pm}_{i,1}]\pm\frac{a_{i,j}}{2}(\{h_{j,0},x^{\pm}_{i,1}\}-\{h_{i,0},x^{\pm}_{i,1}\})\mp a_{j,i}m_{j,i}\frac{\varepsilon_1-\varepsilon_2}{2}x^{\pm}_{i,1},$$

for all $i,j\in I$ *such that* $a_{i,i}\neq 0$.

(9) *For all* $i,j\in I$, *we have*

$$[h_{i,2},h_{j,0}]=0$$

in $\widetilde{Y}_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m|n))$.

(10) *Suppose that* $i,j\in I$ *such that* $a_{i,i}=2$ *and* $a_{i,j}=-1$. *Then,*

$$[h_{i,2},h_{i,1}]=0$$

holds in $\widetilde{Y}_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m|n))$.

Proof. We only prove (1)-(5) since the proof of (6) (resp. (7), (8), (9)) is same as that of Lemma 2.33 (resp. Lemma 2.34, Lemma 2.35, Proposition 2.36).

The proofs of (1) and (2) are the same as those of Lemma 2.22 and Lemma 2.26 in [21]. In the case where $i,j\neq 0,m$, the proof of (3) (resp. (4) and (5)) is also the same as that of Lemma 2.23 (resp. Lemma 2.24 and Lemma 2.28) in [21]. We omit it. We only show that (3) holds since (4) and (5) are derived from (3) in a similar way to the one of [21].

Suppose that $i=j=0,m$. First, we show that $[x^+_{i,1},x^+_{i,0}]=[x^+_{i,0},x^+_{i,1}]=0$ holds. Applying $\mathrm{ad}(\widetilde{h}_{i+1,1})$ to (3.24), we have $\pm a_{i,i+1}[x^\pm_{i,1},x^\pm_{i,0}]\pm a_{i,i+1}[x^\pm_{i,0},x^\pm_{i,1}]$. Since $[x^\pm_{i,1},x^\pm_{i,0}]$ is equal to $[x^\pm_{i,0},x^\pm_{i,1}]$, we obtain $[x^\pm_{i,1},x^\pm_{i,0}]=[x^\pm_{i,0},x^\pm_{i,1}]=0$. Next, we show that $[x^\pm_{i,2},x^\pm_{i,0}]=[x^\pm_{i,1},x^\pm_{i,1}]=[x^\pm_{i,0},x^\pm_{i,2}]$ holds. Applying $\mathrm{ad}(\widetilde{h}_{i+1,1})$ to $[x^\pm_{i,1},x^\pm_{i,0}]=[x^\pm_{i,0},x^\pm_{i,1}]=0$, we obtain

$$\pm a_{i,i+1}([x^\pm_{i,2},x^\pm_{i,0}]+[x^\pm_{i,1},x^\pm_{i,1}])=0, \tag{3.32}$$
$$\pm a_{i,i+1}([x^\pm_{i,1},x^\pm_{i,1}]+[x^\pm_{i,0},x^\pm_{i,2}])=0. \tag{3.33}$$

In the case where $j=0,m$ and $i=j+1$, we can prove (5) in a similar way to the one of Lemma 2.28 in [21]. Then, in the similar discussion to that of Lemma 1.4 in [32], there exists $\widehat{h}_{i+1,2}$ such that

$$[\widehat{h}_{i+1,2},x^\pm_{i,0}]=\pm a_{i,i+1}x^\pm_{i,2}.$$

Applying $\mathrm{ad}(\widehat{h}_{i+1,2})$ to (3.24), we obtain

$$\pm a_{i,i+1}([x^\pm_{i,2},x^\pm_{i,0}]+[x^\pm_{i,0},x^\pm_{i,2}])=0. \tag{3.34}$$

Since (3.32), (3.33), and (3.34) are linearly independent, we obtain $[x^\pm_{i,2},x^\pm_{i,0}]=[x^\pm_{i,1},x^\pm_{i,1}]=[x^\pm_{i,0},x^\pm_{i,2}]$. We have proved (3). □

In the case where $a_{i,i}=-2$ and $a_{i,j}=1$, we obtain $[h_{i,2},h_{i,1}]=0$ by changing the proof of Proposition 2.36 of [21] a little.

Lemma 3.35. *Suppose that $i,j\in I$ such that $a_{i,i}=-2$ and $a_{i,j}=1$. Then, we obtain*

$$[h_{i,2},h_{i,1}]=0$$

in $\widetilde{Y}_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m|n))$.

Proof. We change $h_{i,r}$, $x^+_{i,r}$, and $x^-_{i,r}$, which are written in the proof of Proposition 2.36 of [21], into $-h_{i,r}$, $-x^+_{i,r}$, and $x^-_{i,r}$. Then, we obtain $[-h_{i,2},-h_{i,1}]=0$. □

By Lemma 3.31 (10) and Lemma 3.35, we obtain the following lemma in the same way as Proposition 2.39 of [21] since we only need the condition that $a_{i,i}\neq 0$ and $a_{i,j}\neq 0$. We omit the proof.

Lemma 3.36. *Suppose that $i,j\in I$ such that $a_{i,i}\neq 0$ and $a_{i,j}\neq 0$. Then, we have*

$$[h_{j,2},h_{j,1}]=0$$

in $\widetilde{Y}_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m|n))$.

Therefore, we know that $[h_{i,2},h_{i,1}]=0$ holds for all $i\in I$. By using the relation $[h_{i,2},h_{i,1}]=0$, we obtain the following lemma in a similar way as that of Theorem 1.2 in [32] since the proof of these statements needs only the condition that $a_{i,i}\neq 0$.

Lemma 3.37. (1) *The relation (3.2) holds in $\widetilde{Y}_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m|n))$ when $i=j\neq 0,m$.*

(2) *The relation (3.3) holds in $\widetilde{Y}_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m|n))$ when $i=j\neq 0,m$.*

(3) *The relation (3.6) holds in $\widetilde{Y}_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m|n))$ when $i=j\neq 0,m$.*

(4) *The relation (3.5) holds in $\widetilde{Y}_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m|n))$ when $i=j\neq 0,m$.*

Next, we prove the same statement as that of Lemma 3.37 in the case that $i=j=0,m$.

Lemma 3.38. (1) *The relation* (3.6) *holds in* $\widetilde{Y}_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m|n))$ *when* $i=j=0,m$. *In particular,* (3.8) *holds in* $\widetilde{Y}_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m|n))$.

(2) *The relation* (3.3) *holds in* $\widetilde{Y}_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m|n))$ *when* $i=j=0,m$.

(3) *We obtain* $[h_{i,r},x^{\pm}_{i,0}]=0$ *when* $i=0,m$ *in* $\widetilde{Y}_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m|n))$.

(4) *The relation* (3.5) *holds in* $\widetilde{Y}_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m|n))$ *when* $i=j=0,m$.

(5) *The relation* (3.2) *holds in* $\widetilde{Y}_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m|n))$ *when* $i=j=0,m$.

Proof. (1). It is enough to check the equality $[x^{\pm}_{i,r},x^{\pm}_{i,s}]=0$. We only show that $[x^{+}_{i,r},x^{+}_{i,s}]=0$ holds. We can obtain $[x^{-}_{i,r},x^{-}_{i,s}]=0$ in a similar way. We prove (3.6) holds by the induction on $k=r+s$. When $k=0$, it is nothing but (3.24). Applying $\mathrm{ad}(\widetilde{h}_{i+1,1})$ to $[x^{+}_{i,0},x^{+}_{i,0}]=0$, we obtain

$$a_{i,i+1}([x^{+}_{i,1},x^{+}_{i,0}]+[x^{+}_{i,0},x^{+}_{i,1}])=0.$$

Since $[x^{+}_{i,1},x^{+}_{i,0}]=[x^{+}_{i,0},x^{+}_{i,1}]$, we have $[x^{+}_{i,1},x^{+}_{i,0}]=[x^{+}_{i,0},x^{+}_{i,1}]=0$.

Suppose that $[x^{+}_{i,r},x^{+}_{i,s}]=0$ holds for all r,s such that $r+s=k,k+1$. Applying $\mathrm{ad}(\widetilde{h}_{i+1,1})$ to $[x^{+}_{i,u},x^{+}_{i,k+1-u}]=0$, we have

$$[\widetilde{h}_{i+1,1},[x^{+}_{i,u},x^{+}_{i,k+1-u}]]=0. \tag{3.39}$$

By Lemma 3.31 (4) and the induction hypothesis, we have

$$[\widetilde{h}_{i+1,1},[x^{+}_{i,u},x^{+}_{i,k+1-u}]]=a_{i,i+1}([x^{+}_{i,u+1},x^{+}_{i,k+1-u}]+[x^{+}_{i,u},x^{+}_{i,k+2-u}]). \tag{3.40}$$

Since $a_{i,i+1}\neq 0$, we find the relation

$$[x^{+}_{i,u+1},x^{+}_{i,k+1-u}]=-[x^{+}_{i,u},x^{+}_{i,k+2-u}] \tag{3.41}$$

by (3.39) and (3.40). In particular, we obtain

$$[x^{+}_{i,u+2},x^{+}_{i,k-u}]=[x^{+}_{i,u},x^{+}_{i,k+2-u}]. \tag{3.42}$$

Applying $\mathrm{ad}(\widetilde{h}_{i+1,2})$ to $[x^{+}_{i,u},x^{+}_{i,k-u}]=0$, we have

$$[\widetilde{h}_{i+1,2},[x^{+}_{i,u},x^{+}_{i,k-u}]]=0 \tag{3.43}$$

by the induction hypothesis. By Lemma 3.31 (7), Lemma 3.36 and the induction hypothesis, we have

$$[\widetilde{h}_{i+1,2},[x^{+}_{i,u},x^{+}_{i,k-u}]]=a_{i,i+1}([x^{+}_{i,u+2},x^{+}_{i,k-u}]+[x^{+}_{i,u},x^{+}_{i,k+2-u}]). \tag{3.44}$$

Since $a_{i,i+1}\neq 0$, we obtain the relation

$$[x^{+}_{i,u+2},x^{+}_{i,k-u}]=-[x^{+}_{i,u},x^{+}_{i,k+2-u}]. \tag{3.45}$$

by (3.43) and (3.44). Since (3.45) and (3.42) are linearly independent, we have shown that $[x^{+}_{i,u},x^{+}_{i,k+2-u}]=0$ holds.

(2) We prove the statement by the induction on $r+s=k$. When $k=0$, it is nothing but (3.24). Suppose that $[x^{+}_{i,r},x^{-}_{i,s}]=h_{i,r+s}$ for all r,s such that $r+s\leq k$. Then, we have the following claim.

Claim 3.46. (a) For all r,s, we obtain

$$[h_{i,r+1},x^{+}_{i+1,s}]-[h_{i,r},x^{+}_{i+1,s+1}]=a_{i,i+1}\frac{\varepsilon_1+\varepsilon_2}{2}\{h_{i,r},x^{+}_{i+1,s}\}-b_{i,i+1}\frac{\varepsilon_1-\varepsilon_2}{2}[h_{i,r},x^{+}_{i+1,s}]. \tag{3.47}$$

(b) For all $r+s=k-1$, we obtain

$$[h_{i,r+1},x^{-}_{i+1,s}]-[h_{i,r},x^{-}_{i+1,s+1}]=-a_{i,i+1}\frac{\varepsilon_1+\varepsilon_2}{2}\{h_{i,r},x^{-}_{i+1,s}\}-b_{i,i+1}\frac{\varepsilon_1-\varepsilon_2}{2}[h_{i,r},x^{-}_{i+1,s}]. \tag{3.48}$$

Proof. (a) By the definition of $h_{i,r}$, we have

$$[h_{i,r+1}, x^+_{i+1,s}] - [h_{i,r}, x^+_{i+1,s+1}] = [[x^+_{i,r+1}, x^-_{i,0}], x^+_{i+1,s}] - [[x^+_{i,r}, x^-_{i,0}], x^+_{i+1,s+1}].$$

By the Jacobi identity and Lemma 3.31 (4), we obtain

$$[h_{i,r+1}, x^+_{i+1,s}] - [h_{i,r}, x^+_{i+1,s+1}] = [\{[x^+_{i,r+1}, x^+_{i+1,s}] - [x^+_{i,r}, x^+_{i+1,s+1}]\}, x^-_{i,0}].$$

By Lemma 3.31 (4), we have

$$\begin{aligned}&[h_{i,r+1}, x^+_{i+1,s}] - [h_{i,r}, x^+_{i+1,s+1}]\\&\qquad = [\pm a_{i,i+1}\frac{\varepsilon_1+\varepsilon_2}{2}\{x^+_{i,r}, x^+_{i+1,s}\} - b_{i,i+1}\frac{\varepsilon_1-\varepsilon_2}{2}[x^+_{i,r}, x^+_{i+1,s}], x^-_{i,0}].\end{aligned}$$

By Lemma 3.31 (4), we obtain

$$[h_{i,r+1}, x^+_{i+1,s}] - [h_{m,r}, x^+_{m+1,s+1}] = \pm a_{i,i+1}\frac{\varepsilon_1+\varepsilon_2}{2}\{h_{i,r}, x^+_{i+1,s}\} - b_{i,i+1}\frac{\varepsilon_1-\varepsilon_2}{2}[h_{i,r}, x^+_{i+1,s}].$$

(b) By the assumption that $[x^+_{i,p}, x^-_{i,q}] = h_{i,p+q}$ holds for all $p+q \leq k$, we have

$$[h_{i,r+1}, x^-_{i+1,s}] - [h_{i,r}, x^-_{i+1,s+1}] = [[x^+_{i,r}, x^-_{i,1}], x^-_{i+1,s}] - [[x^+_{i,r}, x^-_{i,0}], x^-_{i+1,s+1}]$$

since $r+1 \leq k$. Similar discussion to (a), we have

$$[h_{i,r+1}, x^-_{i+1,s}] - [h_{i,r}, x^-_{i+1,s+1}] = [x^+_{i,r}, \{[x^-_{i,1}, x^-_{i+1,s}] - [x^-_{i,0}, x^-_{i+1,s+1}]\}].$$

By Lemma 3.31 (4), we obtain

$$\begin{aligned}&[h_{i,r+1}, x^-_{i+1,s}] - [h_{i,r}, x^-_{i+1,s+1}]\\&\qquad = [x^+_{i,r}, -a_{i,i+1}\frac{\varepsilon_1+\varepsilon_2}{2}\{x^-_{i,0}, x^-_{i+1,s}\} - b_{i,i+1}\frac{\varepsilon_1-\varepsilon_2}{2}[x^-_{i,0}, x^-_{i+1,s}]].\end{aligned}$$

Then, by Lemma 3.31 (4), we have

$$[h_{i,r+1}, x^-_{i+1,s}] - [h_{i,r}, x^-_{i+1,s+1}] = -a_{i,i+1}\frac{\varepsilon_1+\varepsilon_2}{2}\{h_{i,r}, x^-_{i+1,s}\} - b_{i,i+1}\frac{\varepsilon_1-\varepsilon_2}{2}[h_{i,r}, x^-_{i+1,s}].$$

□

By the similar discussion to Lemma 1.4 in [32], there exists $\widetilde{h}_{i,k}$ such that

$$\begin{gathered}\widetilde{h}_{i,k} = h_{i,k} + \text{polynomial of } \{h_{i,t} \mid 0 \leq t \leq k-1\},\\ [\widetilde{h}_{i,k}, x^+_{i+1,1}] = a_{i,i+1}x^+_{i+1,k+1}, \quad [\widetilde{h}_{i,k}, x^-_{i+1,0}] = -a_{i,i+1}x^-_{i+1,k}.\end{gathered}$$

Claim 3.49. The following equation holds;

$$[\widetilde{h}_{i+1,1}, h_{i,k}] = 0. \tag{3.50}$$

Proof. By the assumption that $[x^+_{i,p}, x^-_{i,q}] = h_{i,k}$ holds for all $p+q \leq k$ we have

$$[\widetilde{h}_{i+1,1}, h_{i,s}] = [[\widetilde{h}_{i+1,1}, x^+_{i,s}], x^-_{i,0}] + [x^+_{i,s}, [\widetilde{h}_{i+1,1}, x^-_{i,0}]] = 0$$

for all $s < k$. Thus, it is enough to show that $[\widetilde{h}_{i,k}, h_{i+1,1}] = 0$ holds. By the definition of $h_{i+1,1}$, we obtain

$$\begin{aligned}[\widetilde{h}_{i,k}, h_{i+1,1}] &= [\widetilde{h}_{i,k}, [x^+_{i+1,1}, x^-_{i+1,0}]]\\ &= a_{i,i+1}[x^+_{i+1,k+1}, x^-_{i+1,0}] - a_{i,i+1}[x^+_{i+1,1}, x^-_{i+1,k}]. \end{aligned}\tag{3.51}$$

By Lemma 3.37, it is equal to zero. □

Applying $\mathrm{ad}(\widetilde{h}_{i+1,1})$ to $[x^+_{i,r}, x^-_{i,k-r}] = h_{i,k}$, we obtain

$$[\widetilde{h}_{i+1,1}, [x^+_{i,r}, x^-_{i,k-r}]] = [\widetilde{h}_{i+1,1}, h_{i,k}] \tag{3.52}$$

by the induction hypothesis. By Lemma 3.31 (4), we can rewrite (3.52) as

$$a_{i,i+1}([x^+_{i,r+1}, x^-_{i,k-r}] - [x^+_{i,r}, x^-_{i,k-r+1}]) = [\widetilde{h}_{i+1,1}, h_{i,k}] = 0. \tag{3.53}$$

It is nothing but the statement.

(3) We only show the statement for $+$. The other case is proven in a similar way. By (2), $[h_{i,r}, x^+_{i,0}]$ is equal to $[[x^+_{i,r}, x^-_{i,0}], x^+_{i,0}]$. By (1) and the Jacobi identity, we have

$$[[x^+_{i,r}, x^-_{i,0}], x^+_{i,0}] = [x^+_{i,r}, [x^-_{i,0}, x^+_{i,0}]]. \tag{3.54}$$

The right hand side of (3.54) is equal to $[x^+_{i,r}, h_{i,0}]$. By Lemma 3.26 (1), the right hand side is equal to zero since $a_{i,i} = 0$.

(4) It is enough to check the equality $[h_{i,r}, x^\pm_{i,s}] = 0$. We only show the statement for $+$. The other case is proven in a similar way. We prove by the induction on s. When $s = 0$, it is nothing but (3). Suppose that $[h_{i,r}, x^+_{i,s}] = 0$ holds. Applying $\mathrm{ad}(\widetilde{h}_{i+1,1})$ to $[h_{i,r}, x^+_{i,s}] = 0$, we find the equality

$$[\widetilde{h}_{i+1,1}, [h_{i,r}, x^+_{i,s}]] = 0 \tag{3.55}$$

by the induction hypothesis. By the proof of (2), we obtain $[\widetilde{h}_{i+1,1}, h_{i,n}] = 0$. Thus, the right hand side of (3.55) is equal to $[h_{i,r}, [\widetilde{h}_{i+1,1}, x^+_{i,s}]]$. By Lemma 3.31 (4), we obtain

$$[h_{i,r}, [\widetilde{h}_{i+1,1}, x^+_{i,s}]] = a_{i,i+1}[h_{i,r}, (x^+_{i,s+1} - \frac{\varepsilon_1 - \varepsilon_2}{2} b_{i+1,i} x^+_{i,s})]. \tag{3.56}$$

By the induction hypothesis, the right hand side of (3.56) is equal to $a_{i,i+1}[h_{i,r}, x^+_{i,s+1}]$. Since $a_{i,i+1} \neq 0$, we obtain $[h_{i,r}, x^+_{i,s+1}] = 0$.

(5) By (2), $[h_{i,r}, h_{i,s}]$ is equal to $[h_{i,r}, [x^+_{i,s}, x^-_{i,0}]]$. By the Jacobi identity, we have

$$[h_{i,r}, [x^+_{i,s}, x^-_{i,0}]] = [[h_{i,r}, x^+_{i,s}], x^-_{i,0}] + [x^+_{i,s}, [h_{i,r}, x^-_{i,0}]].$$

By (4), the right hand side is equal to zero. We have shown the relation $[h_{i,r}, h_{i,s}] = 0$. □

We obtain the relation (3.6) by Lemma 3.31 (2), Lemma 3.37 (3), and Lemma 3.38 (1). We also find that the relation (3.3) holds by Lemma 3.31 (2), Lemma 3.37 (2), and Lemma 3.38 (2).

In the same way as that of Theorem 1.2 in [32], we obtain the defining relations (3.5), (3.2), and (3.7). Thus, we omit the proof.

Lemma 3.57. (1) *The relations* (3.5) *and* (3.2) *hold in* $\widetilde{Y}_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m|n))$ *when* $i \neq j$.

(2) *The relation* (3.7) *holds for all* $i, j \in I$ *in* $\widetilde{Y}_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m|n))$.

We remark that the relation (3.2) holds by Lemma 3.37 (1), Lemma 3.38 (5), and Lemma 3.57 (1). We also find that the relation (3.5) holds by Lemma 3.37 (4), Lemma 3.38 (4), and Lemma 3.57 (1).

Now, it is enough to show that (3.8) and (3.9) are deduced from (3.17)-(3.25). However, we have already obtained (3.8), since (3.8) is equivalent to (3.6) when $i = j = 0, m$. Thus, to accomplish the proof, we only need to show that (3.9) holds.

Lemma 3.58. *The relation* (3.9) *holds for* $i = 0, m$ *in* $\widetilde{Y}_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m|n))$.

Proof. We prove by the induction on $k = r+s$. When $k = 0$, it is nothing but (3.25). Suppose that (3.25) holds for all r, s such that $r + s = k$. Applying $\mathrm{ad}(\widetilde{h}_{i+2,1})$ to $[[x^\pm_{i-1,r}, x^\pm_{i,0}], [x^\pm_{i,0}, x^\pm_{i+1,s}]] = 0$, we obtain

$$a_{i-2,i-1}[[x^\pm_{i-1,r+1}, x^\pm_{i,0}], [x^\pm_{i,0}, x^\pm_{i+1,s}]] = 0.$$

Similarly, Applying $\mathrm{ad}(\widetilde{h}_{i+2,1})$ to $[[x^{\pm}_{i-1,r},x^{\pm}_{i,0}],[x^{\pm}_{i,0},x^{\pm}_{i+1,s}]]=0$, we have

$$a_{i+2,i+1}[[x^{\pm}_{i-1,r},x^{\pm}_{i,0}],[x^{\pm}_{i,0},x^{\pm}_{i+1,s+1}]]=0.$$

Thus, we have shown that (3.9) holds for all r,s such that $r+s=k+1$. □

This completes the proof of Theorem 3.16.

By Theorem 3.16, we also obtain the minimalistic presentation of $Y_{\varepsilon_1,\varepsilon_2}(\widetilde{\mathfrak{sl}}(m|n))$.

Theorem 3.59. *Suppose that $m,n\geq 2$ and $m\neq n$. Then, $Y_{\varepsilon_1,\varepsilon_2}(\widetilde{\mathfrak{sl}}(m|n))$ is isomorphic to the super algebra generated by $x^+_{i,r},x^-_{i,r},h_{i,r}$ ($i\in\{0,1,\cdots,m+n-1\},r=0,1$) subject to the defining relations* (3.17)-(3.25) *and*

$$[d,h_{i,r}]=0,\quad [d,x^+_{i,r}]=\begin{cases}x^+_{i,r} & \text{if } i=0,\\ 0 & \text{if } i\neq 0,\end{cases}\quad [d,x^-_{i,r}]=\begin{cases}-x^-_{i,r} & \text{if } i=0,\\ 0 & \text{if } i\neq 0,\end{cases}\tag{3.60}$$

where the generators $x^{\pm}_{m,r}$ and $x^{\pm}_{0,r}$ are odd and all other generators are even.

The relation (3.12) is derived from (3.60) in a similar way to the one of Lemma 3.26. We omit the proof.

4 Coproduct for the Affine Super Yangian

In this section, we define the coproduct for the affine super Yangian $Y_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m|n))$. We recall the definition of standard degreewise completion (see [33]).

Definition 4.1. Let $A=\bigoplus_{i\in\mathbb{Z}}A(i)$ be a graded algebra. For all $i\in\mathbb{Z}$, we set a topology on $A(i)$ such that for $a\in A(i)$ the set

$$\{a+\sum_{r>N}A(i-r)\cdot A(r)\mid N\in\mathbb{Z}_{\geq 0}\}$$

forms a fundamental system of open neighborhoods of a. The standard degreewise completion of A is $\bigoplus_{i\in\mathbb{Z}}\widehat{A}(i)$ where $\widehat{A}(i)$ is the completion of the space $A(i)$. By the definition of $\widehat{A}(i)$, we find that

$$\widehat{A}=\bigoplus_{i\in\mathbb{Z}}\varprojlim_{N}A(i)/\sum_{r>N}A(i-r)\cdot A(r).$$

Let us set the degree on $Y_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m|n))$ determined by

$$\deg(h_{i,r})=0,\quad \deg(x^+_{i,r})=\begin{cases}1 \text{ if } i=0,\\ 0 \text{ if } i\neq 0,\end{cases}\qquad \deg(x^-_{i,r})=\begin{cases}-1 & \text{if } i=0,\\ 0 & \text{if } i\neq 0.\end{cases}\tag{4.2}$$

Then, $Y_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m|n))$ and $Y_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m|n))^{\otimes 2}$ become the graded algebra. We define $\widehat{Y}_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m|n))$ (resp. $Y_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m|n))\widehat{\otimes}Y_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m|n)))$ as the standard degreewise completion of $Y_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m|n))$ (resp. $Y_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m|n))^{\otimes 2}$) in the sense of Definition 4.1.

We prepare some notations. There exists a homomorphism from $\widetilde{\mathfrak{sl}}(m|n)$ to $Y_{\varepsilon_1,\varepsilon_2}(\widetilde{\mathfrak{sl}}(m|n))$ determined by $\Phi(h_i)=h_{i,0}$, $\Phi(x^{\pm}_i)=x^{\pm}_{i,0}$, and $\Phi(d)=d$. We sometimes denote $\Phi(x)$ by x in order to simplify the notation. In particular, we denote $\Phi(x^p_\alpha)$ by x^p_α for all $\alpha\in\Delta$. By Theorem 5.1, we note that $\dim(\Phi(\mathfrak{g}_\alpha))=1$ for all $\alpha\in\Delta_{\mathrm{re}}$.

Theorem 4.3. *The linear map* $\Delta\colon Y_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m|n)) \to Y_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m|n))\widehat{\otimes}Y_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m|n))$ *uniquely determined by*

$$\Delta(h_{i,0}) = h_{i,0}\otimes 1 + 1\otimes h_{i,0}, \quad \Delta(x^{\pm}_{i,0}) = x^{\pm}_{i,0}\otimes 1 + 1\otimes x^{\pm}_{i,0},$$
$$\Delta(h_{i,1}) = h_{i,1}\otimes 1 + 1\otimes h_{i,1} + (\varepsilon_1+\varepsilon_2)h_{i,0}\otimes h_{i,0} - (\varepsilon_1+\varepsilon_2)\sum_{\alpha\in\Delta_+}\sum_{1\leq k_\alpha\leq dim\mathfrak{g}_\alpha}(\alpha,\alpha_i)x^{k_\alpha}_{-\alpha}\otimes x^{k_\alpha}_{\alpha} \tag{4.4}$$

is an algebra homomorphism. Moreover, Δ *satisfies the coassociativity.*

The rest of this section is devoted to the proof of Theorem 4.3. The outline of the proof is similar to that of Theorem 4.9 of [21]. In [21], the analogy of the Drinfeld J presentation is considered in order to prove the existence of the coproduct for the affine Yangian. We construct elements similar to those constructed in (3.7) of [21]

Definition 4.5. We set

$$J(h_i) = h_{i,1} + v_i, \qquad J(x^{\pm}_i) = x^{\pm}_{i,1} + w^{\pm}_i,$$

where

$$v_i = \frac{\varepsilon_1+\varepsilon_2}{2}\sum_{\alpha\in\Delta_+}\sum_{1\leq k_\alpha\leq \mathrm{dim}\mathfrak{g}_\alpha}(\alpha,\alpha_i)x^{k_\alpha}_{-\alpha}x^{k_\alpha}_{\alpha} - \frac{\varepsilon_1+\varepsilon_2}{2}h_i^2,$$
$$w^+_i = -\frac{\varepsilon_1+\varepsilon_2}{2}\sum_{\alpha\in\Delta_+}\sum_{1\leq k_\alpha\leq \mathrm{dim}\mathfrak{g}_\alpha}[x^+_i, x^{k_\alpha}_{-\alpha}]x^{k_\alpha}_{\alpha}, \quad w^-_i = \frac{\varepsilon_1+\varepsilon_2}{2}\sum_{\alpha\in\Delta_+}\sum_{1\leq k_\alpha\leq \mathrm{dim}\mathfrak{g}_\alpha}x^{k_\alpha}_{-\alpha}[x^{k_\alpha}_{\alpha}, x^-_i].$$

Then, $J(h_i)$ and $J(x^{\pm}_i)$ are elements of $\widehat{Y}_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m|n))$.

Next, we prove the similar results to Lemma 3.9 and Proposition 3.21 in [21]. In fact, they are (4.8)-(4.11) and (4.27). We prepare one lemma in order to obtain (4.8)-(4.11) and (4.27). It is an analogy of Proposition 2.4 of [26].

Lemma 4.6 ([36], Lemma 18.4.1)**.** *For all* $\alpha,\beta\in\Delta_+$*, we obtain*

$$\sum_{1\leq k_\beta\leq dim\mathfrak{g}_\beta}[x^{k_\beta}_{\beta}, z]\otimes x^{k_\beta}_{-\beta} = \sum_{1\leq k_\alpha\leq dim\mathfrak{g}_\alpha}x^{k_\alpha}_{\alpha}\otimes[z, x^{k_\alpha}_{-\alpha}]$$

if $z\in\mathfrak{g}_{\beta-\alpha}$*.*

Lemma 4.7. *The following relations hold:*

$$[J(h_i), h_j] = 0, \tag{4.8}$$
$$[J(h_i), x^{\pm}_j] = \pm(\alpha_i,\alpha_j)J(x^{\pm}_j) \mp a_{i,j}b_{i,j}\frac{\varepsilon_1-\varepsilon_2}{2}x^{\pm}_{j,0}, \tag{4.9}$$
$$[J(x^{\pm}_i), x^{\pm}_j] = [x^{\pm}_i, J(x^{\pm}_j)] - \frac{\varepsilon_1-\varepsilon_2}{2}b_{i,j}[x^{\pm}_{i,0}, x^{\pm}_{j,0}], \tag{4.10}$$
$$[J(x^{\pm}_i), x^{\mp}_j] = [x^{\pm}_i, J(x^{\mp}_j)] = \delta_{i,j}J(h_i). \tag{4.11}$$

Proof. Since $h_{i,1}$ commutes with h_j by (3.2) and v_i commutes with h_j by the definition of v_i, we obtain (4.8). We only show the other relations hold for $+$. In a similar way, we obtain them for $-$. First, we prove (4.9) holds for $+$. By (3.21), the left hand side of (4.9) is equal to

$$\begin{aligned}&[\widetilde{h}_{i,1} + v_i + \frac{\varepsilon_1+\varepsilon_2}{2}h^2_{i,0}, x^+_j]\\ &= a_{i,j}(x^+_{j,1} - b_{i,j}\frac{\varepsilon_1-\varepsilon_2}{2}x^+_{j,0}) + [\frac{\varepsilon_1+\varepsilon_2}{2}\sum_{\alpha,\beta\in\Delta_+}\sum_{1\leq k_\alpha\leq \mathrm{dim}\mathfrak{g}_\alpha}(\alpha,\alpha_i)x^{k_\alpha}_{-\alpha}x^{k_\alpha}_{\alpha}, x^+_j].\end{aligned} \tag{4.12}$$

By direct computation, the second term of the right hand side of (4.12) is equal to

$$\begin{aligned}&\frac{\varepsilon_1+\varepsilon_2}{2}\sum_{\alpha\in\Delta_+}\sum_{1\leq k_\alpha\leq\dim\mathfrak{g}_\alpha}(\alpha,\alpha_i)x^{k_\alpha}_{-\alpha}[x^{k_\alpha}_{\alpha},x^+_j]\\&\qquad+\frac{\varepsilon_1+\varepsilon_2}{2}\sum_{\alpha\in\Delta_+}\sum_{1\leq k_\alpha\leq\dim\mathfrak{g}_\alpha}(-1)^{p(\alpha)p(\alpha_j)}(\alpha,\alpha_i)[x^{k_\alpha}_{-\alpha},x^+_j]x^{k_\alpha}_{\alpha}.\end{aligned}\tag{4.13}$$

By Lemma 4.6, (4.13) is equal to

$$\begin{aligned}&\frac{\varepsilon_1+\varepsilon_2}{2}\sum_{\alpha\in\Delta_+}\sum_{1\leq k_\alpha\leq\dim\mathfrak{g}_\alpha}(\alpha-\alpha_j,\alpha_i)[x^+_j,x^{k_\alpha}_{-\alpha}]x^{k_\alpha}_{\alpha}\\&\qquad+\frac{\varepsilon_1+\varepsilon_2}{2}\sum_{\alpha\in\Delta_+}\sum_{1\leq k_\alpha\leq\dim\mathfrak{g}_\alpha}(-1)^{p(\alpha)p(\alpha_j)}(\alpha,\alpha_i)[x^{k_\alpha}_{-\alpha},x^+_j]x^{k_\alpha}_{\alpha}.\end{aligned}\tag{4.14}$$

Since $(-1)^{p(\alpha)p(\alpha_j)}[x^{k_\alpha}_{-\alpha},x^+_j]+[x^+_j,x^{k_\alpha}_{-\alpha}]=0$ holds, the sum of the first and second terms of (4.14) is equal to $-\dfrac{\varepsilon_1+\varepsilon_2}{2}\displaystyle\sum_{\alpha\in\Delta_+}\sum_{1\leq k_\alpha\leq\dim\mathfrak{g}_\alpha}(\alpha_j,\alpha_i)[x^+_j,x^{k_\alpha}_{-\alpha}]x^{k_\alpha}_{\alpha}$. Thus, we obtain

$$[J(h_i),x^+_j]=a_{i,j}(x^+_{j,1}-b_{i,j}\frac{\varepsilon_1-\varepsilon_2}{2}x^+_{j,0})-\frac{\varepsilon_1+\varepsilon_2}{2}\sum_{\alpha\in\Delta_+}\sum_{1\leq k_\alpha\leq\dim\mathfrak{g}_\alpha}(\alpha_j,\alpha_i)[x^+_j,x^{k_\alpha}_{-\alpha}]x^{k_\alpha}_{\alpha}.$$

Thus, we have obtained (4.9) for $+$.

Next, we show that (4.10) holds for $+$. By the definition of $J(x^+_i)$, $[J(x^+_i),x^+_j]-[x^+_i,J(x^+_j)]$ is equal to

$$[x^+_{i,1},x^+_{j,0}]-[x^+_{i,0},x^+_{j,1}]+[w^+_i,x^+_j]-[x^+_i,w^+_j].$$

By (3.22), $[x^+_{i,1},x^+_{j,0}]-[x^+_{i,0},x^+_{j,1}]$ is equal to $\dfrac{\varepsilon_1+\varepsilon_2}{2}a_{i,j}\{x^+_{i,0},x^+_{j,0}\}-\dfrac{\varepsilon_1-\varepsilon_2}{2}b_{i,j}[x^+_{i,0},x^+_{j,0}]$. By the definition of w^+_i, we obtain

$$\begin{aligned}&[w^+_i,x^+_j]-[x^+_i,w^+_j]\\&=-\frac{\varepsilon_1+\varepsilon_2}{2}\sum_{\alpha\in\Delta_+}\sum_{1\leq k_\alpha\leq\dim\mathfrak{g}_\alpha}[x^+_i,x^{k_\alpha}_{-\alpha}][x^{k_\alpha}_{\alpha},x^+_j]\\&\quad-\frac{\varepsilon_1+\varepsilon_2}{2}\sum_{\alpha\in\Delta_+}\sum_{1\leq k_\alpha\leq\dim\mathfrak{g}_\alpha}(-1)^{p(\alpha)p(\alpha_j)}[[x^+_i,x^{k_\alpha}_{-\alpha}],x^+_j]x^{k_\alpha}_{\alpha}\\&\quad+\frac{\varepsilon_1+\varepsilon_2}{2}\sum_{\alpha\in\Delta_+}\sum_{1\leq k_\alpha\leq\dim\mathfrak{g}_\alpha}[x^+_i,[x^+_j,x^{k_\alpha}_{-\alpha}]]x^{k_\alpha}_{\alpha}\\&\quad+\frac{\varepsilon_1+\varepsilon_2}{2}\sum_{\alpha\in\Delta_+}\sum_{1\leq k_\alpha\leq\dim\mathfrak{g}_\alpha}(-1)^{p(\alpha)p(\alpha_i)+p(\alpha_j)p(\alpha_i)}[x^+_j,x^{k_\alpha}_{-\alpha}][x^+_i,x^{k_\alpha}_{\alpha}].\end{aligned}\tag{4.15}$$

By Lemma 4.6, we find the equality

$$\begin{aligned}&\text{the first term of the right hand side of (4.15)}\\&=-\frac{\varepsilon_1+\varepsilon_2}{2}\sum_{\alpha\in\Delta_+}\sum_{1\leq k_\alpha\leq\dim\mathfrak{g}_\alpha}[x^+_i,[x^+_j,x^{k_\alpha}_{-\alpha}]]x^{k_\alpha}_{\alpha}+\frac{\varepsilon_1+\varepsilon_2}{2}[x^+_i,h_j]x^+_j.\end{aligned}\tag{4.16}$$

We also find the relation

the fourth term of the right hand side of (4.15)

$$= \frac{\varepsilon_1+\varepsilon_2}{2}\sum_{\alpha\in\Delta_+}\sum_{1\leq k_\alpha\leq\dim\mathfrak{g}_\alpha}(-1)^{p(\alpha)p(\alpha_i)+p(\alpha_j)p(\alpha_i)}[x_j^+,[x_{-\alpha}^{k_\alpha},x_i^+]]x_\alpha^{k_\alpha}$$
$$+\frac{\varepsilon_1+\varepsilon_2}{2}(-1)^{p(\alpha_i)p(\alpha_j)}[x_j^+,h_i]x_i^+ \tag{4.17}$$

by Lemma 4.6. Applying (4.16) and (4.17) to (4.15), we obtain

$$[w_i^+,x_j^+]-[x_i^+,w_j^+]=\frac{\varepsilon_1+\varepsilon_2}{2}[x_i^+,h_j]x_j^+ +\frac{\varepsilon_1+\varepsilon_2}{2}(-1)^{p(\alpha_i)p(\alpha_j)}[x_j^+,h_i]x_i^+.$$

Since $m,n\geq 2$, there exists no i,j such that $a_{i,j}\neq 0$ and $p(\alpha_i)p(\alpha_j)=1$. Thus, we obtain

$$\frac{\varepsilon_1+\varepsilon_2}{2}[x_i^+,h_j]x_j^+ +\frac{\varepsilon_1+\varepsilon_2}{2}(-1)^{p(\alpha_i)p(\alpha_j)}[x_j^+,h_i]x_i^+ = -\frac{\varepsilon_1+\varepsilon_2}{2}a_{i,j}\{x_i^+,x_j^+\}.$$

Hence, we have obtained

$$[J(x_i^+),x_j^+]-[x_i^+,J(x_j^+)]=-\frac{\varepsilon_1-\varepsilon_2}{2}b_{i,j}[x_{i,0}^+,x_{j,0}^+].$$

Finally, we show that $[J(x_i^+),x_j^-]=\delta_{i,j}J(h_i)$ holds. By the definiton of $J(x_i^+)$, $[J(x_i^+),x_j^-]$ is equal to $[x_{i,1}^+,x_{j,0}^-]+[w_i^+,x_{j,0}^-]$. By (3.3), $[x_{i,1}^+,x_{j,0}^-]$ is $\delta_{i,j}h_{i,1}$. By direct computation, we have

$$\begin{aligned}
&[w_i^+,x_{j,0}^-]\\
&=-\frac{\varepsilon_1+\varepsilon_2}{2}\sum_{\alpha\in\Delta_+}\sum_{1\leq k_\alpha\leq\dim\mathfrak{g}_\alpha}[x_i^+,x_{-\alpha}^{k_\alpha}][x_\alpha^{k_\alpha},x_j^-]\\
&\quad-\frac{\varepsilon_1+\varepsilon_2}{2}\sum_{\alpha\in\Delta_+}\sum_{1\leq k_\alpha\leq\dim\mathfrak{g}_\alpha}(-1)^{p(\alpha)p(\alpha_j)}[[x_i^+,x_{-\alpha}^{k_\alpha}],x_j^-]x_\alpha^{k_\alpha}.
\end{aligned}\tag{4.18}$$

By Lemma 4.6, we have

$$\begin{aligned}
&\text{the first term of the right hand side of (4.18)}\\
&=-\frac{\varepsilon_1+\varepsilon_2}{2}\sum_{\alpha\in\Delta_+}\sum_{1\leq k_\alpha\leq\dim\mathfrak{g}_\alpha}[x_i^+,[x_j^-,x_{-\alpha}^{k_\alpha}]]x_\alpha^{k_\alpha}-\frac{\varepsilon_1+\varepsilon_2}{2}\delta_{i,j}h_i^2.
\end{aligned}\tag{4.19}$$

By the Jacobi identity, we find the equality

$$\begin{aligned}
[x_i^+,[x_j^-,x_{-\alpha}^{k_\alpha}]]&=-(-1)^{p(\alpha)p(\alpha_j)}[x_i^+,[x_{-\alpha}^{k_\alpha},x_j^-]]\\
&=-(-1)^{p(\alpha)p(\alpha_j)}[[x_i^+,x_{-\alpha}^{k_\alpha}],x_j^-]x_\alpha^{k_\alpha}-(-1)^{p(\alpha)p(\alpha_j)}(-1)^{p(\alpha)p(\alpha_i)}[x_{-\alpha}^{k_\alpha},[x_i^+,x_j^-]]x_\alpha^{k_\alpha}.
\end{aligned}\tag{4.20}$$

Thus, we obtain

$$\begin{aligned}
&[w_i^+,x_{j,0}^-]\\
&=\frac{\varepsilon_1+\varepsilon_2}{2}\sum_{\alpha\in\Delta_+}\sum_{1\leq k_\alpha\leq\dim\mathfrak{g}_\alpha}(-1)^{p(\alpha)p(\alpha_j)}[[x_i^+,x_{-\alpha}^{k_\alpha}],x_j^-]x_\alpha^{k_\alpha}\\
&\quad+\frac{\varepsilon_1+\varepsilon_2}{2}\delta_{i,j}\sum_{\alpha\in\Delta_+}\sum_{1\leq k_\alpha\leq\dim\mathfrak{g}_\alpha}(-1)^{p(\alpha)p(\alpha_j)}(-1)^{p(\alpha)p(\alpha_j)}[x_{-\alpha}^{k_\alpha},[x_i^+,x_j^-]]x_\alpha^{k_\alpha}-\frac{\varepsilon_1+\varepsilon_2}{2}\delta_{i,j}h_i^2\\
&\quad-\frac{\varepsilon_1+\varepsilon_2}{2}\sum_{\alpha\in\Delta_+}\sum_{1\leq k_\alpha\leq\dim\mathfrak{g}_\alpha}(-1)^{p(\alpha)p(\alpha_j)}[[x_i^+,x_{-\alpha}^{k_\alpha}],x_j^-]x_\alpha^{k_\alpha}\\
&=\frac{\varepsilon_1+\varepsilon_2}{2}\sum_{\alpha\in\Delta_+}\sum_{1\leq k_\alpha\leq\dim\mathfrak{g}_\alpha}\delta_{i,j}(\alpha_i,\alpha)x_{-\alpha}^{k_\alpha}x_\alpha^{k_\alpha}-\frac{\varepsilon_1+\varepsilon_2}{2}\delta_{i,j}h_i^2,
\end{aligned}$$

where the first equality is due to (4.19) and the second equality is due to (4.20). Then, we have shown that $[J(x_i^+),x_i^-]=\delta_{i,j}J(h_i)$. Similarly, we can obtain $[x_i^+,J(x_j^-)]=\delta_{i,j}J(h_i)$. This completes the proof. □

By (4.8)-(4.11), we obtain the following convenient relation.

Corollary 4.21. (1) *When $i \neq j, j \pm 1$, $[J(x_i^{\pm}), x_j^{\pm}] = 0$ holds.*

(2) *Suppose that $j < i-1$. We have the following relation;*

$$\begin{aligned}
&\mathrm{ad}(J(x_i^{\pm})) \prod_{i+1 \leq k \leq m+n-1} \mathrm{ad}(x_k^{\pm}) \prod_{0 \leq k \leq j-1} \mathrm{ad}(x_k^{\pm})(x_j^{\pm}) \\
&= \mathrm{ad}(x_i^{\pm}) \, \mathrm{ad}(J(x_{i+1}^{\pm})) \prod_{i+2 \leq k \leq m+n-1} \mathrm{ad}(x_k^{\pm}) \prod_{0 \leq k \leq j-1} \mathrm{ad}(x_k^{\pm})(x_j^{\pm}) \\
&\quad - b_{i,i+1} \frac{\varepsilon_1 - \varepsilon_2}{2} \prod_{i \leq k \leq m+n-1} \mathrm{ad}(x_k^{\pm}) \prod_{0 \leq k \leq j-1} \mathrm{ad}(x_k^{\pm})(x_j^{\pm}).
\end{aligned}$$

(3) *For all $\alpha \in \sum_{1 \leq l \leq m+n-1} \mathbb{Z}_{\geq 0} \alpha_i$ and $x_{\pm\alpha} \in \mathfrak{g}_{\pm\alpha}$, there exists a number $d_{i,j}^{\alpha}$ such that*

$$(\alpha_j, \alpha)[J(h_i), x_{\pm\alpha}] - (\alpha_i, \alpha)[J(h_j), x_{\pm\alpha}] = \pm d_{i,j}^{\alpha} x_{\pm\alpha}.$$

(4) *Suppose that $j < i-1$. We have*

$$\begin{aligned}
&[J(h_s), \prod_{i \leq k \leq m+n-1} \mathrm{ad}(x_k^{\pm}) \prod_{0 \leq k \leq j-1} \mathrm{ad}(x_k^{\pm})(x_j^{\pm})] \\
&= \pm(\alpha_s, \alpha) \prod_{i \leq k \leq m+n-1} \mathrm{ad}(x_k^{\pm}) \prod_{0 \leq k \leq j-1} \mathrm{ad}(x_k^{\pm}) J(x_j^{\pm}) \\
&\quad \pm c_2 \prod_{i \leq k \leq m+n-1} \mathrm{ad}(x_k^{\pm}) \prod_{0 \leq k \leq j-1} \mathrm{ad}(x_k^{\pm})(x_j^{\pm}),
\end{aligned}$$

where $\alpha = \sum_{i \leq k \leq m+n-1} \alpha_k + \sum_{0 \leq k \leq j} \alpha_k$ and c_2 is a complex number.

Proof. We only show the relations for $+$. The other case is proven in a similar way.

(1) By the definition of the commutator relations of $\widehat{\mathfrak{sl}}(m|n)$, $[x_i^+, x_j^+] = 0$ holds when $i \neq j, j \pm 1$. There exists an index p such that $a_{i,p} \neq 0$ and $a_{j,p} = 0$. Appling $\mathrm{ad}(J(h_p))$ to $[x_i^+, x_j^+] = 0$, we obtain

$$a_{i,p}([J(x_i^+), x_j^+] - b_{i,p} \frac{\varepsilon_1 - \varepsilon_2}{2} [x_i^+, x_j^+]) = 0$$

by (4.9). Since $a_{i,p} \neq 0$, we have shown that $[J(x_i^+), x_j^+] = 0$ holds.

(2) By (1), the left hand side is equal to

$$\mathrm{ad}([J(x_i^+), x_{i+1}^+]) \prod_{i+2 \leq k \leq m+n-1} \mathrm{ad}(x_k^+) \prod_{0 \leq k \leq j-1} \mathrm{ad}(x_k^+)(x_j^+).$$

By (4.11), it is equal to

$$\begin{aligned}
&\mathrm{ad}([x_i^+, J(x_{i+1}^+)]) \prod_{i+2 \leq k \leq m+n-1} \mathrm{ad}(x_k^+) \prod_{0 \leq k \leq j-1} \mathrm{ad}(x_k^+)(x_j^+) \\
&\quad - b_{i,i+1} \frac{\varepsilon_1 - \varepsilon_2}{2} \mathrm{ad}([x_i^+, x_{i+1}^+]) \prod_{i+2 \leq k \leq m+n-1} \mathrm{ad}(x_k^+) \prod_{0 \leq k \leq j-1} \mathrm{ad}(x_k^+)(x_j^+). \qquad (4.22)
\end{aligned}$$

By the Jacobi identity, the first term of (4.22) is equal to

$$\mathrm{ad}(x_i^+) \, \mathrm{ad}(J(x_{i+1}^+)) \prod_{i+2 \leq k \leq m+n-1} \mathrm{ad}(x_k^+) \prod_{0 \leq k \leq j-1} \mathrm{ad}(x_k^+)(x_j^+).$$

This completes the proof.

(3) It is enough to assume that $x_{\pm\alpha}=\prod_{s\leq k\leq t-1}\mathrm{ad}(x^{\pm}_k)x^{\pm}_t$. By (4.9), we have

$$\begin{aligned}
[J(h_i),x_{\pm\alpha}]&=\pm\delta(s\geq i+1\geq t)a_{i,i+1}\prod_{s\leq k\leq i}\mathrm{ad}(x^{\pm}_k)J(x^{\pm}_{i+1})\prod_{i+2\leq k\leq t-1}\mathrm{ad}(x^{\pm}_k)x^{\pm}_t\\
&\quad\pm\delta(s\geq i\geq t)a_{i,i}\prod_{s\leq k\leq i-1}\mathrm{ad}(x^{\pm}_k)J(x^{\pm}_i)\prod_{i+1\leq k\leq t-1}\mathrm{ad}(x^{\pm}_k)x^{\pm}_t\\
&\quad\pm\delta(s\geq i-1\geq t)a_{i,i-1}\prod_{s\leq k\leq i-2}\mathrm{ad}(x^{\pm}_k)J(x^{\pm}_{i-1})\prod_{i+1\leq k\leq t-1}\mathrm{ad}(x^{\pm}_k)x^{\pm}_t\\
&\quad\pm d^1_i(\alpha_i,\alpha)\prod_{s\leq k\leq t-1}\mathrm{ad}(x^{\pm}_k)x^{\pm}_t,
\end{aligned}$$

where d^1_i is a complex number. By a discussion similar to the one in the proof of (2), we find that there exists a complex number d^2_i such that the sum of the first three terms is equal to

$$\pm(\alpha_i,\alpha)\prod_{s\leq k\leq t-1}\mathrm{ad}(x^{\pm}_k)J(x^{\pm}_t)\pm d^2_i(\alpha_i,\alpha)\prod_{s\leq k\leq t-1}\mathrm{ad}(x^{\pm}_k)x^{\pm}_t.$$

Then, we obtain

$$(\alpha_j,\alpha)[J(h_i),x_{\pm\alpha}]-(\alpha_k,\alpha)[J(h_j),x_{\pm\alpha}]=\pm\{(\alpha_j,\alpha)(d^1_i+d^2_i)-(\alpha_i,\alpha)(d^1_j+d^2_j)\}x_{\pm\alpha}.$$

We complete the proof.

(4) It is proven in a similar way to the one in the proof of (3). □

Next, in order to obtain (4.27), we prepare $\{\tau_i\}_{i\neq 0,m}$, which are automorphisms of the affine super Yangian. Let us set $\{s_i\}_{i\neq 0,m}$ as an automorphism of Δ such that $s_i(\alpha)=\alpha-\dfrac{2(\alpha_i,\alpha)}{(\alpha_i,\alpha_i)}\alpha_i$. By the definition of $\widehat{\mathfrak{sl}}(m|n)$, we can rewrite s_i explicitly as follows;

$$s_i(\alpha_j)=\begin{cases}-\alpha_j & \text{if } i=j,\\ \alpha_i+\alpha_j & \text{if } j=i\pm 1,\\ \alpha_j & \text{otherwise.}\end{cases}$$

It is called a simple reflection. We also define $\{\tau_i\}_{i\neq 0,m}$ as an operator on the affine super Yangian determined by

$$\tau_i(x)=\exp(\mathrm{ad}(x^+_i))\exp\Big(-\mathrm{ad}(x^-_i)\Big)\exp(\mathrm{ad}(x^+_i))x. \tag{4.23}$$

By the defining relation (3.7), τ_i is well-defined as an operator on the affine super Yangian. The following lemma is well-known (see [26]).

Lemma 4.24. (1) *The action of τ_i preserves the inner product κ.*

(2) *For all $\alpha\in\Delta$, $\tau_i(\mathfrak{g}_\alpha)=\mathfrak{g}_{s_i(\alpha)}$.*

Then, in a similar way as that of Lemma 3.17 and 3.19 of [21], we can compute the action of τ_i on $J(h_j)$ and write it explicitly.

Lemma 4.25. *When $i\neq 0,m$, we obtain*

$$\tau_i(J(h_j))=J(h_j)-\frac{2(\alpha_i,\alpha_j)}{(\alpha_i,\alpha_i)}J(h_i)+a_{i,j}b_{j,i}(\varepsilon_1-\varepsilon_2)h_i.$$

Since $\dim\mathfrak{g}_\alpha=1$ for all $\alpha\in\Delta^{\mathrm{re}}$, we sometimes denote $x^{k_\alpha}_{-\alpha}$ and $x^{k_\alpha}_{\alpha}$ as $x_{-\alpha}$ and x_α for all $\alpha\in\Delta^{\mathrm{re}}_+$.

Proposition 4.26. *For $i, j \in I$ and a positive real root α, the following equation holds;*

$$(\alpha_j, \alpha)[J(h_i), x_\alpha] - (\alpha_i, \alpha)[J(h_j), x_\alpha] = c^\alpha_{i,j} x_\alpha, \tag{4.27}$$

where $c^\alpha_{i,j}$ is a complex number such that $c^\alpha_{i,j} = -c^{-\alpha}_{i,j}$.

Proof. We divide the proof into two cases; one is that α is even, the other is that α is odd..

Case 1, α is even.

Suppose that α is even. Then, there exists $s \in \mathbb{Z}$ such that α is an element of $\sum_{1\leq l\leq m-1} \mathbb{Z}\alpha_i + s\delta$ or $\sum_{m+1\leq l\leq m+n-1} \mathbb{Z}\alpha_i + s\delta$. We only prove the case where $\alpha \in \sum_{1\leq l\leq m-1} \mathbb{Z}_{\geq 0}\alpha_i + \mathbb{Z}_{\geq 0}\delta$. The other cases are proven in a similar way.

First, we prove the case where $\alpha = \alpha_k + s\delta$, where $k \neq 0, m$.

Claim 4.28. Suppose that $\alpha = \alpha_k + s\delta$ such that $k \neq 0, m$. Then, we have

$$[J(h_i), x_\alpha] = \frac{(\alpha_i, \alpha_k)}{(\alpha_k, \alpha_k)}[J(h_k), x_\alpha] + d_\alpha x_\alpha, \qquad [J(h_i), x_{-\alpha}] = \frac{(\alpha_i, \alpha_k)}{(\alpha_k, \alpha_k)}[J(h_k), x_{-\alpha}] - d_\alpha x_{-\alpha},$$

where d_α is a complex number.

Proof. Let us set

$$x^1_{\pm\delta} = [x^\pm_k, \prod_{k+1\leq p\leq m+n-1} \mathrm{ad}(x^\pm_p) \prod_{0\leq p\leq k-2} \mathrm{ad}(x^\pm_p)(x^\pm_{k-1})].$$

It is enough to suppose that $x_{\pm\alpha} = \mathrm{ad}(x^1_{\pm\delta})^s x^\pm_k$ since $\mathrm{ad}(x^1_{\pm\delta})^s x^\pm_k$ is nonzero. By the Jacobi identity, we obtain

$$\begin{aligned}
&[(\alpha_k, \alpha)J(h_i) - (\alpha_i, \alpha)J(h_k), \mathrm{ad}(x^1_{\pm\delta})^s x^\pm_k] \\
&= \sum_{0\leq t\leq s-1} \mathrm{ad}(x^1_{\pm\delta})^t \,\mathrm{ad}([(\alpha_k, \alpha)J(h_i) - (\alpha_i, \alpha)J(h_k), x^1_{\pm\delta}])\,\mathrm{ad}(x^1_{\pm\delta})^{s-1-t} x^\pm_k \\
&\qquad + \mathrm{ad}(x^1_{\pm\delta})^s[(\alpha_k, \alpha)J(h_i) - (\alpha_i, \alpha)J(h_k), x^\pm_k].
\end{aligned} \tag{4.29}$$

By (4.9), $[(\alpha_k, \alpha)J(h_i)-(\alpha_i, \alpha)J(h_k), x^\pm_k]$ can be written as $\pm f_k x^\pm_k$, where f_k is a complex number. Then, we have

$$\begin{aligned}
&[J(h_i), x^1_{\pm\delta}] \\
&= [x^\pm_k, [(\alpha_k, \alpha)J(h_i) - (\alpha_i, \alpha)J(h_k), \prod_{k+1\leq p\leq m+n-1} \mathrm{ad}(x^\pm_p) \prod_{0\leq p\leq k-2} \mathrm{ad}(x^\pm_p)(x^\pm_{k-1})]] \pm f_k x^1_{\pm\delta}.
\end{aligned}$$

By Corollary 4.21 (4), we can rewrite the first term as

$$\begin{aligned}
&\pm (\alpha_k, \alpha)(\alpha_i, \alpha)[x^\pm_k, \prod_{k+1\leq p\leq m+n-1} \mathrm{ad}(x^\pm_p) \prod_{0\leq p\leq k-2} \mathrm{ad}(x^\pm_p) J(x^\pm_{k-1})] \\
&\quad \mp (\alpha_k, \alpha)(\alpha_i, \alpha)[x^\pm_k, \prod_{k+1\leq p\leq m+n-1} \mathrm{ad}(x^\pm_p) \prod_{0\leq p\leq k-2} \mathrm{ad}(x^\pm_p) J(x^\pm_{k-1})] \pm g_k x^1_{\pm\delta} \\
&= \pm g_k x^1_{\pm\delta},
\end{aligned}$$

where g_k is a complex number. We have obtained the statement. □

Now, let us consider the case where α is a general even root. Any even root $\alpha = \sum_{0\leq k\leq l} \alpha_{p+k}$ can be written as $\prod_{0\leq k\leq l-1} s_{p+k}(\alpha_{p+l})$ by the explicit presentation of s_i. Let us prove that the statement of Proposition 4.26 holds by the induction on l. When $l=1$, it is nothing but Claim 4.28. Assume that (4.27) holds when $l=q$. We set α and β as $\prod_{0\le k\le q} s_{p+k}(\alpha_{p+q+1})$ and $\prod_{1\le k\le q} s_{p+k}(\alpha_{p+q+1})$. Suppose that x_β is a nonzero element of $\mathfrak{g}_\beta$. By Lemma 4.24, $\mathfrak{g}_\alpha$ contains a nonzero element $\tau_{s_p}(x_\beta)$. Thus, we obtain

$$\begin{aligned}&(\alpha_j,\alpha)[J(h_i),\tau_{s_p}(x_{\pm\beta})]-(\alpha_i,\alpha)[J(h_j),\tau_{s_p}(x_{\pm\beta})]\\ =&\,\tau_{s_p}\Big\{(\alpha_j,\alpha)\big[J(h_i)-\frac{2(\alpha_i,\alpha_p)}{(\alpha_p,\alpha_p)}J(h_p),x_{\pm\beta}\big]-(\alpha_i,\alpha)\big[J(h_j)-\frac{2(\alpha_i,\alpha_p)}{(\alpha_p,\alpha_p)}J(h_p),(x_{\pm\beta})\big]\Big\}\\ &\mp\{(\alpha_j,\alpha)a_{p,i}b_{i,p}-(\alpha_i,\alpha)a_{p,j}b_{j,p}\}(\varepsilon_1-\varepsilon_2)x_{\pm\alpha}\end{aligned}\tag{4.30}$$

by Lemma 4.25. Let us suppose that $(\alpha_t,\beta)\neq 0$. Then, by the induction hypothesis, we find the relation

$$[J(h_u),x_{\pm\beta}]=\pm\frac{(\alpha_u,\beta)}{(\alpha_t,\beta)}[J(h_t),x_{\pm\beta}]\pm c^\beta_{u,t}x_{\pm\beta}.\tag{4.31}$$

Applying (4.31) to (4.30), we obtain

$$\begin{aligned}\big[J(h_i)-\frac{2(\alpha_i,\alpha_p)}{(\alpha_p,\alpha_p)}J(h_p),x_{\pm\beta}\big]&=\pm\Big\{\frac{(\alpha_i,\beta)}{(\alpha_t,\beta)}-\frac{2(\alpha_i,\alpha_p)}{(\alpha_p,\alpha_p)}\cdot\frac{(\alpha_p,\beta)}{(\alpha_t,\beta)}\Big\}([J(h_t),x_{\pm\beta}]+c^\beta_{i,t}x_{\pm\beta})\\ &=\pm\frac{\Big(\alpha_i,\beta-\frac{2(\alpha_i,\alpha_p)}{(\alpha_p,\alpha_p)}\alpha_p\Big)}{(\alpha_t,\beta)}([J(h_t),x_{\pm\beta}]+c^\beta_{i,t}x_{\pm\beta}).\end{aligned}$$

By the definition of s_p, α is equal to $\beta-\frac{2(\alpha_i,\alpha_p)}{(\alpha_p,\alpha_p)}\alpha_p$. Then, we have

$$[J(h_i)-\frac{2(\alpha_i,\alpha_p)}{(\alpha_p,\alpha_p)}J(h_p),x_{\pm\beta}]=\pm\frac{(\alpha,\alpha_i)}{(\alpha_t,\beta)}([J(h_t),x_{\pm\beta}]+c^\beta_{i,t}x_{\pm\beta}).\tag{4.32}$$

Similarly, we find the relation

$$[J(h_j)-\frac{2(\alpha_j,\alpha_p)}{(\alpha_p,\alpha_p)}J(h_p),x_{\pm\beta}]=\pm\frac{(\alpha,\alpha_j)}{(\alpha_t,\beta)}([J(h_t),x_{\pm\beta}]+c^\beta_{j,t}x_{\pm\beta}).\tag{4.33}$$

Appling (4.32) and (4.33) to the right hand side of (4.30),

$$\begin{aligned}&(\alpha_j,\alpha)[J(h_i),\tau_{s_p}(x_{\pm\beta})]-(\alpha_i,\alpha)[J(h_j),\tau_{s_p}(x_{\pm\beta})]\\ =&\pm\tau_{s_p}\Big\{(\alpha_j,\alpha)\frac{(\alpha,\alpha_i)}{(\alpha_t,\beta)}c^\beta_{i,t}x_{\pm\beta}-(\alpha_i,\alpha)\frac{(\alpha,\alpha_j)}{(\alpha_t,\beta)}c^\beta_{j,t}x_{\pm\beta}\}\\ &\mp\{(\alpha_j,\alpha)a_{p,i}b_{i,p}-(\alpha_i,\alpha)a_{p,j}b_{j,p}\}(\varepsilon_1-\varepsilon_2)x_{\pm\alpha}.\end{aligned}$$

This completes the proof of the case where α is even.

Case 2, α is odd.

Here after, we suppose that m is greater than 3. The other case is proven in a similar way. First, we consider the case where $\alpha=\sum_{1\le l\le m-1}\alpha_i+\alpha_m+s\delta$.

Claim 4.34. (1) When $i\neq 0,1,m,m+1$, $[J(h_i),x_{\pm\alpha}]=\pm c^i_\alpha x_{\pm\alpha}$, where c_α is a complex number.

(2) We obtain the following equations;

$$[J(h_0),x_{\pm\alpha}]=\frac{(\alpha_0,\alpha)}{(\alpha_1,\alpha)}[J(h_1),x_{\pm\alpha}]\pm d_{0,1}x_{\pm\alpha},\tag{4.35}$$

$$[J(h_m),x_{\pm\alpha}]=\frac{(\alpha_m,\alpha)}{(\alpha_1,\alpha)}[J(h_1),x_{\pm\alpha}]\pm d_{m,1}x_{\pm\alpha},\tag{4.36}$$

$$[J(h_{m+1}),x_{\pm\alpha}]=\frac{(\alpha_{m+1},\alpha)}{(\alpha_m,\alpha)}[J(h_m),x_{\pm\alpha}]\pm d_{m,m+1}x_{\pm\alpha},\tag{4.37}$$

where $d_{0,1}$, $d_{m,1}$, and $d_{m,m+1}$ are complex numbers.

Proof. (1) When $i \neq 0,1,2,m,m+1$, we set $x^2_{\pm\delta} = [x_1^{\pm}, \prod_{2\leq p\leq m+n-1} \mathrm{ad}(x_p^{\pm})(x_0^{\pm})]$. It is sufficient to assume that

$$x_{\pm\alpha} = \mathrm{ad}(x^2_{\pm\delta})^s \sum_{1\leq l\leq m-1} \mathrm{ad}(x_i^{\pm})(x_m^{\pm})$$

since the right hand side is nonzero. In a similar way as that of Claim 4.28, we also have

$$[J(h_i), x^2_{\pm\delta}] = \pm h_\delta x^2_{\pm\delta}, \tag{4.38}$$

$$[J(h_i), \sum_{1\leq l\leq m-1} \mathrm{ad}(x_i^{\pm})(x_m^{\pm})] = \pm i_\alpha \sum_{1\leq l\leq m-1} \mathrm{ad}(x_i^{\pm})(x_m^{\pm}), \tag{4.39}$$

where h_δ and i_α are complex numbers. Thus, we find the equality

$$[J(h_i), x_\alpha^{k_\alpha}] = \pm(sh_\delta + i_\alpha)\,\mathrm{ad}(x^2_{\pm\delta})^s \sum_{1\leq l\leq m-1} \mathrm{ad}(x_i^{\pm})(x_m^{\pm})$$

by the Jacobi identity, (4.38), and (4.39). We have proved the statement when $i \neq 0,1,2,m,m+1$. When $i=2$, we set $x^3_{\pm\delta}$ as

$$[x^{\pm}_{m+1}, \prod_{m+2\leq p\leq m+n-1} \mathrm{ad}(x_p^{\pm}) \prod_{0\leq p\leq m-1} \mathrm{ad}(x_p^{\pm})\,\mathrm{ad}(x_m^{\pm})].$$

It is enough to assume that

$$x_{\pm\alpha} = \mathrm{ad}(x^3_{\pm\delta})^s \sum_{1\leq l\leq m-1} \mathrm{ad}(x_i^{\pm})(x_m^{\pm})$$

since the right hand side is nonzero. In a similar way as that of Claim 4.28, we also have

$$[J(h_i), x^3_{\pm\delta}] = \pm j_\delta x^3_{\pm\delta},$$

$$[J(h_i), \sum_{1\leq l\leq m-1} \mathrm{ad}(x_i^{\pm})(x_m^{\pm})] = \pm k_\alpha \sum_{1\leq l\leq m-1} \mathrm{ad}(x_i^{\pm})(x_m^{\pm}),$$

where j_δ and k_α are complex numbers. Thus, we find the relation

$$[J(h_i), x_\alpha] = \pm(sj_\delta + k_\alpha)\,\mathrm{ad}(x^2_{\pm\delta})^s \sum_{1\leq l\leq m-1} \mathrm{ad}(x_i^{\pm})(x_m^{\pm})$$

by the Jacobi identity, (4) and (4). We have proved the statement when $i=2$.

(2) First, we prove that (4.35) holds. By the definition of α, $x_{\pm\alpha}$ can be written as $[x_{\pm\beta}, x_m^{\pm}]$, where $x_{\pm\beta}$ is a nonzero element of $\mathfrak{g}_{\alpha-\alpha_m}$. Since $[J(h_0), x_m^{\pm}]$ and $[J(h_1), x_m^{\pm}]$ is equal to zero by (4.9), we obtain

$$[J(h_0), x_{\pm\alpha}] = [[J(h_0), x_{\pm\beta}], x_m^{\pm}], \tag{4.40}$$

$$[J(h_1), x_{\pm\alpha}] = [[J(h_1), x_{\pm\beta}], x_m^{\pm}]. \tag{4.41}$$

Then, because β is even, we have

$$[[J(h_0), x_{\pm\beta}], x_m^{\pm}] = \frac{(\alpha_0,\beta)}{(\alpha_1,\beta)}[[J(h_1), x_{\pm\beta}], x_m^{\pm}] + \frac{(\alpha_0,\beta)}{(\alpha_1,\beta)}[x_{\pm\beta}, x_m^{\pm}] \tag{4.42}$$

by Case 1. By(4.40), (4.41), and (4.42), we find the equality

$$[J(h_0), x_{\pm\alpha}] = \frac{(\alpha_0,\beta)}{(\alpha_1,\beta)}[J(h_1), x_{\pm\alpha}] + \frac{(\alpha_0,\beta)}{(\alpha_1,\beta)}[x_{\pm\beta}, x_m^{\pm}].$$

Thus we have shown that (4.35) holds. Similarly, we obtain (4.36) since $[J(h_m), x_m^{+}] = 0$ holds.

Finally, we prove that (4.37) holds. We set $x^4_{\pm\delta}=[x^{\pm}_1,\prod_{2\leq p\leq m+n-1}\mathrm{ad}(x^{\pm}_p)(x^{\pm}_0)]$. It is enough to check the relation under the assumption that $x_{\pm\alpha}=\mathrm{ad}(x^4_{\pm\delta})^s\prod_{1\leq p\leq m-1}\mathrm{ad}(x^{\pm}_p)(x^{\pm}_m)$ since the right hand side is nonzero. Then, we obtain

$$\begin{aligned}
&[J(h_m),x_{\pm\alpha}]\\
&=\sum_{1\leq t\leq s}\mathrm{ad}(x^4_{\pm\delta})^{t-1}\,\mathrm{ad}([J(h_m),x^4_{\pm\delta}])\,\mathrm{ad}(x^4_{\pm\delta})^{s-t}\prod_{1\leq p\leq m-1}\mathrm{ad}(x^{\pm}_p)(x^{\pm}_m)\\
&\qquad+[J(h_m),\mathrm{ad}(x^4_{\pm\delta})^s\prod_{1\leq p\leq m-1}\mathrm{ad}(x^{\pm}_p)(x^{\pm}_m)] \qquad (4.43)\\
&[J(h_{m+1}),x_{\pm\alpha}]\\
&=\sum_{1\leq t\leq s}\mathrm{ad}(x^4_{\pm\delta})^{t-1}\,\mathrm{ad}([J(h_{m+1}),x^4_{\pm\delta}])\,\mathrm{ad}(x^4_{\pm\delta})^{s-t}\prod_{1\leq p\leq m-1}\mathrm{ad}(x^{\pm}_p)(x^{\pm}_m)\\
&\qquad+[J(h_{m+1}),\mathrm{ad}(x^4_{\pm\delta})^s\prod_{1\leq p\leq m-1}\mathrm{ad}(x^{\pm}_p)(x^{\pm}_m)] \qquad (4.44)
\end{aligned}$$

by the Jacobi identity. First, we rewrite the first term of the right hand side of (4.43) and (4.44). By the assumption m is greater than 3, $[J(h_m),x^{\pm}_1]=0$ holds by (4.9). Then, in a similar way as that of Claim 4.28, we find the equalities

$$[J(h_m),x^4_{\pm\delta}]=\pm t_\delta x^4_{\pm\delta}, \qquad (4.45)$$
$$[J(h_{m+1}),x^4_{\pm\delta}]=\pm u_\delta x^4_{\pm\delta}, \qquad (4.46)$$

where t_δ and u_δ are complex numbers. Then, we obtain

$$\begin{aligned}
&\text{the first term of the right hand side of (4.43)}\\
&\quad=\pm t_\delta\,\mathrm{ad}(x^4_{\pm\delta})^s\prod_{1\leq p\leq m-1}\mathrm{ad}(x^{\pm}_p)(x^{\pm}_m), \qquad (4.47)\\
&\text{the first term of the right hand side of (4.44)}\\
&\quad=\pm u_\delta\,\mathrm{ad}(x^4_{\pm\delta})^s\prod_{1\leq p\leq m-1}\mathrm{ad}(x^{\pm}_p)(x^{\pm}_m) \qquad (4.48)
\end{aligned}$$

by (4.45) and (4.46). Next, we rewrite the second term of the right hand side of (4.43) and (4.44). By (4.9), we obtain

$$\begin{aligned}
&\text{the second term of the right hand side of (4.43)}\\
&\quad=\mathrm{ad}(x^4_{\pm\delta})^s\prod_{1\leq p\leq m-2}\mathrm{ad}(x^{\pm}_p)[J(h_m),[x^{\pm}_{m-1},x^{\pm}_m]], \qquad (4.49)\\
&\text{the second term of the right hand side of (4.44)}\\
&\quad=\mathrm{ad}(x^4_{\pm\delta})^s\prod_{1\leq p\leq m-2}\mathrm{ad}(x^{\pm}_p)[J(h_{m+1}),[x^{\pm}_{m-1},x^{\pm}_m]].. \qquad (4.50)
\end{aligned}$$

By (4.9) and (4.10), we find that

$$\begin{aligned}
&[J(h_m),[x^{\pm}_{m-1},x^{\pm}_m]]\\
&=\pm a_{m,m-1}[J(x^{\pm}_{m-1}),x^{\pm}_m]\mp a_{m,m-1}b_{m,m-1}\frac{\varepsilon_1-\varepsilon_2}{2}[x^{\pm}_{m-1},x^{\pm}_m]\\
&=\pm a_{m,m-1}[x^{\pm}_{m-1},J(x^{\pm}_m)]\mp a_{m,m-1}(b_{m-1,m}+b_{m,m-1})\frac{\varepsilon_1-\varepsilon_2}{2}[x^{\pm}_{m-1},x^{\pm}_m], \qquad (4.51)\\
&[J(h_{m+1}),[x^{\pm}_{m-1},x^{\pm}_m]]
\end{aligned}$$

$$= \pm a_{m+1,m}[x^{\pm}_{m-1}, J(x^{\pm}_m)] \mp a_{m+1,m}b_{m,m+1}\frac{\varepsilon_1-\varepsilon_2}{2}[x^{\pm}_{m-1}, x^{\pm}_m]. \tag{4.52}$$

Since $a_{m,m-1} = (\alpha, \alpha_m)$ and $a_{m+1,m} = (\alpha, \alpha_{m+1})$, by (4.51) and (4.52), we obtain

$$(\alpha, \alpha_{m+1})[J(h_m), [x^{\pm}_{m-1}, x^{\pm}_m]] - (\alpha, \alpha_m)[J(h_{m+1}), [x^{\pm}_{m-1}, x^{\pm}_m]] = \pm u_\alpha [x^{\pm}_{m-1}, x^{\pm}_m], \tag{4.53}$$

where u_α is a complex number. Thus, we know that

$$\begin{aligned}&(\alpha, \alpha_{m+1})(\text{the second term of the right hand side of (4.43)})\\ &\qquad -(\alpha, \alpha_m)(\text{the second term of the right hand side of (4.44)})\\ &= u_\alpha \operatorname{ad}(x^4_{\pm\delta})^s \prod_{1\leq p\leq m-1} \operatorname{ad}(x^{\pm}_p)(x^{\pm}_m). \end{aligned}\tag{4.54}$$

holds. By (4.47), (4.48), and (4.54), we have

$$\begin{aligned}&(\alpha, \alpha_{m+1})[J(h_m), x_{\pm\alpha}] - (\alpha, \alpha_m)[J(h_{m+1}), x_{\pm\alpha}]\\ &= \pm(s(\alpha, \alpha_{m+1})t_\delta - s(\alpha, \alpha_m)u_\delta + u_\alpha) \operatorname{ad}(x^4_{\pm\delta})^s \prod_{1\leq p\leq m-1} \operatorname{ad}(x^{\pm}_p)(x^{\pm}_m).\end{aligned}$$

Then, we have obtained (4.37). □

Next, let us consider the case where α is a general odd root. We only show the case where $\alpha \in \alpha_m + \sum_{\substack{1\leq t\leq m+n-1,\\ t\neq m}} \mathbb{Z}_{\geq 0}\alpha_t + s\delta$. The other case is proven in a similar way.

Since $\alpha \in \alpha_m + \sum_{\substack{1\leq t\leq m+n-1,\\ t\neq m}} \mathbb{Z}_{\geq 0}\alpha_t + s\delta$, α can be written as $\prod_{1\leq t\leq p} s_{i_t}(\sum_{1\leq i\leq m} \alpha_i + \alpha_m)$. Then, we prove the statement by the induction on p. When $p = 0$, it is nothing but Claim 4.34. Other cases are proven in a similar way as that of Case 1. □

We easily obtain the following corollary.

Corollary 4.55. *The following equations hold;*

$$[J(h_i), \widetilde{v_j}] + [\widetilde{v_i}, J(h_j)] = 0, \tag{4.56}$$
$$[J(h_i), J(h_j)] + [\widetilde{v_i}, \widetilde{v_j}] = 0, \tag{4.57}$$

where $\widetilde{v_i} = v_i + \dfrac{\varepsilon_1+\varepsilon_2}{2}{h_i}^2$.

Proof. First, we show that (4.56) holds. Since $\widetilde{v_i} = \frac{\varepsilon_1+\varepsilon_2}{2}\sum_{\alpha\in\Delta^{\mathrm{re}}_+}(\alpha_j, \alpha)x^{k_\alpha}_{-\alpha}x^{k_\alpha}_\alpha$ holds, we obtain

$$\begin{aligned}&[J(h_i), \widetilde{v_j}] + [\widetilde{v_i}, J(h_j)]\\ &= \frac{\varepsilon_1+\varepsilon_2}{2}\sum_{\alpha\in\Delta^{\mathrm{re}}_+}(\alpha_j, \alpha)[J(h_i), x_{-\alpha}]x_\alpha + \frac{\varepsilon_1+\varepsilon_2}{2}\sum_{\alpha\in\Delta^{\mathrm{re}}_+}(\alpha_j, \alpha)x_{-\alpha}[J(h_i), x_\alpha]\\ &\quad + \frac{\varepsilon_1+\varepsilon_2}{2}\sum_{\alpha\in\Delta^{\mathrm{re}}_+}(\alpha_i, \alpha)[x_{-\alpha}, J(h_j)]x_\alpha + \frac{\varepsilon_1+\varepsilon_2}{2}\sum_{\alpha\in\Delta^{\mathrm{re}}_+}(\alpha_i, \alpha)x_{-\alpha}[x_\alpha, J(h_j)].\end{aligned}\tag{4.58}$$

By Proposition 4.26, there exists $c^\alpha_{i,j} \in \mathbb{C}$ such that

$$\frac{\varepsilon_1+\varepsilon_2}{2}\sum_{\alpha\in\Delta^{\mathrm{re}}_+}(\alpha_j, \alpha)[J(h_i), x_{-\alpha}]x_\alpha + \frac{\varepsilon_1+\varepsilon_2}{2}\sum_{\alpha\in\Delta^{\mathrm{re}}_+}(\alpha_i, \alpha)[x_{-\alpha}, J(h_j)]x_\alpha$$

$$= -\frac{\varepsilon_1+\varepsilon_2}{2}\sum_{\alpha\in\Delta_+^{\mathrm{re}}} c_{i,j}^{\alpha} x_{-\alpha}x_{\alpha} \tag{4.59}$$

and

$$\begin{aligned}&\frac{\varepsilon_1+\varepsilon_2}{2}\sum_{\alpha\in\Delta_+^{\mathrm{re}}}(\alpha_j,\alpha)x_{-\alpha}[J(h_i),x_\alpha]+\frac{\varepsilon_1+\varepsilon_2}{2}\sum_{\alpha\in\Delta_+^{\mathrm{re}}}(\alpha_i,\alpha)x_{-\alpha}[x_\alpha,J(h_j)]\\&=\frac{\varepsilon_1+\varepsilon_2}{2}\sum_{\alpha\in\Delta_+^{\mathrm{re}}} c_{i,j}^{\alpha} x_{-\alpha}x_{\alpha}.\end{aligned} \tag{4.60}$$

Therefore, applying (4.59) and (4.60) to (4.58), we have obtained the relation (4.56). By the defining relation (3.3), we find the equality

$$[J(h_i)-\widetilde{v_i}, J(h_j)-\widetilde{v_j}] = [h_{i,1},h_{j,1}]=0. \tag{4.61}$$

On the other hand, we find the relation

$$[J(h_i)-\widetilde{v_i}, J(h_j)-\widetilde{v_j}] = [J(h_i),J(h_j)]-[\widetilde{v_i},J(h_j)]-[J(h_i),\widetilde{v_j}]+[\widetilde{v_i},\widetilde{v_j}].$$

By (4.56), the right hand side of (4) is equal to the left hand side of (4.57). Thus, by (4.61), we have found that (4.57) holds. □

Now, we are in position to obtain the proof of Theorem 4.3. To simplify the notation, we set $\square(x)$ as $x\otimes 1+1\otimes x$ for all $x\in Y_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m|n))$.

Proof of Theorem 4.3. It is enough to check that Δ is compatible with (3.17)-(3.25), which are the defining relations of the minimalistic presentation of the affine super Yangian. Since the restriction of Δ to $\widehat{\mathfrak{sl}}(m|n)$ is nothing but the usual coproduct of $\widehat{\mathfrak{sl}}(m|n)$, Δ is compatible with (3.18), (3.23), (3.24), and (3.25). We also know that Δ is compatible with (3.20) since $\Delta(x_{i,1}^{\pm})$ is defined as

$$\begin{cases}\pm\dfrac{1}{a_{i,i}}[\Delta(\widetilde{h}_{i,1}),\Delta(x_{i,0}^{\pm})] & \text{if}\quad i\neq m,0,\\ \pm\dfrac{1}{a_{i+1,i}}[\Delta(\widetilde{h}_{i+1,1}),\Delta(x_{i,0}^{\pm})]+b_{i+1,i}\dfrac{\varepsilon_1-\varepsilon_2}{2}\Delta(x_{i,0}^{\pm}) & \text{if}\quad i=m,0,\end{cases}$$

and $\Delta(\widetilde{h}_{i+1,1})$ and $\Delta(\widetilde{h}_{i,1})$ commute with $\Delta(h_{j,0})$ by the definition. We find that the defining relation (3.19) (resp. (3.21), (3.22)) is equivalent to (4.11) (resp. (4.9), (4.10)) by the proof of Lemma 4.7. It is easy to show that Δ is compatible with (4.11), (4.9), and (4.10) in the same way as that of Theorem 4.9 of [21]. Thus, it is enough to show that Δ is compatible with (3.17). By the definition of $J(h_i)$, we obtain

$$\begin{aligned}&[\Delta(h_{i1}),\Delta(h_{j1})]\\&=[\Delta(J(h_i))-\Delta(\widetilde{v_i}),\Delta(J(h_j))-\Delta(\widetilde{v_j})]\\&=[\Delta(J(h_i)),\Delta(J(h_j))]+[\Delta(\widetilde{v_i}),\Delta(\widetilde{v_j})]-[\Delta(J(h_i)),\Delta(\widetilde{v_j})]-[\Delta(\widetilde{v_i}),\Delta(J(h_j))],\end{aligned} \tag{4.62}$$

where $\widetilde{v_i}=v_i+\dfrac{\varepsilon_1+\varepsilon_2}{2}h_i^2$. It is enough to show that

$$[\Delta(J(h_i)),\Delta(J(h_j))]+[\Delta(\widetilde{v_i}),\Delta(\widetilde{v_j})]=0 \tag{4.63}$$

and

$$[\Delta(J(h_i)),\Delta(\widetilde{v_j})]+[\Delta(\widetilde{v_i}),\Delta(J(h_j))]=0 \tag{4.64}$$

hold. We only show that (4.63) holds. The outline of the proof of (4.64) is the same as that of Theorem 4.9 of [21]. In order to simplify the computation, we define

$$\Omega_+=\sum_{1\leq k\leq \dim\mathfrak{h}} u^k\otimes u_k+\sum_{\alpha\in\Delta_+}\sum_{1\leq k_\alpha\leq\dim\mathfrak{g}_\alpha}(-1)^{p(\alpha)}x_\alpha^{k_\alpha}\otimes x_{-\alpha}^{k_\alpha},$$

$$\Omega_- = \sum_{\alpha\in\Delta_+}\sum_{1\leq k_\alpha\leq \dim\mathfrak{g}_\alpha} x^{k_\alpha}_{-\alpha}\otimes x^{k_\alpha}_{\alpha},$$
$$\Omega = \sum_{1\leq k\leq \dim\mathfrak{h}} u^k\otimes u_k + \sum_{\alpha\in\Delta_+}\sum_{1\leq k_\alpha\leq \dim\mathfrak{g}_\alpha}((-1)^{p(\alpha)}x^{k_\alpha}_{\alpha}\otimes x^{k_\alpha}_{-\alpha} + x^{k_\alpha}_{-\alpha}\otimes x^{k_\alpha}_{\alpha}),$$

where $\{u^k\}$ and $\{u_k\}$ are basis of $\mathfrak{h}$ such that $\kappa(u_k,u^l)=\delta_{k,l}$. By the definition of $J(h_i)$, it is easy to obtain

$$\Delta(J(h_i)) = \square(J(h_i)) + \frac{\varepsilon_1+\varepsilon_2}{2}[h_{i,0}\otimes 1,\Omega] \tag{4.65}$$

since we have

$$\begin{aligned}\Delta(xy) &= (x\otimes 1+1\otimes x)(y\otimes 1+1\otimes y)\\ &= (-1)^{p(1)p(y)}xy\otimes 1 + (-1)^{p(x)p(1)}1\otimes xy + (-1)^{p(1)p(1)}x\otimes y + (-1)^{p(x)p(y)}y\otimes x\end{aligned}$$

by the relation $(x\otimes y)(z\otimes w) = (-1)^{p(y)p(z)}xz\otimes yw$ for all homogeneous elements x,y,z,w. Thus, by (4.65), we obtain

$$\begin{aligned}&[\Delta(J(h_i)),\Delta(J(h_j))]\\ &= \square([J(h_i),J(h_j)]) + \frac{\varepsilon_1+\varepsilon_2}{2}[\square(J(h_i)),[h_{j,0}\otimes 1,\Omega]]\\ &\quad - \frac{\varepsilon_1+\varepsilon_2}{2}[\square(J(h_j)),[h_{i,0}\otimes 1,\Omega]] + \frac{(\varepsilon_1+\varepsilon_2)^2}{4}[[h_{i,0}\otimes 1,\Omega],[h_{j,0}\otimes 1,\Omega]].\end{aligned}$$

First, we prove that

$$\frac{\varepsilon_1+\varepsilon_2}{2}[\square(J(h_i)),[h_{j,0}\otimes 1,\Omega]] - \frac{\varepsilon_1+\varepsilon_2}{2}[\square(J(h_j)),[h_{i,0}\otimes 1,\Omega]] = 0 \tag{4.66}$$

holds. Since $[h_{j,0}\otimes 1,\Omega] = \sum_{\alpha\in\Delta^{\mathrm{re}}_+}(\alpha,\alpha_i)(x_{-\alpha}\otimes x_\alpha - x_\alpha\otimes x_{-\alpha})$ holds, we have

$$\begin{aligned}&[\square(J(h_i)),[h_{j,0}\otimes 1,\Omega]] - [\square(J(h_j)),[h_{i,0}\otimes 1,\Omega]]\\ &= \sum_{\alpha\in\Delta^{\mathrm{re}}_+}(\alpha,\alpha_j)((-1)^{p(\alpha)}[J(h_i),x_\alpha]\otimes x_{-\alpha} - (-1)^{p(\alpha)}x_\alpha\otimes[J(h_i),x_{-\alpha}])\\ &\qquad\qquad + [J(h_i),x_{-\alpha}]\otimes x_\alpha - x_{-\alpha}\otimes[J(h_i),x_\alpha])\\ &\quad - \sum_{\alpha\in\Delta^{\mathrm{re}}_+}(\alpha,\alpha_i)((-1)^{p(\alpha)}[J(h_j),x_\alpha]\otimes x_{-\alpha} - (-1)^{p(\alpha)}x_\alpha\otimes[J(h_j),x_{-\alpha}])\\ &\qquad\qquad + [J(h_j),x_{-\alpha}]\otimes x_\alpha - x_{-\alpha}\otimes[J(h_j),x_\alpha])\\ &= \sum_{\alpha\in\Delta^{\mathrm{re}}_+}(\alpha,\alpha_j)(\alpha,\alpha_i)c^\alpha_{i,j}((-1)^{p(\alpha)}x_\alpha\otimes x_{-\alpha} - (-1)^{p(\alpha)}x_\alpha\otimes x_{-\alpha} + x_{-\alpha}\otimes x_\alpha - x_{-\alpha}\otimes x_\alpha)\\ &\quad - \sum_{\alpha\in\Delta^{\mathrm{re}}_+}(\alpha,\alpha_i)(\alpha,\alpha_j)c^\alpha_{j,i}((-1)^{p(\alpha)}x_\alpha\otimes x_{-\alpha} - (-1)^{p(\alpha)}x_\alpha\otimes x_{-\alpha} + x_{-\alpha}\otimes x_\alpha - x_{-\alpha}\otimes x_\alpha)\\ &= 0.\end{aligned}$$

where the third equality is due to Proposition 4.26. Therefore (4.66) holds. Since $\Delta(\widetilde{v}_i) = \square(\widetilde{v}_i) - \frac{\varepsilon_1+\varepsilon_2}{2}[h_{i,0}\otimes 1,\Omega_+-\Omega_-]$ holds, we obtain

$$\begin{aligned}&[\Delta(\widetilde{v}_i),\Delta(\widetilde{v}_j)]\\ &= \square([\widetilde{v}_i,\widetilde{v}_j]) + \frac{\varepsilon_1+\varepsilon_2}{2}(-[\square(\widetilde{v}_i),[h_{j,0}\otimes 1,\Omega_+-\Omega_-]] + [\square(\widetilde{v}_j),[h_{i,0}\otimes 1,\Omega_+-\Omega_-]])\\ &\quad + \frac{(\varepsilon_1+\varepsilon_2)^2}{4}[[h_{i,0}\otimes 1,\Omega_+-\Omega_-],[h_{j,0}\otimes 1,\Omega_+-\Omega_-]].\end{aligned}$$

Using this along with $\Omega=\Omega_+ +\Omega_-$ and (4.66), we find the equality

$$\begin{aligned}
&[\Delta(J(h_i)),\Delta(J(h_j))]+[\Delta(\widetilde{v}_i),\Delta(\widetilde{v}_j)]\\
&=\square([J(h_i),J(h_j)]+[\widetilde{v}_i,\widetilde{v}_j])\\
&\quad+\frac{\varepsilon_1+\varepsilon_2}{2}(-[\square(\widetilde{v}_i),[h_{j,0}\otimes 1,\Omega_+-\Omega_-]]+[\square(\widetilde{v}_j),[h_{i,0}\otimes 1,\Omega_+-\Omega_-]])\\
&\quad+\frac{(\varepsilon_1+\varepsilon_2)^2}{2}([[h_{i,0}\otimes 1,\Omega_+],[h_{j,0}\otimes 1,\Omega_+]]+[[h_{i,0}\otimes 1,\Omega_-],[h_{j,0}\otimes 1,\Omega_-]]).
\end{aligned}\tag{4.67}$$

By the same way as the one of Theorem 4.9 in [21], we can check that the sum of the last four terms of the right hand side of (4.67) vanishes. By Corollary 4.55, $\square([J(h_i),J(h_j)]+[\widetilde{v}_i,\widetilde{v}_j])=0$ holds. The coassocitivity is proven in a similar way to the one of [21]. We complete the proof. □

By setting the degree on $Y_{\varepsilon_1,\varepsilon_2}(\widetilde{\mathfrak{sl}}(m|n))$ determined by (4.2) and $\deg(d)=0$, we can define the $\widehat{Y}_{\varepsilon_1,\varepsilon_2}(\widetilde{\mathfrak{sl}}(m|n))$ (resp. $Y_{\varepsilon_1,\varepsilon_2}(\widetilde{\mathfrak{sl}}(m|n))\widehat{\otimes}Y_{\varepsilon_1,\varepsilon_2}(\widetilde{\mathfrak{sl}}(m|n))$) as the degreewise completion of $Y_{\varepsilon_1,\varepsilon_2}(\widetilde{\mathfrak{sl}}(m|n))$ (resp. $Y_{\varepsilon_1,\varepsilon_2}(\widetilde{\mathfrak{sl}}(m|n))^{\otimes 2}$) in the sense of [33]. We regard a represenation of $Y_{\varepsilon_1,\varepsilon_2}(\widetilde{\mathfrak{sl}}(m|n))$ as that of $\widetilde{\mathfrak{sl}}(m|n)$ via Φ. By Theorem 4.3, we easily obtain the following corollary.

Corollary 4.68. *The linear map $\Delta\colon Y_{\varepsilon_1,\varepsilon_2}(\widetilde{\mathfrak{sl}}(m|n))\to Y_{\varepsilon_1,\varepsilon_2}(\widetilde{\mathfrak{sl}}(m|n))\widehat{\otimes}Y_{\varepsilon_1,\varepsilon_2}(\widetilde{\mathfrak{sl}}(m|n))$ uniquely determined by*

$$\Delta(h_{i,0})=h_{i,0}\otimes 1+1\otimes h_{i,0},\quad \Delta(x^\pm_{i,0})=x^\pm_{i,0}\otimes 1+1\otimes x^\pm_{i,0}\quad \Delta(d)=d\otimes 1+1\otimes d,$$
$$\Delta(h_{i,1})=h_{i,1}\otimes 1+1\otimes h_{i,1}+(\varepsilon_1+\varepsilon_2)h_{i,0}\otimes h_{i,0}-(\varepsilon_1+\varepsilon_2)\sum_{\alpha\in\Delta_+}\sum_{1\leq k_\alpha\leq dim\mathfrak{g}_\alpha}(\alpha,\alpha_i)x^{k_\alpha}_{-\alpha}\otimes x^{k_\alpha}_{\alpha}$$

is an algebra homomorphism. Moreover, Δ satisfies the coassociativity.

In particular, Δ defines an action on $Y_{\varepsilon_1,\varepsilon_2}(\widetilde{\mathfrak{sl}}(m|n))$ on $V\otimes W$ for any $Y_{\varepsilon_1,\varepsilon_2}(\widetilde{\mathfrak{sl}}(m|n))$-modules V,W which are in the category $\mathcal{O}$ as $\widetilde{\mathfrak{sl}}(m|n)$-modules.

5 Evaluation map for the Affine Super Yangian

Since the definition of the affine super Yangian is very complicated, it is not clear whether the affine super Yangian is trivial or not. In this section, we construct the non-trivial homomorphism from the affine super Yangian to the completion of $U(\widehat{\mathfrak{gl}}(m|n))$. In this section, we define a Lie superalgebra $\widehat{\mathfrak{gl}}(m|n)^{\mathrm{str}}=\mathfrak{gl}(m|n)\otimes\mathbb{C}[t,t^{-1}]\oplus\mathbb{C}\tilde{c}\oplus\mathbb{C}z$ whose commutator relations are given by

$$[x\otimes t^u,y\otimes t^v]=\begin{cases}[x,y]\otimes t^{u+v}+\delta_{u+v,0}u\mathrm{str}(xy)\tilde{c}\ \text{ if } x,y\in\mathfrak{sl}(m|n),\\ [e_{a,b},e_{i,i}]\otimes t^{u+v}+\delta_{u+v,0}u\mathrm{str}(E_{a,b}E_{i,i})\tilde{c}+\delta_{u+v,0}\delta_{a,b}u(-1)^{p(a)+p(i)}z\\ \qquad\qquad\qquad\qquad\text{if } x=e_{a,b},\ y=e_{i,i},\end{cases}$$

z and $\tilde{c}$ are central elements of $\widehat{\mathfrak{gl}}(m|n)$.

For all $s\in\mathbb{Z}$, we denote $E_{i,j}\otimes t^s$ by $E_{i,j}(s)$. We also set the grading of $U(\widehat{\mathfrak{gl}}(m|n))/U(\widehat{\mathfrak{gl}}(m|n))(z-1)$ as $\deg(X(s))=s$ and $\deg(c)=0$. We introduce a completion of $U(\widehat{\mathfrak{gl}}(m|n)^{\mathrm{str}})/U(\widehat{\mathfrak{gl}}(m|n)^{\mathrm{str}})(z-1)$ following [33] and [21]. For all $s\in\mathbb{Z}$, we denote $E_{i,j}\otimes t^s$ by $E_{i,j}(s)$. We also set the grading of $U(\widehat{\mathfrak{gl}}(m|n)^{\mathrm{str}})/U(\widehat{\mathfrak{gl}}(m|n)^{\mathrm{str}})(z-1)$ as $\deg(X(s))=s$ and $\deg(c)=0$. Then, we find that $U(\widehat{\mathfrak{gl}}(m|n)^{\mathrm{str}})/U(\widehat{\mathfrak{gl}}(m|n)^{\mathrm{str}})(z-1)$ becomes a graded algebra and we denote the set of the degree d elements of $U(\widehat{\mathfrak{gl}}(m|n)^{\mathrm{str}})/U(\widehat{\mathfrak{gl}}(m|n)^{\mathrm{str}})(z-1)$ by $U(\widehat{\mathfrak{gl}}(m|n)^{\mathrm{str}})_d$. We obtain the completion

$$U(\widehat{\mathfrak{gl}}(m|n)^{\mathrm{str}})_{\mathrm{comp}}=\bigoplus_{d\in\mathbb{Z}}U(\widehat{\mathfrak{gl}}(m|n)^{\mathrm{str}})_{\mathrm{comp},d},$$

where

$$U(\widehat{\mathfrak{gl}}(m|n)^{\mathrm{str}})_d=\varprojlim_{N}U(\widehat{\mathfrak{gl}}(m|n)^{\mathrm{str}})_d/\sum_{r>N}U(\widehat{\mathfrak{gl}}(m|n)^{\mathrm{str}})_{d-r}U(\widehat{\mathfrak{gl}}(m|n)^{\mathrm{str}})_r.$$

Let us state the main result of this section. In order to simplify the notation, we denote $\varepsilon_1+\varepsilon_2$ as $\hbar$.

Theorem 5.1. *Assume $c\hbar=(-m+n)\varepsilon_1$ and $z=1$. Let υ be a complex number. Then, there exists an algebra homomorphism $\mathrm{ev}_\upsilon\colon Y_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m|n))\to U(\widehat{\mathfrak{gl}}(m|n))_{\mathrm{comp},+}$ uniquely determined by*

$$\mathrm{ev}_\upsilon(x_{i,0}^+)=x_i^+,\quad \mathrm{ev}_\upsilon(x_{i,0}^-)=x_i^-,\quad \mathrm{ev}_\upsilon(h_{i,0})=h_i, \tag{5.2}$$

$$\mathrm{ev}_\upsilon(x_{i,1}^+)=\begin{cases}(\upsilon-(m-n)\varepsilon_1)x_0^+ +\hbar\displaystyle\sum_{s\geq0}\sum_{k=1}^{m+n}(-1)^{p(k)}E_{m+n,k}(-s)E_{k,1}(s+1) & \text{if } i=0,\\[2ex] (\upsilon-(i-2\delta(i\geq m+1)(i-m))\varepsilon_1)x_i^+\\ \quad+\hbar\displaystyle\sum_{s\geq0}\sum_{k=1}^{i}(-1)^{p(k)}E_{i,k}(-s)E_{k,i+1}(s)\\ \quad\quad+\hbar\displaystyle\sum_{s\geq0}\sum_{k=i+1}^{m+n}(-1)^{p(k)}E_{i,k}(-s-1)E_{k,i+1}(s+1)\\ & \text{if } i\neq0,\end{cases} \tag{5.3}$$

$$\mathrm{ev}_\upsilon(x_{i,1}^-)=\begin{cases}(\upsilon-(m-n)\varepsilon_1)x_0^- -\hbar\displaystyle\sum_{s\geq0}\sum_{k=1}^{m+n}(-1)^{p(k)}E_{1,k}(-s-1)E_{k,m+n}(s) & \text{if } i=0,\\[2ex] (\upsilon-(i-2\delta(i\geq m+1)(i-m))\varepsilon_1)x_i^-\\ \quad+(-1)^{p(i)}\hbar\displaystyle\sum_{s\geq0}\sum_{k=1}^{i}(-1)^{p(k)}E_{i+1,k}(-s)E_{k,i}(s)\\ \quad\quad+(-1)^{p(i)}\hbar\displaystyle\sum_{s\geq0}\sum_{k=i+1}^{m+n}(-1)^{p(k)}E_{i+1,k}(-s-1)E_{k,i}(s+1)\\ & \text{if } i\neq0,\end{cases} \tag{5.4}$$

$$
\mathrm{ev}_\upsilon(h_{i,1}) = \begin{cases}
(\upsilon-(m-n)\varepsilon_1)h_0+\hbar E_{m+n,m+n}(E_{1,1}-c)\\
\quad -\hbar\displaystyle\sum_{s\geq 0}\sum_{k=1}^{m+n}(-1)^{p(k)}E_{m+n,k}(-s)E_{k,m+n}(s)\\
\quad\quad -\hbar\displaystyle\sum_{s\geq 0}\sum_{k=1}^{m+n}(-1)^{p(k)}E_{1,k}(-s-1)E_{k,1}(s+1)\\
\hfill \text{if } i=0,\\
(\upsilon-(i-2\delta(i\geq m+1)(i-m))\varepsilon_1)h_i-(-1)^{p(E_{i,i+1})}\hbar E_{i,i}E_{i+1,i+1}\\
\quad +\hbar(-1)^{p(i)}\displaystyle\sum_{s\geq 0}\sum_{k=1}^{i}(-1)^{p(k)}E_{i,k}(-s)E_{k,i}(s)\\
\quad\quad +\hbar(-1)^{p(i)}\displaystyle\sum_{s\geq 0}\sum_{k=i+1}^{m+n}(-1)^{p(k)}E_{i,k}(-s-1)E_{k,i}(s+1)\\
\quad\quad\quad -\hbar(-1)^{p(i+1)}\displaystyle\sum_{s\geq 0}\sum_{k=1}^{i}(-1)^{p(k)}E_{i+1,k}(-s)E_{k,i+1}(s)\\
\quad\quad\quad\quad -\hbar(-1)^{p(i+1)}\displaystyle\sum_{s\geq 0}\sum_{k=i+1}^{m+n}(-1)^{p(k)}E_{i+1,k}(-s-1)E_{k,i+1}(s+1)\\
\hfill \text{if } i\neq 0.
\end{cases} \tag{5.5}
$$

The outline of the proof is the same as that of [30]. It is enough to check that ev_υ is compatible with (3.17)-(3.25), which are the defining relations of the minimalistic presentation of the affine super Yangian. When we restrict ev_υ to $\widehat{\mathfrak{sl}}(m|n)$, ev_υ is an identity map on $\widehat{\mathfrak{sl}}(m|n)$. Thus, ev_υ is compatible with (3.18), (3.20), (3.23)-(3.25).

We set a anti-automorphism $\omega\colon U(\widehat{\mathfrak{gl}}(m|n))\to U(\widehat{\mathfrak{gl}}(m|n))$ as

$$
\omega(X\otimes t^r)=(-1)^rX^T\otimes t^r,\qquad \omega(c)=c,
$$

where X^T is a transpose of a matrix X. Then, the compatibility of ev_υ with (3.21) and (3.22) for $-$ are deduced from those for $+$ by applying the anti-automorphism ω since we have $\omega(\mathrm{ev}_\upsilon(h_{i,1}))=\mathrm{ev}_\upsilon(h_{i,1})$ and $\omega(\mathrm{ev}_\upsilon(x^+_{i,1}))=(-1)^{p(i)}\mathrm{ev}_\upsilon(x^-_{i,1})$. Therefore, it is enough to check the following lemma.

Lemma 5.6. *The following equations hold;*

$$
[\mathrm{ev}_\upsilon(x^+_{i,1}),\mathrm{ev}_\upsilon(x^-_{j,0})]=\delta_{i,j}\,\mathrm{ev}_\upsilon(h_{i,1}), \tag{5.7}
$$

$$
[\mathrm{ev}_\upsilon(\widetilde{h}_{i,1}),x^+_j]=a_{i,j}(\mathrm{ev}_\upsilon(x^+_{j,1})-b_{i,j}\frac{\varepsilon_1-\varepsilon_2}{2}x^+_j), \tag{5.8}
$$

$$
[\mathrm{ev}_\upsilon(x^+_{i,1}),x^+_j]-[x^+_i,\mathrm{ev}_\upsilon(x^+_{j,1})]=a_{i,j}\frac{\varepsilon_1+\varepsilon_2}{2}\{x^+_i,x^+_j\}-b_{i,j}\frac{\varepsilon_1-\varepsilon_2}{2}[x^+_i,x^+_j], \tag{5.9}
$$

$$
[\mathrm{ev}_\upsilon(h_{i,1}),\mathrm{ev}_\upsilon(h_{j,1})]=0. \tag{5.10}
$$

The rest of the paper is devoted to the proof of Lemma 5.6.

5.1 The proof of (5.7)

We prepare one claim before starting the proof.

Claim 5.11. The following relations hold;

$$
[\sum_{s\geq p}\sum_{k=1}^{a}(-1)^{p(k)}E_{i,k}(-s)E_{k,j}(s),E_{x,y}] \tag{5.12}
$$

$$= \delta_{j,x}\sum_{s\geq p}\sum_{k=1}^{a}(-1)^{p(k)}E_{i,k}(-s)E_{k,y}(s) - (-1)^{p(E_{i,j})p(E_{x,y})}\sum_{s\geq p}\sum_{k=1}^{a}(-1)^{p(k)}E_{x,k}(-s)E_{k,j}(s)$$

$$+ \{\delta(x \leq a < y) - \delta(x > a \geq y)\}\sum_{s\geq p}(-1)^{p(x)+p(E_{x,j})p(E_{x,y})}E_{i,y}(-s)E_{x,j}(s), \tag{5.13}$$

$$[\sum_{s\geq p}\sum_{k=a}^{m+n}(-1)^{p(k)}E_{i,k}(-s)E_{k,j}(s), E_{x,y}]$$

$$= \delta_{j,x}\sum_{s\geq p}\sum_{k=a}^{m+n}(-1)^{p(k)}E_{i,k}(-s)E_{k,y}(s) - (-1)^{p(E_{i,j})p(E_{x,y})}\sum_{s\geq p}\sum_{k=a}^{m+n}(-1)^{p(k)}E_{x,k}(-s)E_{k,j}(s)$$

$$+ \{\delta(x \geq a > y) - \delta(x < a \leq y)\}\sum_{s\geq p}(-1)^{p(x)+p(E_{x,j})p(E_{x,y})}E_{i,y}(-s)E_{x,j}(s). \tag{5.14}$$

Proof. We prove only (5.13) since (5.14) is proven in a similar way. By direct computation, the first term of (5.13) is equal to

$$\begin{aligned}
&\delta_{j,x}\sum_{s\geq p}\sum_{k=1}^{a}(-1)^{p(k)}E_{i,k}(-s)E_{k,y}(s)\\
&\quad - \delta(y \leq a)\sum_{s\geq p}(-1)^{p(y)+p(E_{y,j})p(E_{x,y})}E_{i,y}(-s)E_{x,j}(s)\\
&\quad + \delta(x \leq a)\sum_{s\geq p}(-1)^{p(x)+p(E_{x,j})p(E_{x,y})}E_{i,y}(-s)E_{x,j}(s)\\
&\quad - (-1)^{p(E_{i,j})p(E_{x,y})}\sum_{s\geq p}\sum_{k=1}^{a}(-1)^{p(k)}E_{x,k}(-s)E_{k,j}(s).
\end{aligned} \tag{5.15}$$

Since $p(y) + p(E_{y,j})p(E_{x,y}) = p(x) + p(E_{x,j})p(E_{x,y})$, the sum of the second and third terms of (5.15) is equal to

$$\{\delta(x \leq a < y) - \delta(x > a \geq y)\}\sum_{s\geq p}(-1)^{p(x)+p(E_{x,j})p(E_{x,y})}E_{i,y}(-s)E_{x,j}(s).$$

Then, we obtain (5.14). □

Suppose that $i, j \neq 0$. Other cases are proven in a similar way. By the definition of $\mathrm{ev}_\upsilon(x^+_{i,1})$, we obtain

$$\begin{aligned}
&[\mathrm{ev}_\upsilon(x^+_{i,1}), \mathrm{ev}_\upsilon(x^-_{j,0})]\\
&= [(\upsilon - (i - 2\delta(i \geq m+1)(i-m))\varepsilon_1)x^+_i, (-1)^{p(j)}E_{j+1,j}]\\
&\quad + [\hbar\sum_{s\geq 0}\sum_{k=1}^{i}(-1)^{p(k)}E_{i,k}(-s)E_{k,i+1}(s), (-1)^{p(j)}E_{j+1,j}]\\
&\quad + [\hbar\sum_{s\geq 0}\sum_{k=i+1}^{m+n}(-1)^{p(k)}E_{i,k}(-s-1)E_{k,i+1}(s+1), (-1)^{p(j)}E_{j+1,j}].
\end{aligned} \tag{5.16}$$

By (5.13), $[\hbar\sum_{s\geq 0}\sum_{k=1}^{i}(-1)^{p(k)}E_{i,k}(-s)E_{k,i+1}(s), (-1)^{p(j)}E_{j+1,j}]$, the second term of the right hand side of (5.16), is equal to

$$[\hbar\sum_{s\geq 0}\sum_{k=1}^{i}(-1)^{p(k)}E_{i,k}(-s)E_{k,i+1}(s), (-1)^{p(j)}E_{j+1,j}]$$

$$
\begin{aligned}
&= \delta_{i,j}\hbar\sum_{s\geq 0}\sum_{k=1}^{i}(-1)^{p(i)+p(k)}E_{i,k}(-s)E_{k,i}(s)\\
&\quad - \delta_{i,j}\hbar\sum_{s\geq 0}\sum_{k=1}^{i}(-1)^{p(k)+p(i)+p(E_{i,i+1})}E_{i+1,k}(-s)E_{k,i+1}(s)\\
&\quad - \delta_{i,j}\hbar\sum_{s\geq 0}(-1)^{p(E_{i,i+1})p(E_{i,i+1})}E_{i,i}(-s)E_{i+1,i+1}(s).
\end{aligned}\tag{5.17}
$$

Similarly, by (5.14), $[\hbar\sum_{s\geq 0}\sum_{k=i+1}^{m+n}(-1)^{p(k)}E_{i,k}(-s-1)E_{k,i+1}(s+1),(-1)^{p(j)}E_{j+1,j}]$, the third term of the right hand side of (5.16), is equal to

$$
\begin{aligned}
&[\hbar\sum_{s\geq 0}\sum_{k=i+1}^{m+n}(-1)^{p(k)}E_{i,k}(-s-1)E_{k,i+1}(s+1),(-1)^{p(j)}E_{j+1,j}]\\
&= \hbar\sum_{s\geq 0}\sum_{k=i+1}^{m+n}\delta_{i,j}(-1)^{p(k)+p(i)}E_{i,k}(-s-1)E_{k,i}(s+1)\\
&\quad - \hbar\sum_{s\geq 0}\sum_{k=i+1}^{m+n}\delta_{i,j}(-1)^{p(k)+p(i)+p(E_{i,i+1})}E_{i+1,k}(-s-1)E_{k,i+1}(s+1)\\
&\quad + \hbar\sum_{s\geq 0}\delta_{i,j}(-1)^{p(i+1)+p(i)}E_{i,i}(-s-1)E_{i+1,i+1}(s+1).
\end{aligned}\tag{5.18}
$$

We can rewrite the sum of the last term of (5.17) and the last term of (5.18). Since $p(E_{i,i+1}) = p(i)+p(i+1)$ holds, we obtain

$$
\begin{aligned}
&- \hbar\sum_{s\geq 0}(-1)^{p(E_{i,i+1})p(E_{i,i+1})}E_{i,i}(-s)E_{i+1,i+1}(s)\\
&+ \hbar\sum_{s\geq 0}(-1)^{p(i+1)+p(i)+p(E_{i+1,i+1})p(E_{i,i+1})}E_{i,i}(-s-1)E_{i+1,i+1}(s+1)\\
&= -\hbar\sum_{s\geq 0}(-1)^{p(E_{i,i+1})}E_{i,i}(-s)E_{i+1,i+1}(s) + \hbar\sum_{s\geq 0}(-1)^{p(E_{i,i+1})}E_{i,i}(-s-1)E_{i+1,i+1}(s+1)\\
&= -\hbar(-1)^{p(E_{i,i+1})}E_{i,i}E_{i+1,i+1}.
\end{aligned}\tag{5.19}
$$

Thus, we have shown that $[\mathrm{ev}_\upsilon(x_{i,1}^+),\mathrm{ev}_\upsilon(x_{j,0}^-)] = \delta_{i,j}\,\mathrm{ev}_\upsilon(h_{i,1})$ holds by (5.17), (5.18) and (5.19).

5.2 The proof of (5.8)

We only show the case where $i,j\neq 0$ and when $i=0$ and $j\neq 0$. The other case is proven in a similar way.

Case 1, $i,j\neq 0$.

First, we show the case where $i,j\neq 0$. By the definition of $\mathrm{ev}_\upsilon(h_{i,1})$, we obtain

$$
\begin{aligned}
&[\mathrm{ev}_\upsilon(\widetilde{h}_{i,1}),\mathrm{ev}_\upsilon(x_{j,0}^+)]\\
&= [(\upsilon-(i-2\delta(i\geq m+1)(i-m))\varepsilon_1)h_i - \frac{1}{2}\hbar((E_{i,i})^2+(E_{i+1,i+1})^2),E_{j,j+1}]\\
&\quad + [\hbar(-1)^{p(i)}\sum_{s\geq 0}\sum_{k=1}^{i}(-1)^{p(k)}E_{i,k}(-s)E_{k,i}(s),E_{j,j+1}]\\
&\quad + [\hbar(-1)^{p(i)}\sum_{s\geq 0}\sum_{k=i+1}^{m+n}(-1)^{p(k)}E_{i,k}(-s-1)E_{k,i}(s+1),E_{j,j+1}]
\end{aligned}
$$

$$
\begin{aligned}
&- [\hbar(-1)^{p(i+1)}\sum_{s\geq 0}\sum_{k=1}^{i}(-1)^{p(k)}E_{i+1,k}(-s)E_{k,i+1}(s), E_{j,j+1}]\\
&- [\hbar(-1)^{p(i+1)}\sum_{s\geq 0}\sum_{k=i+1}^{m+n}(-1)^{p(k)}E_{i+1,k}(-s-1)E_{k,i+1}(s+1), E_{j,j+1}]. \qquad (5.20)
\end{aligned}
$$

Let us compute these terms respectively. By direct computation, the first term of the right hand side of (5.20) is equal to

$$
\begin{aligned}
&[(\upsilon-(i-2\delta(i\geq m+1)(i-m))\varepsilon_1)h_i-\frac{1}{2}\hbar((E_{i,i})^2+(E_{i+1,i+1})^2), E_{j,j+1}]\\
&=(\upsilon-(i-2\delta(i\geq m+1)(i-m))\varepsilon_1)a_{i,j}x_j^+\\
&\quad-\frac{\hbar}{2}(\delta_{i,j}(\{E_{i,i+1},E_{i,i}\}-\{E_{i,i+1},E_{i+1,i+1}\})-\delta_{i,j+1}\{E_{i-1,i},E_{i,i}\}+\delta_{i+1,j}\{E_{i+1,i+2},E_{i+1,i+1}\}).
\end{aligned}
\tag{5.21}
$$

By (5.13) and (5.14), we also find that the sum of the second and third terms of the right hand side of (5.20) is equal to

$$
\begin{aligned}
&[\hbar(-1)^{p(i)}\sum_{s\geq 0}\sum_{k=1}^{i}(-1)^{p(k)}E_{i,k}(-s)E_{k,i}(s), E_{j,j+1}]\\
&\qquad\qquad+[\hbar(-1)^{p(i)}\sum_{k=i+1}^{m+n}(-1)^{p(k)}E_{i,k}(-s-1)E_{k,i}(s+1), E_{j,j+1}]\\
&=\hbar(-1)^{p(i)}\sum_{s\geq 0}\sum_{k=1}^{i}\delta_{i,j}(-1)^{p(k)}E_{i,k}(-s)E_{k,i+1}(s)\\
&\quad-\hbar(-1)^{p(i)}\sum_{s\geq 0}\sum_{k=1}^{i}\delta_{i,j+1}(-1)^{p(k)}E_{j,k}(-s)E_{k,i}(s)\\
&\quad+\hbar(-1)^{p(i)}\sum_{s\geq 0}\delta_{i,j}(-1)^{p(i)}E_{i,i+1}(-s)E_{i,i}(s)\\
&\quad+\hbar(-1)^{p(i)}\sum_{s\geq 0}\sum_{k=i+1}^{m+n}\delta_{i,j}(-1)^{p(k)}E_{i,k}(-s-1)E_{k,j+1}(s+1)\\
&\quad-\hbar(-1)^{p(i)}\sum_{s\geq 0}\sum_{k=i+1}^{m+n}\delta_{i,j+1}(-1)^{p(k)}E_{j,k}(-s-1)E_{k,i}(s+1)\\
&\quad-\hbar(-1)^{p(i)}\sum_{s\geq 0}\delta_{i,j}(-1)^{p(i+1)+p(E_{i,i+1})p(E_{i+1,i})}E_{i,i+1}(-s-1)E_{i,i}(s+1). \qquad (5.22)
\end{aligned}
$$

By a direct computation, we obtain

$$
\text{the sum of the third and 6-th terms of (5.22)} = \hbar\delta_{i,j}E_{i,i+1}E_{i,i}. \tag{5.23}
$$

Next, let us rewrite the sum of the first and 4-th terms of (5.22). By the definition of $\mathrm{ev}_\upsilon(x_{i,1}^+)$, we obtain

$$
\begin{aligned}
&\text{the first term of (5.22)} + \text{the 4-th term of (5.22)}\\
&=\delta_{i,j}(\mathrm{ev}_\upsilon(x_{i,1}^+)-(\upsilon-(i-2\delta(i\geq m+1)(i-m))\varepsilon_1)x_i^+).
\end{aligned}
\tag{5.24}
$$

By the definition of $\mathrm{ev}_\upsilon(x_{i,1}^+)$, we also obtain

$$
\text{the second term of (5.22)} + \text{the 5-th term of (5.22)}
$$

$$
\begin{aligned}
&= -\delta_{i,j+1}\hbar(-1)^{p(i)}\sum_{s\geq 0}\sum_{k=1}^{j}(-1)^{p(k)}E_{j,k}(-s)E_{k,i}(s)\\
&\quad -\delta_{i,j+1}\hbar(-1)^{p(i)}\sum_{s\geq 0}\sum_{k=j+1}^{m+n}(-1)^{p(k)}E_{j,k}(-s-1)E_{k,i}(s+1)\\
&\quad -\hbar\delta_{i,j+1}E_{j,i}E_{i,i}\\
&= -\delta_{i,j+1}(-1)^{p(i)}(\mathrm{ev}_\upsilon(x^+_{j,1})-(\upsilon-(j-2\delta(i\geq m+1)(j-m))\varepsilon_1)x^+_j)-\hbar\delta_{i,j+1}E_{j,i}E_{i,i}. \quad (5.25)
\end{aligned}
$$

Therefore, by (5.23), (5.24) and (5.25), the sum of first, second and third terms of the right hand side of (5.20) is equal to

$$
\begin{aligned}
&(\upsilon-(i-2\delta(i\geq m+1)(i-m))\varepsilon_1)a_{i,j}x^+_j\\
&\quad -\frac{\hbar}{2}(\delta_{i,j}(\{E_{i,i+1},E_{i,i}\}-\{E_{i,i+1},E_{i+1,i+1}\})-\delta_{i,j+1}\{E_{i-1,i},E_{i,i}\}+\delta_{i+1,j}\{E_{i+1,i+2},E_{i+1,i+1}\})\\
&\quad +\hbar\delta_{i,j}E_{i,i+1}E_{i,i}+(-1)^{p(i)}\delta_{i,j}(\mathrm{ev}_\upsilon(x^+_{i,1})-(\upsilon-(i-2\delta(i\geq m+1)(i-m))\varepsilon_1)x^+_i)\\
&\quad -(-1)^{p(i)}\delta_{i,j+1}(\mathrm{ev}_\upsilon(x^+_{j,1})-(\upsilon-(j-2\delta(i\geq m+1)(j-m))\varepsilon_1)x^+_j). \quad (5.26)
\end{aligned}
$$

Similarly to (5.26), we find that the sum of the 4-th and 5-th terms of the right hand side of (5.20) is equal to

$$
\begin{aligned}
&-\hbar\delta_{j,i}E_{i+1,i+1}E_{i,i+1}-(-1)^{p(i+1)}\delta_{i+1,j}(\mathrm{ev}_\upsilon(x^+_{j,1})-(\upsilon-(i-2\delta(j\geq m+1)(j-m))\varepsilon_1)x^+_j)\\
&\quad +\delta_{i+1,j}\hbar E_{i+1,i+1}E_{i+1,j+1}+\delta_{i,j}(-1)^{p(i+1)}(\mathrm{ev}_\upsilon(x^+_{i,1})-(\upsilon-(i-2\delta(i\geq m+1)(i-m))\varepsilon_1)x^+_i). \quad (5.27)
\end{aligned}
$$

Then, $[\mathrm{ev}_\upsilon(\widetilde{h}_{i,1}),\mathrm{ev}_\upsilon(x^+_{j,0})]$ is equal to the sum of (5.21), (5.26) and (5.27).

$$
\begin{aligned}
&(\upsilon-(i-2\delta(i\geq m+1)(i-m))\varepsilon_1)a_{i,j}x^+_j\\
&\quad -\frac{\hbar}{2}(\delta_{i,j}(\{E_{i,i+1},E_{i,i}\}-\{E_{i,i+1},E_{i+1,i+1}\})-\delta_{i,j+1}\{E_{i-1,i},E_{i,i}\}+\delta_{i+1,j}\{E_{i+1,i+2},E_{i+1,i+1}\})\\
&\quad +(\upsilon-(i-2\delta(i\geq m+1)(i-m))\varepsilon_1)a_{i,j}x^+_j\\
&\quad -\frac{\hbar}{2}(\delta_{i,j}(\{E_{i,i+1},E_{i,i}\}-\{E_{i,i+1},E_{i+1,i+1}\})-\delta_{i,j+1}\{E_{i-1,i},E_{i,i}\}+\delta_{i+1,j}\{E_{i+1,i+2},E_{i+1,i+1}\})\\
&\quad +\hbar\delta_{i,j}E_{i,i+1}E_{i,i}+(-1)^{p(i)}\delta_{i,j}(\mathrm{ev}_\upsilon(x^+_{i,1})-(\upsilon-(i-2\delta(i\geq m+1)(i-m))\varepsilon_1)x^+_i)\\
&\quad -(-1)^{p(i)}\delta_{i,j+1}(\mathrm{ev}_\upsilon(x^+_{j,1})-(\upsilon-(j-2\delta(i\geq m+1)(j-m))\varepsilon_1)x^+_j)\\
&\quad -\hbar\delta_{j,i}E_{i+1,i+1}E_{i,i+1}-(-1)^{p(i+1)}\delta_{i+1,j}(\mathrm{ev}_\upsilon(x^+_{j,1})-(\upsilon-(i-2\delta(j\geq m+1)(j-m))\varepsilon_1)x^+_j)\\
&\quad +\delta_{i+1,j}\hbar E_{i+1,i+1}E_{i+1,j+1}+\delta_{i,j}(-1)^{p(i+1)}(\mathrm{ev}_\upsilon(x^+_{i,1})-(\upsilon-(i-2\delta(i\geq m+1)(i-m))\varepsilon_1)x^+_i).
\end{aligned}
$$

By (5.21), (5.26) and (5.27), when $i\neq j,j\pm 1$, $[\mathrm{ev}_\upsilon(\widetilde{h}_{i,1}),\mathrm{ev}_\upsilon(x^+_{j,0})]$ is zero. Provided that $i=j$, $[\mathrm{ev}_\upsilon(\widetilde{h}_{i,1}),\mathrm{ev}_\upsilon(x^+_{i,0})]$ is equal to

$$
\begin{aligned}
&(\upsilon-(i-2\delta(i\geq m+1)(i-m))\varepsilon_1)a_{i,i}x^+_i-\frac{\hbar}{2}(\{E_{i,i+1},E_{i,i}\}-\{E_{i,i+1},E_{i+1,i+1}\})\\
&\quad +\hbar E_{i,i+1}E_{i,i}+(-1)^{p(i)}(\mathrm{ev}_\upsilon(x^+_{i,1})-(\upsilon-(i-2\delta(i\geq m+1)(i-m))\varepsilon_1)x^+_i)\\
&\quad -\hbar E_{i+1,i+1}E_{i,i+1}+(-1)^{p(i+1)}(\mathrm{ev}_\upsilon(x^+_{i,1})-(\upsilon-(i-2\delta(i\geq m+1)(i-m))\varepsilon_1)x^+_i). \quad (5.28)
\end{aligned}
$$

Since $a_{i,i}=(-1)^{p(i)}+(-1)^{p(i+1)}$ holds, we have

$$
(\upsilon-(i-2\delta(i\geq m+1)(i-m))\varepsilon_1)a_{i,i}x^+_i-(-1)^{p(i)}(\upsilon-(i-2\delta(i\geq m+1)(i-m))\varepsilon_1)x^+_i
$$

$$- (-1)^{p(i+1)}(\upsilon - (i - 2\delta(i \geq m+1)(i-m))\varepsilon_1)x_i^+ = 0$$

and

$$\begin{aligned}
&-\frac{\hbar}{2}(\{E_{i,i+1}, E_{i,i}\} - \{E_{i,i+1}, E_{i+1,i+1}\}) + \hbar E_{i,i+1}E_{i,i} - \hbar E_{i+1,i+1}E_{i,i+1}\\
&= -\frac{\hbar}{2}(E_{i,i}E_{i,i+1} - E_{i,i+1}E_{i,i} + E_{i+1,i+1}E_{i,i+1} - E_{i,i+1}E_{i+1,i+1})\\
&= -\frac{\hbar}{2}(E_{i,i+1} - E_{i,i+1}) = 0.
\end{aligned}$$

Then, we find that $[\mathrm{ev}_\upsilon(\widetilde{h}_{i,1}), \mathrm{ev}_\upsilon(x_{i,0}^+)]$ is equal to $a_{i,i}\,\mathrm{ev}_\upsilon(x_{i,1}^+)$.

When $i = j+1$, $[\mathrm{ev}_\upsilon(\widetilde{h}_{i,1}), \mathrm{ev}_\upsilon(x_{j,0}^+)]$ is equal to

$$\begin{aligned}
&(\upsilon - (i - 2\delta(i \geq m+1)(i-m))\varepsilon_1)a_{i,j}x_j^+ + \frac{\hbar}{2}\{E_{i-1,i}, E_{i,i}\}\\
&- (-1)^{p(i)}(\mathrm{ev}_\upsilon(x_{j,1}^+) - (\upsilon - (j - 2\delta(i \geq m+1)(j-m))\varepsilon_1)x_j^+) - \hbar E_{j,i}E_{i,i}. \qquad (5.29)
\end{aligned}$$

Since $a_{i,j} = -(-1)^{p(i)}$ holds, we have

$$\frac{\hbar}{2}\{E_{i-1,i}, E_{i,i}\} - \hbar E_{j,i}E_{i,i} = \frac{\hbar}{2}[E_{i,i}, E_{i-1,i}] = -\frac{\hbar}{2}E_{i-1,i}$$

and

$$\begin{aligned}
&(\upsilon - (i - 2\delta(i \geq m+1)(i-m))\varepsilon_1)a_{i,j}x_j^+ + (-1)^{p(i)}(\upsilon - (j - 2\delta(i \geq m+1)(j-m)\varepsilon_1)x_j^+\\
&= \varepsilon_1 x_j^+.
\end{aligned}$$

Then, we find that $[\mathrm{ev}_\upsilon(\widetilde{h}_{i,1}), \mathrm{ev}_\upsilon(x_{j,0}^+)]$ is equal to $a_{i,i-1}(\mathrm{ev}_\upsilon(x_{i-1}^+) + a_{i,i-1}\dfrac{\varepsilon_1 - \varepsilon_2}{2}E_{i-1,i})$.

When $i = j-1$, $[\mathrm{ev}_\upsilon(\widetilde{h}_{i,1}), \mathrm{ev}_\upsilon(x_{j,0}^+)]$ is equal to

$$\begin{aligned}
&(\upsilon - (i - 2\delta(i \geq m+1)(i-m))\varepsilon_1)a_{i,j}x_j^+ - \frac{\hbar}{2}(\{E_{i+1,i+2}, E_{i+1,i+1}\})\\
&- (-1)^{p(i+1)}(\mathrm{ev}_\upsilon(x_{j,1}^+) - (\upsilon - (i - 2\delta(j \geq m+1)(j-m))\varepsilon_1)x_j^+ + \hbar E_{i+1,i+1}E_{i+1,j+1}. \qquad (5.30)
\end{aligned}$$

Since $a_{i,j} = -(-1)^{p(j)}$ holds, we have

$$-\frac{\hbar}{2}\{E_{i+1,i+2}, E_{i+1,i+1}\} + \hbar E_{i+1,i+1}E_{i+1,j+1} = \frac{\hbar}{2}[E_{i+1,i+1}, E_{i+1,i+2}] = \frac{\hbar}{2}E_{i+1,i+2}$$

and

$$\begin{aligned}
&(\upsilon - (i - 2\delta(i \geq m+1)(i-m))\varepsilon_1)a_{i,j}x_j^+ + (-1)^{p(i+1)}(\upsilon - (i - 2\delta(j \geq m+1)(j-m))\varepsilon_1)x_j^+\\
&= -\varepsilon_1 E_{i+1,i+2}
\end{aligned}$$

Then, $[\mathrm{ev}_\upsilon(\widetilde{h}_{i,1}), \mathrm{ev}_\upsilon(x_{i+1,0}^+)]$ is equal to $a_{i,i+1}(\mathrm{ev}_\upsilon(x_{i+1,1}^+) - a_{i,i+1}\dfrac{\varepsilon_1 - \varepsilon_2}{2}E_{i+1,i+2})$.

Case 2, $i = 0$ and $j \neq 0$.

By the definition of ev_υ, we obtain

$$\begin{aligned}
&[\mathrm{ev}_\upsilon(\widetilde{h}_{0,1}), \mathrm{ev}_\upsilon(x_{j,0}^+)]\\
&= [(\upsilon - (m-n)\varepsilon_1)h_0 - \frac{1}{2}\hbar((E_{m+n,m+n})^2 + (E_{1,1} - c)^2), E_{j,j+1}]\\
&\quad - [\hbar\sum_{s \geq 0}\sum_{k=1}^{m+n}(-1)^{p(k)}E_{m+n,k}(-s)E_{k,m+n}(s), E_{j,j+1}]
\end{aligned}$$

$$-[\hbar\sum_{s\geq 0}\sum_{k=1}^{m+n}(-1)^{p(k)}E_{1,k}(-s-1)E_{k,1}(s+1),E_{j,j+1}]. \tag{5.31}$$

By direct computation, the first term of (5.31) is equal to

$$\begin{aligned}&(\upsilon-(m-n)\varepsilon_1)a_{0,j}x_j^+\\&\quad-\frac{\hbar}{2}\Big(-\delta_{m+n-1,j}(\{E_{m+n,m+n},E_{m+n-1,m+n}\}+\delta_{1,j}\{E_{1,2},(E_{1,1}-c)\})\Big)\end{aligned} \tag{5.32}$$

We also find that the second term of (5.31) is equal to

$$\begin{aligned}&\hbar\sum_{s\geq 0}(-1)^{p(j+1)+p(E_{j,j+1})p(E_{j+1,m+n})}E_{m+n,j+1}(-s)E_{j,m+n}(s)\\&\quad-\hbar\sum_{s\geq 0}(-1)^{p(j)+p(E_{j,j+1})p(E_{j,m+n})}E_{m+n,j+1}(-s)E_{j,m+n}(s)\\&\quad+\hbar\sum_{s\geq 0}\sum_{k=1}^{m+n}\delta_{m+n,j+1}(-1)^{p(k)}E_{m+n-1,k}(-s)E_{k,m+n}(s).\end{aligned} \tag{5.33}$$

By direct computation, we also know that the third term of (5.31) is equal to

$$\begin{aligned}&-\hbar\sum_{s\geq 0}\sum_{k=1}^{m+n}\delta_{1,j}(-1)^{p(k)}E_{1,k}(-s-1)E_{k,2}(s+1)\\&\quad+\hbar\sum_{s\geq 0}(-1)^{p(j+1)+p(E_{j,j+1})p(E_{j+1,1})}E_{1,j+1}(-s-1)E_{j,1}(s+1)\\&\quad-\hbar\sum_{s\geq 0}(-1)^{p(j)+p(E_{j,j+1})p(E_{j,1})}E_{1,j+1}(-s-1)E_{j,1}(s+1).\end{aligned} \tag{5.34}$$

First, we show that the sum of the first and second terms of (5.33) is equal to zero. By direct computation, we have

$$\begin{aligned}&\text{the first term of (5.33)}+\text{the second term of (5.33)}\\&=\hbar\sum_{s\geq 0}(-1)^{p(j+1)+p(E_{j,j+1})p(E_{j+1,m+n})}E_{m+n,j+1}(-s)E_{j,m+n}(s)\\&\quad-\hbar\sum_{s\geq 0}(-1)^{p(j)+p(E_{j,j+1})p(E_{j,m+n})}E_{m+n,j+1}(-s)E_{j,m+n}(s)\\&=0.\end{aligned} \tag{5.35}$$

Similarly, by direct computation, we also obtain

$$\begin{aligned}&\text{the second term of (5.34)}+\text{the third term of (5.34)}\\&=\hbar\sum_{s\geq 0}(-1)^{p(j+1)+p(E_{j,j+1})p(E_{j+1,1})}E_{1,j+1}(-s-1)E_{j,1}(s+1)\\&\quad-\hbar\sum_{s\geq 0}(-1)^{p(j)+p(E_{j,j+1})p(E_{j,1})}E_{1,j+1}(-s-1)E_{j,1}(s+1)\\&=0.\end{aligned} \tag{5.36}$$

Next, we rewrite the third term of (5.33). By direct computation, we have

$$\begin{aligned}&\text{the third term of (5.33)}\\&=\hbar\sum_{s\geq 0}\sum_{k=1}^{m+n-1}\delta_{m+n,j+1}(-1)^{p(k)}E_{m+n-1,k}(-s)E_{k,m+n}(s)\end{aligned}$$

$$
\begin{aligned}
&+\hbar\sum_{s\geq 0}\delta_{m+n,j+1}(-1)^{p(m+n)}E_{m+n-1,m+n}(-s-1)E_{m+n,m+n}(s+1)\\
&+\hbar\delta_{m+n,j+1}(-1)^{p(m+n)}E_{m+n-1,m+n}E_{m+n,m+n}\\
=\,&\delta_{m+n,j+1}(\mathrm{ev}_\upsilon(x^+_{m+n-1,1})-(\upsilon-(m-n+1)\varepsilon_1)x^+_{m+n-1})\\
&+\hbar\delta_{m+n,j+1}(-1)^{p(m+n)}E_{m+n-1,m+n}E_{m+n,m+n}. \qquad (5.37)
\end{aligned}
$$

Similarly, we rewrite the first term of (5.34) as follows;

$$
\begin{aligned}
&\text{the first term of (5.34)}\\
=\,&-\hbar\sum_{s\geq 0}\delta_{1,j}E_{1,1}(-s)E_{1,2}(s)-\hbar\sum_{s\geq 0}\sum_{k=2}^{m+n}\delta_{1,j}(-1)^{p(k)}E_{1,k}(-s-1)E_{k,2}(s+1)\\
&+\hbar\sum_{s\geq 0}\delta_{1,j}E_{1,1}E_{1,2}\\
=\,&-\delta_{j,1}(\mathrm{ev}_\upsilon(x^+_{1,1})-(\upsilon-\varepsilon_1)x^+_1)+\hbar\delta_{j,1}E_{1,1}E_{1,2}. \qquad (5.38)
\end{aligned}
$$

Then, by (5.31), (5.35), (5.36), (5.37), and (5.38), we can rewrite $[\mathrm{ev}_\upsilon(\widetilde{h}_{0,1}),x^+_{j,0}]$ as

$$
\begin{aligned}
&(\upsilon-(m-n)\varepsilon_1)a_{0,j}x^+_j-\frac{\hbar}{2}(-\delta_{m+n,j+1}\{E_{m+n,m+n},E_{j,m+n}\}+\delta_{1,j}\{E_{1,j+1},(E_{1,1}-c)\})\\
&+\delta_{m+n,j+1}(\mathrm{ev}_\upsilon(x^+_{m+n-1,1})-(\upsilon-(m-n-1)\varepsilon_1)x^+_{m+n-1})\\
&+\hbar\delta_{m+n,j+1}(-1)^{p(m+n)}E_{m+n-1,m+n}E_{m+n,m+n}\\
&-\delta_{j,1}(\mathrm{ev}_\upsilon(x^+_{1,1}-(\upsilon-\varepsilon_1)x^+_1)+\hbar\delta_{j,1}E_{1,1}E_{1,2}. \qquad (5.39)
\end{aligned}
$$

By (5.39), when $j\neq 0,1,m+n-1$, $[\mathrm{ev}_\upsilon(\widetilde{h}_{0,1}),\mathrm{ev}_\upsilon(x^+_{j,0})]$ is equal to zero. When $j=m+n-1$, $[\mathrm{ev}_\upsilon(\widetilde{h}_{0,1}),\mathrm{ev}_\upsilon(x^+_{j,0})]$ is equal to

$$
\begin{aligned}
&(\upsilon-(m-n)\varepsilon_1)x^+_{m+n-1}+\frac{\hbar}{2}\{E_{m+n,m+n},E_{j,m+n}\}\\
&+\mathrm{ev}_\upsilon(x^+_{m+n-1,1})-(\upsilon-(m-n+1)\varepsilon_1)x^+_{m+n-1}+\hbar(-1)^{p(m+n)}E_{m+n-1,m+n}E_{m+n,m+n}.
\end{aligned}
$$

Since

$$
\begin{aligned}
&\frac{\hbar}{2}\{E_{m+n,m+n},E_{m+n-1,m+n}\}+\hbar(-1)^{p(m+n)}E_{m+n-1,m+n}E_{m+n,m+n}\\
=\,&\frac{\hbar}{2}[E_{m+n,m+n},E_{m+n-1,m+n}]-\frac{\hbar}{2}E_{m+n-1,m+n}.
\end{aligned}
$$

holds, $[\mathrm{ev}_\upsilon(\widetilde{h}_{0,1}),\mathrm{ev}_\upsilon(x^+_{m+n-1,0})]$ is equal to

$$
a_{m+n-1,0}(\mathrm{ev}_\upsilon(x^+_{m+n-1,1})+a_{m+n-1,0}\frac{\varepsilon_1-\varepsilon_2}{2}E_{m+n-1,m+n}).
$$

By (5.39), when $j=1$, $[\mathrm{ev}_\upsilon(\widetilde{h}_{0,1}),\mathrm{ev}_\upsilon(x^+_{1,0})]$ can be written as

$$
-(\upsilon-(m-n)\varepsilon_1)x^+_1-\frac{\hbar}{2}\{E_{1,j+1},(E_{1,1}-c)\}-\mathrm{ev}_\upsilon(x^+_{1,1})-(\upsilon-\varepsilon_1)x^+_1+\hbar E_{1,1}E_{1,2}.
$$

Since

$$
\hbar E_{1,1}E_{1,2}-\frac{\hbar}{2}\{E_{1,2},(E_{1,1}-c)\}=\frac{\hbar}{2}[E_{1,1},E_{1,2}]+\hbar cE_{1,2}=(\frac{\hbar}{2}+\hbar c)E_{1,2}
$$

holds, $[\mathrm{ev}_\upsilon(\widetilde{h}_{0,1}),x^+_{1,0}]=a_{0,1}(\mathrm{ev}_\upsilon(x^+_{1,1})-a_{0,1}\frac{\varepsilon_1-\varepsilon_2}{2}\mathrm{ev}_\upsilon(x^+_{1,0}))$ is equivalent to the relation $c\hbar=(m-n)\varepsilon_1$. It is nothing but assumption. This completes the proof of the case $j\neq 0$ and $i=0$.

Other cases are proven in the same way. Thus, we show that

$$[\mathrm{ev}_\upsilon(\widetilde{h}_{i,1}), \mathrm{ev}_\upsilon(x^+_{j,0})] = a_{i,j}(\mathrm{ev}_\upsilon(x^+_{j,1}) - b_{i,j}\frac{\varepsilon_1-\varepsilon_2}{2}\mathrm{ev}_\upsilon(x^+_{j,0}))$$

holds.

5.3 The proof of (5.9)

We only show the case where $i,j\neq 0$ and $i=0, j\neq 0$. The other case is proven in a similar way.

Case 1, $i,j\neq 0$

Suppose that $i,j\neq 0$. First, we let us compute $[\mathrm{ev}_\upsilon(x^+_{i,1}), \mathrm{ev}_\upsilon(x^+_{j,0})]$. By the definition of $\mathrm{ev}_\upsilon(x^+_{i,1})$, we have

$$\begin{aligned}
&[\mathrm{ev}_\upsilon(x^+_{i,1}), \mathrm{ev}_\upsilon(x^+_{j,0})]\\
&=[(\upsilon-(i-2\delta(i\geq m+1)(i-m))\varepsilon_1)x^+_i, E_{j,j+1}]\\
&\quad+[\hbar\sum_{s\geq 0}\sum_{k=1}^{i}(-1)^{p(k)}E_{i,k}(-s)E_{k,i+1}(s), E_{j,j+1}]\\
&\quad+[\hbar\sum_{s\geq 0}\sum_{k=i+1}^{m+n}(-1)^{p(k)}E_{i,k}(-s-1)E_{k,i+1}(s+1), E_{j,j+1}].
\end{aligned}\tag{5.40}$$

By direct computation, the second term of (5.40) is equal to

$$\begin{aligned}
&[\hbar\sum_{s\geq 0}\sum_{k=1}^{i}(-1)^{p(k)}E_{i,k}(-s)E_{k,i+1}(s), E_{j,j+1}]\\
&=\hbar\sum_{s\geq 0}\sum_{k=1}^{i}\delta_{i+1,j}(-1)^{p(k)}E_{i,k}(-s)E_{k,j+1}(s)\\
&\quad-\hbar\sum_{s\geq 0}\sum_{k=1}^{i}\delta_{i,j+1}(-1)^{p(k)+p(E_{i+1,i})p(E_{j,j+1})}E_{j,k}(-s)E_{k,i+1}(s)\\
&\quad+\hbar\sum_{s\geq 0}\delta_{j,i}(-1)^{p(i)+p(E_{i,i+1})p(E_{i,i+1})}E_{i,i+1}(-s)E_{i,i+1}(s).
\end{aligned}\tag{5.41}$$

We also find that the third term of (5.40) is equal to

$$\begin{aligned}
&[\hbar\sum_{s\geq 0}\sum_{k=i+1}^{m+n}(-1)^{p(k)}E_{i,k}(-s-1)E_{k,i+1}(s+1), E_{j,j+1}]\\
&=\hbar\sum_{s\geq 0}\sum_{k=i+1}^{m+n}\delta_{i+1,j}(-1)^{p(k)}E_{i,k}(-s-1)E_{k,j+1}(s+1)\\
&\quad-\hbar\sum_{s\geq 0}\sum_{k=i+1}^{m+n}\delta_{i,j+1}(-1)^{p(k)+p(E_{i+1,i})p(E_{j,j+1})}E_{j,k}(-s-1)E_{k,i+1}(s+1)\\
&\quad-\hbar\sum_{s\geq 0}\delta_{i,j}(-1)^{p(i+1)+p(E_{i+1,i})p(E_{i,i+1})}E_{i,i+1}(-s-1)E_{i,i+1}(s+1).
\end{aligned}\tag{5.42}$$

Thus, we can rewrite $[\mathrm{ev}_\upsilon(x^+_{i,1}), \mathrm{ev}_\upsilon(x^+_{j,0})]$ as

$$[(\upsilon-(i-2\delta(i\geq m+1)(i-m))\varepsilon_1)x^+_i, E_{j,j+1}]$$

$$
\begin{aligned}
&+\hbar\sum_{s\geq 0}\sum_{k=1}^{i}\delta_{i+1,j}(-1)^{p(k)}E_{i,k}(-s)E_{k,j+1}(s)\\
&-\hbar\sum_{s\geq 0}\sum_{k=1}^{i}\delta_{i,j+1}(-1)^{p(k)+(p(E_{i+1,k})+p(E_{k,i}))p(E_{j,j+1})}E_{j,k}(-s)E_{k,i+1}(s)\\
&+\hbar\sum_{s\geq 0}\sum_{k=i+1}^{m+n}\delta_{i+1,j}(-1)^{p(k)}E_{i,k}(-s-1)E_{k,j+1}(s+1)\\
&-\hbar\sum_{s\geq 0}\sum_{k=i+1}^{m+n}\delta_{i,j+1}(-1)^{p(k)+(p(E_{i+1,k})+p(E_{k,i}))p(E_{j,j+1})}E_{j,k}(-s-1)E_{k,i+1}(s+1)\\
&+\hbar\sum_{s\geq 0}\delta_{j,i}(-1)^{p(i)+p(E_{i,i+1})p(E_{j,i+1})}E_{i,i+1}(-s)E_{i,i+1}(s)\\
&-\hbar\sum_{s\geq 0}\delta_{i,j}(-1)^{p(i+1)+p(E_{i+1,i})p(E_{i,i+1})}E_{i,i+1}(-s-1)E_{i,i+1}(s+1).
\end{aligned}\tag{5.43}
$$

Next, let us compute $[\mathrm{ev}_\upsilon(x^+_{i,0}),\mathrm{ev}_\upsilon(x^+_{j,1})]$. Since it is equal to

$$
-(-1)^{p(E_{i,i+1})p(E_{j,j+1})}[\mathrm{ev}_\upsilon(x^+_{j,1}),\mathrm{ev}_\upsilon(x^+_{i,0})],
$$

we can rewrite $[\mathrm{ev}_\upsilon(x^+_{i,0}),\mathrm{ev}_\upsilon(x^+_{j,1})]$ as

$$
\begin{aligned}
&[E_{i,i+1},(\upsilon-(j-2\delta(j\geq m+1)(j-m))\varepsilon_1)x^+_j]\\
&\quad-\hbar\sum_{s\geq 0}\sum_{k=1}^{j}\delta_{i,j+1}(-1)^{p(k)+p(E_{i,i+1})(p(E_{j,k})+p(E_{k,j+1}))}E_{j,k}(-s)E_{k,i+1}(s)\\
&\quad+\hbar\sum_{s\geq 0}\sum_{k=1}^{j}\delta_{i+1,j}(-1)^{p(k)}E_{i,k}(-s)E_{k,j+1}(s)\\
&\quad-\hbar\sum_{s\geq 0}\sum_{k=j+1}^{m+n}\delta_{i,j+1}(-1)^{p(k)+p(E_{i,i+1})(p(E_{j,k})+p(E_{k,j+1}))}E_{j,k}(-s-1)E_{k,i+1}(s+1)\\
&\quad+\hbar\sum_{s\geq 0}\sum_{k=j+1}^{m+n}\delta_{i+1,j}(-1)^{p(k)}E_{i,k}(-s-1)E_{k,j+1}(s+1)-\hbar\sum_{s\geq 0}\delta_{i,j}(-1)^{p(i)}E_{i,i+1}(-s)E_{i,i+1}(s)\\
&\quad+\hbar\sum_{s\geq 0}\delta_{i,j}(-1)^{p(i+1)}E_{i,i+1}(-s-1)E_{i,i+1}(s+1).
\end{aligned}\tag{5.44}
$$

By (5.43) and (5.44), when $i\neq j,j\pm 1$, $[\mathrm{ev}_\upsilon(x^+_{i,1}),\mathrm{ev}_\upsilon(x^+_{j,0})]-[\mathrm{ev}_\upsilon(x^+_{i,0}),\mathrm{ev}_\upsilon(x^+_{j,1})]$ is equal to zero.

When $i=j$, $[\mathrm{ev}_\upsilon(x^+_{i,1}),\mathrm{ev}_\upsilon(x^+_{j,0})]-[\mathrm{ev}_\upsilon(x^+_{i,0}),\mathrm{ev}_\upsilon(x^+_{j,1})]$ is equal to

$$
\begin{aligned}
&[(\upsilon-(i-2\delta(i\geq m+1)(i-m))\varepsilon_1)x^+_i,E_{j,j+1}]\\
&\quad-[E_{i,i+1},(\upsilon-(j-2\delta(j\geq m+1)(j-m))\varepsilon_1)x^+_j]\\
&\quad+\hbar\sum_{s\geq 0}(-1)^{p(i+1)}E_{i,i+1}(-s)E_{i,i+1}(s)-\hbar\sum_{s\geq 0}(-1)^{p(i)}E_{i,i+1}(-s-1)E_{i,i+1}(s+1)\\
&\quad+\hbar\sum_{s\geq 0}(-1)^{p(i)}E_{i,i+1}(-s)E_{i,i+1}(s)-\hbar\sum_{s\geq 0}(-1)^{p(i+1)}E_{i,i+1}(-s-1)E_{i,i+1}(s+1).
\end{aligned}\tag{5.45}
$$

Since $[x^+_i,x^+_i]=0$ holds, the first and second term are zero. We also obtain

$$
\text{the third term of (5.45)}+\text{the 4-th term of (5.45)}=\hbar(-1)^{p(i)}E_{i,i+1}E_{i,i+1}
$$

and

$$\text{the 5-th term of (5.45)} + \text{the 6-th term of (5.45)} = \hbar(-1)^{p(i+1)}E_{i,i+1}E_{i,i+1}$$

by direct computation. Thus, we have

$$[\mathrm{ev}_\upsilon(x_{i,1}^+), \mathrm{ev}_\upsilon(x_{j,0}^+)] - [\mathrm{ev}_\upsilon(x_{i,0}^+), \mathrm{ev}_\upsilon(x_{j,1}^+)] = \hbar a_{i,i}E_{i,i+1}E_{i,i+1}$$

since $a_{i,i} = (-1)^{p(i)} + (-1)^{p(i+1)}$ holds.

When $i = j-1$, $[\mathrm{ev}_\upsilon(x_{i,1}^+), \mathrm{ev}_\upsilon(x_{j,0}^+)] - [\mathrm{ev}_\upsilon(x_{i,0}^+), \mathrm{ev}_\upsilon(x_{j,1}^+)]$ is equal to

$$\begin{aligned}
&[(\upsilon - (i - 2\delta(i \geq m+1)(i-m))\varepsilon_1)x_i^+, E_{j,j+1}] - [E_{i,i+1}, (\upsilon - (j - 2\delta(j \geq m+1)(j-m))\varepsilon_1)x_j^+]\\
&\quad + \hbar\sum_{s\geq 0}\sum_{k=1}^{i}\delta_{i+1,j}(-1)^{p(k)}E_{i,k}(-s)E_{k,j+1}(s)\\
&\quad + \hbar\sum_{s\geq 0}\sum_{k=i+1}^{m+n}\delta_{i+1,j}(-1)^{p(k)}E_{i,k}(-s-1)E_{k,j+1}(s+1)\\
&\quad - \hbar\sum_{s\geq 0}\sum_{k=1}^{j}\delta_{i+1,j}(-1)^{p(k)}E_{i,k}(-s)E_{k,j+1}(s)\\
&\quad - \hbar\sum_{s\geq 0}\sum_{k=j+1}^{m+n}\delta_{i+1,j}(-1)^{p(k)}E_{i,k}(-s-1)E_{k,j+1}(s+1). \qquad (5.46)
\end{aligned}$$

By direct computation, we obtain

$$\text{the third term of (5.46)} + \text{the 5-th term of (5.46)} = -\hbar\sum_{s\geq 0}(-1)^{p(i+1)}E_{i,i+1}(-s)E_{i+1,i+2}(s)$$

and

$$\begin{aligned}
&\text{the 4-th term of (5.45)} + \text{the 6-th term of (5.45)}\\
&= \hbar\sum_{s\geq 0}(-1)^{p(i+1)}E_{i,i+1}(-s-1)E_{i+1,i+2}(s+1).
\end{aligned}$$

Then, $[\mathrm{ev}_\upsilon(x_{i,1}^+), \mathrm{ev}_\upsilon(x_{j,0}^+)] - [\mathrm{ev}_\upsilon(x_{i,0}^+), \mathrm{ev}_\upsilon(x_{j,1}^+)]$ is equal to

$$\begin{aligned}
&[(\upsilon - (i - 2\delta(i \geq m+1)(i-m))\varepsilon_1)x_i^+, E_{j,j+1}] - [E_{i,i+1}, (\upsilon - (j - 2\delta(j \geq m+1)(j-m))\varepsilon_1)x_j^+]\\
&- \hbar\sum_{s\geq 0}(-1)^{p(i+1)}E_{i,i+1}(-s)E_{i+1,i+2}(s) + \hbar\sum_{s\geq 0}(-1)^{p(i+1)}E_{i,i+1}(-s-1)E_{i+1,i+2}(s+1).
\end{aligned}$$

Since $a_{i,i+1} = -(-1)^{p(i+1)}$ holds, we have

$$\begin{aligned}
&- \hbar\sum_{s\geq 0}(-1)^{p(i+1)}E_{i,i+1}(-s)E_{i+1,i+2}(s) + \hbar\sum_{s\geq 0}(-1)^{p(i+1)}E_{i,i+1}(-s-1)E_{i+1,i+2}(s+1)\\
&= -(-1)^{p(i+1)}\hbar E_{i,i+1}E_{i+1,i+2} = a_{i,i+1}\frac{\hbar}{2}\{E_{i,i+1}, E_{i+1,i+2}\} + a_{i,i+1}\frac{\hbar}{2}[E_{i,i+1}, E_{i+1,i+2}]
\end{aligned}$$

and

$$\begin{aligned}
&[(\upsilon - (i - 2\delta(i \geq m+1)(i-m))\varepsilon_1)x_i^+, E_{j,j+1}] - [E_{i,i+1}, (\upsilon - (j - 2\delta(j \geq m+1)(j-m))\varepsilon_1)x_j^+]\\
&\qquad = (-1)^{p(i+1)}\varepsilon_1[E_{i,i+1}, E_{i+1,i+2}] = -a_{i,i+1}\varepsilon_1[E_{i,i+1}, E_{i+1,i+2}].
\end{aligned}$$

Then, $[\mathrm{ev}_\upsilon(x^+_{i,1}),\mathrm{ev}_\upsilon(x^+_{j,0})]-[\mathrm{ev}_\upsilon(x^+_{i,0}),\mathrm{ev}_\upsilon(x^+_{j,1})]$ is equal to

$$a_{i,i+1}\frac{\hbar}{2}\{E_{i,i+1},E_{i+1,i+2}\}-a_{i,i+1}\frac{\varepsilon_1-\varepsilon_2}{2}[E_{i,i+1},E_{i+1,i+2}].$$

When $i=j+1$, $[\mathrm{ev}_\upsilon(x^+_{i,1}),\mathrm{ev}_\upsilon(x^+_{j,0})]-[\mathrm{ev}_\upsilon(x^+_{i,0}),\mathrm{ev}_\upsilon(x^+_{j,1})]$ is equal to

$$\begin{aligned}
&[(\upsilon-(i-2\delta(i\geq m+1)(i-m))\varepsilon_1)x^+_i,E_{j,j+1}]-[E_{i,i+1},(\upsilon-(j-2\delta(j\geq m+1)(j-m))\varepsilon_1)x^+_j]\\
&\quad-\hbar\sum_{s\geq 0}\sum_{k=1}^{i}\delta_{i,j+1}(-1)^{p(k)+p(E_{i+1,i})p(E_{j,j+1})}E_{j,k}(-s)E_{k,i+1}(s)\\
&\quad-\hbar\sum_{s\geq 0}\sum_{k=i+1}^{m+n}\delta_{i,j+1}(-1)^{p(k)+p(E_{i+1,i})p(E_{j,j+1})}E_{j,k}(-s-1)E_{k,i+1}(s+1)\\
&\quad+\hbar\sum_{s\geq 0}\sum_{k=1}^{j}\delta_{i,j+1}(-1)^{p(k)+p(E_{i+1,i})p(E_{j,j+1})}E_{j,k}(-s)E_{k,i+1}(s)\\
&\quad+\hbar\sum_{s\geq 0}\sum_{k=j+1}^{m+n}\delta_{i,j+1}(-1)^{p(k)+p(E_{i+1,i})p(E_{j,j+1})}E_{j,k}(-s-1)E_{k,i+1}(s+1).
\end{aligned}\tag{5.47}$$

By direct computation, we find that

$$\text{the 4-th term of (5.45)}+\text{the 6-th term of (5.45)}=-\hbar\sum_{s\geq 0}(-1)^{p(i)}E_{i-1,i}(-s)E_{i,i+1}(s)$$

and

$$\text{the 4-th term of (5.45)}+\text{the 6-th term of (5.45)}=\hbar\sum_{s\geq 0}(-1)^{p(i)}E_{i-1,i}(-s-1)E_{i,i+1}(s+1).$$

hold. Since $a_{i,i-1}=-(-1)^{p(i)}$ holds, we have

$$\begin{aligned}
&-\hbar\sum_{s\geq 0}(-1)^{p(i)}E_{i-1,i}(-s)E_{i,i+1}(s)+\hbar\sum_{s\geq 0}(-1)^{p(i)}E_{i-1,i}(-s-1)E_{i,i+1}(s+1)\\
&=-\hbar(-1)^{p(i)}E_{i-1,i}E_{i,i+1}=\frac{\hbar}{2}a_{i-1,i}\{E_{i,i+1},E_{i-1,i}\}-\frac{\hbar}{2}a_{i-1,i}[E_{i,i+1},E_{i-1,i}]
\end{aligned}$$

and

$$\begin{aligned}
&[(\upsilon-(i-2\delta(i\geq m+1)(i-m))\varepsilon_1)x^+_i,E_{j,j+1}]-[E_{i,i+1},(\upsilon-(j-2\delta(j\geq m+1)(j-m))\varepsilon_1)x^+_j]\\
&\quad=-(-1)^{p(i)}\varepsilon_1[E_{i,i+1},E_{i-1,i}]=a_{i,i-1}\varepsilon_1[E_{i,i+1},E_{i-1,i}].
\end{aligned}$$

holds, $[\mathrm{ev}_\upsilon(x^+_{i,1}),\mathrm{ev}_\upsilon(x^+_{j,0})]-[\mathrm{ev}_\upsilon(x^+_{i,0}),\mathrm{ev}_\upsilon(x^+_{j,1})]$ is equal to

$$\begin{aligned}
&[(\upsilon-(i-2\delta(i\geq m+1)(i-m))\varepsilon_1)x^+_i,E_{j,j+1}]-[E_{i,i+1},(\upsilon-(j-2\delta(j\geq m+1)(j-m))\varepsilon_1)x^+_j]\\
&-\hbar(-1)^{p(i)}\{E_{i-1,i},E_{i,i+1}\}+(-1)^{p(i)}\frac{\hbar}{2}E_{i-1,i+1}.
\end{aligned}$$

Therefore, it is equal to $-a_{i,i-1}\frac{\hbar}{2}\{E_{i-1,i},E_{i,i+1}\}+a_{i,i-1}\frac{\varepsilon_1-\varepsilon_2}{2}E_{i-1,i+1}$.

Case 2, $i\neq 0$ and $j=0$

Suppose that $i\neq 0$. First, we compute $[\mathrm{ev}_\upsilon(x^+_{i,1}),\mathrm{ev}_\upsilon(x^+_{0,0})]$. By the definition of ev_υ, we obtain

$$[\mathrm{ev}_\upsilon(x^+_{i,1}),\mathrm{ev}_\upsilon(x^+_{0,0})]$$

$$
\begin{aligned}
&= [(\upsilon - (i - 2\delta(i \geq m+1)(i-m))\varepsilon_1)x_i^+, E_{m+n,1}(1)]\\
&\quad + [\hbar \sum_{s\geq 0}\sum_{k=1}^{i} (-1)^{p(k)} E_{i,k}(-s)E_{k,i+1}(s), E_{m+n,1}(1)]\\
&\quad + [\hbar \sum_{s\geq 0}\sum_{k=i+1}^{m+n} (-1)^{p(k)} E_{i,k}(-s-1)E_{k,i+1}(s+1), E_{m+n,1}(1)].
\end{aligned}\tag{5.48}
$$

By direct computation, the second term of (5.48) is equal to

$$
\begin{aligned}
&\hbar \sum_{s\geq 0}\sum_{k=1}^{m+n-1} \delta_{m+n,i+1}(-1)^{p(k)} E_{m+n-1,k}(-s)E_{k,1}(s+1)\\
&\quad - \hbar \sum_{s\geq 0} (-1)^{p(E_{m+n,1})p(E_{1,i})+p(1)} E_{i,1}(-s)E_{m+n,i}(s+1)\\
&\quad - \hbar \sum_{s\geq 0} \delta_{1,i} E_{m+n,1}(1-s)E_{1,2}(s)
\end{aligned}\tag{5.49}
$$

and the third term of (5.48) is equal to

$$
\begin{aligned}
&\hbar \sum_{s\geq 0}\sum_{k=i+1}^{m+n} (-1)^{p(m+n)} E_{m+n-1,m+n}(-s-1)E_{m+n,1}(s+2)\\
&\quad + \hbar \sum_{s\geq 0} (-1)^{p(E_{m+n,1})p(E_{m+n,i})+p(m+n)} E_{i,1}(-s)E_{m+n,i}(s+1)\\
&\quad - \hbar \sum_{s\geq 0}\sum_{k=2}^{m+n} \delta_{1,i}(-1)^{p(k)} E_{m+n,k}(-s)E_{k,1}(s+1) + \delta_{i,1} c E_{m+n,2}(1).
\end{aligned}\tag{5.50}
$$

Next, we rewrite the sum of the second term of (5.49) and the second term of (5.50) as follows;

$$
\text{the second term of (5.49)} + \text{the second term of (5.50)} = 0.
$$

Therefore, $[\mathrm{ev}_\upsilon(x_{i,1}^+), \mathrm{ev}_\upsilon(x_{0,0}^+)]$ is equal to

$$
\begin{aligned}
&[(\upsilon - i\varepsilon_1)x_i^+, E_{m+n,1}(1)] + \hbar \sum_{s\geq 0}\sum_{k=1}^{m+n-1} \delta_{m+n,i+1}(-1)^{p(k)} E_{m+n-1,k}(-s)E_{k,1}(s+1)\\
&\quad - \hbar \sum_{s\geq 0} \delta_{1,i} E_{m+n,1}(1-s)E_{1,2}(s)\\
&\quad + \hbar \sum_{s\geq 0} \delta_{m+n,i+1}(-1)^{p(m+n)} E_{m+n-1,m+n}(-s-1)E_{m+n,1}(s+2)\\
&\quad - \hbar \sum_{s\geq 0}\sum_{k=2}^{m+n} \delta_{1,i}(-1)^{p(k)} E_{m+n,k}(-s)E_{k,1}(s+1) + \delta_{i,1} c E_{m+n,2}(1).
\end{aligned}\tag{5.51}
$$

Next, let us compute $[\mathrm{ev}_\upsilon(x_{i,0}^+), \mathrm{ev}_\upsilon(x_{0,1}^+)]$. By direct computation, we have

$$
\begin{aligned}
&[\mathrm{ev}_\upsilon(x_{i,0}^+), \mathrm{ev}_\upsilon(x_{0,1}^+)]\\
&= [E_{i,i+1}, (\upsilon - (m-n)\varepsilon_1)x_0^+] + [E_{i,i+1}, \hbar \sum_{s\geq 0}\sum_{k=1}^{m+n} (-1)^{p(k)} E_{m+n,k}(-s)E_{k,1}(s+1)].
\end{aligned}\tag{5.52}
$$

By direct computation, the second term of (5.52) is equal to

$$
\hbar \sum_{s\geq 0}\sum_{k=1}^{m+n} \delta_{m+n,i+1}(-1)^{p(k)} E_{m+n-1,k}(-s)E_{k,1}(s+1)
$$

$$
\begin{aligned}
&-\hbar\sum_{s\geq 0}(-1)^{p(i)+p(E_{i,i+1})p(E_{m+n,i})}E_{m+n,i+1}(-s)E_{i,1}(s+1)\\
&+\hbar\sum_{s\geq 0}(-1)^{p(i+1)+p(E_{i,i+1})p(E_{m+n,i+1})}E_{m+n,i+1}(-s)E_{i,1}(s+1)\\
&-\hbar\sum_{s\geq 0}\sum_{k=1}^{m+n}\delta_{1,i}(-1)^{p(k)+p(E_{1,2})p(E_{m+n,1})}E_{m+n,k}(-s)E_{k,i+1}(s+1).
\end{aligned}\tag{5.53}
$$

The sum of the second term of (5.53) and the third term of (5.53) is equal to zero. Thus, $[\mathrm{ev}_\upsilon(x_{i,0}^+),\mathrm{ev}_\upsilon(x_{0,1}^+)]$ is equal to

$$
\begin{aligned}
&[E_{i,i+1},(\upsilon-(m-n)\varepsilon_1)x_0^+]+\hbar\sum_{s\geq 0}\sum_{k=1}^{m+n}\delta_{m+n,i+1}(-1)^{p(k)}E_{m+n-1,k}(-s)E_{k,1}(s+1)\\
&\quad-\hbar\sum_{s\geq 0}\sum_{k=1}^{m+n}\delta_{1,i}(-1)^{p(k)+p(E_{1,2})p(E_{m+n,1})}E_{m+n,k}(-s)E_{k,2}(s+1).
\end{aligned}\tag{5.54}
$$

Therefore, when $i\neq 0,1,m+n-1$, $[\mathrm{ev}_\upsilon(x_{i,1}^+),\mathrm{ev}_\upsilon(x_{0,0}^+)]-[\mathrm{ev}_\upsilon(x_{i,0}^+),\mathrm{ev}_\upsilon(x_{0,1}^+)]$ is zero. When $i=1$, $[\mathrm{ev}_\upsilon(x_{1,1}^+),\mathrm{ev}_\upsilon(x_{0,0}^+)]-[\mathrm{ev}_\upsilon(x_{1,0}^+),\mathrm{ev}_\upsilon(x_{0,1}^+)]$ is equal to

$$
\begin{aligned}
&[(\upsilon-\varepsilon_1)x_1^+,E_{m+n,1}(1)]-[E_{1,2},(\upsilon-(m-n)\varepsilon_1)x_0^+]\\
&\quad-\hbar\sum_{s\geq 0}E_{m+n,1}(1-s)E_{1,2}(s)-\hbar\sum_{s\geq 0}\sum_{k=2}^{m+n}(-1)^{p(k)}E_{m+n,k}(-s)E_{k,2}(s+1)\\
&\quad+cE_{m+n,2}(1)+\hbar\sum_{s\geq 0}\sum_{k=1}^{m+n}(-1)^{p(k)}E_{m+n,k}(-s)E_{k,2}(s+1).
\end{aligned}\tag{5.55}
$$

By direct computation, we obtain

$$
\begin{aligned}
&\text{the third term of (5.55)}+\text{the 4-th term of (5.55)}+\text{the 6-th term of (5.55)}\\
&=-\hbar E_{m+n,1}(1)E_{1,2}(0)=-\frac{\hbar}{2}\{E_{1,2}(0),E_{m+n,1}(1)\}+\frac{\hbar}{2}[E_{1,2}(0),E_{m+n,1}(1)].
\end{aligned}
$$

Moreover, by direct computation, we obtain

$$
[(\upsilon-\varepsilon_1)x_1^+,E_{m+n,1}(1)]-[E_{1,2},(\upsilon-(m-n)\varepsilon_1)x_0^+]=(m-n-1)\varepsilon_1[x_1^+,E_{m+n,1}(1)].
$$

Therefore, $[\mathrm{ev}_\upsilon(x_{1,1}^+),\mathrm{ev}_\upsilon(x_{0,0}^+)]-[\mathrm{ev}_\upsilon(x_{1,0}^+),\mathrm{ev}_\upsilon(x_{0,1}^+)]$ is equal to

$$
-\frac{\hbar}{2}\{E_{1,2}(0),E_{m+n,1}(1)\}-\frac{\varepsilon_1-\varepsilon_2}{2}[x_1^+,E_{m+n,1}(1)].
$$

by the assumption $\hbar c=(m-n)\varepsilon_1$.

When $i=m+n-1$, $[\mathrm{ev}_\upsilon(x_{m+n-1,1}^+),\mathrm{ev}_\upsilon(x_{0,0}^+)]-[\mathrm{ev}_\upsilon(x_{m+n-1,0}^+),\mathrm{ev}_\upsilon(x_{0,1}^+)]$ is equal to

$$
\begin{aligned}
&[(\upsilon-(m-n+1)\varepsilon_1)x_{m+n-1}^+,E_{m+n,1}(1)]-[E_{m+n-1,m+n},(\upsilon-(m-n)\varepsilon_1)x_0^+]\\
&+\hbar\sum_{s\geq 0}\sum_{k=1}^{m+n-1}(-1)^{p(k)}E_{m+n-1,k}(-s)E_{k,1}(s+1)\\
&+\hbar\sum_{s\geq 0}\sum_{k=m+n}^{m+n}(-1)^{p(k)}E_{m+n-1,k}(-s-1)E_{k,1}(s+2)\\
&-\hbar\sum_{s\geq 0}\sum_{k=1}^{m+n}(-1)^{p(k)}E_{m+n-1,k}(-s)E_{k,1}(s+1).
\end{aligned}
$$

By direct computation, we obtain

$$\begin{aligned}
&\text{the third term of (5.55)} + \text{the 4-th term of (5.55)} + \text{the 5-th term of (5.55)}\\
&=\hbar E_{m+n-1,m+n}(0)E_{m+n,1}(1)\\
&=\frac{\hbar}{2}\{E_{m+n-1,m+n}(0),E_{m+n,1}(1)\}+\frac{\hbar}{2}[E_{m+n-1,m+n}(0),E_{m+n,1}(1)].
\end{aligned}$$

Moreover, by direct computation, we have

$$[(\upsilon-(m-n+1)\varepsilon_1)x_i^+,E_{m+n,1}(1)]-[E_{i,i+1},(\upsilon-(m-n)\varepsilon_1)x_0^+]=-\varepsilon_1[x_i^+,x_0^+]$$

Then, $[\mathrm{ev}_\upsilon(x^+_{m+n-1,1}),\mathrm{ev}_\upsilon(x^+_{0,0})]-[\mathrm{ev}_\upsilon(x^+_{m+n-1,0}),\mathrm{ev}_\upsilon(x^+_{0,1})]$ is equal to

$$\frac{\hbar}{2}\{E_{m+n-1,m+n}(0),E_{m+n,1}(1)\}-\frac{\varepsilon_1-\varepsilon_2}{2}[x^+_{m+n-1},x^+_0].$$

This completes the proof of (5.9).

5.4 The proof of (5.10)

Finally, we show $[\mathrm{ev}_\upsilon(h_{i,1}),\mathrm{ev}_\upsilon(h_{j,1})]=0$. Suppose that $i,j\neq 0$. It is enough to show the case where $i<j$. We set

$$\begin{aligned}
A_i&=\sum_{s\geq 0}\sum_{k=1}^{i}(-1)^{p(k)}E_{i,k}(-s)E_{k,i}(s), &B_i&=\sum_{s\geq 0}\sum_{k=i+1}^{m+n}(-1)^{p(k)}E_{i,k}(-s-1)E_{k,i}(s+1),\\
C_i&=\sum_{s\geq 0}\sum_{k=1}^{i}(-1)^{p(k)}E_{i+1,k}(-s)E_{k,i+1}(s), &D_i&=\sum_{s\geq 0}\sum_{k=i+1}^{m+n}(-1)^{p(k)}E_{i+1,k}(-s-1)E_{k,i+1}(s+1).
\end{aligned}$$

Then, by the definition of $\mathrm{ev}_\upsilon(h_{i,1})$, we have

$$\begin{aligned}
&[\mathrm{ev}_\upsilon(h_{i,1}),\mathrm{ev}_\upsilon(h_{j,1})]\\
&=(-1)^{p(i)+p(j)}\{[A_i,A_j]+[B_i,A_j]+[B_i,B_j]+[A_i,B_j]\}\\
&\quad+(-1)^{p(i)+p(j+1)}\{[A_i,C_j]+[B_i,C_j]+[B_i,D_j]+[A_i,D_j]\}\\
&\quad+(-1)^{p(i+1)+p(j)}\{[C_i,A_j]+[D_i,B_j]+[D_i,A_j]+[C_i,B_j]\}\\
&\quad+(-1)^{p(i+1)+p(j+1)}\{[C_i,C_j]+[D_i,C_j]+[D_i,D_j]+[C_i,D_j]\}.
\end{aligned}$$

By the definition of A_i, B_i, C_i, and D_i, we obtain

$$[A_i,B_j]=[A_i,D_j]=0.$$

Thus, it is enough to show the following lemma.

Lemma 5.56. *The following relations hold;*

$$\begin{gathered}
[A_i,A_j]+[B_i,A_j]+[B_i,B_j]=0,\\
[A_i,C_j]+[B_i,C_j]+[B_i,D_j]+[A_i,D_j]=0,\\
[C_i,A_j]+[D_i,B_j]+[D_i,B_j]+[C_i,B_j]=0,\\
[C_i,C_j]+[D_i,C_j]+[D_i,D_j]=0.
\end{gathered}$$

Proof. We only show that $[A_i,A_j]+[B_i,A_j]+[B_i,B_j]=0$ holds. Other relations are obtained in the same way. By direct computation, we can rewrite $[A_i,A_j]$ as follows;

$$[A_i,A_j]$$

$$
\begin{aligned}
&= -\sum_{s,t\geq 0}\sum_{k=1}^{i}(-1)^{p(k)+p(i)+p(E_{i,k})p(E_{j,i})}E_{j,k}(-s-t)E_{i,j}(t)E_{k,i}(s)\\
&\quad+\sum_{s,t\geq 0}\sum_{k=1}^{i}\sum_{l=1}^{j}\delta_{k,l}(-1)^{p(E_{i,k})p(E_{j,k})}E_{j,k}(-t)E_{i,j}(-s+t)E_{k,i}(s)\\
&\quad-\sum_{s,t\geq 0}\sum_{k=1}^{i}\sum_{l=1}^{j}\delta_{k,l}(-1)^{p(E_{k,i})p(E_{j,k})}E_{i,k}(-s)E_{j,i}(s-t)E_{k,j}(t)\\
&\quad+\sum_{s,t\geq 0}\sum_{k=1}^{i}(-1)^{p(k)+p(i)+p(E_{k,i})p(E_{i,j})}E_{i,k}(-s)E_{j,i}(-t)E_{k,j}(s+t)\\
&\quad+\sum_{s\geq 0}(sE_{i,i}(-s)E_{j,j}(s)-sE_{j,j}(-s)E_{i,i}(s)). \qquad (5.57)
\end{aligned}
$$

Since we find two relations

$$
\begin{aligned}
&\text{the second term of (5.57)}\\
&=\sum_{s,t\geq 0}\sum_{k=1}^{i}\sum_{l=1}^{j}\delta_{k,l}(-1)^{p(E_{i,k})p(E_{j,k})}E_{j,k}(-s-t)E_{i,j}(t)E_{k,i}(s)\\
&\quad+\sum_{s,t\geq 0}\sum_{k=1}^{i}\sum_{l=1}^{j}\delta_{k,l}(-1)^{p(E_{i,k})p(E_{j,k})}E_{j,k}(-s)E_{i,j}(-t-1)E_{k,i}(s+t+1),\\
&\text{the third term of (5.57)}\\
&=\sum_{s,t\geq 0}\sum_{k=1}^{i}\sum_{l=1}^{j}\delta_{k,l}(-1)^{p(E_{k,i})p(E_{j,l})}E_{i,k}(-s-t-1)E_{j,i}(s+1)E_{k,j}(t)\\
&\quad+\sum_{s,t\geq 0}\sum_{k=1}^{i}\sum_{l=1}^{j}\delta_{k,l}(-1)^{p(E_{k,i})p(E_{k,j})}E_{i,k}(-s)E_{j,i}(-t)E_{k,j}(s+t),
\end{aligned}
$$

we have

$$
\begin{aligned}
&[A_i,A_j]\\
&= -\sum_{s,t\geq 0}\sum_{k=1}^{i}(-1)^{p(k)+p(i)+p(E_{i,k})p(E_{j,i})}E_{j,k}(-s-t)E_{i,j}(t)E_{k,i}(s)\\
&\quad+\sum_{s,t\geq 0}\sum_{k=1}^{i}\sum_{l=1}^{j}\delta_{k,l}(-1)^{p(E_{i,k})p(E_{j,k})}E_{j,k}(-s-t)E_{i,j}(t)E_{k,i}(s)\\
&\quad+\sum_{s,t\geq 0}\sum_{k=1}^{i}\sum_{l=1}^{j}\delta_{k,l}(-1)^{p(E_{i,k})p(E_{j,k})}E_{j,k}(-s)E_{i,j}(-t-1)E_{k,i}(s+t+1)\\
&\quad-\sum_{s,t\geq 0}\sum_{k=1}^{i}\sum_{l=1}^{j}\delta_{k,l}(-1)^{p(E_{k,i})p(E_{j,l})}E_{i,k}(-s-t-1)E_{j,i}(s+1)E_{k,j}(t)\\
&\quad-\sum_{s,t\geq 0}\sum_{k=1}^{i}\sum_{l=1}^{j}\delta_{k,l}(-1)^{p(E_{k,i})p(E_{k,j})}E_{i,k}(-s)E_{j,i}(-t)E_{k,j}(s+t)\\
&\quad+\sum_{s,t\geq 0}\sum_{k=1}^{i}(-1)^{p(k)+p(i)+p(E_{k,i})p(E_{i,j})}E_{i,k}(-s)E_{j,i}(-t)E_{k,j}(s+t)
\end{aligned}
$$

$$+\sum_{s\geq 0}(sE_{i,i}(-s)E_{j,j}(s)-sE_{j,j}(-s)E_{i,i}(s)). \tag{5.58}$$

We simplify the right hand side of (5.58). By direct computation, we obtain

$$\begin{aligned}
&\text{the first term of (5.58)} + \text{the second term of (5.58)}\\
&=-\sum_{s,t\geq 0}\sum_{k=1}^{i}(-1)^{p(k)+p(i)+p(E_{i,k})p(E_{j,i})}E_{j,k}(-s-t)E_{i,j}(t)E_{k,i}(s)\\
&\quad+\sum_{s,t\geq 0}\sum_{k=1}^{i}(-1)^{p(E_{i,k})p(E_{j,k})}E_{j,k}(-s-t)E_{i,j}(t)E_{k,i}(s)\\
&=0
\end{aligned} \tag{5.59}$$

since $p(k)+p(i)+p(E_{i,k})p(E_{j,i})=p(E_{i,k})p(E_{j,k})$. Similarly, we have

$$\text{the 4-th term of (5.58)} + \text{the 6-th term of (5.58)} = 0. \tag{5.60}$$

By (5.59) and (5.60), we find the equality

$$\begin{aligned}
&[A_i,A_j]\\
&=\sum_{s,t\geq 0}\sum_{k=1}^{i}\sum_{l=1}^{j}\delta_{k,l}(-1)^{p(k)+p(l)+p(E_{i,k})p(E_{j,l})}E_{j,l}(-s)E_{i,j}(-t-1)E_{k,i}(s+t+1)\\
&\quad-\sum_{s,t\geq 0}\sum_{k=1}^{i}\sum_{l=1}^{j}\delta_{k,l}(-1)^{p(k)+p(l)+p(E_{k,i})p(E_{j,l})}E_{i,k}(-s-t-1)E_{j,i}(s+1)E_{l,j}(t)\\
&\quad+\sum_{s\geq 0}(sE_{i,i}(-s)E_{j,j}(s)-sE_{j,j}(-s)E_{i,i}(s)).
\end{aligned}$$

Computing the parity, we obtain

$$\begin{aligned}
&[A_i,A_j]\\
&=\sum_{s,t\geq 0}\sum_{k=1}^{i}(-1)^{p(E_{i,k})p(E_{j,k})}E_{j,k}(-s)E_{i,j}(-t-1)E_{k,i}(s+t+1)\\
&\quad-\sum_{s,t\geq 0}\sum_{k=1}^{i}(-1)^{p(E_{k,i})p(E_{j,k})}E_{i,k}(-s-t-1)E_{j,i}(s+1)E_{k,j}(t)\\
&\quad+\sum_{s\geq 0}(sE_{i,i}(-s)E_{j,j}(s)-sE_{j,j}(-s)E_{i,i}(s)).
\end{aligned} \tag{5.61}$$

Similarly, by direct computation, we have

$$\begin{aligned}
&[B_i,B_j]\\
&=\sum_{s,t\geq 0}\sum_{l=j+1}^{m+n}(-1)^{p(j)+p(l)+p(E_{j,l})p(E_{j,i})}E_{i,l}(-s-t-2)E_{j,i}(s+1)E_{l,j}(t+1)\\
&\quad-\sum_{s,t\geq 0}\sum_{k=i+1}^{m+n}\sum_{l=j+1}^{m+n}\delta_{k,l}(-1)^{p(E_{j,k})p(E_{k,i})}E_{i,k}(-s-t-2)E_{j,i}(s+1)E_{k,j}(t+1)\\
&\quad-\sum_{s,t\geq 0}\sum_{k=i+1}^{m+n}\sum_{l=j+1}^{m+n}\delta_{k,l}(-1)^{p(E_{j,k})p(E_{k,i})}E_{i,k}(-s-1)E_{j,i}(-t)E_{k,j}(s+t+1)
\end{aligned}$$

$$+\sum_{s,t\geq 0}\sum_{k=i+1}^{m+n}\sum_{l=j+1}^{m+n}\delta_{k,l}(-1)^{p(E_{k,i})p(E_{k,j})}E_{j,l}(-s-t-1)E_{i,j}(t)E_{k,i}(s+1)$$
$$+\sum_{s,t\geq 0}\sum_{k=i+1}^{m+n}\sum_{l=j+1}^{m+n}\delta_{k,l}(-1)^{p(E_{k,i})p(E_{k,j})}E_{j,k}(-t-1)E_{i,j}(-s-1)E_{k,i}(s+t+2)$$
$$-\sum_{s,t\geq 0}\sum_{l=j+1}^{m+n}(-1)^{p(j)+p(l)+p(E_{j,i})p(E_{l,j})}E_{j,l}(-t-1)E_{i,j}(-s-1)E_{l.i}(s+t+2). \quad (5.62)$$

We simplify the right hand side of (5.62). By direct computation, we obtain

$$\text{the first term of (5.62)} + \text{the second term of (5.62)} = 0 \quad (5.63)$$

and

$$\text{the 5-th term of (5.62)} + \text{the 6-th term of (5.62)} = 0 \quad (5.64)$$

By (5.63) and (5.64), we find the equality

$$\begin{aligned}&[B_i, B_j]\\
&=-\sum_{s,t\geq 0}\sum_{k=i+1}^{m+n}\sum_{l=j+1}^{m+n}\delta_{k,l}(-1)^{p(E_{j,k})p(E_{k,i})}E_{i,k}(-s-1)E_{j,i}(-t)E_{k,j}(s+t+1)\\
&\quad+\sum_{s,t\geq 0}\sum_{k=i+1}^{m+n}\sum_{l=j+1}^{m+n}\delta_{k,l}(-1)^{p(E_{k,i})p(E_{k,j})}E_{j,k}(-s-t-1)E_{i,j}(t)E_{k,i}(s+1)\\
&=-\sum_{s,t\geq 0}\sum_{l=j+1}^{m+n}(-1)^{p(E_{j,l})p(E_{l,i})}E_{i,l}(-s-1)E_{j,i}(-t)E_{l,j}(s+t+1)\\
&\quad+\sum_{s,t\geq 0}\sum_{l=j+1}^{m+n}(-1)^{p(E_{l,i})p(E_{l,j})}E_{j,l}(-s-t-1)E_{i,j}(t)E_{l,i}(s+1).\end{aligned} \quad (5.65)$$

By direct computation, we also obtain

$$\begin{aligned}&[B_i, A_j]\\
&=\sum_{s,t\geq 0}\sum_{l=1}^{j}(-1)^{p(j)+p(l)}E_{i,l}(-s-t-1)E_{l,j}(t)E_{j,i}(s+1)\\
&\quad-\sum_{s,t\geq 0}\sum_{k=i+1}^{m+n}(-1)^{p(k)+p(i)+p(E_{i,k})p(E_{j,i})}E_{j,k}(-s-t-1)E_{i,j}(t)E_{k,i}(s+1)\\
&\quad+\sum_{s,t\geq 0}\sum_{k=i+1}^{m+n}\sum_{l=1}^{j}\delta_{k,l}(-1)^{p(E_{i,k})p(E_{j,l})}E_{j,l}(-s-t-1)E_{i,j}(t)E_{k,i}(s+1)\\
&\quad+\sum_{s,t\geq 0}\sum_{k=i+1}^{m+n}\sum_{l=1}^{j}\delta_{k,l}(-1)^{p(E_{i,k})p(E_{j,l})}E_{j,l}(-s)E_{i,j}(-t-1)E_{k,i}(s+t+1)\\
&\quad-\sum_{s,t\geq 0}\sum_{k=i+1}^{m+n}\sum_{l=1}^{j}\delta_{k,l}(-1)^{p(E_{k,i})p(E_{j,l})}E_{i,k}(-s-1)E_{j,i}(-t)E_{l,j}(s+t+1)\\
&\quad-\sum_{s,t\geq 0}\sum_{k=i+1}^{m+n}\sum_{l=1}^{j}\delta_{k,l}(-1)^{p(E_{k,i})p(E_{j,l})}E_{i,k}(-s-t-1)E_{j,i}(s+1)E_{l,j}(t)\end{aligned}$$

$$+\sum_{s,t\geq 0}\sum_{k=i+1}^{m+n}(-1)^{p(k)+p(i)+p(E_{k,i})p(E_{i,j})}E_{i,k}(-s-1)E_{j,i}(-t)E_{k,j}(s+t+1)$$
$$-\sum_{s,t\geq 0}\sum_{l=1}^{j}(-1)^{p(j)+p(l)}E_{i,j}(-s-1)E_{j,l}(-t)E_{l,i}(s+t+1). \tag{5.66}$$

Let us simplify the right hand side of (5.66). We prepare the following four relations by direct computation;

$$\begin{aligned}
&\text{the second term of (5.66)} + \text{the third term of (5.66)}\\
&=-\sum_{s,t\geq 0}\sum_{k=i+1}^{m+n}(-1)^{p(k)+p(i)+p(E_{i,k})p(E_{j,i})}E_{j,k}(-s-t-1)E_{i,j}(t)E_{k,i}(s+1)\\
&\quad+\sum_{s,t\geq 0}\sum_{k=i+1}^{m+n}\sum_{l=1}^{j}\delta_{k,l}(-1)^{p(E_{i,k})p(E_{j,k})}E_{j,k}(-s-t-1)E_{i,j}(t)E_{k,i}(s+1)\\
&=-\sum_{s,t\geq 0}\sum_{k=j+1}^{m+n}(-1)^{p(E_{i,k})p(E_{j,i})}E_{j,k}(-s-t-1)E_{i,j}(t)E_{k,i}(s+1),
\end{aligned}\tag{5.67}$$

$$\begin{aligned}
&\text{the first term of (5.66)} + \text{the 6-th term of (5.66)}\\
&=\sum_{s,t\geq 0}\sum_{l=1}^{j}(-1)^{p(j)+p(l)}E_{i,l}(-s-t-1)E_{l,j}(t)E_{j,i}(s+1)\\
&\quad-\sum_{s,t\geq 0}\sum_{l=1}^{j}\sum_{k=i+1}^{m+n}\delta_{k,l}(-1)^{p(E_{k,i})p(E_{j,k})}E_{i,k}(-s-t-1)E_{j,i}(s+1)E_{k,j}(t)\\
&=\sum_{s,t\geq 0}\sum_{k=1}^{i}(-1)^{p(E_{k,i})p(E_{j,k})}E_{i,k}(-s-t-1)E_{j,i}(s+1)E_{k,j}(t)\\
&\quad+\sum_{s,t\geq 0}\sum_{l=1}^{j}(-1)^{p(j)+p(l)}E_{i,l}(-s-t-1)[E_{l,j}(t),E_{j,i}(s+1)]\\
&=\sum_{s,t\geq 0}\sum_{k=1}^{i}(-1)^{p(E_{k,i})p(E_{j,k})}E_{i,k}(-s-t-1)E_{j,i}(s+1)E_{k,j}(t)\\
&\quad+\sum_{s,t\geq 0}\sum_{l=1}^{j}(-1)^{p(j)+p(l)}E_{i,l}(-s-t-1)E_{l,i}(s+t+1)\\
&\quad-\sum_{s,t\geq 0}E_{i,i}(-s-t-1)E_{j,j}(s+t+1),
\end{aligned}\tag{5.68}$$

$$\begin{aligned}
&\text{the 4-th term of (5.66)} + \text{the 8-th term of (5.66)}\\
&=\sum_{s,t\geq 0}\sum_{k=i+1}^{m+n}\sum_{l=1}^{j}\delta_{k,l}(-1)^{p(E_{i,k})p(E_{j,k})}E_{j,k}(-s)E_{i,j}(-t-1)E_{k,i}(s+t+1)\\
&\quad-\sum_{s,t\geq 0}\sum_{l=1}^{j}(-1)^{p(k)+p(l)}E_{i,j}(-s-1)E_{j,l}(-t)E_{l,i}(s+t+1)\\
&=-\sum_{s,t\geq 0}\sum_{l=1}^{i}(-1)^{p(E_{i,l})p(E_{j,l})}E_{j,l}(-s)E_{i,j}(-t-1)E_{l,i}(s+t+1)
\end{aligned}$$

$$
\begin{aligned}
&-\sum_{s,t\geq 0}\sum_{l=1}^{j}(-1)^{p(j)+p(l)}[E_{i,j}(-s-1),E_{j,l}(-t)]E_{l,i}(s+t+1)\\
=&-\sum_{s,t\geq 0}\sum_{l=1}^{i}(-1)^{p(E_{i,l})p(E_{j,l})}E_{j,l}(-s)E_{i,j}(-t-1)E_{l,i}(s+t+1)\\
&-\sum_{s,t\geq 0}\sum_{l=1}^{j}(-1)^{p(j)+p(l)}E_{i,l}(-s-t-1)E_{l,i}(s+t+1)\\
&+\sum_{s,t\geq 0}E_{j,j}(-s-t-1)E_{i,i}(s+t+1), \qquad (5.69)
\end{aligned}
$$

the 5-th term of (5.66) + the 7-th term of (5.66)

$$
\begin{aligned}
=&-\sum_{s,t\geq 0}\sum_{k=i+1}^{m+n}\sum_{l=1}^{j}\delta_{k,l}(-1)^{p(E_{k,i})p(E_{j,k})}E_{i,k}(-s-1)E_{j,i}(-t)E_{k,j}(s+t+1)\\
&+\sum_{s,t\geq 0}\sum_{k=i+1}^{m+n}(-1)^{p(k)+p(i)+p(E_{k,i})p(E_{i,j})}E_{i,k}(-s-1)E_{j,i}(-t)E_{k,j}(s+t+1)\\
=&\sum_{s,t\geq 0}\sum_{k=j+1}^{m+n}(-1)^{p(E_{k,i})p(E_{j,k})}E_{i,k}(-s-1)E_{j,i}(-t)E_{k,j}(s+t+1). \qquad (5.70)
\end{aligned}
$$

Thus, by (5.67)-(5.70), we have

$$
\begin{aligned}
&[B_i,A_j]\\
=&-\sum_{s,t\geq 0}\sum_{k=j+1}^{m+n}(-1)^{p(E_{i,k})p(E_{j,i})}E_{j,k}(-s-t-1)E_{i,j}(t)E_{k,i}(s+1)\\
&+\sum_{s,t\geq 0}\sum_{k=1}^{i}(-1)^{p(E_{k,i})p(E_{j,k})}E_{i,k}(-s-t-1)E_{j,i}(s+1)E_{k,j}(t)\\
&-\sum_{s,t\geq 0}\sum_{l=1}^{i}(-1)^{p(E_{i,l})p(E_{j,l})}E_{j,l}(-s)E_{i,j}(-t-1)E_{l,i}(s+t+1)\\
&+\sum_{s,t\geq 0}\sum_{k=j+1}^{m+n}(-1)^{p(E_{k,i})p(E_{j,k})}E_{i,k}(-s-1)E_{j,i}(-t)E_{k,j}(s+t+1)\\
&-\sum_{s\geq 0}(sE_{i,i}(-s)E_{j,j}(s)-sE_{j,j}(-s)E_{i,i}(s)). \qquad (5.71)
\end{aligned}
$$

Adding (5.61), (5.65), and (5.71), we obtain $[A_i,A_j]+[B_i,A_j]+[B_i,B_j]=0$. □

This completes the proof of Lemma 5.6.

6 The surjectivity of the evaluation map

In this section, we show that the image of ev_υ is dense in the completion of $U(\widehat{\mathfrak{gl}}(m|n))$ provided that $\varepsilon_1\neq 0$. By the definition of ev_υ, the image of ev_υ contains h_i and $x_i^{\pm}$. Since h_i and $x_i^{\pm}$ are generators of $\widehat{\mathfrak{sl}}(m|n)$, the image of ev_υ contains $\widehat{\mathfrak{sl}}(m|n)$. Thus, it is enough to prove that the image of ev_υ contains $E_{i,i}(s)$ for all $1\leq i\leq m+n$ and $s\in\mathbb{Z}$.

First, we show that the image of ev_υ contains $E_{i,i}(0)$.

Theorem 6.1. *We obtain*

$$\begin{aligned}
&\mathrm{ev}_\upsilon(\sum_{0\leq i\leq m+n-1}\tilde{h}_{i,1})\\
&=(\upsilon-(m-n)\varepsilon_1)h_0+\sum_{1\leq i\leq m+n-1}(\upsilon-(i-2\delta(i\geq m+1)(i-m))\varepsilon_1)h_i-c\hbar E_{m+n,m+n}.
\end{aligned}\tag{6.2}$$

Proof. By the definition of $\mathrm{ev}_\upsilon(h_{i,1})$, we obtain

$$\mathrm{ev}_\upsilon(\tilde{h}_{i,1})=\begin{cases}
(\upsilon-(m-n)\varepsilon_1)h_0-\frac{1}{2}\hbar E_{m+n,m+n}^2-\frac{1}{2}\hbar E_{1,1}^2-c\hbar E_{m+n,m+n}\\
\quad-\hbar\sum_{s\geq0}\sum_{k=1}^{m+n}(-1)^{p(k)}E_{m+n,k}(-s)E_{k,m+n}(s)\\
\quad\quad-\hbar\sum_{s\geq0}\sum_{k=1}^{m+n}(-1)^{p(k)}E_{1,k}(-s-1)E_{k,1}(s+1)\\
\hfill\text{if } i=0,\\
(\upsilon-(i-2\delta(i\geq m+1)(i-m))\varepsilon_1)h_i-\frac{1}{2}\hbar E_{i,i}^2-\frac{1}{2}\hbar E_{i+1,i+1}^2\\
\quad+\hbar(-1)^{p(i)}\sum_{s\geq0}\sum_{k=1}^{i}(-1)^{p(k)}E_{i,k}(-s)E_{k,i}(s)\\
\quad\quad+\hbar(-1)^{p(i)}\sum_{s\geq0}\sum_{k=i+1}^{m+n}(-1)^{p(k)}E_{i,k}(-s-1)E_{k,i}(s+1)\\
\quad\quad\quad-\hbar(-1)^{p(i+1)}\sum_{s\geq0}\sum_{k=1}^{i}(-1)^{p(k)}E_{i+1,k}(-s)E_{k,i+1}(s)\\
\quad\quad\quad\quad-\hbar(-1)^{p(i+1)}\sum_{s\geq0}\sum_{k=i+1}^{m+n}(-1)^{p(k)}E_{i+1,k}(-s-1)E_{k,i+1}(s+1)\\
\hfill\text{if } i\neq0.
\end{cases}$$

Then, we rewrite the left hand side of (6.2) as

$$\begin{aligned}
&(\upsilon-(m-n)\varepsilon_1)h_0+\sum_{1\leq i\leq m+n-1}(\upsilon-(i-2\delta(i\geq m+1)(i-m))\varepsilon_1)h_i\\
&\quad-\frac{\hbar}{2}(E_{1,1}^2+E_{m+n,m+n}^2)-c\hbar E_{m+n,m+n}-\sum_{1\leq i\leq m+n-1}\frac{\hbar}{2}(E_{i,i}^2+E_{i+1,i+1}^2)\\
&\quad+\sum_{1\leq i\leq m+n}\hbar(-1)^{p(i)}\sum_{s\geq0}\sum_{k=1}^{i}(-1)^{p(k)}E_{i,k}(-s)E_{k,i}(s)\\
&\quad+\sum_{1\leq i\leq m+n}\hbar(-1)^{p(i)}\sum_{s\geq0}\sum_{k=i+1}^{m+n}(-1)^{p(k)}E_{i,k}(-s-1)E_{k,i}(s+1)\\
&\quad-\sum_{0\leq i\leq m+n-1}\hbar(-1)^{p(i+1)}\sum_{s\geq0}\sum_{k=1}^{i}(-1)^{p(k)}E_{i+1,k}(-s)E_{k,i+1}(s)\\
&\quad-\sum_{0\leq i\leq m+n-1}\hbar(-1)^{p(i+1)}\sum_{s\geq0}\sum_{k=i+1}^{m+n}(-1)^{p(k)}E_{i+1,k}(-s-1)E_{k,i+1}(s+1).
\end{aligned}\tag{6.3}$$

Adding the first and third terms of (6.3), we have

$$\sum_{1\leq i\leq m+n}\hbar(-1)^{p(i)}\sum_{s\geq0}\sum_{k=1}^{i}(-1)^{p(k)}E_{i,k}(-s)E_{k,i}(s)$$

$$
\begin{aligned}
&- \sum_{1\le j\le m+n} \hbar(-1)^{p(j)} \sum_{s\ge 0}\sum_{k=1}^{j-1} (-1)^{p(k)} E_{j,k}(-s)E_{k,j}(s)\\
&= \hbar \sum_{1\le i\le m+n}\sum_{s\ge 0} E_{i,i}(-s)E_{i,i}(s). \qquad (6.4)
\end{aligned}
$$

Adding the second and 4-th terms of (6.3), we obtain

$$
\begin{aligned}
&\sum_{1\le i\le m+n} \hbar(-1)^{p(i)} \sum_{s\ge 0}\sum_{k=i+1}^{m+n} (-1)^{p(k)} E_{i,k}(-s-1)E_{k,i}(s+1)\\
&\quad - \sum_{1\le j\le m+n} \hbar(-1)^{p(j)} \sum_{s\ge 0}\sum_{k=j}^{m+n} (-1)^{p(k)} E_{j,k}(-s-1)E_{k,j}(s+1)\\
&= -\hbar \sum_{1\le i\le m+n}\sum_{s\ge 0} E_{i,i}(-s-1)E_{i,i}(s+1). \qquad (6.5)
\end{aligned}
$$

Applying (6.4) and (6.5) to (6.3), we find that the left hand side of (6.2) is equal to

$$
\begin{aligned}
&(\upsilon-(m-n)\varepsilon_1)h_0 + \sum_{1\le i\le m+n-1} (\upsilon-(i-2\delta(i\ge m+1)(i-m))\varepsilon_1)h_i\\
&\quad - \frac{\hbar}{2}(E_{1,1}^2+E_{m+n,m+n}^2) - c\hbar E_{m+n,m+n} - \sum_{1\le i\le m+n-1}\frac{\hbar}{2}(E_{i,i}^2+E_{i+1,i+1}^2) + \hbar\sum_{1\le i\le m+n} E_{i,i}^2.
\end{aligned}
$$

By direct computation, it is equal to

$$
(\upsilon-(m-n)\varepsilon_1)h_0 + \sum_{1\le i\le m+n-1} (\upsilon-(i-2\delta(i\ge m+1)(i-m))\varepsilon_1)h_i - c\hbar E_{m+n,m+n}.
$$

Thus, we have obtained Theorem 6.1. □

Since h_i is contained in the image of ev_υ, the image of ev_υ contains $c\hbar E_{m+n,m+n}$.

Corollary 6.6. *The image of* ev_υ *contains* $E_{m+n,m+n}$ *provided that* $\hbar c\neq 0$.

Next, let us show that the completion of the image of ev_υ contains $E_{i,i}(s)$ $(s\neq 0)$.

Theorem 6.7. *For all* $i\neq 0$, *we obtain*

$$
\begin{aligned}
&[\mathrm{ev}_\upsilon(h_{i,1}), ((-1)^{p(i)}E_{i,i}-(-1)^{p(i+1)}E_{i+1,i+1})t^a]\\
&= \hbar\sum_{s\ge 0}\delta_{s+a,0} scE_{i,i}(-s) - \hbar\sum_{s\ge 0}\delta_{-s+a,0} scE_{i,i}(s)\\
&\quad + \hbar\sum_{s\ge 0}\delta_{s+1+a,0}(s+1)cE_{i+1,i+1}(-s-1) - \hbar\sum_{s\ge 0}\delta_{-s-1+a,0}(s+1)cE_{i+1,i+1}(s+1)\\
&\quad + \text{sum of elements of the completion of } U(\widehat{\mathfrak{sl}}(m|n)).
\end{aligned}
$$

Proof. The proof is done by direct computation. By the definition of $\mathrm{ev}_\upsilon(h_{i,1})$, we have

$$
\begin{aligned}
&[\mathrm{ev}_\upsilon(h_{i,1}), ((-1)^{p(i)}E_{i,i}-(-1)^{p(i+1)}E_{i+1,i+1})t^a]\\
&= [(\upsilon-(i-2\delta(i\ge m+1)(i-m))\varepsilon_1)h_i, ((-1)^{p(i)}E_{i,i}-(-1)^{p(i+1)}E_{i+1,i+1})t^a]\\
&\quad - (-1)^{p(E_{i,i+1})}\hbar[E_{i,i}E_{i+1,i+1}, ((-1)^{p(i)}E_{i,i}-(-1)^{p(i+1)}E_{i+1,i+1})t^a]\\
&\quad + [\hbar(-1)^{p(i)}\sum_{s\ge 0}\sum_{k=1}^{i}(-1)^{p(k)}E_{i,k}(-s)E_{k,i}(s), ((-1)^{p(i)}E_{i,i}-(-1)^{p(i+1)}E_{i+1,i+1})t^a]
\end{aligned}
$$

$$+[\hbar(-1)^{p(i)}\sum_{s\geq 0}\sum_{k=i+1}^{m+n}(-1)^{p(k)}E_{i,k}(-s-1)E_{k,i}(s+1),((-1)^{p(i)}E_{i,i}-(-1)^{p(i+1)}E_{i+1,i+1})t^a]$$

$$-[\hbar(-1)^{p(i+1)}\sum_{s\geq 0}\sum_{k=1}^{i}(-1)^{p(k)}E_{i+1,k}(-s)E_{k,i+1}(s),((-1)^{p(i)}E_{i,i}-(-1)^{p(i+1)}E_{i+1,i+1})t^a]$$

$$-[\hbar(-1)^{p(i+1)}\sum_{s\geq 0}\sum_{k=i+1}^{m+n}(-1)^{p(k)}E_{i+1,k}(-s-1)E_{k,i+1}(s+1), ((-1)^{p(i)}E_{i,i}-(-1)^{p(i+1)}E_{i+1,i+1})t^a]. \quad (6.8)$$

We can rewrite each terms of the right hand side of (6.8). By an easy computation, we find that the first two terms of the right hand side of (6.8). Other terms are computed as follows.

Claim 6.9. (1) The 4-th and 5-th terms of the right hand side of (6.8) are elements of the completion of $U(\widehat{\mathfrak{sl}}(m|n))$.

(2) We can rewrite the third term of the right hand side of (6.8) as

$$\hbar(-1)^{p(i)}\sum_{s\geq 0}\delta_{s+a,0}scE_{i,i}(-s)-\hbar\sum_{s\geq 0}\delta_{-s+a,0}scE_{i,i}(s)$$
$$+\text{ an element of the completion of } U(\widehat{\mathfrak{sl}}(m|n)). \quad (6.10)$$

(3) We can rewrite 6-th term of the right hand side of (6.8) as

$$\hbar\sum_{s\geq 0}\delta_{s+1+a,0}(s+1)cE_{i+1,i+1}(-s-1)-\hbar\sum_{s\geq 0}\delta_{-s-1+a,0}(s+1)cE_{i+1,i+1}(s+1)$$
$$+\text{ an element of the completion of } U(\widehat{\mathfrak{sl}}(m|n)). \quad (6.11)$$

Assuming Claim 6.9, we obtain Theorem 6.7 by adding (6.10) and (6.11). In order to complete the proof of Theorem 6.7, we prove Claim 6.9.

the proof of Claim 6.9. (1) The proof is due to direct computation. First we prove the 4-th case. We can rewrite the 4-th term of the right hand side of (6.8) as follows;

$$\hbar(-1)^{p(i)}\sum_{s\geq 0}\sum_{k=i+1}^{m+n}(-1)^{p(k)}E_{i,k}(-s-1)[E_{k,i}(s+1),((-1)^{p(i)}E_{i,i}-(-1)^{p(i+1)}E_{i+1,i+1})t^a]$$
$$+\hbar(-1)^{p(i)}\sum_{s\geq 0}\sum_{k=i+1}^{m+n}(-1)^{p(k)}[E_{i,k}(-s-1),((-1)^{p(i)}E_{i,i}-(-1)^{p(i+1)}E_{i+1,i+1})t^a]E_{k,i}(s+1). \quad (6.12)$$

We rewrite each terms of the right hand side of (6.12). By direct computation, we can rewrite the first term of the right hand side of (6.12) as

$$\hbar\sum_{s\geq 0}\sum_{k=i+1}^{m+n}(-1)^{p(k)}E_{i,k}(-s-1)E_{k,i}(s+1+a)$$
$$+\hbar(-1)^{p(i)+p(i+1)}\sum_{s\geq 0}(-1)^{p(i+1)}E_{i,i+1}(-s-1)E_{i+1,i}(s+1+a). \quad (6.13)$$

By direct computation, we can rewrite the second term of the right hand side of (6.12) as

$$-\hbar(-1)^{p(i)}\sum_{s\geq 0}E_{i,i+1}(-s-1+a)E_{i+1,i}(s+1)$$

$$-\hbar\sum_{s\geq 0}\sum_{k=i+1}^{m+n}(-1)^{p(k)}E_{i,k}(-s-1+a)E_{k,i}(s+1). \tag{6.14}$$

Adding (6.13) and (6.14), we obtain

$$\begin{aligned}
&\text{the 4-th term of the right hand side of (6.8)}\\
&=\hbar\sum_{s\geq 0}\sum_{k=i+1}^{m+n}(-1)^{p(k)}E_{i,k}(-s-1)E_{k,i}(s+1+a)+\hbar(-1)^{p(i)}\sum_{s\geq 0}E_{i,i+1}(-s-1)E_{i+1,i}(s+1+a)\\
&\quad-\hbar(-1)^{p(i)}\sum_{s\geq 0}E_{i,i+1}(-s-1+a)E_{i+1,i}(s+1)\\
&\quad-\hbar\sum_{s\geq 0}\sum_{k=i+1}^{m+n}(-1)^{p(k)}E_{i,k}(-s-1+a)E_{k,i}(s+1),
\end{aligned} \tag{6.15}$$

Since all of the terms of the right hand side of (6.15) are elements of the completion of $U(\widehat{\mathfrak{sl}}(m|n))$, the 4-th term of the right hand side of (6.8) is an element of the completion of $U(\widehat{\mathfrak{sl}}(m|n))$.

Next, we prove the 5-th case. Let us rewrite the 5-th term of the right hand side of (6.8) as follows;

$$\begin{aligned}
&-\hbar(-1)^{p(i+1)}\sum_{s\geq 0}\sum_{k=1}^{i}(-1)^{p(k)}E_{i+1,k}(-s)[E_{k,i+1}(s),((-1)^{p(i)}E_{i,i}-(-1)^{p(i+1)}E_{i+1,i+1})t^a]\\
&\quad-\hbar(-1)^{p(i+1)}\sum_{s\geq 0}\sum_{k=1}^{i}(-1)^{p(k)}[E_{i+1,k}(-s),((-1)^{p(i)}E_{i,i}-(-1)^{p(i+1)}E_{i+1,i+1})t^a]E_{k,i+1}(s).
\end{aligned} \tag{6.16}$$

We rewrite each terms of the right hand side of (6.16). By direct computation, we can rewrite the first term of the right hand side of (6.16) as

$$\hbar\sum_{s\geq 0}\sum_{k=1}^{i}(-1)^{p(k)}E_{i+1,k}(-s)E_{k,i+1}(s+a)+\hbar(-1)^{p(i+1)}\sum_{s\geq 0}E_{i+1,i}(-s)E_{i,i+1}(s+a). \tag{6.17}$$

By direct computation, we can also rewrite the first term of the right hand side of (6.16) as

$$-\hbar(-1)^{p(i+1)}\sum_{s\geq 0}E_{i+1,i}(-s+a)E_{i,i+1}(s)-\hbar\sum_{s\geq 0}\sum_{k=1}^{i}(-1)^{p(k)}E_{i+1,k}(a-s)E_{k,i+1}(s). \tag{6.18}$$

Adding (6.17) and (6.18), we have

$$\begin{aligned}
&\text{the 5-th term of the right hand side of (6.8)}\\
&=\hbar\sum_{s\geq 0}\sum_{k=1}^{i}(-1)^{p(k)}E_{i+1,k}(-s)E_{k,i+1}(s+a)+\hbar(-1)^{p(i+1)}\sum_{s\geq 0}E_{i+1,i}(-s)E_{i,i+1}(s+a)\\
&\quad-\hbar(-1)^{p(i+1)}\sum_{s\geq 0}E_{i+1,i}(-s+a)E_{i,i+1}(s)-\hbar\sum_{s\geq 0}\sum_{k=1}^{i}(-1)^{p(k)}E_{i+1,k}(a-s)E_{k,i+1}(s).
\end{aligned} \tag{6.19}$$

Since all of the terms of the right hand side of (6.19) are elements of the completion of $U(\widehat{\mathfrak{sl}}(m|n))$, the 5-th term of the right hand side of (6.8) is an element of the completion of $U(\widehat{\mathfrak{sl}}(m|n))$.

(2) The proof is due to direct computation. Let us rewrite the third term of the right hand side of (6.8) as follows;

$$\hbar(-1)^{p(i)}\sum_{s\geq 0}\sum_{k=1}^{i}(-1)^{p(k)}E_{i,k}(-s)[E_{k,i}(s),((-1)^{p(i)}E_{i,i}-(-1)^{p(i+1)}E_{i+1,i+1})t^a]$$

$$+\hbar(-1)^{p(i)}\sum_{s\geq 0}\sum_{k=1}^{i}(-1)^{p(k)}[E_{i,k}(-s),((-1)^{p(i)}E_{i,i}-(-1)^{p(i+1)}E_{i+1,i+1})t^a]E_{k,i}(s). \quad (6.20)$$

We rewrite each terms of the right hand side of (6.20). By direct computation, we can rewrite the first term of the right hand side of (6.20) as

$$\begin{aligned}
&\hbar\sum_{s\geq 0}\sum_{k=1}^{i}(-1)^{p(k)}E_{i,k}(-s)E_{k,i}(s+a)-\hbar(-1)^{p(i)}\sum_{s\geq 0}E_{i,i}(-s)E_{i,i}(s+a)\\
&\quad+\hbar\sum_{s\geq 0}\delta_{s+a,0}scE_{i,i}(-s)+\hbar\sum_{s\geq 0}\delta_{s+a,0}sE_{i,i}(-s)-\hbar\sum_{s\geq 0}\delta_{s+a,0}sE_{i,i}(-s)\\
&=\hbar\sum_{s\geq 0}\sum_{k=1}^{i-1}(-1)^{p(k)}E_{i,k}(-s)E_{k,i}(s+a)+\hbar\sum_{s\geq 0}\delta_{s+a,0}scE_{i,i}(-s). \qquad (6.21)
\end{aligned}$$

Similarly, we can rewrite the second term of the right hand side of (6.20) as

$$\begin{aligned}
&\hbar(-1)^{p(i)}\sum_{s\geq 0}E_{i,i}(-s+a)E_{i,i}(s)-\hbar\sum_{s\geq 0}\sum_{k=1}^{i}(-1)^{p(k)}E_{i,k}(-s+a)E_{k,i}(s)\\
&\quad-\hbar\sum_{s\geq 0}\delta_{-s+a,0}scE_{i,i}(s)-\hbar\sum_{s\geq 0}\delta_{-s+a,0}sE_{i,i}(s)+\hbar\sum_{s\geq 0}\delta_{-s+a,0}sE_{i,i}(s)\\
&=-\hbar\sum_{s\geq 0}\sum_{k=1}^{i-1}(-1)^{p(k)}E_{i,k}(-s+a)E_{k,i}(s)-\hbar\sum_{s\geq 0}\delta_{-s+a,0}scE_{i,i}(s). \qquad (6.22)
\end{aligned}$$

Adding (6.21) and (6.22), we obtain

$$\begin{aligned}
&\text{the third term of the right hand side of (6.8)}\\
&=\hbar\sum_{s\geq 0}\sum_{k=1}^{i-1}(-1)^{p(k)}E_{i,k}(-s)E_{k,i}(s+a)-\hbar\sum_{s\geq 0}\sum_{k=1}^{i-1}(-1)^{p(k)}E_{i,k}(-s+a)E_{k,i}(s)\\
&\quad+\hbar(-1)^{p(i)}\sum_{s\geq 0}\delta_{s+a,0}scE_{i,i}(-s)-\hbar\sum_{s\geq 0}\delta_{-s+a,0}scE_{i,i}(s). \qquad (6.23)
\end{aligned}$$

Since the first two terms of the right hand side of (6.23) are elements of the completion of $U(\widehat{\mathfrak{sl}}(m|n))$, we have obtained (6.10).

(3) We rewrite the 6-th term of the right hand side of (6.8) as follows;

$$\begin{aligned}
&-\hbar(-1)^{p(i+1)}\sum_{s\geq 0}\sum_{k=i+1}^{m+n}(-1)^{p(k)}E_{i+1,k}(-s-1)\\
&\qquad\qquad[E_{k,i+1}(s+1),((-1)^{p(i)}E_{i,i}-(-1)^{p(i+1)}E_{i+1,i+1})t^a]\\
&\quad-\hbar(-1)^{p(i+1)}\sum_{s\geq 0}\sum_{k=i+1}^{m+n}(-1)^{p(k)}[E_{i+1,k}(-s-1),\\
&\qquad\qquad((-1)^{p(i)}E_{i,i}-(-1)^{p(i+1)}E_{i+1,i+1})t^a]E_{k,i+1}(s+1). \qquad (6.24)
\end{aligned}$$

We compute each terms of the right hand side of (6.24). By direct computation, we can rewrite the first term of the right hand side of (6.24) as

$$\hbar\sum_{s\geq 0}\sum_{k=i+1}^{m+n}(-1)^{p(k)}E_{i+1,k}(-s-1)E_{k,i+1}(s+1+a)$$

$$
\begin{aligned}
&-\hbar(-1)^{p(i+1)}\sum_{s\geq 0}E_{i+1,i+1}(-s-1)E_{i+1,i+1}(s+1+a)+\hbar\sum_{s\geq 0}(s+1)c\delta_{s+1+a,0}E_{i+1,i+1}(-s-1)\\
&-\hbar\sum_{s\geq 0}(s+1)\delta_{s+1+a,0}E_{i+1,i+1}(-s-1)+\hbar\sum_{s\geq 0}(s+1)\delta_{s+1+a,0}E_{i+1,i+1}(-s-1)\\
=&\ \hbar\sum_{s\geq 0}\sum_{k=i+2}^{m+n}(-1)^{p(k)}E_{i+1,k}(-s-1)E_{k,i+1}(s+1+a)+\hbar\sum_{s\geq 0}\delta_{s+1+a,0}(s+1)cE_{i+1,i+1}(-s-1).
\end{aligned}
\tag{6.25}
$$

By direct computation, we can also rewrite the second term of the right hand side of (6.24) as

$$
\begin{aligned}
&\hbar(-1)^{p(i+1)}\sum_{s\geq 0}E_{i+1,i+1}(-s-1+a)E_{i+1,i+1}(s+1)\\
&-\hbar\sum_{s\geq 0}\sum_{k=i+1}^{m+n}(-1)^{p(k)}E_{i+1,k}(-s-1+a)E_{k,i+1}(s+1)-\hbar\sum_{s\geq 0}\delta_{-s-1+a,0}(s+1)cE_{i+1,i+1}(s+1)\\
&+\hbar\sum_{s\geq 0}\delta_{-s-1+a,0}(s+1)E_{i+1,i+1}(s+1)-\hbar\sum_{s\geq 0}\delta_{-s-1+a,0}(s+1)E_{i+1,i+1}(s+1)\\
=&-\hbar\sum_{s\geq 0}\sum_{k=i+2}^{m+n}(-1)^{p(k)}E_{i+1,k}(-s-1+a)E_{k,i+1}(s+1)\\
&-\hbar\sum_{s\geq 0}\delta_{-s-1+a,0}(s+1)cE_{i+1,i+1}(s+1).
\end{aligned}
\tag{6.26}
$$

Adding (6.25) and (6.26), we have

$$
\begin{aligned}
&\text{the 6-th term of the right hand side of (6.8)}\\
&=\hbar\sum_{s\geq 0}\sum_{k=i+2}^{m+n}(-1)^{p(k)}E_{i+1,k}(-s-1)E_{k,i+1}(s+1+a)\\
&\quad-\hbar\sum_{s\geq 0}\sum_{k=i+2}^{m+n}(-1)^{p(k)}E_{i+1,k}(-s-1+a)E_{k,i+1}(s+1)\\
&\quad+\hbar\sum_{s\geq 0}\delta_{s+1+a,0}(s+1)cE_{i+1,i+1}(-s-1)-\hbar\sum_{s\geq 0}\delta_{-s-1+a,0}(s+1)cE_{i+1,i+1}(s+1).
\end{aligned}
\tag{6.27}
$$

Since the first two terms of the right hand side of (6.27) are elements of the completion of $U(\widehat{\mathfrak{sl}}(m|n))$, we have obtained (6.11). □

This completes the proof of Theorem 6.7. □

By the assumption that $m,n\geq 2$ and $m\neq n$, we can take $1\leq i\leq m+n-1$ such that $p(i)=p(i+1)$. By Theorem 6.7, The completion of the image of ev_υ contains $\hbar c(E_{i,i}+E_{i+1,i+1})t^a$ for all $a\neq 0$. Provided that $\hbar c\neq 0$, the completion of the image of ev_υ contains $(E_{i,i}+E_{i+1,i+1})t^a$. By the assumption that $p(i)=p(i+1)$, $(E_{i,i}+E_{i+1,i+1})t^a$ is not contained in $\widehat{\mathfrak{sl}}(m|n)$. Thus, we obtain the following corollary.

Corollary 6.28. *The completion of the image of* ev_υ *contains* $E_{i,i}t^a$ *for all* $a\neq 0$ *provided that* $\hbar c\neq 0$.

By the assumption that $\hbar c=-(m-n)\varepsilon_1$, we find that $\hbar c$ is nonzero if and only if $\varepsilon_1\neq 0$. Under the assumption that by Corollary 6.6 and Corollary 6.28, the image of ev_υ contains $E_{i,i}t^s$ for all $s\in\mathbb{Z}$. Thus, we have the following theorem.

Theorem 6.29. *Provided that* $\varepsilon_1\neq 0$, *the image of* ev_υ *is dense in* $U(\widehat{\mathfrak{gl}}(m|n))_{\mathrm{comp}}$.

7 Another presentation of affine super Yangians

There exists another presentation of the affine super Yangian.

Proposition 7.1. *Suppose that $m,n\geq 2$ and $m\neq n$. The affine super Yangian $Y_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m|n))$ is isomorphic to the associative superalgebra generated by $X^+_{i,r}, X^-_{i,r}, H_{i,r}$ $(i\in\{0,1,\cdots,m+n-1\}, r=0,1)$ subject to the following defining relations:*

$$[H_{i,r},H_{j,s}]=0, \tag{7.2}$$
$$[X^+_{i,0},X^-_{j,0}]=\delta_{ij}H_{i,0}, \tag{7.3}$$
$$[X^+_{i,1},X^-_{j,0}]=\delta_{ij}H_{i,1}=[X^+_{i,0},X^-_{j,1}], \tag{7.4}$$
$$[H_{i,0},X^\pm_{j,r}]=\pm a_{ij}X^\pm_{j,r}, \tag{7.5}$$
$$[\tilde{H}_{i,1},X^\pm_{j,0}]=\pm a_{ij}\left(X^\pm_{j,1}\right),\ \textit{if } (i,j)\neq(0,m+n-1),(m+n-1,0), \tag{7.6}$$
$$[\tilde{H}_{0,1},X^\pm_{m+n-1,0}]=\mp(-1)^{p(m+n)}\left(X^\pm_{m+n-1,1}-(\varepsilon+\frac{m-n}{2}\hbar)X^\pm_{m+n-1,0}\right), \tag{7.7}$$
$$[\tilde{H}_{m+n-1,1},X^\pm_{0,0}]=\mp(-1)^{p(m+n)}\left(X^\pm_{0,1}+(\varepsilon+\frac{m-n}{2}\hbar)X^\pm_{0,0}\right), \tag{7.8}$$
$$[X^\pm_{i,1},X^\pm_{j,0}]-[X^\pm_{i,0},X^\pm_{j,1}]=\pm a_{ij}\frac{\hbar}{2}\{X^\pm_{i,0},X^\pm_{j,0}\}\ \textit{if } (i,j)\neq(0,m+n-1),(m+n-1,0), \tag{7.9}$$
$$\begin{aligned}&[X^\pm_{0,1},X^\pm_{m+n-1,0}]-[X^\pm_{0,0},X^\pm_{m+n-1,1}]\\&\quad=\pm(-1)^{p(m+n)}\frac{\hbar}{2}\{X^\pm_{0,0},X^\pm_{m+n-1,0}\}-(\varepsilon+\frac{m-n}{2}\hbar)[X^\pm_{0,0},X^\pm_{m+n-1,0}],\end{aligned} \tag{7.10}$$
$$(\mathrm{ad}\,X^\pm_{i,0})^{1+|a_{ij}|}(X^\pm_{j,0})=0\quad(i\neq j), \tag{7.11}$$
$$[X^\pm_{i,0},X^\pm_{i,0}]=0\ (i=0,m), \tag{7.12}$$
$$[[X^\pm_{i-1,0},X^\pm_{i,0}],[X^\pm_{i,0},X^\pm_{i+1,0}]]=0\ (i=0,m), \tag{7.13}$$

where $\hbar=\varepsilon_1+\varepsilon_2$, $\tilde{H}_{i,1}=H_{i,1}-\dfrac{\hbar}{2}H^2_{i,0}$, $\varepsilon=-(m-n)\varepsilon_2$, the generators $X^\pm_{m,r}$ and $X^\pm_{0,r}$ are odd and all other generators are even and we define $X^\pm_{-1,0}$ as $X^\pm_{m+n-1,0}$.

Proof. The homomorphism Ψ from $Y_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m|n))$ to the superalgebra defined in Proposition 7.1 is given by

$$\Psi(h_{i,0})=H_{i,0},\quad \Psi(x^\pm_{i,0})=X^\pm_{i,0},$$
$$\Psi(h_{i,1})=\begin{cases}H_{0,1} & \text{if } i=0,\\ H_{i,1}-\dfrac{i-2\delta(i>m)(i-m)}{2}(\varepsilon_1-\varepsilon_2)H_{i,0} & \text{if } i\neq 0,\end{cases}$$

where

$$\delta(i>m)=\begin{cases}1 & \text{if } i>m,\\ 0 & \text{if } i\leq m.\end{cases}$$

It is clear that Ψ is an isomorphism. □

Now, we can write down the image of $\{H_{i,r},X^\pm_{i,r}\mid r=0,1\}$ via the evaluation map.

Theorem 7.14 (Ueda [45], Proposition 5.2)**.** *Set $\tilde{c}=\dfrac{(-m+n)\varepsilon_1}{\hbar}$. Then, there exists an algebra homomorphism*

$$\mathrm{ev}_0\colon Y_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m|n))\to U(\widehat{\mathfrak{gl}}(m|n)^{str})_{\mathrm{comp}}$$

uniquely determined by

$$\mathrm{ev}_0(X^+_{i,0})=x^+_i,\quad \mathrm{ev}_0(X^-_{i,0})=x^-_i,\quad \mathrm{ev}_0(H_{i,0})=h_i,$$

$$\mathrm{ev}_0(H_{i,1})=\begin{cases}\hbar\tilde{c}h_0-(-1)^{p(m+n)}\hbar E_{m+n,m+n}(E_{1,1}-\tilde{c})\\ \quad+(-1)^{p(m+n)}\hbar\displaystyle\sum_{s\geq 0}\sum_{k=1}^{m+n}(-1)^{p(k)}E_{m+n,k}(-s)E_{k,m+n}(s)\\ \quad-\hbar\displaystyle\sum_{s\geq 0}\sum_{k=1}^{m+n}(-1)^{p(k)}E_{1,k}(-s-1)E_{k,1}(s+1)\\ \hfill\text{if } i=0,\\ -\dfrac{(i-2\delta(i\geq m+1)(i-m))}{2}\hbar h_i-(-1)^{p(E_{i,i+1})}\hbar E_{i,i}E_{i+1,i+1}\\ \quad+\hbar(-1)^{p(i)}\displaystyle\sum_{s\geq 0}\sum_{k=1}^{i}(-1)^{p(k)}E_{i,k}(-s)E_{k,i}(s)\\ \quad+\hbar(-1)^{p(i)}\displaystyle\sum_{s\geq 0}\sum_{k=i+1}^{m+n}(-1)^{p(k)}E_{i,k}(-s-1)E_{k,i}(s+1)\\ \quad-\hbar(-1)^{p(i+1)}\displaystyle\sum_{s\geq 0}\sum_{k=1}^{i}(-1)^{p(k)}E_{i+1,k}(-s)E_{k,i+1}(s)\\ \quad-\hbar(-1)^{p(i+1)}\displaystyle\sum_{s\geq 0}\sum_{k=i+1}^{m+n}(-1)^{p(k)}E_{i+1,k}(-s-1)E_{k,i+1}(s+1)\\ \hfill\text{if } i\neq 0,\end{cases}$$

$$\mathrm{ev}_0(X^+_{i,1})=\begin{cases}\hbar\tilde{c}x^+_0+\hbar\displaystyle\sum_{s\geq 0}\sum_{k=1}^{m+n}(-1)^{p(k)}E_{m+n,k}(-s)E_{k,1}(s+1)\\ \hfill\text{if } i=0,\\ -\dfrac{i-2\delta(i\geq m+1)(i-m)}{2}\hbar x^+_i+\hbar\displaystyle\sum_{s\geq 0}\sum_{k=1}^{i}(-1)^{p(k)}E_{i,k}(-s)E_{k,i+1}(s)\\ \quad+\hbar\displaystyle\sum_{s\geq 0}\sum_{k=i+1}^{m+n}(-1)^{p(k)}E_{i,k}(-s-1)E_{k,i+1}(s+1)\\ \hfill\text{if } i\neq 0,\end{cases}$$

$$\mathrm{ev}_0(X^-_{i,1})=\begin{cases}\hbar\tilde{c}x^-_0+(-1)^{p(m+n)}\hbar\displaystyle\sum_{s\geq 0}\sum_{k=1}^{m+n}(-1)^{p(k)}E_{1,k}(-s-1)E_{k,m+n}(s),\\ \hfill\text{if } i=0,\\ -\dfrac{i-2\delta(i\geq m+1)(i-m)}{2}\hbar x^-_i+(-1)^{p(i)}\hbar\displaystyle\sum_{s\geq 0}\sum_{k=1}^{i}(-1)^{p(k)}E_{i+1,k}(-s)E_{k,i}(s)\\ \quad+(-1)^{p(i)}\hbar\displaystyle\sum_{s\geq 0}\sum_{k=i+1}^{m+n}(-1)^{p(k)}E_{i+1,k}(-s-1)E_{k,i}(s+1)\quad\text{if } i\neq 0.\end{cases}$$

It was shown in [48] that the image of ev_0 is dense in $U(\widehat{\mathfrak{gl}}(m|n)^{\mathrm{str}})_{\mathrm{comp}}$ in the case when $\varepsilon_1\neq 0$.

Remark 7.15. In [45], the evaluation map was defined in terms of the generators $h_{i,r}$ and $x^{\pm}_{i,r}$ $(r = 0, 1)$.

In the non-super case, the affine Yangian was defined in Definition 3.2 of [19] and Definition 2.3 of [20] as follows.

Definition 7.16. Suppose that $m \geq 3$ and set two $m \times m$-matrices $(a_{i,j})$ and $(m_{i,j})$ as

$$a_{ij} = \begin{cases} 2 & \text{if } i = j, \\ -1 & \text{if } i = j \pm 1, \\ -1 & \text{if } (i,j) = (0, m-1), (m-1, 0), \\ 0 & \text{otherwise,} \end{cases} \quad m_{i,j} = \begin{cases} 1 & \text{if } i = j-1, \\ -1 & \text{if } i = j+1, \\ 1 & \text{if } (i,j) = (0, m-1), \\ -1 & \text{if } (i,j) = (m-1, 0), \\ 0 & \text{otherwise.} \end{cases}$$

The affine Yangian $Y_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m))$ is the associative algebra over $\mathbb{C}$ generated by $x^{+}_{i,r}, x^{-}_{i,r}$, $h_{i,r}$ $(i \in \{0, 1, \cdots, m-1\}, r \in \mathbb{Z}_{\geq 0})$ with parameters $\varepsilon_1, \varepsilon_2 \in \mathbb{C}$ subject to the defining relations (3.2)-(3.7).

Similarly to Proposition 7.1, the affine Yangian $Y_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m))$ also has a presentation whose generators are $H_{i,r}, X^{\pm}_{i,r}$ $(0 \leq i \leq m-1,\ r = 0, 1)$.

Proposition 7.17. *The affine Yangian $Y_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m))$ is isomorphic to the associative algebra generated by $X^{+}_{i,r}, X^{-}_{i,r}, H_{i,r}$ $(i \in \{0, 1, \cdots, m-1\}, r = 0, 1)$ subject to the defining relations* (7.2)-(7.5), (7.11) *and*

$$[\tilde{H}_{i,1}, X^{\pm}_{j,0}] = \pm a_{ij} \left(X^{\pm}_{j,1}\right), \text{ if } (i,j) \neq (0, m-1), (m-1, 0), \tag{7.18}$$

$$[\tilde{H}_{0,1}, X^{\pm}_{m-1,0}] = \mp \left(X^{\pm}_{m-1,1} - (\varepsilon + \frac{m}{2}\hbar) X^{\pm}_{m-1,0}\right), \tag{7.19}$$

$$[\tilde{H}_{m-1,1}, X^{\pm}_{0,0}] = \mp \left(X^{\pm}_{0,1} + (\varepsilon + \frac{m}{2}\hbar) X^{\pm}_{0,0}\right), \tag{7.20}$$

$$[X^{\pm}_{i,1}, X^{\pm}_{j,0}] - [X^{\pm}_{i,0}, X^{\pm}_{j,1}] = \pm a_{ij} \frac{\hbar}{2} \{X^{\pm}_{i,0}, X^{\pm}_{j,0}\} \text{ if } (i,j) \neq (0, m-1), (m-1, 0), \tag{7.21}$$

$$\begin{aligned} &[X^{\pm}_{0,1}, X^{\pm}_{m-1,0}] - [X^{\pm}_{0,0}, X^{\pm}_{m-1,1}] \\ &\qquad = \pm \frac{\hbar}{2} \{X^{\pm}_{0,0}, X^{\pm}_{m-1,0}\} - (\varepsilon + \frac{m}{2}\hbar) [X^{\pm}_{0,0}, X^{\pm}_{m-1,0}], \end{aligned} \tag{7.22}$$

where $\hbar = \varepsilon_1 + \varepsilon_2$, $\tilde{H}_{i,1} = H_{i,1} - \frac{\hbar}{2} H^2_{i,0}$, and $\varepsilon = -m\varepsilon_2$.

The evaluation map for the affine Yangian $Y_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m))$ was constructed in Section 6 of [20] and Theorem 3.8 of [30]. In fact, the evaluation map of [20] and [30] was defined in the same formula as that of Theorem 7.14 by setting $n = 0$ and assuming all of the parity is equal to zero. In the non-super case, the surjectivity of the evaluation map was shown in Theorem 4.18 of [29].

8 Generators of rectangular W-superalgebras of type A

We fix some notations for vertex algebras. For a vertex algebra V, we denote the generating field associated with $v \in V$ by $v(z) = \sum_{n \in \mathbb{Z}} v_{(n)} z^{-n-1}$. We also denote the OPE of V by

$$u(z)v(w) \sim \sum_{s \geq 0} \frac{(u_{(s)}v)(w)}{(z-w)^{s+1}}$$

for all $u, v \in V$. We denote the identity vector (resp. the translation operator) by $|0\rangle$ (resp. ∂).

First, we recall the definition of rectangular W-superalgebras of type A (see [27], [28], and [1]). Let us set

$$\mathfrak{g}=\mathfrak{gl}(ml|nl)=\bigoplus_{\substack{1\leq i,j\leq m+n\\ 1\leq s,t\leq l}}\mathbb{C}e_{(s-1)(m+n)+i,(t-1)(m+n)+j},$$

where $e_{(s-1)(m+n)+i,(t-1)(m+n)+j}$ is the unit matrix whose parity is $p(i)+p(j)$. Since $\mathfrak{gl}(ml|nl)$ is isomorphic to $\mathfrak{gl}(m|n)\otimes\mathfrak{gl}(l)$ as a graded vector space, we identify $e_{(s-1)(m+n)+i,(t-1)(m+n)+j}\in\mathfrak{gl}(ml|nl)$ with $e_{i,j}\otimes e_{s,t}\in\mathfrak{gl}(m|n)\otimes\mathfrak{gl}(l)$. We set a parity of $e_{i,j}\in\mathfrak{gl}(m|n)$ as $p(i)+p(j)$. We take an even nilpotent element $f=\sum_{s=1}^{l-1}\sum_{i=1}^{m+n}e_{s(m+n)+i,(s-1)(m+n)+i}\in\mathfrak{gl}(ml|nl)$ and fix $k\in\mathbb{C}$. We also take $(\ |\ )$ as a supersymmetric invariant inner product of $\mathfrak{g}$ such that

$$(u|v)=\begin{cases}k\,\mathrm{str}(uv) & \text{if } u \text{ or } v \text{ is an element of } \mathfrak{sl}(ml|nl),\\ k\,\mathrm{str}(uv)+(-1)^{p(i)+p(j)}(1-c) & \text{if } u=e_{i,i}\otimes e_{r_1,r_1}, v=e_{j,j}\otimes e_{r_2,r_2},\end{cases}\tag{8.1}$$

where c is a complex number and str is a supertrace of $\mathfrak{gl}(ml|nl)$. We set

$$\mathfrak{g}_t=\bigoplus_{\substack{1\leq i,j\leq m+n\\ 0\leq s\leq l-1\\ 0\leq s+t\leq l-1}}\mathbb{C}e_{s(m+n)+i,(s+t)(m+n)+j}.$$

and fix a $\mathfrak{sl}_2$-triple (x,e,f) such that

$$\mathfrak{g}_t=\{y\in\mathfrak{g}\mid[x,y]=ty\}.$$

Let us set

$$S=\{(i,j,s,t)\mid 1\leq i,j\leq m+n,\ 0\leq s,s+t\leq l-1\},$$
$$S_+=\{(i,j,s,t)\mid 1\leq i,j\leq m+n,\ 0\leq s,s+t\leq l-1,t\geq 1\}.$$

For all $\beta=(i,j,s,t)\in S$, we also set u_β as $e_{s(m+n)+i,(s+t)(m+n)+j}$ and $p(\beta)$ as the parity of u_β. Then, we have

$$\mathfrak{g}=\bigoplus_{\beta\in S}\mathbb{C}u_\beta,\qquad \mathfrak{g}_{\geq 0}=\bigoplus_{t\geq 0}\mathfrak{g}_t=\bigoplus_{\beta\in S_+}\mathbb{C}u_\beta.$$

Moreover, let $\mathfrak{b}$ be $\bigoplus_{j\leq 0}\mathfrak{g}_j$, which is a subalgebra of $\mathfrak{g}$. We define κ as an inner product of $\mathfrak{b}$ such that

$$\kappa(u,v)=(u|v)+\frac{1}{2}\big(\kappa_{\mathfrak{g}}(u,v)-\kappa_{\mathfrak{g}_0}(p_0(u),p_0(v))\big)\text{ for all }u,v\in\mathfrak{b},$$

where $p_0\colon\mathfrak{b}\to\mathfrak{g}_0$ is the projection map and $\kappa_{\mathfrak{g}}$ (resp. $\kappa_{\mathfrak{g}_0}$) is the Killing form on $\mathfrak{g}$ (resp. $\mathfrak{g}_0$). By the definition of κ, we have

$$\begin{aligned}&\kappa(e_{s_1(m+n)+i_1,t_1(m+n)+j_1},e_{s_2(m+n)+i_2,t_2(m+n)+j_2})\\ &=\delta_{s_1,t_2}\delta_{t_1,s_2}\delta_{i_1,j_2}\delta_{j_1,i_2}(-1)^{p(i_1)}(k+(l-1)(m-n))\\ &\qquad-\delta_{s_1,t_1}\delta_{s_2,t_2}\delta_{i_1,j_1}\delta_{i_2,j_2}(-1)^{p(i_1)+p(i_2)}(c-\delta_{s_1,s_2}).\end{aligned}$$

Let $\hat{\mathfrak{b}}$ be the Lie superalgebra $\mathfrak{b}\otimes\mathbb{C}[t^{\pm1}]\oplus\mathbb{C}y$ whose commutator relations are

$$[at^u,bt^v]=[a,b]t^{u+v}+\delta_{u+v,0}u\kappa(a,b)y,$$
$$y\text{ is a central element.}$$

We also set a left $\hat{\mathfrak{b}}$-module $V^\kappa(\mathfrak{b})$ as $U(\hat{\mathfrak{b}})/U(\hat{\mathfrak{b}})(\mathfrak{b}[t]\oplus\mathbb{C}(y-1))\cong U(\mathfrak{b}[t^{-1}]t^{-1})$. Then, it has a vertex algebra structure whose identity vector is 1 and the generating field $(ut^{-1})(z)$ is equal to $\sum_{s\in\mathbb{Z}}(ut^s)z^{-s-1}$ for all $u\in\mathfrak{b}$. We call $V^\kappa(\mathfrak{b})$ the universal affine vertex algebra associated with $(\mathfrak{b},\kappa)$.

In order to simplify the notation, we denote the generating field $(ut^{-1})(z)$ as $u(z)$. By the definition of $V^\kappa(\mathfrak{b})$, generating fields $u(z)$ and $v(z)$ satisfy the OPE

$$u(z)v(w)\sim\frac{[u,v](w)}{z-w}+\frac{\kappa(u,v)}{(z-w)^2}\tag{8.2}$$

for all $u,v\in\mathfrak{b}$.

We set a Lie superalgebra $\mathfrak{a}_{m,n}=\bigoplus_{u_\beta\in\mathfrak{g}_{\leq 0}}\mathbb{C}J^{(u_\beta)}\oplus\bigoplus_{u_\beta\in\mathfrak{g}_{<0}}\mathbb{C}\psi_{(u_\beta)}$ with the following commutator relations;

$$[J^{(u)},J^{(v)}]=J^{([u,v])},\quad [J^{(e_{i,j})},\psi_{e_{s,t}}]=\delta_{j,s}\psi_{e_{i,t}}-\delta_{i,t}(-1)^{p(e_{i,j})(p(e_{s,t})+1)}\psi_{e_{s,j}},\quad [\psi_u,\psi_v]=0,$$

where the parity of $J^{(u_\beta)}$ (resp. ψ_{u_β}) is equal to $p(\beta)$ (resp. $p(\beta)+1$) and we denote $\sum_{u_\beta\in\mathfrak{g}_{\leq 0}}a_\beta J^{(u_\beta)}$ (resp. $\sum_{u_\beta\in\mathfrak{g}_{<0}}a_\beta\psi_{(u_\beta)}$) by $J^{(\sum_{u_\beta\in\mathfrak{g}_{\leq 0}}a_\beta u_\beta)}$ (resp. $\psi_{\sum_{u_\beta\in\mathfrak{g}_{<0}}a_\beta u_\beta}$) for all $a_\beta\in\mathbb{C}$. We define an affinization of $\mathfrak{a}_{m,n}$ by using the inner product on $\mathfrak{a}_{m,n}$ such that

$$\kappa_{m,n}(J^{(u)},J^{(v)})=\kappa(u,v),\quad \kappa_{m,n}(J^{(u)},\psi_v)=\kappa_{m,n}(\psi_u,\psi_v)=0.$$

By (8.2), $V^{\kappa_{m,n}}(\mathfrak{a}_{m,n})$ contains $V^\kappa(\mathfrak{b})$. We identify $ut^{-1}\in V^\kappa(\mathfrak{b})$ with $J^{(u)}t^{-1}\in V^{\kappa_{m,n}}(\mathfrak{a}_{m,n})$.

For all $u\in\mathfrak{a}_{m,n}$, let $u[-s]$ be ut^{-s}. In this section, we regard $V^{\kappa_{m,n}}(\mathfrak{a}_{m,n})$ (resp. $V^\kappa(\mathfrak{b})$) as a non-associative superalgebra whose product $\cdot$ is defined by

$$u[-t]\cdot v[-s]=(u[-t])_{(-1)}v[-s].$$

We sometimes omit $\cdot$ and denote $\psi_{e_{(v+w)(m+n)+j,v(m+n)+i}}[s]$ by $\psi_{(v+w)(m+n)+j,v(m+n)+i}[s]$ in order to simplify the notation. A rectangular W-superalgebra $\mathcal{W}^k(\mathfrak{gl}(lm|ln),(l^{(m|n)}))$ can be realized as the subalgebra of $V^{\kappa_{m,n}}(\mathfrak{a}_{m,n})$ ([27] and [28]) as follows.

Let us set α as $k+(l-1)(m-n)$. We can define an odd differential $d_0\colon V^\kappa(\mathfrak{b})\to V^{\kappa_{m,n}}(\mathfrak{a}_{m,n})$ determined by

$$d_0 1=0,\tag{8.3}$$
$$[d_0,\partial]=0,\tag{8.4}$$

$$\begin{aligned}
&[d_0,e_{(s-1)(m+n)+j,(t-1)(m+n)+i}[-1]]\\
&=\sum_{\substack{t<a\leq s,\\1\leq r\leq m+n}}(-1)^{p(e_{i,j})+p(e_{i,r})p(e_{r,j})}e_{(a-1)(m+n)+r,(t-1)(m+n)+i}[-1]\psi_{(s-1)(m+n)+j,(a-1)(m+n)+r}[-1]\\
&\quad-\sum_{\substack{t\leq a<s,\\1\leq r\leq m+n}}(-1)^{p(e_{i,r})p(e_{r,j})}\psi_{(a-1)(m+n)+r,(t-1)(m+n)+i}[-1]e_{(s-1)(m+n)+j,(a-1)(m+n)+r}[-1]\\
&\quad+\delta(s<t)(-1)^{p(j)}\alpha\psi_{(s-1)(m+n)+j,(t-1)(m+n)+i}[-2]\\
&\quad+(-1)^{p(j)}\psi_{s(m+n)+j,(t-1)(m+n)+i}[-1]-\psi_{(s-1)(m+n)+j,(t-2)(m+n)+i}[-1].
\end{aligned}\tag{8.5}$$

Definition 8.6 (Kac-Roan-Wakimoto [23], Theorem 2.4)**.** The rectangular W-superalgebra associated with a Lie superalgebra $\mathfrak{gl}(m|n)$ and a nilpotent element $f=\sum_{s=1}^{l-1}\sum_{i=1}^{m+n}e_{s(m+n)+i,(s-1)(m+n)+i}$ is the vertex subalgebra defined by

$$\mathcal{W}^k(\mathfrak{gl}(ml|nl),(l^{(m|n)}))=\{y\in V^\kappa(\mathfrak{b})\subset V^{\kappa_{m,n}}(\mathfrak{a}_{m,n})\mid d_0(y)=0\}.$$

We denote the rectangular W-superalgebra associated with a Lie superalgebra $\mathfrak{gl}(m|n)$ and a nilpotent element f by $\mathcal{W}^k(\mathfrak{gl}(lm|ln),(l^{(m|n)}))$. The rest of this section is devoted to the construction of two kinds of elements $W^{(1)}_{i,j}$ and $W^{(2)}_{i,j}$, which are generators of $\mathcal{W}^k(\mathfrak{gl}(lm|ln),(l^{(m|n)}))$.

We regard $V^\kappa(\mathfrak{b})\otimes\mathbb{C}[\tau]$ and $V^{\kappa_{m,n}}(\mathfrak{a}_{m,n})\otimes\mathbb{C}[\tau]$ as non-associative superalgebras whose defining relations are given by

$$u[-t]\cdot v[-s]=(u[-t])_{(-1)}v[-s],\ [\tau,u[-s]]=su[-s],$$

where τ is an even element. Let $\widetilde{d}^{m,n}_0\colon V^{\kappa_{m,n}}(\mathfrak{a}_{m,n})\otimes\mathbb{C}[\tau]\to V^{\kappa_{m,n}}(\mathfrak{a}_{m,n})\otimes\mathbb{C}[\tau]$ be the odd differential determined by

$$\widetilde{d}^{m,n}_0 1=0,\quad [\widetilde{d}^{m,n}_0,u[-s]]=[d_0,u[-s]],\quad [\widetilde{d}^{m,n}_0,\tau]=0.$$

First, let us recall how to construct generators of the principal W-algebra $\mathcal{W}^k(\mathfrak{gl}(l),(l^1))$ ([3], Section 2). We denote by $T(C)$ a non-associative free algebra associated with a vector space C and by $\mathfrak{gl}(l)_{\leq 0}$ the Lie algebra $\bigoplus_{1\leq j\leq i\leq l}\mathbb{C}e_{i,j}$. In the principal case, $\mathfrak{b}$ is equal to $\mathfrak{gl}(l)_{\leq 0}$. By Definition 8.6, the principal W-algebra can be defined as

$$\mathcal{W}^k(\mathfrak{gl}(l),(l^1))=\{x\in V^\kappa(\mathfrak{gl}(l)_{\leq 0})\otimes\mathbb{C}[\tau]\mid d_0(x)=0\}.$$

Similarly to $V^\kappa(\mathfrak{b})\otimes\mathbb{C}[\tau]$, we define a non-associative algebra $T(\mathfrak{gl}(l)_{\leq 0}[t^{-1}]t^{-1})\otimes\mathbb{C}[\tau]$. Let us set π as $k+l-1$ and an $l\times l$ matrix $B=(b_{i,j})_{1\leq i,j\leq l}$ as

$$\begin{bmatrix}\pi\tau+e_{1,1}[-1] & -1 & 0 & \cdots & 0\\ e_{2,1}[-1] & \pi\tau+e_{2,2}[-1] & -1 & \cdots & 0\\ \vdots & \vdots & \ddots & & \vdots\\ e_{l-1,1}[-1] & e_{l-1,2}[-1] & \cdots & \pi\tau+e_{l-1,l-1}[-1] & -1\\ e_{l,1}[-1] & e_{l,2}[-1] & \cdots & e_{l,l-1}[-1] & \pi\tau+e_{l,l}[-1]\end{bmatrix}\tag{8.7}$$

whose entries are elements of $T(\mathfrak{gl}(l)_{\leq 0}[t^{-1}]t^{-1})\otimes\mathbb{C}[\tau]$. For any matrix $A=(a_{i,j})_{1\leq i,j\leq s}$, we define $\mathrm{cdet}(A)$ as

$$\sum_{\sigma\in\mathfrak{S}_s}\mathrm{sgn}(\sigma)a_{\sigma(1),1}\bigl(a_{\sigma(2),2}(a_{\sigma(3),3}\cdots a_{\sigma(s-1),s-1})a_{\sigma(s),s}\bigr)\in T(\mathfrak{gl}(l)_{\leq 0}[t^{-1}]t^{-1})\otimes\mathbb{C}[\tau].$$

By the commutator relation of $T(\mathfrak{gl}(l)_{\leq 0}[t^{-1}]t^{-1})\otimes\mathbb{C}[\tau]$, we can rewrite $\mathrm{cdet}(B)$ as $\sum_{r=0}^{l}\widetilde{W}^{(r)}(\pi\tau)^{l-r}$ such that $\widetilde{W}^{(r)}\in T(\mathfrak{gl}(l)_{\leq 0}[t^{-1}]t^{-1})$. Let p be the projection map from $T(\mathfrak{gl}(l)_{\leq 0}[t^{-1}]t^{-1})$ to $V^\kappa(\mathfrak{gl}(l)_{\leq 0})=U(\mathfrak{gl}(l)_{\leq 0}[t^{-1}]t^{-1})$ and $W^{(r)}$ be $p(\widetilde{W}^{(r)})$. Proving that $[\widetilde{d}^{1,0}_0,p(\mathrm{cdet}(B))]=0$, we obtain the following theorem (see Theorem 2.1 of [3]).

Theorem 8.8. *The W-superalgebra $\mathcal{W}^k(\mathfrak{gl}(l),(l^1))$ is generated by $\{W^{(r)}\}_{1\leq r\leq l}$.*

Remark 8.9. In [3], the tensor algebra $T(C)$ should have been defined as a non-associative superalgebra as above since $V(\mathfrak{g}_{\leq 0})$ is non-associative.

Let $A_{1,0}$ be a quotient algebra of $T(\mathfrak{a}_{1,0}[t^{-1}]t^{-1})\otimes\mathbb{C}[\tau]$ subjected to the relation

$$(e_{a,1}[-1]\psi_{i,a}[-1])\mathrm{cdet}(C^{l-i})-e_{a,1}[-1](\psi_{i,a}[-1]\mathrm{cdet}(C^{l-i}))=0\text{ for all }1\leq a\leq i,$$

where C^{l-i} is a submatrix of B consisting of the last $(l-i)$ rows and columns. Constructing a homomorphism

$$D\colon T(\mathfrak{gl}(l)_{\leq 0}[t^{-1}]t^{-1})\otimes\mathbb{C}[\tau]\to A_{1,0}$$

determined by

$$D(e_{s,u}[-1]) = \sum_{u<a\leq s} e_{a,u}[-1]\psi_{s,a}[-1] - \sum_{u\leq a<s} \psi_{a,u}[-1]e_{s,a}[-1]$$
$$+ \delta(s<u)\pi\psi_{s,u}[-2] + \psi_{s,u+1}[-1] - \psi_{s-1,u}[-1],$$

we obtain the relation $D(\mathrm{cdet}(B)) = 0$ in the way similar to the one of Theorem 2.1 of [3].

We regard $\mathfrak{gl}(m|n)$ as an associative superalgebra whose product $\cdot$ is determined by $e_{i,j}\cdot e_{s,u} = \delta_{j,s}e_{i,u}$. Then, we obtain a non-associative superalgebra $\mathfrak{gl}(m|n)\otimes V^\kappa(\mathfrak{b})\otimes\mathbb{C}[\tau]$. We construct a homomorphism

$$T\colon T(\mathfrak{gl}(l)_{\leq 0}[t^{-1}]t^{-1})\otimes\mathbb{C}[\tau] \to \mathfrak{gl}(m|n)\otimes V^\kappa(\mathfrak{b})\otimes\mathbb{C}[\tau]$$

determined by

$$T_{i,j}(x) = (-1)^{p(i)}x\otimes e_{i,j} \in \mathfrak{gl}(l)_{\leq 0}[t^{-1}]t^{-1}\otimes\mathfrak{gl}(m|n) = \mathfrak{b}[t^{-1}]t^{-1},\quad T(\tau)=\tau,$$

where $T_{i,j}(x)$ is defined as $e_{j,i}\otimes T_{i,j}(x) = T(x)$. Since T is a homomorphism, we obtain

$$T_{i,j}(xy) = \sum_{r=1}^{m+n} (-1)^{p(e_{i,r})p(e_{j,r})} T_{r,i}(x)T_{j,r}(y).$$

By the commutator relation of $V^\kappa(\mathfrak{b})$ and $\mathbb{C}[\tau]$, $W^{(r)}_{i,j}\in V^\kappa(\mathfrak{b})$ is defined by

$$T_{j,i}(\mathrm{cdet}(B)) = \sum_{r=0}^{l} (-1)^{p(j)} W^{(r)}_{i,j}(\alpha\tau)^{l-r}, \tag{8.10}$$

where B is defined by replacing π in (8.7) with α.

Theorem 8.11. *For all $m,n\geq 0$ such that $m\neq n$, the W-superalgebra $\mathcal{W}^k(\mathfrak{gl}(ml|nl),(l^{(m|n)}))$ is freely generated by $\{W^{(r)}_{i,j}\mid 1\leq r\leq l, 1\leq i,j\leq m+n\}$.*

Remark 8.12. In the case when $n=0$, Theorem 8.11 is shown in Theorem 3.1 of [3].

Proof. Under the assumption that π is equal to α, we denote $A_{1,0}$ (resp. D) as $\bar{A}_{1,0}$ (resp. $\bar{D}$). We construct a homomorphism $T^p\colon \bar{A}_{1,0}\to\mathfrak{gl}(m|n)\otimes V^{\kappa_{m,n}}(\mathfrak{a}_{m,n})\otimes\mathbb{C}[\tau]$ determined by

$$T^p_{i,j}(e_{s,w}[u]) = (-1)^{p(j)}e_{(s-1)(m+n)+i,(w-1)(m+n)+j}[u],$$
$$T^p_{i,j}(\psi_{s,w}[u]) = \psi_{(s-1)(m+n)+i,(w-1)(m+n)+j}[u],\quad T^p(\tau)=\tau,$$

where $T^p_{i,j}(x)$ is defined as $e_{j,i}\otimes T^p_{i,j}(x) = T^p(x)$. Since T^p is a homomorphism, we obtain

$$T^p_{i,j}(e_{s,w}[-1]\psi_{u,v}[-1]) = \sum_{r=1}^{m+n}(-1)^{p(e_{i,r})p(e_{j,r})}T^p_{r,j}(e_{s,w}[-1])T^p_{i,r}(\psi_{u,v}[-1]),$$
$$T^p_{i,j}(\psi_{u,v}[-1]e_{s,w}[-1]) = \sum_{r=1}^{m+n}(-1)^{p(e_{i,r})+p(e_{i,r})p(e_{j,r})}T^p_{r,j}(\psi_{u,v}[-1])T^p_{i,r}(e_{s,w}[-1]).$$

By the definition of $T_{j,i}$ and d_0, we have

$$\begin{aligned}
&[\widetilde{d}_0^{m,n}, T_{j,i}(e_{s,w})]\\
&= [\widetilde{d}_0^{m,n}, (-1)^{p(j)}e_{(s-1)(m+n)+j,(w-1)(m+n)+i}[-1]]\\
&= \sum_{\substack{w<a\leq s,\\ 1\leq r\leq m+n}} (-1)^{p(i)+p(e_{i,r})p(e_{j,r})} e_{(a-1)(m+n)+r,(w-1)(m+n)+i}[-1]\psi_{(s-1)(m+n)+j,(a-1)(m+n)+r}[-1]
\end{aligned}$$

$$
\begin{aligned}
&- \sum_{\substack{w\le a<s,\\ 1\le r\le m+n}} (-1)^{\gamma}\psi_{(a-1)(m+n)+r,(w-1)(m+n)+i}[-1]e_{(s-1)(m+n)+j,(a-1)(m+n)+r}[-1]\\
&+ \delta(s<w)\alpha\psi_{(s-1)(m+n)+j,(w-1)(m+n)+i}[-2]\\
&+ \psi_{s(m+n)+j,(w-1)(m+n)+i}[-1] - \psi_{(s-1)(m+n)+j,(w-2)(m+n)+i}[-1]\\
=& T^{p}_{j,i}([\bar{D}, e_{s,w}]),
\end{aligned} \tag{8.13}
$$

where $\gamma = p(j) + p(e_{i,r})p(e_{j,r})$. Thus, the relation $[\widetilde{d}_0^{m,n}, T_{j,i}(a)] = T^p_{j,i}([\bar{D}, a])$ holds for all $a \in T(\mathfrak{gl}(l)_{\le 0})$. Then, we obtain

$$
[\widetilde{d}_0^{m,n}, T_{j,i}(\mathrm{cdet}(B))] = T^p_{j,i}([\bar{D}, \mathrm{cdet}(B)]). \tag{8.14}
$$

Since $[\bar{D}, \mathrm{cdet}(B)] = 0$ holds by the proof of Theorem 2.1 of [3], the right hand side of (8.14) is equal to zero. Thus, we have obtained the relation $[d_0, W^{(r)}_{i,j}] = 0$. The rest of the proof is same as [3]. □

In particular, by (8.10), we have

$$
W^{(1)}_{i,j} = \sum_{1\le s\le l} e_{(s-1)(m+n)+j,(s-1)(m+n)+i}[-1], \tag{8.15}
$$

$$
\begin{aligned}
W^{(2)}_{i,j} =& \sum_{1\le s\le l-1} e_{s(m+n)+j,(s-1)(m+n)+i}[-1] + \alpha \sum_{1\le s\le l} (s-1)e_{(s-1)(m+n)+j,(s-1)(m+n)+i}[-2]\\
&+ \sum_{\substack{r_1<r_2\\ 1\le t\le m+n}} (-1)^{p(t)+p(e_{i,t})p(e_{j,t})} e^{(r_1)}_{t,i}[-1]e^{(r_2)}_{j,t}[-1],
\end{aligned} \tag{8.16}
$$

where we set $e^{(r)}_{j,i}$ as $e_{(r-1)(m+n)+j,(r-1)(m+n)+i}$.

Theorem 8.17. *The rectangular W-superalgebra $\mathcal{W}^k(\mathfrak{gl}(ml|nl), (l^{(m|n)}))$ is generated by $W^{(1)}_{i,j}$ and $W^{(2)}_{i,j}$ $(1 \le i,j \le m+n)$ provided that $\alpha = k + (l-1)(m-n) \neq 0$, $m \neq n$ and $m+n \ge 2$.*

Theorem 8.17 is proved in the appendix A.

Remark 8.18. In the case when $(m,n) = (1,0)$ or $(0,1)$, the elements $W^{(1)}_{i,i+1}$ or $W^{(2)}_{i,i+1}$ do not exist. This is the reason why we need the condition that $m+n \ge 2$ in Theorem 8.17.

9 OPEs of rectangular W-superalgebras

First, let us recall the definition of the universal enveloping algebras of vertex algebras. For all vertex algebra V, let $L(V)$ be the Borchards Lie algebra, that is,

$$
L(V) = V \otimes \mathbb{C}[t,t^{-1}]/\mathrm{Im}(\partial \otimes \mathrm{id} + \mathrm{id} \otimes \frac{d}{dt}), \tag{9.1}
$$

where the commutation relation is given by

$$
[ut^a, vt^b] = \sum_{r\ge 0} \binom{a}{r} (u_{(r)}v)t^{a+b-r}
$$

for all $u, v \in V$ and $a, b \in \mathbb{Z}$. Now, we define the universal enveloping algebra of V.

Definition 9.2 (Frenkel-Zhu [14], Matsuo-Nagatomo-Tsuchiya [33])**.** We set $\mathcal{U}(V)$ as the quotient algebra of the standard degreewise completion of the universal enveloping algebra of $L(V)$ by the completion of the two-sided ideal generated by

$$
(u_{(a)}v)t^b - \sum_{i\ge 0} \binom{a}{i} (-1)^i (ut^{a-i}vt^{b+i} - (-1)^{p(u)p(v)}(-1)^a vt^{a+b-i}ut^i), \tag{9.3}
$$

$$|0\rangle t^{-1} - 1, \tag{9.4}$$

where $|0\rangle$ is the identity vector of V. We call $\mathcal{U}(V)$ the universal enveloping algebra of V.

Lemma 9.5 (Kac-Roan-Wakimoto [23], Theorem 2.4)**.** *There exists a homomorphism from the universal enveloping algebra of* $\widehat{\mathfrak{gl}}(m|n)^\kappa$ *to* $\mathcal{U}(\mathcal{W}^k(\mathfrak{gl}(ml|nl),(l^{(m|n)})))$ *determined by*

$$\xi(E_{i,j}t^s) = W^{(1)}_{j,i}t^s, \quad \xi(\tilde{c}) = l\alpha t^{-1}, \quad \xi(x) = 1.$$

In order to construct a homomorphism from the affine super Yangian to the universal enveloping algebra of W-superalgebras in Section 6, we need to compute the following terms;

$$(W^{(1)}_{i,j})_{(u)}W^{(2)}_{s,t} \ (u \geq 0), \quad (W^{(2)}_{i,i})_{(0)}W^{(2)}_{j,j}, \quad (W^{(2)}_{i,i})_{(1)}W^{(2)}_{j,j}.$$

First, we compute $(W^{(1)}_{i,j})_{(u)}W^{(2)}_{s,t}$ $(u \geq 0)$. By direct computation, we obtain the below two lemmas. We omit the proof.

Lemma 9.6. *We obtain*

$$(W^{(1)}_{u,v})_{(0)}W^{(2)}_{i,j} = \delta_{j,u}W^{(2)}_{i,v} - \delta_{i,v}(-1)^{p(e_{u,v})p(e_{i,j})}W^{(2)}_{u,j}.$$

Lemma 9.7. *The following equations hold;*

$$\begin{aligned}
(W^{(1)}_{v,w})_{(1)}W^{(2)}_{i,j} &= \delta_{j,v}(l-1)\alpha W^{(1)}_{i,w} - \delta_{v,w}(-1)^{p(w)}(l-1)(lc-1)W^{(1)}_{i,j},\\
(W^{(1)}_{v,w})_{(2)}W^{(2)}_{i,j} &= l(l-1)\alpha\kappa(e_{w,v},e_{j,i}),\\
(W^{(1)}_{v,w})_{(s)}W^{(2)}_{i,j} &= 0 \ (\text{for all } s \geq 3).
\end{aligned}$$

Corollary 9.8. *The following equation holds;*

$$\begin{aligned}
&[W^{(1)}_{v,w}t^s, W^{(2)}_{i,j}t^u]\\
&= \delta_{j,v}W^{(2)}_{i,w}t^{s+u} - \delta_{i,w}(-1)^{p(e_{v,w})p(e_{i,j})}W^{(2)}_{v,j}t^{s+u}\\
&\quad + \delta_{j,v}s(l-1)\alpha W^{(1)}_{i,w}t^{s+u-1} - \delta_{v,w}(-1)^{p(w)}(l-1)(lc-1)sW^{(1)}_{i,j}t^{s+u-1}\\
&\quad + \frac{s(s-1)}{2}l(l-1)\alpha\kappa(e_{w,v},e_{j,i})t^{s+u-2}.
\end{aligned}$$

The following assertion is also shown by direct calculation.

Lemma 9.9. *We obtain*

$$\begin{aligned}
(W^{(2)}_{i,i})_{(0)}W^{(2)}_{j,j} &= (-1)^{p(i)}(W^{(1)}_{i,j})_{(-1)}W^{(2)}_{j,i} - (-1)^{p(j)}(W^{(1)}_{j,i})_{(-1)}W^{(2)}_{i,j} - (\delta_{i,j}\alpha + (-1)^{p(i)})\partial W^{(2)}_{j,j}\\
&\quad + (-1)^{p(j)}(l-1)\alpha(W^{(1)}_{j,i})_{(-1)}\partial W^{(1)}_{i,j} - \{(l-1)^2c-(l-1)\}(W^{(1)}_{j,j})_{(-1)}\partial W^{(1)}_{i,i}\\
&\quad + \delta_{i,j}\frac{l(l-1)}{2}\alpha^2\partial^2 W^{(1)}_{i,i} + (-1)^{p(j)}\frac{l(l-1)}{2}\alpha\partial^2 W^{(1)}_{i,i} - (-1)^{p(j)}\frac{l(l-1)^2}{2}c\alpha\partial^2 W^{(1)}_{i,i}\\
&\quad + \frac{1}{2}(-1)^{p(i)}(l-1)\alpha\partial^2 W^{(1)}_{j,j} - \frac{1}{2}(-1)^{p(j)}(l-1)\alpha\partial^2 W^{(1)}_{i,i}
\end{aligned}$$

and

$$\begin{aligned}
(W^{(2)}_{i,i})_{(1)}W^{(2)}_{j,j} &= -\{(l-1)^2c-(l-1)\}(W^{(1)}_{j,j})_{(-1)}W^{(1)}_{i,i} - 2\delta_{i,j}\alpha W^{(2)}_{i,i} - (-1)^{p(i)}W^{(2)}_{j,j}\\
&\quad - (-1)^{p(j)}W^{(2)}_{i,i} + (-1)^{p(j)}(l-1)\alpha(W^{(1)}_{j,i})_{(-1)}W^{(1)}_{i,j} + \delta_{i,j}l(l-1)\alpha^2\partial W^{(1)}_{i,i}\\
&\quad + (-1)^{p(j)}l(l-1)\alpha\partial W^{(1)}_{i,i} - (-1)^{p(j)}l(l-1)^2c\alpha\partial W^{(1)}_{i,i}\\
&\quad + (-1)^{p(i)}(l-1)\alpha\partial W^{(1)}_{j,j} - (-1)^{p(j)}(l-1)\alpha\partial W^{(1)}_{i,i}.
\end{aligned}$$

Remark 9.10. In [41], Rapčák defined two kinds of elements of rectangular W-superalgebras of type A, which are called $U_{1,i,j}$ and $U_{2,i,j}$ $(1 \leq i,j \leq m+n)$ under the assumption that $c=0$. The element $U_{r,i,j}$ is corresponding to $(-1)^{p(i)p(j)}W^{(r)}_{i,j}$ $(r=1,2)$, where $J^{(x)}_{a,b}$ in [41] is corresponding to $(-1)^{p(a)p(b)}e_{b,a}$ in this paper.

10 Affine super Yangians and rectangular W-superalgebras

In this section, we prove the main result of this paper. Here after, we assume that $m \neq n$ and set

$$\varepsilon_1 = \frac{\alpha}{m-n}, \quad \varepsilon_2 = -1 - \frac{\alpha}{m-n}$$

and fix an invariant inner product on $\mathfrak{gl}(m|n)$ such that $c = 0$ (see (8.1)).

Theorem 10.1. *There exists an algebra homomorphism*

$$\Phi\colon Y_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m|n)) \to \mathcal{U}(\mathcal{W}^k(\mathfrak{gl}(ml|nl),(l^{(m|n)})))$$

determined by

$$\Phi(H_{i,0}) = \begin{cases} (-1)^{p(m+n)} W^{(1)}_{m+n,m+n} - W^{(1)}_{1,1} + l\alpha & (i=0),\\ (-1)^{p(i)} W^{(1)}_{i,i} - (-1)^{p(i+1)} W^{(1)}_{i+1,i+1} & (i \neq 0),\end{cases}$$

$$\Phi(X^+_{i,0}) = \begin{cases} W^{(1)}_{1,m+n} t & (i=0),\\ W^{(1)}_{i+1,i} & (i\neq 0),\end{cases} \quad \Phi(X^-_{i,0}) = \begin{cases} (-1)^{p(m+n)} W^{(1)}_{m+n,1} t^{-1} & (i=0),\\ (-1)^{p(i)} W^{(1)}_{i,i+1} & (i \neq 0),\end{cases}$$

$$\Phi(H_{i,1}) = \begin{cases} (-1)^{p(m+n)} W^{(2)}_{m+n,m+n} t - W^{(2)}_{1,1} t + (-1)^{p(m+n)} (l-1)\alpha W^{(1)}_{m+n,m+n} \\ \quad -l\alpha \Phi(H_{0,0}) + (-1)^{p(m+n)} W^{(1)}_{m+n,m+n}(W^{(1)}_{1,1} - l\alpha) \\ \quad -(-1)^{p(m+n)} \displaystyle\sum_{s\geq 0}\sum_{u=1}^{m+n} (-1)^{p(u)} W^{(1)}_{u,m+n} t^{-s} W^{(1)}_{m+n,u} t^s \\ \quad +\displaystyle\sum_{s\geq 0}\sum_{u=1}^{m+n} (-1)^{p(u)} W^{(1)}_{u,1} t^{-s-1} W^{(1)}_{1,u} t^{s+1}, \\ \hfill i=0, \\ (-1)^{p(i)} W^{(2)}_{i,i} t - (-1)^{p(i+1)} W^{(2)}_{i+1,i+1} t \\ \quad + \dfrac{i - 2\delta(i \geq m+1)(i-m)}{2} \Phi(H_{i,0}) + (-1)^{p(E_{i,i+1})} W^{(1)}_{i,i} W^{(1)}_{i+1,i+1} \\ \quad -(-1)^{p(i)} \displaystyle\sum_{s\geq 0}\sum_{u=1}^{i} (-1)^{p(u)} W^{(1)}_{u,i} t^{-s} W^{(1)}_{i,u} t^s \\ \quad -(-1)^{p(i)} \displaystyle\sum_{s\geq 0}\sum_{u=i+1}^{m+n} (-1)^{p(u)} W^{(1)}_{u,i} t^{-s-1} W^{(1)}_{i,u} t^{s+1} \\ \quad +(-1)^{p(i+1)} \displaystyle\sum_{s\geq 0}\sum_{u=1}^{i} (-1)^{p(u)} W^{(1)}_{u,i+1} t^{-s} W^{(1)}_{i+1,u} t^s \\ \quad +(-1)^{p(i+1)} \displaystyle\sum_{s\geq 0}\sum_{u=i+1}^{m+n} (-1)^{p(u)} W^{(1)}_{u,i+1} t^{-s-1} W^{(1)}_{i+1,u} t^{s+1} \\ \hfill i \neq 0, \end{cases}$$

$$\Phi(X^+_{i,1}) = \begin{cases} W^{(2)}_{1,m+n} t^2 + (l-1)\alpha W^{(1)}_{1,m+n} t - l\alpha \Phi(X^+_{0,0}) - \displaystyle\sum_{s\geq 0}\sum_{u=1}^{m+n} (-1)^{p(u)} W^{(1)}_{u,m+n} t^{-s} W^{(1)}_{1,u} t^{s+1} \\ \hfill \textit{if } i=0, \\ W^{(2)}_{i+1,i} t + \dfrac{i - 2\delta(i\geq m+1)(i-m)}{2} \Phi(X^+_{i,0}) \\ \quad - \displaystyle\sum_{s\geq 0}\sum_{u=1}^{i} (-1)^{p(u)} W^{(1)}_{u,i} t^{-s} W^{(1)}_{i+1,u} t^s - \sum_{s\geq 0}\sum_{u=i+1}^{m+n} (-1)^{p(u)} W^{(1)}_{u,i} t^{-s-1} W^{(1)}_{i+1,u} t^{s+1} \\ \hfill \textit{if } i \neq 0, \end{cases}$$

$$\Phi(X_{i,1}^-)=\begin{cases}(-1)^{p(m+n)}W^{(2)}_{m+n,1}-l\alpha\Phi(X_{0,0}^-)\\ \quad -(-1)^{p(m+n)}\displaystyle\sum_{s\geq 0}\sum_{u=1}^{m+n}(-1)^{p(u)}W^{(1)}_{1,u}t^{-s-1}W^{(1)}_{m+n,u}t^s, & \text{if } i=0,\\ (-1)^{p(i)}W^{(2)}_{i,i+1}t+\dfrac{i-2\delta(i\geq m+1)(i-m)}{2}\Phi(X_{i,0}^-)\\ \quad -(-1)^{p(i)}\displaystyle\sum_{s\geq 0}\sum_{u=1}^{i}(-1)^{p(u)}W^{(1)}_{u,i+1}t^{-s}W^{(1)}_{i,u}t^s\\ \quad -(-1)^{p(i)}\displaystyle\sum_{s\geq 0}\sum_{u=i+1}^{m+n}(-1)^{p(u)}W^{(1)}_{u,i+1}t^{-s-1}W^{(1)}_{i,u}t^{s+1} & \text{if } i\neq 0.\end{cases}$$

Proof. It is enough to show that Φ is compatible with the defining relations (7.2)-(7.13). By Lemma 9.5, we find that Φ is compatible with (7.3), (7.11), (7.12) and (7.13). Thus, it is enough to show that Φ is compatible with (7.2) and (7.4)-(7.10). We divide the proof into two piecies, that is, Claim 10.3 and Claim 10.16 below. In Claim 10.3, we show that Φ is compatible with (7.4)-(7.10). In Claim 10.16, we prove that Φ is compatible with (7.2).

In order to prove Claims 10.3 and 10.16, we relate Φ with the evaluation map of the afiine super Yangian. We set $\widetilde{\mathrm{ev}}(H_{i,s})$ and $\widetilde{\mathrm{ev}}(X^{\pm}_{i,s})$ $(s=0,1)$ as

$$\widetilde{\mathrm{ev}}(H_{i,0})=\Phi(H_{i,0}),\quad \widetilde{\mathrm{ev}}(X^{\pm}_{i,0})=\Phi(X^{\pm}_{i,0}),$$

$$\widetilde{\mathrm{ev}}(H_{i,1})=\begin{cases}\Phi(H_{0,1})-\{(-1)^{p(m+n)}W^{(2)}_{m+n,m+n}t-W^{(2)}_{1,1}t+(-1)^{p(m+n)}(l-1)\alpha W^{(1)}_{m+n,m+n}\}\\ \hfill \text{if } i=0,\\ \Phi(H_{i,1})-\{(-1)^{p(i)}W^{(2)}_{i,i}t-(-1)^{p(i+1)}W^{(2)}_{i+1,i+1}t\} \text{ if } i\neq 0,\end{cases}$$

$$\widetilde{\mathrm{ev}}(X^{+}_{i,1})=\begin{cases}\Phi(X^+_{i,1})-\{W^{(2)}_{1,m+n}t^2+(l-1)\alpha W^{(1)}_{1,m+n}t\} & \text{if } i=0,\\ \Phi(X^+_{i,1})-W^{(2)}_{i+1,i}t & \text{if } i\neq 0,\end{cases}$$

$$\widetilde{\mathrm{ev}}(X^{-}_{i,1})=\begin{cases}\Phi(X^-_{i,1})-(-1)^{p(m+n)}W^{(2)}_{m+n,1} & \text{if } i=0,\\ \Phi(X^-_{i,1})-(-1)^{p(i)}W^{(2)}_{i,i+1}t & \text{if } i\neq 0.\end{cases}$$

We define $\widehat{\mathfrak{gl}}(m|n)^\kappa$ as a Lie superalgebra $\mathfrak{gl}(m|n)\otimes\mathbb{C}[t^{\pm 1}]\oplus\mathbb{C}\tilde{c}\oplus\mathbb{C}x$ whose commutator relations are

$$\tilde{c}\text{ is a central element,}$$
$$[u\otimes t^a,v\otimes t^b]=[u,v]\otimes t^{a+b}+\delta_{a+b,0}a\,\mathrm{str}(uv)\tilde{c},\ \text{if } u \text{ or } v\in\mathfrak{sl}(m|n)$$
$$[E_{i,i}\otimes t^a,E_{j,j}\otimes t^b]=\delta_{a+b,0}a\,\mathrm{str}(E_{i,i}E_{j,j})\tilde{c}-\delta_{a+b,0}al(lc-1)(-1)^{p(i)+p(j)}x.$$

We note that $\widehat{\mathfrak{gl}}(m|n)^\kappa$ is the same as $\widehat{\mathfrak{gl}}(m|n)^{\mathrm{str}}$ except of the inner product on the diagonal part. By Lemma 9.5, we can prove that $\widetilde{\mathrm{ev}}$ is compatible with (7.3)-(7.13) which are parts of the defining relations of the affine super Yangian $Y_{\frac{l\alpha}{m-n},-1-\frac{l\alpha}{m-n}}(\widehat{\mathfrak{sl}}(m|n))$ in a way similar to the proof of the existence of the evaluation map (see Theorem 5.2 in [45]). This is summarized as the following lemma.

Lemma 10.2. *Let us set*

$$\tilde{\varepsilon}_1=\frac{l\alpha}{m-n},\qquad \tilde{\varepsilon}_2=-1-\frac{l\alpha}{m-n}.$$

Then, $\widetilde{\mathrm{ev}}$ *is compatible with* (7.3)-(7.13) *which are parts of the defining relations of the affine super Yangian* $Y_{\tilde{\varepsilon}_1,\tilde{\varepsilon}_2}(\widehat{\mathfrak{sl}}(m|n))$.

We remark that $\widetilde{\mathrm{ev}}$ is not an algebra homomorphism since $[\widetilde{\mathrm{ev}}(H_{i,1}),\widetilde{\mathrm{ev}}(H_{j,1})]$ is not equal to zero. See (10.32) below for the details.

Claim 10.3. For all $i,j\in\{0,1,\cdots,m+n-1\}$, Φ is compatible with (7.4)-(7.10).

Proof. We only show that Φ is compatible with (7.7). The other cases are proven in a similar way. It is enough to show that

$$\begin{aligned}&[\Phi(\tilde{H}_{0,1}),\Phi(X^+_{m+n-1,0})]\\&=-(-1)^{p(m+n)}\{\Phi(X^+_{m+n-1,1})-\left((m-n)+\alpha-\frac{m-n}{2}\right)W^{(1)}_{m+n,m+n-1}\},\end{aligned}\tag{10.4}$$

$$\begin{aligned}&[\Phi(\tilde{H}_{0,1}),\Phi(X^-_{m+n-1,0})]\\&=(-1)^{p(m+n)}\{\Phi(X^-_{m+n-1,1})-(-1)^{p(m+n-1)}\left((m-n)+\alpha-\frac{m-n}{2}\right)W^{(1)}_{m+n-1,m+n}\}.\end{aligned}\tag{10.5}$$

By the definition of Φ, we can rewrite the left hand side of (10.4) as

$$\begin{aligned}&[\Phi(\tilde{H}_{0,1}),\Phi(X^+_{m+n-1,0})]\\&=-[W^{(1)}_{m+n,m+n-1},(-1)^{p(m+n)}W^{(2)}_{m+n,m+n}t]+[W^{(1)}_{m+n,m+n-1},W^{(2)}_{1,1}t]\\&\quad-[W^{(1)}_{m+n,m+n-1},(-1)^{p(m+n)}(l-1)\alpha W^{(1)}_{m+n,m+n}]+[\widetilde{\mathrm{ev}}(\tilde{H}_{0,1}),\widetilde{\mathrm{ev}}(X^+_{m+n-1,0})].\end{aligned}\tag{10.6}$$

By Corollary 9.8, we obtain

$$-[W^{(1)}_{m+n,m+n-1},(-1)^{p(m+n)}W^{(2)}_{m+n,m+n}t]=-(-1)^{p(m+n)}W^{(2)}_{m+n,m+n-1}t,\tag{10.7}$$

$$[W^{(1)}_{m+n,m+n-1},W^{(2)}_{1,1}t]=0.\tag{10.8}$$

By Corollary 9.8, we have

$$-[W^{(1)}_{m+n,m+n-1},(-1)^{p(m+n)}(l-1)\alpha W^{(1)}_{m+n,m+n}]=-(-1)^{p(m+n)}(l-1)\alpha W^{(1)}_{m+n,m+n-1}.\tag{10.9}$$

By Lemma 10.2, we also obtain

$$\begin{aligned}&[\widetilde{\mathrm{ev}}(\tilde{H}_{0,1}),\widetilde{\mathrm{ev}}(X^+_{m+n-1,0})]\\&=-(-1)^{p(m+n)}\widetilde{\mathrm{ev}}(X^+_{m+n-1,1})+(-1)^{p(m+n)}\left((m-n)+l\alpha-\frac{m-n}{2}\right)W^{(1)}_{m+n,m+n-1}.\end{aligned}\tag{10.10}$$

The identity (10.4) follows by applying (10.7)-(10.10) to (10.6). We can prove that Φ is compatible with (7.8) in a similar way.

Similarly, by the definition of Φ, we obtain

$$\begin{aligned}&[\Phi(\tilde{H}_{0,1}),(-1)^{p(m+n-1)}W^{(1)}_{m+n-1,m+n}]\\&=-[(-1)^{p(m+n-1)}W^{(1)}_{m+n-1,m+n},(-1)^{p(m+n)}W^{(2)}_{m+n,m+n}t]\\&\quad-[(-1)^{p(m+n-1)}W^{(1)}_{m+n-1,m+n},-W^{(2)}_{1,1}t]\\&\quad-[(-1)^{p(m+n-1)}W^{(1)}_{m+n-1,m+n},(-1)^{p(m+n)}(l-1)\alpha W^{(1)}_{m+n,m+n}]\\&\quad+[\widetilde{\mathrm{ev}}(\tilde{H}_{0,1}),\widetilde{\mathrm{ev}}(X^-_{m+n-1,0})].\end{aligned}\tag{10.11}$$

By Corollary 9.8, we obtain

$$-[(-1)^{p(m+n-1)}W^{(1)}_{m+n-1,m+n},(-1)^{p(m+n)}W^{(2)}_{m+n,m+n}t]=(-1)^{p(m+n-1)+p(m+n)}W^{(2)}_{m+n-1,m+n}t,\tag{10.12}$$

$$-[(-1)^{p(m+n-1)}W^{(1)}_{m+n-1,m+n},W^{(2)}_{1,1}t]=0.\tag{10.13}$$

By Lemma 9.5, we have

$$\begin{aligned}&-[(-1)^{p(m+n-1)}W^{(1)}_{m+n-1,m+n},(-1)^{p(m+n)}(l-1)\alpha W^{(1)}_{m+n,m+n}]\\&=(-1)^{p(m+n-1)+p(m+n)}(l-1)\alpha W^{(1)}_{m+n-1,m+n}.\end{aligned}\tag{10.14}$$

By Lemma 10.2, we obtain

$$\begin{aligned}&[\widetilde{\mathrm{ev}}(\tilde{H}_{0,1}),\widetilde{\mathrm{ev}}(X^-_{m+n-1,0})]\\&=(-1)^{p(m+n)}\widetilde{\mathrm{ev}}(X^-_{m+n-1,1})\\&\qquad-(-1)^{p(m+n)}\left((m-n)+l\alpha-\frac{m-n}{2}\right)\left((-1)^{p(m+n-1)}W^{(1)}_{m+n-1,m+n}\right).\end{aligned}\tag{10.15}$$

The identity (10.5) follows by applying (10.12)-(10.15) to (10.11). Thus, we have shown that Φ is compatible with (7.7). □

Finally, we prove that Φ is compatible with (7.2).

Claim 10.16. The following equation holds for all $i,j\in\{0,1,\cdots,m+n-1\}$, $r,s\in\{0,1\}$;

$$[\Phi(H_{i,r}),\Phi(H_{j,s})]=0.$$

Proof. By Lemma 9.5, we obtain $[\Phi(H_{i,0}),\Phi(H_{j,0})]=0$. In the similar way as that of Claim 10.3, we have $[\Phi(H_{i,0}),\Phi(H_{j,1})]=0$. Thus, it is enough to show that $[\Phi(H_{i,1}),\Phi(H_{j,1})]=0$. We only show the case when $i,j\neq 0$ and $i>j$. The other case is proven in a similar way. In order to simplify the notation, we set

$$\begin{aligned}X_i=&-(-1)^{p(i)}\sum_{s\geq 0}\sum_{u=1}^{i}(-1)^{p(u)}W^{(1)}_{u,i}t^{-s}W^{(1)}_{i,u}t^s\\&-(-1)^{p(i)}\sum_{s\geq 0}\sum_{u=i+1}^{m+n}(-1)^{p(u)}W^{(1)}_{u,i}t^{-s-1}W^{(1)}_{i,u}t^{s+1}.\end{aligned}$$

By the definition of $\widetilde{\mathrm{ev}}$, we obtain

$$\begin{aligned}\widetilde{\mathrm{ev}}(H_{i,1})&=\frac{i-2\delta(i\geq m+1)(i-m)}{2}((-1)^{p(i)}W^{(1)}_{i,i}-(-1)^{p(i+1)}W^{(1)}_{i+1,i+1})\\&\quad+(-1)^{p(E_{i,i+1})}W^{(1)}_{i,i}W^{(1)}_{i+1,i+1}+X_i-X_{i+1}-(W^{(1)}_{i+1,i+1})^2\\&=X_i-X_{i+1}+(\text{the term generated by }\{W^{(1)}_{i,i}t^0|1\leq i\leq m+n\}).\end{aligned}\tag{10.17}$$

By Lemma 9.5, Lemma 9.6 and (10.17), we obtain

$$[\widetilde{\mathrm{ev}}(H_{i,1}),\widetilde{\mathrm{ev}}(H_{j,1})]=[X_i-X_{i+1},X_j-X_{j+1}],\tag{10.18}$$

$$\begin{aligned}&[\widetilde{\mathrm{ev}}(H_{i,1}),((-1)^{p(j)}W^{(2)}_{j,j}-(-1)^{p(j+1)}W^{(2)}_{j+1,j+1})t]\\&\qquad=[X_i-X_{i+1},((-1)^{p(j)}W^{(2)}_{j,j}-(-1)^{p(j+1)}W^{(2)}_{j+1,j+1})t].\end{aligned}\tag{10.19}$$

We remark that $[\widetilde{\mathrm{ev}}(H_{i,1}),\widetilde{\mathrm{ev}}(H_{j,1})]$ is not equal to zero since the inner products on the diagonal parts of $\widehat{\mathfrak{gl}}(m|n)^\kappa$ and $\widehat{\mathfrak{gl}}(m|n)^{\mathrm{str}}$ are different.

By (10.18), (10.19), and the definition of Φ, we obtain

$$\begin{aligned}&[\Phi(H_{i,1}),\Phi(H_{j,1})]\\&=[((-1)^{p(i)}W^{(2)}_{i,i}-(-1)^{p(i+1)}W^{(2)}_{i+1,i+1})t,((-1)^{p(j)}W^{(2)}_{j,j}-(-1)^{p(j+1)}W^{(2)}_{j+1,j+1})t]\end{aligned}$$

$$+ [X_i - X_{i+1}, ((-1)^{p(j)} W^{(2)}_{j,j} - (-1)^{p(j+1)} W^{(2)}_{j+1,j+1})t]$$
$$+ [((-1)^{p(i)} W^{(2)}_{i,i} - (-1)^{p(i+1)} W^{(2)}_{i+1,i+1})t, X_j - X_{j+1}] + [X_i - X_{i+1}, X_j - X_{j+1}].$$

Thus, it is enough to show the relation

$$[(-1)^{p(i)} W^{(2)}_{i,i} t, (-1)^{p(j)} W^{(2)}_{j,j} t] + [X_i, (-1)^{p(j)} W^{(2)}_{j,j} t] + [(-1)^{p(i)} W^{(2)}_{i,i} t, X_j] + [X_i, X_j] = 0 \quad (10.20)$$

holds for all $i, j \in \{1, \cdots, m+n\}$, $i \geq j$. Let us compute each terms of the left hand side of (10.20). First, we compute the first term of the left hand side of (10.20). By Lemma 9.9, we obtain

$$\begin{aligned}
&(-1)^{p(i)+p(j)} [W^{(2)}_{i,i} t, W^{(2)}_{j,j} t] \\
&= (-1)^{p(i)+p(j)} (W^{(2)}_{i,i})_{(0)} W^{(2)}_{j,j} t^2 + (-1)^{p(i)+p(j)} (W^{(2)}_{i,i})_{(1)} W^{(2)}_{j,j} t \\
&= (-1)^{p(j)} (W^{(1)}_{i,j})_{(-1)} W^{(2)}_{j,i} t^2 - (-1)^{p(i)} (W^{(1)}_{j,i})_{(-1)} W^{(2)}_{i,j} t^2 - \delta_{i,j} \alpha \partial W^{(2)}_{j,j} t^2 \\
&\quad - (-1)^{p(j)} \partial W^{(2)}_{j,j} t^2 + (-1)^{p(i)} (l-1) \alpha (W^{(1)}_{j,i})_{(-1)} \partial W^{(1)}_{i,j} t^2 \\
&\quad - (-1)^{p(i)+p(j)} \{(l-1)^2 c - (l-1)\} (W^{(1)}_{j,j})_{(-1)} \partial W^{(1)}_{i,i} t^2 \\
&\quad + \delta_{i,j} \frac{l(l-1)}{2} \alpha^2 \partial^2 W^{(1)}_{i,i} t^2 + (-1)^{p(i)} \frac{l(l-1)}{2} \alpha \partial^2 W^{(1)}_{i,i} t^2 \\
&\quad - (-1)^{p(i)} \frac{l(l-1)^2}{2} c \alpha \partial^2 W^{(1)}_{i,i} t^2 + \frac{1}{2} (-1)^{p(j)} (l-1) \alpha \partial^2 W^{(1)}_{j,j} t^2 - \frac{1}{2} (-1)^{p(i)} (l-1) \alpha \partial^2 W^{(1)}_{i,i} t^2 \\
&\quad - (-1)^{p(i)+p(j)} \{(l-1)^2 c - (l-1)\} (W^{(1)}_{j,j})_{(-1)} W^{(1)}_{i,i} t - 2\delta_{i,j} \alpha W^{(2)}_{i,i} t \\
&\quad - (-1)^{p(j)} W^{(2)}_{j,j} t - (-1)^{p(i)} W^{(2)}_{i,i} t + (-1)^{p(i)} (l-1) \alpha (W^{(1)}_{j,i})_{(-1)} W^{(1)}_{i,j} t \\
&\quad + \delta_{i,j} l(l-1) \alpha^2 \partial W^{(1)}_{i,i} t + (-1)^{p(j)} l(l-1) \alpha \partial W^{(1)}_{i,i} t - (-1)^{p(i)} l(l-1)^2 c \alpha \partial W^{(1)}_{i,i} t \\
&\quad - (-1)^{p(i)} (l-1) \alpha \partial W^{(1)}_{i,i} t + (-1)^{p(j)} (l-1) \alpha \partial W^{(1)}_{j,j} t.
\end{aligned}$$

We can rewrite it as

$$\begin{aligned}
&- (-1)^{p(i)} W^{(2)}_{i,i} t + (-1)^{p(j)} W^{(2)}_{j,j} t + (-1)^{p(j)} (W^{(1)}_{i,j})_{(-1)} W^{(2)}_{j,i} t^2 - (-1)^{p(i)} (W^{(1)}_{j,i})_{(-1)} W^{(2)}_{i,j} t^2 \\
&\quad + (-1)^{p(i)} (l-1) \alpha (W^{(1)}_{j,i})_{(-1)} \partial W^{(1)}_{i,j} t^2 + (-1)^{p(i)} (l-1) \alpha (W^{(1)}_{j,i})_{(-1)} W^{(1)}_{i,j} t \\
&\quad - (-1)^{p(i)+p(j)} \{(l-1)^2 c - (l-1)\} \Big((W^{(1)}_{j,j})_{(-1)} \partial W^{(1)}_{i,i} t^2 + (W^{(1)}_{j,j})_{(-1)} W^{(1)}_{i,i} t \Big) \qquad (10.21)
\end{aligned}$$

since six relations

$$-\delta_{i,j} \alpha \partial W^{(2)}_{j,j} t^2 - 2\delta_{i,j} \alpha W^{(2)}_{j,j} t = 0, \qquad (10.22)$$

$$\begin{aligned}
&-(-1)^{p(i)} W^{(2)}_{i,i} t - (-1)^{p(j)} W^{(2)}_{j,j} t - (-1)^{p(j)} \partial W^{(2)}_{j,j} t^2 \\
&\qquad = -((-1)^{p(i)} W^{(2)}_{i,i} t - (-1)^{p(j)} W^{(2)}_{j,j} t), \qquad (10.23)
\end{aligned}$$

$$\delta_{i,j} \frac{l(l-1)}{2} \alpha^2 \partial^2 W^{(1)}_{i,i} t^2 + \delta_{i,j} l(l-1) \alpha^2 \partial W^{(1)}_{i,i} t = 0, \qquad (10.24)$$

$$(-1)^{p(i)} \frac{l(l-1)}{2} \alpha \partial^2 W^{(1)}_{i,i} t^2 + (-1)^{p(i)} l(l-1) \alpha \partial W^{(1)}_{i,i} t = 0, \qquad (10.25)$$

$$-(-1)^{p(i)} \frac{l(l-1)^2}{2} c \alpha \partial^2 W^{(1)}_{i,i} t^2 - (-1)^{p(i)} l(l-1)^2 c \alpha \partial W^{(1)}_{i,i} t = 0, \qquad (10.26)$$

$$\begin{aligned}
&\frac{1}{2} (-1)^{p(j)} (l-1) \alpha \partial^2 W^{(1)}_{j,j} t^2 - \frac{1}{2} (-1)^{p(i)} (l-1) \alpha \partial^2 W^{(1)}_{i,i} t^2 \\
&\qquad - (-1)^{p(i)} (l-1) \alpha \partial W^{(1)}_{i,i} t + (-1)^{p(j)} (l-1) \alpha \partial W^{(1)}_{j,j} t = 0 \qquad (10.27)
\end{aligned}$$

hold by the definition of the translation operator ∂.

In order to rewrite (10.21), we remark that the following two relations

$$(x_{(-1)}y)t=\sum_{s\geq 0}xt^{-1-s}yt^{s+1}+(-1)^{p(x)p(y)}yt^{-s}xt^{s},\tag{10.28}$$

$$\begin{aligned}(x_{(-1)}\partial y)t^2&=\sum_{s\geq 0}xt^{-1-s}(\partial y)t^{s+2}+(-1)^{p(x)p(y)}(\partial y)t^{1-s}xt^{s}\\&=\sum_{s\geq 0}(-(s+2)xt^{-1-s}yt^{s+1}-(-1)^{p(x)p(y)}(1-s)yt^{-s}xt^{s})\end{aligned}\tag{10.29}$$

hold by (9.3) for all $x,y\in\mathcal{W}^k(\mathfrak{gl}(ml|nl),(l^{(m|n)}))$. By (10.28) and (10.29), we also obtain

$$(x_{(-1)}\partial y)t^2+(x_{(-1)}y)t=\sum_{s\geq 0}(-(1+s)xt^{-1-s}yt^{1+s}+(-1)^{p(x)p(y)}syt^{-s}xt^{s}).\tag{10.30}$$

By (10.28)-(10.30), we can rewrite (10.21) as

$$\begin{aligned}&-(-1)^{p(i)}W^{(2)}_{i,i}t+(-1)^{p(j)}W^{(2)}_{j,j}t\\&+(-1)^{p(j)}\sum_{s\geq 0}W^{(1)}_{i,j}t^{-s-1}W^{(2)}_{j,i}t^{s+2}+(-1)^{p(i)}\sum_{s\geq 0}W^{(2)}_{j,i}t^{1-s}W^{(1)}_{i,j}t^{s}\\&-(-1)^{p(i)}\sum_{s\geq 0}W^{(1)}_{j,i}t^{-s-1}W^{(2)}_{i,j}t^{s+2}-(-1)^{p(j)}\sum_{s\geq 0}W^{(2)}_{i,j}t^{1-s}W^{(1)}_{j,i}t^{s}\\&+(-1)^{p(j)}(l-1)\alpha\sum_{s\geq 0}sW^{(1)}_{i,j}t^{-s}W^{(1)}_{j,i}t^{s}-(-1)^{p(i)}(l-1)\alpha\sum_{s\geq 0}sW^{(1)}_{j,i}t^{-s}W^{(1)}_{i,j}t^{s}\\&-(-1)^{p(i)+p(j)}\{(l-1)^2c-(l-1)\}\sum_{s\geq 0}(-sW^{(1)}_{j,j}t^{-s}W^{(1)}_{i,i}t^{s}+sW^{(1)}_{i,i}t^{-s}W^{(1)}_{j,j}t^{s}).\end{aligned}\tag{10.31}$$

Next, let us compute the last term of (10.20). By a computation similar to the proof of the existence of the evaluation map (see Theorem 5.2 of [45]), it is equal to

$$\begin{aligned}[X_i,X_j]&=-(-1)^{p(i)+p(j)}l(lc-1)\sum_{s\geq 0}s\{W^{(1)}_{i,i}t^{-s}W^{(1)}_{j,j}t^{s}-W^{(1)}_{j,j}t^{-s}W^{(1)}_{i,i}t^{s}\}\\&\quad-(-1)^{p(i)+p(j)}\sum_{s\geq 0}s\{W^{(1)}_{i,i}t^{-s}W^{(1)}_{j,j}t^{s}-W^{(1)}_{j,j}t^{-s}W^{(1)}_{i,i}t^{s}\}.\end{aligned}\tag{10.32}$$

Finally, let us compute the second term and the third term of (10.20). By the definition of X_i, we obtain

$$\begin{aligned}&[X_i,W^{(2)}_{j,j}t]\\&=-(-1)^{p(i)}\sum_{s\geq 0}\sum_{u=1}^{i}(-1)^{p(u)}W^{(1)}_{u,i}t^{-s}[W^{(1)}_{i,u}t^{s},W^{(2)}_{j,j}t]\\&\quad-(-1)^{p(i)}\sum_{s\geq 0}\sum_{u=1}^{i}(-1)^{p(u)}[W^{(1)}_{u,i}t^{-s},W^{(2)}_{j,j}t]W^{(1)}_{i,u}t^{s}\\&\quad-(-1)^{p(i)}\sum_{s\geq 0}\sum_{u=i+1}^{m+n}(-1)^{p(u)}W^{(1)}_{u,i}t^{-s-1}[W^{(1)}_{i,u}t^{s+1},W^{(2)}_{j,j}t]\\&\quad-(-1)^{p(i)}\sum_{s\geq 0}\sum_{u=i+1}^{m+n}(-1)^{p(u)}[W^{(1)}_{u,i}t^{-s-1},W^{(2)}_{j,j}t]W^{(1)}_{i,u}t^{s+1}.\end{aligned}\tag{10.33}$$

By Corollary 9.8, the first term of the right hand side of (10.33) is equal to

$$\begin{aligned}
&-(-1)^{p(i)}\sum_{s\geq 0}\sum_{u=1}^{i}(-1)^{p(u)}W^{(1)}_{u,i}t^{-s}[W^{(1)}_{i,u}t^s,W^{(2)}_{j,j}t]\\
&=-\delta_{i,j}(-1)^{p(i)}\sum_{s\geq 0}\sum_{u=1}^{i}(-1)^{p(u)}W^{(1)}_{u,i}t^{-s}W^{(2)}_{i,u}t^{s+1}+\delta(i\geq j)(-1)^{p(i)+p(j)}\sum_{s\geq 0}W^{(1)}_{j,i}t^{-s}W^{(2)}_{i,j}t^{s+1}\\
&\quad-\delta_{i,j}(-1)^{p(i)}(l-1)\alpha\sum_{s\geq 0}\sum_{u=1}^{i}(-1)^{p(u)}sW^{(1)}_{u,i}t^{-s}W^{(1)}_{i,u}t^s\\
&\quad+(-1)^{p(i)}(l-1)(lc-1)\sum_{s\geq 0}sW^{(1)}_{i,i}t^{-s}W^{(1)}_{j,j}t^s
\end{aligned}\tag{10.34}$$

since by (9.1) $\kappa(e_{i,u},e_{j,j})t^{s-1}$ is equal to zero unless $s=0$. Similarly to (10.34), we rewrite the second, third, and 4-th terms of the right hand side of (10.33). By Corollary 9.8, the second term of the right hand side of (10.33) is equal to

$$\begin{aligned}
&-(-1)^{p(i)}\sum_{s\geq 0}\sum_{u=1}^{i}(-1)^{p(u)}[W^{(1)}_{u,i}t^{-s},W^{(2)}_{j,j}t]W^{(1)}_{i,u}t^s\\
&=-\delta(i\geq j)(-1)^{p(i)+p(j)}\sum_{s\geq 0}W^{(2)}_{j,i}t^{1-s}W^{(1)}_{i,j}t^s+\delta_{i,j}(-1)^{p(i)}\sum_{s\geq 0}\sum_{u=1}^{i}(-1)^{p(u)}W^{(2)}_{u,i}t^{1-s}W^{(1)}_{i,u}t^s\\
&\quad+\delta(i\geq j)(-1)^{p(i)+p(j)}(l-1)\alpha\sum_{s\geq 0}sW^{(1)}_{j,i}t^{-s}W^{(1)}_{i,j}t^s\\
&\quad-(-1)^{p(i)}(l-1)(lc-1)\sum_{s\geq 0}sW^{(1)}_{j,j}t^{-s}W^{(1)}_{i,i}t^s.
\end{aligned}\tag{10.35}$$

By Corollary 9.8, the third term of the right hand side of (10.33) is equal to

$$\begin{aligned}
&-(-1)^{p(i)}\sum_{s\geq 0}\sum_{u=i+1}^{m+n}(-1)^{p(u)}W^{(1)}_{u,i}t^{-s-1}[W^{(1)}_{i,u}t^{s+1},W^{(2)}_{j,j}t]\\
&=-\delta_{i,j}(-1)^{p(i)}\sum_{s\geq 0}\sum_{u=i+1}^{m+n}(-1)^{p(u)}W^{(1)}_{u,i}t^{-s-1}W^{(2)}_{i,u}t^{s+2}\\
&\quad+\delta(i<j)(-1)^{p(i)+p(j)}\sum_{s\geq 0}W^{(1)}_{j,i}t^{-s-1}W^{(2)}_{i,j}t^{s+2}\\
&\quad-\delta_{i,j}(-1)^{p(i)}(l-1)\alpha\sum_{s\geq 0}\sum_{u=i+1}^{m+n}(s+1)(-1)^{p(u)}W^{(1)}_{u,i}t^{-s-1}W^{(1)}_{i,u}t^{s+1}.
\end{aligned}\tag{10.36}$$

By Corollary 9.8, the 4-th term of the right hand side of (10.33) is equal to

$$\begin{aligned}
&-(-1)^{p(i)}\sum_{s\geq 0}\sum_{u=i+1}^{m+n}(-1)^{p(u)}[W^{(1)}_{u,i}t^{-s-1},W^{(2)}_{j,j}t]W^{(1)}_{i,u}t^{s+1}\\
&=-\delta(i<j)(-1)^{p(i)+p(j)}\sum_{s\geq 0}W^{(2)}_{j,i}t^{-s}W^{(1)}_{i,j}t^{s+1}\\
&\quad+\delta_{i,j}(-1)^{p(i)}\sum_{s\geq 0}\sum_{u=i+1}^{m+n}(-1)^{p(u)}W^{(2)}_{u,i}t^{-s}W^{(1)}_{i,u}t^{s+1}\\
&\quad+\delta(i<j)(-1)^{p(i)+p(j)}(l-1)\alpha\sum_{s\geq 0}(s+1)W^{(1)}_{j,i}t^{-s-1}W^{(1)}_{i,j}t^{s+1}.
\end{aligned}\tag{10.37}$$

We prepare some notations. We denote the i-th term of the right hand side of (10.34) (resp. (10.35), (10.36), (10.37)) by $(10.34)_i$ (resp. $(10.35)_i$, $(10.36)_i$, $(10.37)_i$). Let us set

$$\begin{aligned}
A_{i,j} &= (-1)^{p(j)}(10.34)_1 + (-1)^{p(j)}(10.35)_2 + (-1)^{p(j)}(10.36)_1 + (-1)^{p(j)}(10.37)_2 \\
&= -\delta_{i,j}\sum_{s\geq 0}\sum_{u=1}^{i}(-1)^{p(u)}W^{(1)}_{u,i}t^{-s}W^{(2)}_{i,u}t^{s+1} + \delta_{i,j}\sum_{s\geq 0}\sum_{u=1}^{i}(-1)^{p(u)}W^{(2)}_{u,i}t^{1-s}W^{(1)}_{i,u}t^{s} \\
&\quad - \delta_{i,j}\sum_{s\geq 0}\sum_{u=i+1}^{m+n}(-1)^{p(u)}W^{(1)}_{u,i}t^{-s-1}W^{(2)}_{i,u}t^{s+2} + \delta_{i,j}\sum_{s\geq 0}\sum_{u=i+1}^{m+n}(-1)^{p(u)}W^{(2)}_{u,i}t^{-s}W^{(1)}_{i,u}t^{s+1},\\
B_{i,j} &= (-1)^{p(j)}(10.34)_2 + (-1)^{p(j)}(10.35)_1 + (-1)^{p(j)}(10.36)_3 + (-1)^{p(j)}(10.37)_1 \\
&= \delta(i\geq j)(-1)^{p(i)}\sum_{s\geq 0}W^{(1)}_{j,i}t^{-s}W^{(2)}_{i,j}t^{s+1} - \delta(i\geq j)(-1)^{p(i)}\sum_{s\geq 0}W^{(2)}_{j,i}t^{1-s}W^{(1)}_{i,j}t^{s} \\
&\quad + \delta(i<j)(-1)^{p(i)}\sum_{s\geq 0}W^{(1)}_{j,i}t^{-s-1}W^{(2)}_{i,j}t^{s+2} - \delta(i<j)(-1)^{p(i)}\sum_{s\geq 0}W^{(2)}_{j,i}t^{-s}W^{(1)}_{i,j}t^{s+1},\\
C_{i,j} &= (-1)^{p(j)}(10.35)_3 + (-1)^{p(j)}(10.37)_3 \\
&= \delta(i\geq j)(-1)^{p(i)}(l-1)\alpha\sum_{s\geq 0}sW^{(1)}_{j,i}t^{-s}W^{(1)}_{i,j}t^{s} \\
&\quad + \delta(i<j)(-1)^{p(i)}(l-1)\alpha\sum_{s\geq 0}(s+1)W^{(1)}_{j,i}t^{-s-1}W^{(1)}_{i,j}t^{s+1} \\
&= (-1)^{p(i)}(l-1)\alpha\sum_{s\geq 0}sW^{(1)}_{j,i}t^{-s}W^{(1)}_{i,j}t^{s},\\
D_{i,j} &= (-1)^{p(j)}(10.34)_3 + (-1)^{p(j)}(10.36)_3 \\
&= -\delta_{i,j}(l-1)\alpha\sum_{s\geq 0}\sum_{u=1}^{i}s(-1)^{p(u)}W^{(1)}_{u,i}t^{-s}W^{(1)}_{i,u}t^{s} \\
&\quad - \delta_{i,j}(l-1)\alpha\sum_{s\geq 0}\sum_{u=i+1}^{m+n}(s+1)(-1)^{p(u)}W^{(1)}_{u,i}t^{-s-1}W^{(1)}_{i,u}t^{s+1},\\
\tilde{E}_{i,j} &= (-1)^{p(j)}(10.34)_4 + (-1)^{p(j)}(10.35)_4 \\
&= (-1)^{p(i)+p(j)}(l-1)(lc-1)\sum_{s\geq 0}sW^{(1)}_{i,i}t^{-s}W^{(1)}_{j,j}t^{s} \\
&\quad - (-1)^{p(i)+p(j)}(l-1)(lc-1)\sum_{s\geq 0}sW^{(1)}_{j,j}t^{-s}W^{(1)}_{i,i}t^{s}.
\end{aligned}$$

Then, we can rewrite $[X_i, (-1)^{p(j)}W^{(2)}_{j,j}t]$ as $A_{i,j}+B_{i,j}+C_{i,j}+D_{i,j}+\tilde{E}_{i,j}$. By exchanging i and j, we find that $[X_j, (-1)^{p(i)}W^{(2)}_{i,i}t]$ is equal to $A_{j,i}+B_{j,i}+C_{j,i}+D_{j,i}+\tilde{E}_{j,i}$. We find that the left hand side of (10.20) is equal to

$$(10.31) + A_{i,j}+B_{i,j}+C_{i,j}+D_{i,j}+\tilde{E}_{i,j} - (A_{j,i}+B_{j,i}+C_{j,i}+D_{j,i}+\tilde{E}_{j,i}) + (10.32).$$

By the definition of $A_{i,j}$ and $D_{i,j}$, we have

$$A_{i,j}-A_{j,i}=0,\qquad D_{i,j}-D_{j,i}=0. \tag{10.38}$$

By direct computation, we obtain

$$C_{i,j}-C_{j,i}+(10.31)_7+(10.31)_8=0, \tag{10.39}$$

where we denote the i-th term of (10.31) by $(10.31)_i$. Hence, by (10.38), it is enough to obtain the following two relations

$$B_{i,j}-B_{j,i}+(10.31)_1+(10.31)_2+(10.31)_3+(10.31)_4+(10.31)_5+(10.31)_6=0, \tag{10.40}$$

$$\tilde{E}_{i,j}-\tilde{E}_{j,i}+(10.31)_9+(10.32)=0. \tag{10.41}$$

First, we show that (10.40) holds. Let us compute $B_{i,j}-B_{j,i}$. When $i=j$, it is equal to zero and (10.40) holds. Suppose that $i>j$. Then, we can rewrite $B_{i,j}-B_{j,i}$ as

$$\begin{aligned}
&(-1)^{p(i)}\sum_{s\geq 0}W^{(1)}_{j,i}t^{-s}W^{(2)}_{i,j}t^{s+1}-(-1)^{p(i)}\sum_{s\geq 0}W^{(2)}_{j,i}t^{1-s}W^{(1)}_{i,j}t^{s}\\
&\quad-(-1)^{p(j)}\sum_{s\geq 0}W^{(1)}_{i,j}t^{-s-1}W^{(2)}_{j,i}t^{s+2}+(-1)^{p(j)}\sum_{s\geq 0}W^{(2)}_{i,j}t^{-s}W^{(1)}_{j,i}t^{s+1}.
\end{aligned} \tag{10.42}$$

By Corollary 9.8, we obtain

$$\begin{aligned}
&(-1)^{p(i)}\sum_{s\geq 0}W^{(1)}_{j,i}t^{-s}W^{(2)}_{i,j}t^{s+1}+(-1)^{p(j)}\sum_{s\geq 0}W^{(2)}_{i,j}t^{-s}W^{(1)}_{j,i}t^{s+1}\\
&=(-1)^{p(i)}\sum_{s\geq 0}W^{(1)}_{j,i}t^{-s-1}W^{(2)}_{i,j}t^{s+2}+(-1)^{p(j)}\sum_{s\geq 0}W^{(2)}_{i,j}t^{1-s}W^{(1)}_{j,i}t^{s}\\
&\quad+(-1)^{p(i)}W^{(2)}_{i,i}t-(-1)^{p(j)}W^{(2)}_{j,j}t
\end{aligned} \tag{10.43}$$

Appling (10.43) to (10.42), we obtain

$$\begin{aligned}
&B_{i,j}-B_{j,i}\\
&=(-1)^{p(i)}\sum_{s\geq 0}W^{(1)}_{j,i}t^{-s-1}W^{(2)}_{i,j}t^{s+2}-(-1)^{p(i)}\sum_{s\geq 0}W^{(2)}_{j,i}t^{1-s}W^{(1)}_{i,j}t^{s}\\
&\quad-(-1)^{p(j)}\sum_{s\geq 0}W^{(1)}_{i,j}t^{-s-1}W^{(2)}_{j,i}t^{s+2}+(-1)^{p(j)}\sum_{s\geq 0}W^{(2)}_{i,j}t^{1-s}W^{(1)}_{j,i}t^{s}\\
&\quad+(-1)^{p(i)}W^{(2)}_{i,i}t-(-1)^{p(j)}W^{(2)}_{j,j}t.
\end{aligned}$$

We have shown that (10.40) holds.

Finally, let us compute the left hand side of (10.41). By direct computation, we obtain

$$\begin{aligned}
&\tilde{E}_{i,j}-\tilde{E}_{j,i}\\
&=2(-1)^{p(i)+p(j)}(l-1)(lc-1)\sum_{s\geq 0}sW^{(1)}_{i,i}t^{-s}W^{(1)}_{j,j}t^{s}\\
&\quad-2(-1)^{p(i)+p(j)}(l-1)(lc-1)\sum_{s\geq 0}sW^{(1)}_{j,j}t^{-s}W^{(1)}_{i,i}t^{s}.
\end{aligned}$$

It follows that the left hand side of (10.41) is equal to

$$\begin{aligned}
&2(-1)^{p(i)+p(j)}(l-1)(lc-1)\sum_{s\geq 0}sW^{(1)}_{i,i}t^{-s}W^{(1)}_{j,j}t^{s}\\
&\quad-2(-1)^{p(i)+p(j)}(l-1)(lc-1)\sum_{s\geq 0}sW^{(1)}_{j,j}t^{-s}W^{(1)}_{i,i}t^{s}\\
&\quad-(-1)^{p(i)+p(j)}\{(l-1)^2c-(l-1)\}\sum_{s\geq 0}(-sW^{(1)}_{j,j}t^{-s}W^{(1)}_{i,i}t^{s}+sW^{(1)}_{i,i}t^{-s}W^{(1)}_{j,j}t^{s})\\
&\quad-(-1)^{p(i)+p(j)}l(lc-1)\sum_{s\geq 0}s\{W^{(1)}_{i,i}t^{-s}W^{(1)}_{j,j}t^{s}-W^{(1)}_{j,j}t^{-s}W^{(1)}_{i,i}t^{s}\}\\
&\quad-(-1)^{p(i)+p(j)}\sum_{s\geq 0}s\{W^{(1)}_{i,i}t^{-s}W^{(1)}_{j,j}t^{s}-W^{(1)}_{j,j}t^{-s}W^{(1)}_{i,i}t^{s}\}\\
&=-(-1)^{p(i)+p(j)}c\sum_{s\geq 0}(-sW^{(1)}_{j,j}t^{-s}W^{(1)}_{i,i}t^{s}+sW^{(1)}_{i,i}t^{-s}W^{(1)}_{j,j}t^{s}).
\end{aligned}$$

Since $c=0$, this is equal to zero. Thus, (10.41) holds. We have shown that $[\Phi(H_{i,1}),\Phi(H_{j,1})]=0$. □

Since we have proved Claim 10.3 and Claim 10.16, we have proven that Φ is compatible with the defining relations of the affine super Yangian. $\square$

Next, let us show that Φ is essentially surjective when $\alpha\neq 0$.

Theorem 10.44. *The image of Φ is dense in $\mathcal{U}(\mathcal{W}^k(\mathfrak{gl}(ml|nl),(l^{(m|n)})))$ provided that α is nonzero.*

Proof. Suppose that $\alpha\neq 0$. By Theorem 8.17, it is enough to show that the completion of the image of Φ contains $W^{(1)}_{i,j}t^s$ and $W^{(2)}_{i,j}t^s$ for all $1\leq i,j\leq m+n$ and $s\in\mathbb{Z}$.

First, we show that $W^{(1)}_{j,j}t^s$ is contained in the completion of the image of Φ. By the definition of $\Phi(H_{i,0})$ and $\Phi(X^{\pm}_{i,0})$, the image of Φ contains $((-1)^{p(i)}W^{(1)}_{i,i}-(-1)^{p(j)}W^{(1)}_{j,j})t$ and $W^{(1)}_{i,j}t^s$ for all $i\neq j$ and $s\in\mathbb{Z}$. Then, by the definition of $\Phi(H_{i,1})$, the completion of the image of Φ contains

$$\begin{aligned}&((-1)^{p(j)}W^{(2)}_{j,j}-(-1)^{p(j+1)}W^{(2)}_{j+1,j+1})t\\&\quad-\sum_{a\geq 0}W^{(1)}_{j,j}t^{-a}W^{(1)}_{j,j}t^{a}+\sum_{a\geq 0}W^{(1)}_{j+1,j+1}t^{-a-1}W^{(1)}_{j+1,j+1}t^{a+1}-(-1)^{p(e_{j,j+1})}W^{(1)}_{j,j}W^{(1)}_{j+1,j+1}.\end{aligned}$$

We take $1\leq r,q\leq m+n$ such that $q\neq r,r+1$. Then, by Corollary 9.8, we have

$$\begin{aligned}&[W^{(1)}_{q,r}t^{s-1},((-1)^{p(r)}W^{(2)}_{r,r}-(-1)^{p(r+1)}W^{(2)}_{r+1,r+1})t-\sum_{a\geq 0}W^{(1)}_{r,r}t^{-a}W^{(1)}_{r,r}t^{a}\\&\qquad\qquad+\sum_{a\geq 0}W^{(1)}_{r+1,r+1}t^{-a-1}W^{(1)}_{r+1,r+1}t^{a+1}-(-1)^{p(e_{r,r+1})}W^{(1)}_{r,r}W^{(1)}_{r+1,r+1}]\\&=-(-1)^{p(r)}W^{(2)}_{q,r}t^s+\sum_{a\geq 0}W^{(1)}_{q,r}t^{-a+s-1}W^{(1)}_{r,r}t^{a}\\&\qquad+\sum_{a\geq 0}W^{(1)}_{r,r}t^{-a-1}W^{(1)}_{q,r}t^{a+s}-(-1)^{p(e_{r,r+1})}W^{(1)}_{q,r}t^{s-1}W^{(1)}_{r+1,r+1}.\end{aligned}$$

Let us set $\sum\limits_{a\geq 0}W^{(1)}_{q,r}t^{-a+s-1}W^{(1)}_{r,r}t^{a}+\sum\limits_{a\geq 0}W^{(1)}_{r,r}t^{-a-1}W^{(1)}_{q,r}t^{a+s}-(-1)^{p(e_{r,r+1})}W^{(1)}_{q,r}t^{s-1}W^{(1)}_{r+1,r+1}$ as $P^s_{q,r}$. By Lemma 9.5, we obtain

$$\begin{aligned}&[W^{(1)}_{r,q},P^s_{q,r}]-[W^{(1)}_{r,q}t,P^{s-1}_{q,r}]\\&=-(-1)^{p(q)}l\alpha W^{(1)}_{r,r}t^{s-1}+\delta_{s,1}(-1)^{p(e_{r,r+1})+p(q)}l\alpha W^{(1)}_{r+1,r+1}\\&\qquad+(-1)^{p(e_{q,r})}W^{(1)}_{q,r}t^{s-1}W^{(1)}_{r,q}-W^{(1)}_{r,q}W^{(1)}_{q,r}t^{s-1}.\end{aligned}$$

Then, by Lemma 9.5 and Corollary 9.8, we have

$$\begin{aligned}&[W^{(1)}_{r,q},W^{(2)}_{q,r}t^s-(-1)^{p(r)}P^s_{q,r}]-[W^{(1)}_{r,q}t,W^{(2)}_{q,r}t^{s-1}-(-1)^{p(r)}P^{s-1}_{q,r}]\\&=-(l-1)\alpha W^{(1)}_{q,q}t^{s-1}+(-1)^{p(e_{q,r})}l\alpha W^{(1)}_{r,r}t^{s-1}-\delta_{s,1}(-1)^{p(e_{q,r+1})}l\alpha W^{(1)}_{r+1,r+1}\\&\quad-(-1)^{p(q)}W^{(1)}_{q,r}t^{s-1}W^{(1)}_{r,q}+(-1)^{p(r)}W^{(1)}_{r,q}W^{(1)}_{q,r}t^{s-1}\\&=\alpha W^{(1)}_{q,q}t^{s-1}-l\alpha(W^{(1)}_{q,q}t^{s-1}-(-1)^{p(e_{q,r})}W^{(1)}_{r,r}t^{s-1})-\delta_{s,1}(-1)^{p(e_{q,r+1})}l\alpha W^{(1)}_{r+1,r+1}\\&\quad-(-1)^{p(q)}W^{(1)}_{q,r}t^{s-1}W^{(1)}_{r,q}+(-1)^{p(r)}W^{(1)}_{r,q}W^{(1)}_{q,r}t^{s-1}.\end{aligned}\tag{10.45}$$

We find that $\alpha W^{(1)}_{q,q}t^{s-1}-\delta_{s,1}(-1)^{p(e_{q,r+1})}l\alpha W^{(1)}_{r+1,r+1}$ is contained in the completion of the image of Φ by (10.45), we have shown that the completion of the image of Φ contains $W^{(1)}_{q,q}t^s$.

Since we have already shown that the completion of the image of Φ contains $\{W^{(1)}_{i,j}t^s\mid 1\leq i,j\leq m+n\}$, we find that the completion of the image of Φ contains $((-1)^{p(i)}W^{(2)}_{i,i}-(-1)^{p(j)}W^{(2)}_{j,j})t$ and $W^{(2)}_{i,i\pm 1}t$ for all $1\leq i,j\leq m+n$ by the definition of $\Phi(H_{i,1})$ and $\Phi(X^{\pm}_{i,1})$. Since there exists a pair (i,j) such that $p(i)=p(j)$, it is enough to prove that $W^{(2)}_{i,j}t^s$ $(i\neq j)$, $((-1)^{p(i)}W^{(2)}_{i,i}-(-1)^{p(j)}W^{(2)}_{j,j})t^s$, $W^{(1)}_{j,j}t^s$, and $W^{(2)}_{i,i}t^s+W^{(2)}_{j,j}t^s$ are contained in the image of the completion of Φ. Next, we show that the completion of the image of Φ contains $W^{(2)}_{i,j}t^s$ $(i\neq j)$. By Corollary 9.8, we have

$$[W^{(1)}_{i,j}t^{s-1},((-1)^{p(j)}W^{(2)}_{j,j}-(-1)^{p(j+1)}W^{(2)}_{j+1,j+1})t]=-(-1)^{p(j)}W^{(2)}_{i,j}t^s \text{ (if } i\neq j,j+1),$$
$$[W^{(1)}_{j+1,j}t^{s-1},((-1)^{p(j)}W^{(2)}_{j,j}-(-1)^{p(j-1)}W^{(2)}_{j-1,j-1})t]=-(-1)^{p(j)}W^{(2)}_{j+1,j}t^s.$$

Thus, $W^{(2)}_{i,j}t^s$ $(i\neq j)$ is contained in the completion of the image of Φ. By (9.3), we obtain

$$\begin{aligned}
&[((-1)^{p(i)}W^{(2)}_{i,i}-(-1)^{p(j)}W^{(2)}_{j,j})t,((-1)^{p(i)}W^{(2)}_{i,i}-(-1)^{p(j)}W^{(2)}_{j,j})t^s]\\
&\quad-[((-1)^{p(i)}W^{(2)}_{i,i}-(-1)^{p(j)}W^{(2)}_{j,j}),((-1)^{p(i)}W^{(2)}_{i,i}-(-1)^{p(j)}W^{(2)}_{j,j})t^{s+1}]\\
&=((-1)^{p(i)}W^{(2)}_{i,i}-(-1)^{p(j)}W^{(2)}_{j,j})_{(0)}((-1)^{p(i)}W^{(2)}_{i,i}-(-1)^{p(j)}W^{(2)}_{j,j})t^{s+1}\\
&\quad+((-1)^{p(i)}W^{(2)}_{i,i}-(-1)^{p(j)}W^{(2)}_{j,j})_{(1)}((-1)^{p(i)}W^{(2)}_{i,i}-(-1)^{p(j)}W^{(2)}_{j,j})t^{s}\\
&\quad-((-1)^{p(i)}W^{(2)}_{i,i}-(-1)^{p(j)}W^{(2)}_{j,j})_{(0)}((-1)^{p(i)}W^{(2)}_{i,i}-(-1)^{p(j)}W^{(2)}_{j,j})t^{s+1}\\
&=((-1)^{p(i)}W^{(2)}_{i,i}-(-1)^{p(j)}W^{(2)}_{j,j})_{(1)}((-1)^{p(i)}W^{(2)}_{i,i}-(-1)^{p(j)}W^{(2)}_{j,j})t^{s}.
\end{aligned}$$

By Lemma 9.9, provided that $i\neq j$, it is equal to

$$\begin{aligned}
&-2\alpha(W^{(2)}_{i,i}+W^{(2)}_{j,j})t^s-2(-1)^{p(i)}W^{(2)}_{i,i}-2(-1)^{p(j)}W^{(2)}_{j,j}+2((-1)^{p(i)}W^{(2)}_{i,i}+(-1)^{p(j)}W^{(2)}_{j,j})t^s\\
&\quad+(\text{the terms consisting of }\{W^{(1)}_{i,j}\ (1\leq i,j\leq m+n),\ W^{(2)}_{i,j}\ (i\neq j)\})\\
&=-2\alpha(W^{(2)}_{i,i}+W^{(2)}_{j,j})t^s\\
&\quad+(\text{the terms consisting of }\{W^{(1)}_{i,j}\ (1\leq i,j\leq m+n),\ W^{(2)}_{i,j}\ (i\neq j)\}).
\end{aligned}$$

Thus, the completion of the image of Φ contains $W^{(2)}_{i,i}t^s+W^{(2)}_{j,j}t^s$. □

We obtain the following theorem in the similar proof as that of Theorem 10.1 and Theorem 10.44.

Theorem 10.46. *We assume that $m\geq 3$ and $l\geq 2$. Let us set*

$$\varepsilon_1=\frac{k+(l-1)m}{m},\quad \varepsilon_2=-1-\frac{k+(l-1)m}{m}.$$

Then, there exists an algebra homomorphism

$$\Phi\colon Y_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m))\to\mathcal{U}(\mathcal{W}^k(\mathfrak{gl}(ml),(l^m)))$$

determined by the same formula as that of Theorem 10.1 under the assumption that $n=0$. Moreover, the image of Φ is dense in $\mathcal{U}(\mathcal{W}^k(\mathfrak{gl}(ml),(l^m)))$ provided that $k+(l-1)m\neq 0$.

11 Rectangular W-algebras of type D

For all $n\in\mathbb{Z}_{>0}$, let I_n be $\{-n+1,-n+3,\dots,n-1\}$. Then, $\mathfrak{gl}(n)$ has a basis $\{e_{i,j}\mid i,j\in I_n\}$, where $e_{i,j}$ is a matrix unit. Using an $n\times n$ matrix $J_n\in\mathfrak{gl}(n)$ whose (i,j) component is equal to $\delta_{i,-j}$, we can set $\mathfrak{so}(n)$ as $\{x\in\mathfrak{gl}(n)\mid x^TJ_n+J_nx=0\}$, where x^T is the transpose of x. We remark that $\mathfrak{so}(n)$ is not simple but reductive in the sense of this definition. Under this notation, $\mathfrak{so}(n)$ is spanned by the set of matrices $\{f_{i,j}=e_{i,j}-e_{-j,-i}\mid i,j\in I_n\}$.

In this paper, we suppose that l and n are even positive. For all $a\in I_{nl}$, we take $\mathrm{row}(a)\in I_n$ and $\mathrm{col}(a)\in I_l$ such that $a=(\mathrm{col}(a))n+\mathrm{row}(a)$. By the definition of $\mathrm{row}(a)$ and $\mathrm{col}(a)$, we have $\mathrm{row}(-a)=-\mathrm{row}(a)$ and $\mathrm{col}(-a)=-\mathrm{col}(a)$.

We take a nilpotent element f as follows;

$$f = \sum_{\substack{a,b\in I_{nl}\\ \mathrm{row}(a)=\mathrm{row}(b)\\ \mathrm{col}(b)+2=\mathrm{col}(a)\geq 2}} f_{a,b} + \sum_{\substack{a,b\in I_{nl}\\ \mathrm{row}(a)=\mathrm{row}(b)>0\\ \mathrm{col}(b)+2=\mathrm{col}(a)=1}} f_{a,b}.$$

We also set

$$\mathfrak{g}_p = \bigoplus_{\substack{a,b\in I_{nl},\\ \mathrm{col}(b)-\mathrm{col}(a)=p}} \mathbb{C}f_{a,b} \subset \mathfrak{so}(nl).$$

and fix the $\mathfrak{sl}_2$-triple (x,e,f) such that

$$\mathfrak{g}_p = \{y \in \mathfrak{so}(nl) \mid [x,y] = py\}.$$

Let $\mathfrak{b} = \bigoplus_{r\leq 0} \mathfrak{g}_r$ and $\mathfrak{c} = \bigoplus_{r\neq 0} \mathfrak{g}_r$, then $\mathfrak{b}$ and $\mathfrak{c}$ are subalgebras of $\mathfrak{so}(nl)$. We take an invariant inner product on $\mathfrak{so}(nl)$ by

$$\begin{aligned}&(f_{a_1,b_1}, f_{a_2,b_2})\\ &= k(\delta_{a_1,b_2}\delta_{b_1,a_2} - \delta_{a_1+a_2,0}\delta_{b_1+b_2,0}) + \delta_{a_1,b_1}\delta_{a_2,b_2}(\delta(\mathrm{col}(a_1)\,\mathrm{col}(a_2) > 0) - \delta(\mathrm{col}(a_1)\,\mathrm{col}(a_2) < 0)).\end{aligned}$$

We fix some notations about vertex algebras. For a vertex algebra V, we denote the generating field associated with $v \in V$ by $v(z) = \sum_{n\in\mathbb{Z}} v_{(n)}z^{-n-1}$ and the vacuum vector (resp. the translation operator) by $|0\rangle$ (resp. ∂). We also denote the OPE of $u,v \in V$ by

$$u(z)v(w) \sim \sum_{s\geq 0} \frac{(u_{(s)}v)(w)}{(z-w)^{s+1}}.$$

There exists a non-degenerate invariant inner product on $\mathfrak{so}(nl)$ determined by

$$\kappa(f_{a_1,b_1}, f_{a_2,b_2}) = (\delta_{a_1,b_2}\delta_{b_1,a_2} - \delta_{a_1+a_2,0}\delta_{b_1+b_2,0})\alpha + \delta_{a_1,b_1}\delta_{a_2,b_2}(\delta_{\mathrm{col}(a_1),\mathrm{col}(a_2)} - \delta_{\mathrm{col}(a_1)+\mathrm{col}(a_2),0}),$$

where $\alpha = k + (l-1)n - 2$. Let $\widehat{\mathfrak{b}} = \mathfrak{b}[t^{\pm 1}] \oplus \mathbb{C}y$ be the affinization of $\mathfrak{b}$ associated with the inner product κ. We define a left $\widehat{\mathfrak{b}}$-module $V^\kappa(\mathfrak{b})$ as $U(\widehat{\mathfrak{b}})/U(\widehat{\mathfrak{b}})(\mathfrak{b}[t] \oplus \mathbb{C}(y-1)) \cong U(\mathfrak{b}[t^{-1}]t^{-1})$. Then, $V^\kappa(\mathfrak{b})$ has a vertex algebra structure whose vacuum vector is 1 and the generating field $(ut^{-1})(z)$ is equal to $\sum_{s\in\mathbb{Z}}(ut^s)z^{-s-1}$ for all $u \in \mathfrak{b}$. We denote the generating field $(ut^{-1})(z)$ also by $u(z)$. We call $V^\kappa(\mathfrak{b})$ the universal affine vertex algebra associated with $(\mathfrak{b},\kappa)$. By the definition of $V^\kappa(\mathfrak{b})$, generating fields $u(z)$ and $v(z)$ satisfy

$$u(z)v(w) \sim \frac{[u,v](w)}{z-w} + \frac{\kappa(u,v)}{(z-w)^2} \tag{11.1}$$

for all $u,v \in \mathfrak{b}$.

Let $\mathfrak{a}$ be a Lie superalgebra generated by $\{J^{(u)}, \psi_v \mid u \in \mathfrak{b}, v \in \mathfrak{c}\}$ with the following commutator relations;

$$[J^{(u)}, J^{(v)}] = J^{([u,v])}, \quad [J^{(u)}, \psi_v] = \psi_{[u,v]}, \quad [\psi_u, \psi_v] = 0,$$

where $J^{(u)}$ is an even element and ψ_v is an odd element. We define a vertex algebra $V^{\widetilde{\kappa}}(\mathfrak{a})$ associated with a Lie superalgebra $\mathfrak{a}$ and the inner product on $\mathfrak{a}$ determined by

$$\widetilde{\kappa}(J^{(u)}, J^{(v)}) = \kappa(u,v), \quad \widetilde{\kappa}(J^{(u)}, \psi_v) = \widetilde{\kappa}(\psi_u, \psi_v) = 0.$$

In this section, we regard $V^{\widetilde{\kappa}}(\mathfrak{a})$ (resp. $V^{\kappa}(\mathfrak{b})$) as a non-associative superalgebra whose product $\cdot$ is defined by $u \cdot v = u_{(-1)}v$. In order to simplify the notation, we denote $J^{(u)}t^s \in V^{\widetilde{\kappa}}(\mathfrak{a})$ by $u[s]$ and set

$$\widehat{i} = \begin{cases} 0 & \text{if } i \geq 0, \\ 1 & \text{if } i < 0. \end{cases}$$

By [23], $\mathcal{W}^k(\mathfrak{so}(nl), (l^n))$ can be realized as a vertex subalgebra of $V^{\kappa}(\mathfrak{b})$.

Definition 11.2. We define $\mathcal{W}^k(\mathfrak{so}(nl), (l^n))$ as

$$\mathcal{W}^k(\mathfrak{so}(nl), (l^n)) = \{y \in V^{\kappa}(\mathfrak{b}) \mid d_0(y) = 0\},$$

where $d_0 \colon V^{\kappa}(\mathfrak{b}) \to V^{\widetilde{\kappa}}(\mathfrak{a})$ is an odd differential determined by

$$[d_0, 1] = 0, \quad [d_0, \partial] = 0, \tag{11.3}$$

$$\begin{aligned} &[d_0, f_{a,b}[-1]] \\ &= \sum_{\mathrm{col}(b) \leq \mathrm{col}(c) < \mathrm{col}(a)} f_{c,b}[-1]\psi_{f_{a,c}}[-1] - \sum_{\mathrm{col}(b) < \mathrm{col}(c) \leq \mathrm{col}(a)} \psi_{f_{c,b}}[-1] f_{a,c}[-1] \\ &\quad + \alpha\psi_{f_{a,b}}[-2] + \delta(\mathrm{col}(a) > \mathrm{col}(-a) > \mathrm{col}(b))\psi_{f_{a,b}}[-2] + \delta(\mathrm{col}(a) \geq \mathrm{col}(-b) > \mathrm{col}(b))\psi_{f_{a,b}}[-2] \\ &\quad + (-1)^{\widehat{p+2} + (\widehat{p} + \widehat{p+2}) \cdot \widehat{i}} \psi_{f_{a+2n,b}}[-1] - (-1)^{\widehat{q} + (\widehat{q} + \widehat{q-2}) \cdot \widehat{j}} \psi_{f_{a,b-2n}}[-1], \end{aligned} \tag{11.4}$$

where $i = \mathrm{row}(a), j = \mathrm{row}(b), p = \mathrm{col}(a), q = \mathrm{col}(b)$.

Especially, we have

$$[d_0, f_{a,b}[-1]] = (-1)^{\widehat{p+2} + (\widehat{p} + \widehat{p+2}) \cdot \widehat{j}} \psi_{f_{a+2n,b}}[-1] - (-1)^{\widehat{p} + (\widehat{p} + \widehat{p-2}) \cdot \widehat{i}} \psi_{f_{a,b-2n}}[-1] \tag{11.5}$$

provided that $\mathrm{col}(a) = \mathrm{col}(b) = p, \mathrm{row}(a) = j, \mathrm{row}(b) = i$ and

$$\begin{aligned} &[d_0, f_{a,b}[-1]] \\ &= \sum_{\mathrm{col}(c) = \mathrm{col}(b)} f_{c,b}[-1]\psi_{f_{a,c}}[-1] - \sum_{\mathrm{col}(a) = \mathrm{col}(c)} \psi_{f_{c,b}}[-1] f_{a,c}[-1] + \alpha\psi_{f_{a,b}}[-2] \\ &\quad + \delta_{p,1}\psi_{f_{a,b}}[-2] + (-1)^{\widehat{p+2} + (\widehat{p} + \widehat{p+2}) \cdot \widehat{j}} \psi_{f_{a+2n,b}}[-1] - (-1)^{\widehat{p-2} + (\widehat{p-2} + \widehat{p-4}) \cdot \widehat{i}} \psi_{f_{a,b-2n}}[-1], \end{aligned} \tag{11.6}$$

provided that $\mathrm{col}(a) = \mathrm{col}(b) + 2 = p, \mathrm{row}(a) = j, \mathrm{row}(b) = i$.

In the following theorem, we give two kinds of elements of $\mathcal{W}^k(\mathfrak{so}(nl), (l^n))$, which are in fact generators of $\mathcal{W}^k(\mathfrak{so}(nl), (l^n))$ (see Theorem 11.20).

Theorem 11.7. *For $i, j \in I_n$, the rectangular W-algebra $\mathcal{W}^k(\mathfrak{so}(nl), (l^n))$ has the following elements;*

$$\widetilde{W}^{(1)}_{i,j} = \sum_{\substack{\mathrm{row}(a) = j, \mathrm{row}(b) = i, \\ \mathrm{col}(a) = \mathrm{col}(b) = p}} (-1)^{\widehat{p} \cdot (\widehat{j} + \widehat{i})} f_{a,b}[-1],$$

$$\begin{aligned} \widetilde{W}^{(2)}_{i,j} &= \alpha \sum_{\substack{\mathrm{row}(a) = j, \mathrm{row}(b) = i, \\ \mathrm{col}(a) = \mathrm{col}(b) = p}} (-1)^{\widehat{p} \cdot (\widehat{j} + \widehat{i})} \frac{p}{2} f_{a,b}[-2] + \sum_{\substack{\mathrm{row}(a) = j, \mathrm{row}(b) = i, \\ \mathrm{col}(a) = \mathrm{col}(b) + 2 = p}} (-1)^{\widehat{p} + \widehat{p} \cdot \widehat{j} + \widehat{p-2} \cdot \widehat{i}} f_{a,b}[-1] \\ &\quad + \sum_{\substack{\mathrm{row}(a_2) = j, \mathrm{row}(b_1) = i, \\ p = \mathrm{col}(a_1) = \mathrm{col}(b_1) < \mathrm{col}(a_2) = \mathrm{col}(b_2) = q \\ \mathrm{row}(a_1) = \mathrm{row}(b_2) = r}} (-1)^{(\widehat{r} + \widehat{i}) \cdot \widehat{p} + (\widehat{j} + \widehat{r}) \cdot \widehat{q}} f_{a_1,b_1}[-1] f_{a_2,b_2}[-1] \\ &\quad + \frac{1}{2} \sum_{\substack{\mathrm{row}(a) = j, \mathrm{row}(b) = i, \\ \mathrm{col}(a) = \mathrm{col}(b) = p}} (-1)^{\widehat{p} + \widehat{p} \cdot (\widehat{j} + \widehat{i})} f_{a,b}[-2]. \end{aligned}$$

Proof. By Definition 11.2, it is enough to show that $[d_0, \widetilde{W}^{(r)}_{i,j}] = 0$. We only show the case when $r = 2$. The case when $r = 1$ is proven in a similar way. By the definition of $\widetilde{W}^{(2)}_{i,j}$, we have

$$\begin{aligned}
&[d_0, \widetilde{W}^{(2)}_{i,j}] \\
&= [d_0, \alpha \sum_{\substack{\mathrm{row}(a)=j,\mathrm{row}(b)=i,\\ \mathrm{col}(a)=\mathrm{col}(b)=p}} (-1)^{\widehat{p}\cdot(\widehat{j}+\widehat{i})} \frac{p}{2} f_{a,b}[-2]] + [d_0, \sum_{\substack{\mathrm{row}(a)=j,\mathrm{row}(b)=i,\\ \mathrm{col}(a)=\mathrm{col}(b)+2=p}} (-1)^{\widehat{p}+\widehat{p}\cdot\widehat{j}+\widehat{p-2}\cdot\widehat{i}} f_{a,b}[-1]] \\
&\quad + [d_0, \sum_{\substack{\mathrm{row}(a_2)=j,\mathrm{row}(b_1)=i,\\ p=\mathrm{col}(a_1)=\mathrm{col}(b_1)<\mathrm{col}(a_2)=\mathrm{col}(b_2)=q\\ \mathrm{row}(a_1)=\mathrm{row}(b_2)=r}} (-1)^{(\widehat{r}+\widehat{i})\cdot\widehat{p}+(\widehat{j}+\widehat{r})\cdot\widehat{q}} f_{a_1,b_1}[-1] f_{a_2,b_2}[-1]] \\
&\quad + \frac{1}{2}[d_0, \sum_{\substack{\mathrm{row}(a)=j,\mathrm{row}(b)=i,\\ \mathrm{col}(a)=\mathrm{col}(b)=p}} (-1)^{\widehat{p}+\widehat{p}\cdot(\widehat{j}+\widehat{i})} f_{a,b}[-2]]. \qquad (11.8)
\end{aligned}$$

We compute each terms in the right hand side of (11.8). First, we compute the first term of the right hand side of (11.8). By (11.5) and (11.3), we can rewrite it as

$$\begin{aligned}
&\alpha \sum_{\substack{\mathrm{row}(a)=j,\mathrm{row}(b)=i,\\ \mathrm{col}(a)=\mathrm{col}(b)=p}} (-1)^{\widehat{p+2}+\widehat{p+2}\cdot\widehat{j}+\widehat{p}\cdot\widehat{i}} \frac{p}{2} \psi_{f_{a+2n,b}}[-2] \\
&\quad - \alpha \sum_{\substack{\mathrm{row}(a)=j,\mathrm{row}(b)=i,\\ \mathrm{col}(a)=\mathrm{col}(b)=p}} (-1)^{\widehat{p}+\widehat{p-2}\cdot\widehat{i}+\widehat{p}\cdot\widehat{j}} \frac{p}{2} \psi_{f_{a,b-2n}}[-2]. \qquad (11.9)
\end{aligned}$$

Replacing a and b with $a + 2n$ and $b + 2n$, we can rewrite the second term of (11.9) as

$$\alpha \sum_{\substack{\mathrm{row}(a)=j,\mathrm{row}(b)=i,\\ \mathrm{col}(a)=\mathrm{col}(b)=p}} (-1)^{\widehat{p+2}+\widehat{p+2}\cdot\widehat{j}+\widehat{p}\cdot\widehat{i}} \frac{p+2}{2} \psi_{f_{a+2n,b}}[-2] \qquad (11.10)$$

Since $\mathrm{col}(a + 2n) = \mathrm{col}(a) + 2$, we find that

$$\begin{aligned}
&\text{the first term of the right hand side of (11.8)} \\
&= -\alpha \sum_{\substack{\mathrm{row}(a)=j,\mathrm{row}(b)=i,\\ \mathrm{col}(a)=\mathrm{col}(b)=p}} (-1)^{\widehat{p+2}+\widehat{p+2}\cdot\widehat{j}+\widehat{p}\cdot\widehat{i}} \psi_{f_{a+2n,b}}[-2] \qquad (11.11)
\end{aligned}$$

by applying (11.10) to (11.9).

Next, we compute the second term of the right hand side of (11.8). By (11.5), we can rewrite it as

$$\begin{aligned}
&\sum_{\substack{\mathrm{row}(a_2)=j,\mathrm{row}(b_1)=i,\\ p=\mathrm{col}(a_1)=\mathrm{col}(b_1)<\mathrm{col}(a_2)=\mathrm{col}(b_2)=q\\ \mathrm{row}(a_1)=\mathrm{row}(b_2)=r}} (-1)^{\beta_1} \psi_{f_{a_1+2n,b_1}}[-1] f_{a_2,b_2}[-1] \\
&\quad - \sum_{\substack{\mathrm{row}(a_2)=j,\mathrm{row}(b_1)=i,\\ p=\mathrm{col}(a_1)=\mathrm{col}(b_1)\leq\mathrm{col}(a_2)=\mathrm{col}(b_2)=q\\ \mathrm{row}(a_1)=\mathrm{row}(b_2)=r}} (-1)^{\beta_1} \psi_{f_{a_1+2n,b_1}}[-1] f_{a_2,b_2}[-1] \\
&\quad + \sum_{\substack{\mathrm{row}(a_2)=j,\mathrm{row}(b_1)=i,\\ p=\mathrm{col}(a_1)=\mathrm{col}(b_1)\leq\mathrm{col}(a_2)=\mathrm{col}(b_2)=q\\ \mathrm{row}(a_1)=\mathrm{row}(b_2)=r}} (-1)^{\beta_2} f_{a_1,b_1}[-1] \psi_{f_{a_2,b_2-2n}}[-1]
\end{aligned}$$

$$-\sum_{\substack{\mathrm{row}(a_2)=j,\mathrm{row}(b_1)=i,\\ p=\mathrm{col}(a_1)=\mathrm{col}(b_1)<\mathrm{col}(a_2)=\mathrm{col}(b_2)=q\\ \mathrm{row}(a_1)=\mathrm{row}(b_2)=r}}(-1)^{\beta_2}f_{a_1,b_1}[-1]\psi_{f_{a_2,b_2-2n}}[-1], \tag{11.12}$$

where we set

$$\beta_1=(\widehat{r}+\widehat{i})\cdot\widehat{p}+(\widehat{j}+\widehat{r})\cdot\widehat{q}+\widehat{p+2}+(\widehat{p}+\widehat{p+2})\cdot\widehat{r},$$
$$\beta_2=(\widehat{r}+\widehat{i})\cdot\widehat{p}+(\widehat{j}+\widehat{r})\cdot\widehat{q}+\widehat{q}+(\widehat{q}+\widehat{q-2})\cdot\widehat{r}.$$

By a direct computation, we can rewrite the sum of the first two terms of (11.12) as

$$\sum_{\substack{\mathrm{row}(a_2)=j,\mathrm{row}(b_1)=i,\\ \mathrm{col}(a_1)+2=\mathrm{col}(b_1)+2=\mathrm{col}(a_2)=\mathrm{col}(b_2)=q\\ \mathrm{row}(a_1)=\mathrm{row}(b_2)}}(-1)^{\widehat{i}\cdot\widehat{q-2}+\widehat{j}\cdot\widehat{q}+\widehat{q}}\psi_{f_{a_1+2n,b_1}}[-1]f_{a_2,b_2}[-1] \tag{11.13}$$

and the sum of the last two terms of (11.12) as

$$-\sum_{\substack{\mathrm{row}(a_2)=j,\mathrm{row}(b_1)=i,\\ \mathrm{col}(a_1)=\mathrm{col}(b_1)=\mathrm{col}(a_2)-2=\mathrm{col}(b_2)-2=p\\ \mathrm{row}(a_1)=\mathrm{row}(b_2)}}(-1)^{\widehat{i}\cdot\widehat{p}+\widehat{j}\cdot\widehat{p+2}+\widehat{p+2}}f_{a_1,b_1}[-1]\psi_{f_{a_2,b_2-2n}}[-1]. \tag{11.14}$$

Adding (11.13) and (11.14), we have

$$\begin{aligned}&\text{the third term of the right hand side of (11.8)}\\ =&\sum_{\substack{\mathrm{row}(a_2)=j,\mathrm{row}(b_1)=i,\\ \mathrm{col}(a_1)+2=\mathrm{col}(b_1)+2=\mathrm{col}(a_2)=\mathrm{col}(b_2)=q\\ \mathrm{row}(a_1)=\mathrm{row}(b_2)}}(-1)^{\widehat{i}\cdot\widehat{q-2}+\widehat{j}\cdot\widehat{q}+\widehat{q}}\psi_{f_{a_1+2n,b_1}}[-1]f_{a_2,b_2}[-1]]\\ &-\sum_{\substack{\mathrm{row}(a_2)=j,\mathrm{row}(b_1)=i,\\ \mathrm{col}(a_1)=\mathrm{col}(b_1)=\mathrm{col}(a_2)-2=\mathrm{col}(b_2)-2=p\\ \mathrm{row}(a_1)=\mathrm{row}(b_2)}}(-1)^{\widehat{i}\cdot\widehat{p}+\widehat{j}\cdot\widehat{p+2}+\widehat{p+2}}f_{a_1,b_1}[-1]\psi_{f_{a_2,b_2-2n}}[-1].\end{aligned} \tag{11.15}$$

Next, we compute the 4-th term of the right hand side of (11.8). By a direct computation, we obtain

$$\begin{aligned}&\frac{1}{2}[d_0,\sum_{\substack{\mathrm{row}(a)=j,\mathrm{row}(b)=i,\\ \mathrm{col}(a)=\mathrm{col}(b)=p}}(-1)^{\widehat{p}+\widehat{p}\cdot(\widehat{j}+\widehat{i})}f_{a,b}[-2]]\\ &=\frac{1}{2}\sum_{\substack{\mathrm{row}(a)=j,\mathrm{row}(b)=i,\\ \mathrm{col}(a)=\mathrm{col}(b)=p}}(-1)^{\widehat{p+2}+(\widehat{p}+\widehat{p+2})\widehat{j}+\widehat{p}+\widehat{p}\cdot(\widehat{j}+\widehat{i})}\psi_{a+2n,b}[-2]\\ &\quad-\frac{1}{2}\sum_{\substack{\mathrm{row}(a)=j,\mathrm{row}(b)=i,\\ \mathrm{col}(a)=\mathrm{col}(b)=p}}(-1)^{\widehat{p}+(\widehat{p}+\widehat{p-2})\widehat{i}+\widehat{p}+\widehat{p}\cdot(\widehat{j}+\widehat{i})}\psi_{a,b-2n}[-2].\end{aligned} \tag{11.16}$$

By a direct computation, we find that the second term of the right hand side of (11.16) is equal to

$$-\frac{1}{2}\sum_{\substack{\mathrm{row}(a)=j,\mathrm{row}(b)=i,\\ \mathrm{col}(a)=\mathrm{col}(b)=p}}(-1)^{\widehat{p+2}+(\widehat{p+2}+\widehat{p})\widehat{i}+\widehat{p+2}+\widehat{p+2}\cdot(\widehat{j}+\widehat{i})}\psi_{a+2n,b}[-2] \tag{11.17}$$

Then, we have

$$\text{the third term of the right hand side of (11.8)}=-(-1)^{\widehat{i}}\psi_{n+j,-n+i}[-2] \tag{11.18}$$

by applying (11.17) to (11.16).

Finally, we compute the second term of (11.8). By (11.6), we can rewrite the right hand side of the second term of (11.8) as

$$\begin{aligned}
&\sum_{\substack{\mathrm{row}(a)=j,\mathrm{row}(b)=i,\\ \mathrm{col}(a)=\mathrm{col}(b)+2=\mathrm{col}(c)+2=p}} (-1)^{\widehat{p}+\widehat{p}\cdot\widehat{j}+\widehat{p-2}\cdot\widehat{i}} f_{c,b}[-1]\psi_{f_{a,c}}[-1]\\
&-\sum_{\substack{\mathrm{row}(a)=j,\mathrm{row}(b)=i,\\ \mathrm{col}(a)=\mathrm{col}(c)=\mathrm{col}(b)+2=p}} (-1)^{\widehat{p}+\widehat{p}\cdot\widehat{j}+\widehat{p-2}\cdot\widehat{i}} \psi_{f_{c,b}}[-1]f_{a,c}[-1]\\
&+\alpha\sum_{\substack{\mathrm{row}(a)=j,\mathrm{row}(b)=i,\\ \mathrm{col}(a)=\mathrm{col}(b)+2=p}} (-1)^{\widehat{p}+\widehat{p}\cdot\widehat{j}+\widehat{p-2}\cdot\widehat{i}} \psi_{f_{a,b}}[-2]+(-1)^{\widehat{i}}\psi_{f_{n+j,-n+i}}[-2]\\
&+\sum_{\substack{\mathrm{row}(a)=j,\mathrm{row}(b)=i,\\ \mathrm{col}(a)=\mathrm{col}(b)+2=p}} (-1)^{\widehat{p}+\widehat{p+2}+\widehat{p+2}\cdot\widehat{j}+\widehat{p-2}\cdot\widehat{i}} \psi_{f_{a+2n,b}}[-1]\\
&-\sum_{\substack{\mathrm{row}(a)=j,\mathrm{row}(b)=i,\\ \mathrm{col}(a)=\mathrm{col}(b)+2=p}} (-1)^{\widehat{p+2}+\widehat{p}+\widehat{p-2}\cdot\widehat{i}+\widehat{p+2}\cdot\widehat{j}} \psi_{f_{a+2n,b}}[-1].
\end{aligned}\tag{11.19}$$

We can easily find that the sum of the last two terms of (11.19) is equal to zero. We also find that the sum of (11.15)(resp. (11.11), (11.18)) and first and second terms (resp. third term, 4-th term) of (11.19) is equal to zero. □

Theorem 11.20. *Assume that $n\geq 4$ and $\alpha\neq 0$. The rectangular W-algebra $\mathcal{W}^k(\mathfrak{so}(nl),(l^n))$ is generated by $\{\widetilde{W}^{(r)}_{i,j}\mid 1\leq i,j\leq n, r=1,2\}$.*

The proof of Theorem 11.20 is given in the appendix B. We prepare one lemma in order to prove the main theorem.

Lemma 11.21. (1) *The following relations hold;*

$$\begin{aligned}
(\widetilde{W}^{(1)}_{i,j})_{(0)}\widetilde{W}^{(2)}_{v,w}&=\delta_{i,w}\widetilde{W}^{(2)}_{v,j}-\delta_{j,v}\widetilde{W}^{(2)}_{i,w}+(-1)^{\widehat{i}+\widehat{j}}\delta_{i,-v}\widetilde{W}^{(2)}_{-w,j}-(-1)^{\widehat{i}+\widehat{j}}\delta_{j,-w}\widetilde{W}^{(2)}_{i,-v},\\
(\widetilde{W}^{(1)}_{v,w})_{(1)}\widetilde{W}^{(2)}_{i,j}&=\frac{l-1}{2}\alpha(\delta_{j,v}\widetilde{W}^{(1)}_{i,w}+\delta_{i,w}\widetilde{W}^{(1)}_{v,j}+\delta_{-w,j}\widetilde{W}^{(1)}_{i,-v}+\delta_{-v,i}\widetilde{W}^{(1)}_{-w,j})\\
&\quad+\frac{1}{2}(-1)^{p(j)+p(i)}\delta_{v,-i}W^{(1)}_{-j,w}-\frac{1}{2}(-1)^{p(j)+p(v)}\delta_{w,i}W^{(1)}_{-j,-v}\\
&\quad-\frac{1}{2}\delta_{v,j}W^{(1)}_{i,w}+\frac{1}{2}(-1)^{p(v)+p(w)}\delta_{-w,j}W^{(1)}_{i,-v},\\
(\widetilde{W}^{(1)}_{v,w})_{(s)}\widetilde{W}^{(2)}_{i,j}&=0\ (s\geq 2).
\end{aligned}$$

(2) *We define a grading on $V^\kappa(\mathfrak{b})$ by setting $deg(x[-s])=j$ if $x\in\mathfrak{b}\cap\mathfrak{g}_j$. Then, we obtain*

$$(\widetilde{W}^{(2)}_{i,i})_{(1)}\widetilde{W}^{(2)}_{j,j}=(1+\alpha\delta_{i,j}-\alpha(-1)^{\widehat{i}+\widehat{j}}\delta_{i,-j})(\widetilde{W}^{(2)}_{i,i}+\widetilde{W}^{(2)}_{j,j})+\textit{higher terms}.$$

The proof is due to a direct computation. We omit it.

12 Twisted affine Yangian

Let us recall the Drinfeld J presentation of the finite Yangian $Y_\hbar(\mathfrak{g})$. It is the original definition of Drinfeld ([11]).

Definition 12.1 ([10], Section 12)**.** Suppose that $\mathfrak{g}$ is a Kac-Moody Lie algebra of finite type. The Yangian $Y_\hbar(\mathfrak{g})$ is the associative algbra over $\mathbb{C}$ with generators $\{x, J(x)|x\in\mathfrak{g}\}$ subject to the following defining relations:

$$\begin{gathered}xy-yx=[x,y]\text{ for all }x,y\in\mathfrak{g},\\ J(ax+by)=aJ(x)+bJ(y)\text{ for all }a,b\in\mathbb{C},\\ J([x,y])=[x,J(y)],\\ [J(x),J([y,z])]+[J(z),J([x,y])]+[J(y),J([z,x])]=\hbar^2\sum_{a,b,c\in\mathtt{A}}([x,\xi_a],[[y,\xi_b],[z,\xi_c]])\{\xi_a,\xi_b,\xi_c\},\end{gathered}$$

$$\begin{aligned}&[[J(x),J(y)],[z,J(w)]]+[[J(z),J(w)],[x,J(y)]]\\ &\quad=\hbar^2\sum_{a,b,c\in\mathtt{A}}\big(([x,\xi_a],[[y,\xi_b],[[z,w],\xi_c]])+([z,\xi_a],[[w,\xi_b],[[x,y],\xi_c]])\big)\{\xi_a,\xi_b,J(\xi_c)\},\end{aligned}$$

where $(\ ,\ )$ is a non-zero invariant bilinear form, $\{\xi_a\}_{a\in\mathtt{A}}$ is an orthonormal basis of $\mathfrak{g}$ and $\{\xi_a,\xi_b,\xi_c\}=\dfrac{1}{24}\sum_{\pi\in G}\xi_{\pi(a)}\xi_{\pi(b)}\xi_{\pi(c)}$, G being the group of permutations of $\{a,b,c\}$. By Definition 12.1,we note that there exists an isomorphism of $\chi_\hbar\colon Y_\hbar(\mathfrak{g})\to Y_{-\hbar}(\mathfrak{g})$ determined by $x\mapsto x$ and $J(x)\mapsto J(x)$.

Belliard and Regelskis ([4]) gave generators of twisted Yangians in the words of the Drinfeld J presentation.

Theorem 12.2 ([4], Theorem 5.5)**.** *Let $\big(\mathfrak{g},\mathfrak{g}^\theta\big)$ be a symmetric pair of a finite-dimensional simple complex Lie algebra $\mathfrak{g}$ of rank$(\mathfrak{g})\geq 2$ with respect to the involution θ, such that $\mathfrak{g}^\theta$ is the positive eigenspace of θ. Let $\{X_a\}$ (resp. $\{Y_p\}$) be a basis of $\mathfrak{g}^\theta$ (resp. $\{x\in\mathfrak{g}\mid\theta(x)=-x\}$). We decompose the Cartan element of $\mathfrak{g}$ into $C_\mathfrak{k}+C_\mathfrak{m}$, where $C_\mathfrak{k}$ (resp. $C_\mathfrak{m}$) is an element of $U(\mathfrak{k})$ (resp. $\mathbb{C}[\mathfrak{m}]$). Then, the twisted yangian $T_\hbar(\mathfrak{g},\mathfrak{g}^\theta)$ is isomorphic to the subalgebra of $Y_h(\mathfrak{g})$ generated by $\{X_a, B(Y_p)\}$, where*

$$B(Y_p)=J(Y_p)+\frac{\hbar}{4}\Big[Y_p,C_\mathfrak{k}\Big].$$

Belliard and Regelskis also gave the Drinfeld J presentation of twisted Yangians whose generators are $\{X_a, B(Y_p)\}$. Its defining relations contain the relation $[X_a,B(Y_p)]=B([X_a,Y_p])$. By Theorem 12.2, we can realize $T_\hbar(\mathfrak{g},\mathfrak{g}^\theta)$ as a subalgebra of $Y_{-\hbar}(\mathfrak{g})$ via $\chi_\hbar$.

There exists the following symmetric pair decomposition of $\mathfrak{sl}(n)$;

$$\mathfrak{sl}(n)=\bigoplus_{i,j\in I_n}\mathbb{C}(e_{i,j}-(-1)^{\widehat{i}+\widehat{j}}e_{-j,-i})\oplus\Big(\bigoplus_{i,j\in I_n}\mathbb{C}(e_{i,j}+(-1)^{\widehat{i}+\widehat{j}}e_{-j,-i})\cap\mathfrak{sl}(n)\Big),$$

Let $\mathfrak{k}$ be $\bigoplus_{i,j\in I_n}\mathbb{C}(e_{i,j}-(-1)^{\widehat{i}+\widehat{j}}e_{-j,-i})$ and $\mathfrak{m}$ be $\bigoplus_{i,j\in I_n}\mathbb{C}(e_{i,j}+(-1)^{\widehat{i}+\widehat{j}}e_{-j,-i})\cap\mathfrak{sl}(n)$. Setting $H_i=e_{i,i}-e_{i+2,i+2}\in\mathfrak{sl}(n)$, by Theorem 12.2, we can rewrite $B(H_i-H_{-i-2})$ as

$$J(H_i-H_{-i-2})+\frac{\hbar}{8}\Big[H_i-H_{-i-2},\sum_{u>v}(e_{u,v}-(-1)^{\widehat{u}+\widehat{v}}e_{-v,-u})(e_{v,u}-(-1)^{\widehat{u}+\widehat{v}}e_{-u,-v})\Big].$$

In a similar way to Theorem 12.2, we define the twisted affine Yangian of type D. We have a decomposition $\widehat{\mathfrak{sl}}(n)=\widehat{\mathfrak{k}}\oplus\mathfrak{m}\otimes\mathbb{C}[t^{\pm1}]$. We remark that $\widehat{\mathfrak{k}}$ is isomorphic to $\widehat{\mathfrak{so}}(n)$ and $\mathfrak{m}\otimes\mathbb{C}[t^{\pm1}]=[h_i-h_{-i-2},\widehat{\mathfrak{k}}]+[[h_i-h_{-i-2},\widehat{\mathfrak{k}}],\widehat{\mathfrak{k}}]$.

By the similar formula in Section 3 of [21], we can define $J(h_i)$ as an element of $\widetilde{Y}_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(n))$;

$$J(h_i)=h_{i,1}+\frac{\hbar}{2}\sum_{\gamma\in\Delta_{\mathrm{re}}^+}(\alpha_i,\gamma)x_{-\gamma}x_\gamma-\frac{\hbar}{2}h_i^2,$$

where Δ^+_{re} is a set of positive real root of $\widehat{\mathfrak{sl}}(n)$ and x_γ is a root γ element such that $(x_\gamma, x_{-\gamma}) = 1$. By the definition of $J(h_i)$ and Theorem 4.3, we obtain

$$\begin{aligned}\widetilde{\Delta}(J(h_i)) &= \square(J(h_i)) + \frac{\hbar}{2}\sum_{\gamma\in\Delta^+_{\mathrm{re}}}(\alpha_i,\gamma)(x_\gamma\otimes x_{-\gamma} - x_{-\gamma}\otimes x_\gamma)\\ &= \square(J(h_i)) + \frac{\hbar}{2}\sum_{\gamma\in\Delta_{\mathrm{re}}}[h_i, x_\gamma]\otimes x_{-\gamma},\end{aligned} \tag{12.3}$$

where Δ_{re} is a set of real roots of $\widehat{\mathfrak{sl}}(n)$.

Definition 12.4. Let us set $B(h_i - h_{-i-2})$ as

$$\begin{aligned}&J(h_i - h_{-i-2}) + \frac{\hbar}{8}[\sum_{\substack{u<v\\ m\geq 1}}(e_{u,v} - (-1)^{\widehat{u}+\widehat{v}}e_{-v,-u})t^{-m}(e_{v,u} - (-1)^{\widehat{u}+\widehat{v}}e_{-u,-v})t^m, h_i - h_{-i-2}]\\ &\qquad + \frac{\hbar}{8}[\sum_{\substack{u>v\\ m\geq 0}}(e_{u,v} - (-1)^{\widehat{u}+\widehat{v}}e_{-v,-u})t^{-m}(e_{v,u} - (-1)^{\widehat{u}+\widehat{v}}e_{-u,-v})t^m, h_i - h_{-i-2}].\end{aligned} \tag{12.5}$$

We define $TY_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{so}}(n))$ as a subalgebra of $\widetilde{Y}_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(n))$ topologically generated by $\widehat{\mathfrak{k}}$ and $B(h_i - h_{-i-2})$.

By the definition of $TY_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{so}}(n))$, the universal enveloping algebra of $\widehat{\mathfrak{k}}$ can be embedded into $TY_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{so}}(n))$.

Proposition 12.6. *The restriction of the coproduct $\widetilde{\Delta}\colon \widetilde{Y}_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(n)) \to Y_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(n))\widehat{\otimes}Y_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(n))$ gives a coideal structure to $TY_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{so}}(n))$. That is, we have*

$$\widetilde{\Delta}(TY_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{so}}(n))) \subset TY_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{so}}(n))\widehat{\otimes}Y_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(n)),$$

where the completed tensor product $Y_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(n))\widehat{\otimes}TY_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{so}}(n))$ is defined in the same way as $Y_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(n))\widehat{\otimes}Y_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(n))$.

Proof. It is enough to show that $\widetilde{\Delta}(B(h_i - h_{-i-2})) \subset TY_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{so}}(n))\widehat{\otimes}Y_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(n))$. By the definition of $B(h_i - h_{-i-2})$ and (12.3), we find that

$$\widetilde{\Delta}(B(h_i - h_{-i-2})) = \square(B(h_i - h_{-i-2})) + C_i + C_{-i} - C_{i+2} - C_{-i-2} + D_i + D_{-i} - D_{i+2} - D_{-i-2},$$

where

$$\begin{aligned}C_i &= \frac{\hbar}{2}\sum_{\gamma\in\Delta_{\mathrm{re}}}[e_{i,i}, x_\gamma]\otimes x_{-\gamma},\\ D_i &= \frac{\hbar}{8}[\sum_{\substack{u\neq v\\ m\in\mathbb{Z}}}(e_{u,v} - (-1)^{\widehat{u}+\widehat{v}}e_{-v,-u})t^{-m}\otimes(e_{v,u} - (-1)^{\widehat{u}+\widehat{v}}e_{-u,-v})t^m, \square e_{i,i}].\end{aligned}$$

By a direct computation, we obtain

$$C_i = -\frac{\hbar}{2}\sum_{\substack{u\neq i\\ s\in\mathbb{Z}}}e_{u,i}t^{-s}\otimes e_{i,u}t^s + \frac{\hbar}{2}\sum_{\substack{u\neq i\\ s\in\mathbb{Z}}}e_{i,u}t^{s+1}\otimes e_{u,i}t^{-s-1}. \tag{12.7}$$

By a direct computation, we also obtain

$$D_i = -\frac{\hbar}{4}\sum_{\substack{v\neq i\\ s\in\mathbb{Z}}}(-1)^{\widehat{v}+\widehat{i}}e_{-v,-i}t^{-s}\otimes e_{v,i}t^s - \frac{\hbar}{4}\sum_{\substack{v\neq -i\\ s\in\mathbb{Z}}}(-1)^{\widehat{v}+\widehat{-i}}e_{-v,i}t^{-s}\otimes e_{v,-i}t^s$$

$$+\frac{\hbar}{4}\sum_{\substack{u\neq i\\ s\in\mathbb{Z}}}(-1)^{\widehat{u}+\widehat{i}}e_{-i,-u}t^{-s}\otimes e_{i,u}t^s+\frac{\hbar}{4}\sum_{\substack{u\neq -i\\ s\in\mathbb{Z}}}(-1)^{\widehat{u}+\widehat{-i}}e_{i,-u}t^{-s}\otimes e_{-i,u}t^s. \tag{12.8}$$

By setting

$$F_i=-\frac{\hbar}{2}\sum_{\substack{v\neq i\\ s\in\mathbb{Z}}}(-1)^{\widehat{v}+\widehat{i}}e_{-v,-i}t^{-s}\otimes e_{v,i}t^s+\frac{\hbar}{2}\sum_{\substack{u\neq i\\ s\in\mathbb{Z}}}(-1)^{\widehat{u}+\widehat{i}}e_{-i,-u}t^{-s}\otimes e_{i,u}t^s, \tag{12.9}$$

we obtain $D_i+D_{-i}=F_i+F_{-i}$ by a direct computation. We denote the a-th term of the right hand side of (12.7) (resp. (12.9)) by $C_{i,a}$ (resp. $F_{i,a}$). Then, we find that

$$C_{i,1}+F_{i,2}=-\frac{\hbar}{2}\sum_{\substack{u\neq i\\ s\in\mathbb{Z}}}(e_{u,i}t^{-s}-(-1)^{\widehat{u}+\widehat{i}}e_{-i,-u}t^{-s})\otimes e_{i,u}t^s,$$
$$C_{i,2}+F_{i,1}=\frac{\hbar}{2}\sum_{\substack{u\neq i\\ s\in\mathbb{Z}}}(e_{i,u}t^{s}-(-1)^{\widehat{u}+\widehat{i}}e_{-i,-u}t^{s})\otimes e_{u,i}t^{-s}.$$

Since $C_{i,1}+F_{i,2}$ and $C_{i,2}+F_{i,1}$ are contained in $TY_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{so}}(n))\widehat{\otimes}Y_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(n))$, we find that

$$\widetilde{\Delta}(B(h_i-h_{-i-2}))=\square(B(h_i-h_{-i-2}))+C_i+C_{-i}-C_{i+2}-C_{-i-2}+F_i+F_{-i}-F_{i+2}-F_{-i-2}$$

is contained in $TY_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{so}}(n))\widehat{\otimes}Y_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(n))$. □

13 Twisted affine Yangians and rectangular W-algebras of type D

Before starting the homomorohism from twisted affine Yangians to universal enveloping algebras of rectangular W-algebras of type D, we recall the another proof of Theorem 10.1 given in [31]. In [31], we construct Φ by using the coproduct and evaluation map for the affine super Yangian and the Miura map of a rectangular W-superalgebra.

The projection $\mathfrak{gl}(nl)\to\mathfrak{g}_0=\mathfrak{gl}(n)^{\otimes l}$ induces the Miura transformation ([24])

$$\mu\colon \mathcal{W}^k(\mathfrak{gl}(ml|nl),(l^{(m|n)}))\to V^\kappa(\mathfrak{gl}(m|n)^{\otimes l}).$$

The Miura transformation also induces the injective homomorphism ([13], [2])

$$\widetilde{\mu}\colon \mathcal{U}(\mathcal{W}^k(\mathfrak{gl}(ml|nl),(l^{(m|n)})))\to U(\widehat{\mathfrak{gl}}(m|n))^{\otimes l}_{\text{comp}},$$

where $U(\widehat{\mathfrak{gl}}(m|n))^{\otimes l}_{\text{comp}}$ is the standard degreewise completion of $U(\widehat{\mathfrak{gl}}(n))^{\otimes l}$ in the sense of [33]. By the definition of μ, we have

$$\widetilde{\mu}(W^{(1)}_{i,j}t^s)=\sum_{1\leq r\leq l}e^{(r)}_{j,i}t^s \tag{13.1}$$

and

$$\widetilde{\mu}(W^{(2)}_{i,j}t)=\sum_{\substack{1\leq r_1<r_2\leq l'\\ s\in\mathbb{Z}\\ 1\leq u\leq m+n}}(-1)^{p(u)+p(E_{u,i})p(E_{u,j})}e^{(r_1)}_{u,i}t^{-s}e^{(r_2)}_{j,u}t^s-\sum_{1\leq r\leq l}(r-1)\alpha E^{[r]}_{j,i}. \tag{13.2}$$

Theorem 13.3. *[Kodera-Ueda, [31]] There exists an algebra homomorphism*

$$\Phi\colon Y_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m|n))\to\mathcal{U}(\mathcal{W}^k(\mathfrak{gl}(ml|nl),(l^{(m|n)}))),$$

satisfying the commutator diagram

$$
\begin{array}{ccc}
Y_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m|n)) & \xrightarrow{\widehat{\Delta}^l} & \overbrace{Y_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m|n))\widehat{\otimes}\cdots\widehat{\otimes}Y_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m|n))}^{l} \\
\Big\downarrow{\scriptstyle \widetilde{\Phi}} & \circlearrowleft & \Big\downarrow{\scriptstyle (\mathrm{ev}_0\otimes\mathrm{ev}_\alpha\otimes\cdots\otimes\mathrm{ev}_{(l-1)\alpha})} \\
\mathcal{U}(W^k(\mathfrak{gl}(lm|ln),(l^{(m|n)}))) & \xhookrightarrow{\widetilde{\mu}} & \underbrace{U(\widehat{\mathfrak{gl}}(m|n))\widehat{\otimes}\cdots\widehat{\otimes}U(\widehat{\mathfrak{gl}}(m|n))}_{l}.
\end{array}
$$

Since there exists an isomorphism $Y_{\varepsilon_1,\varepsilon_2}(\widehat{\mathfrak{sl}}(m|n)) \to Y_{x\varepsilon_1,x\varepsilon_2}(\widehat{\mathfrak{sl}}(m|n))$ determined by $h_{i,r} \mapsto x^r h_{i,r}$ and $x^{\pm}_{i,r} \mapsto x^r x^{\pm}_{i,r}$ for all $x \neq 0$, it is enough to assume that $\hbar = \varepsilon_1 + \varepsilon_2 = -1$. Setting $\hbar = -1$ in Theorem 13.3, we find that $\widetilde{\Phi}$ is equal to Φ. Thus, the proof of Theorem 13.3 becomes another proof of Theorem 10.1.

Next, we construct a homomorphsim from the twisted affine Yangian to the universal enveloping algebra of the rectangular W-algebra of type D. Let l' be $\dfrac{l}{2}$. There exists an isomorphism $\Psi\colon \mathfrak{g}_0 \to \mathfrak{gl}_n^{\otimes l'} = \mathfrak{L}$ determined by

$$\Phi(f_{a,b}) = e_{\mathrm{row}(a),\mathrm{row}(b)}[-1] \text{ if } \mathrm{col}(a) = \mathrm{col}(b) > 0.$$

We denote $1^{\otimes r-1} \otimes e_{i,j} \otimes 1^{\otimes l'-r}$ by $e^{(r)}_{i,j}$. The projection $\mathfrak{so}(nl) \to \mathfrak{g}_0$ induces the Miura transformation ([24])

$$\mu_D\colon \mathcal{W}^k(\mathfrak{so}(nl),(l^n)) \to V^\Gamma(\mathfrak{L}),$$

where

$$\Gamma(e^{(r_1)}_{a,b}, e^{(r_2)}_{c,d}) = \delta_{a,d}\delta_{b,c}\delta_{r_1,r_2}\alpha + \delta_{a,b}\delta_{c,d}\delta_{r_1,r_2}.$$

The Miura transformation also induces the injective homomorphism ([13], [2])

$$\widetilde{\mu}_D\colon \mathcal{U}(\mathcal{W}^k(\mathfrak{so}(nl),(l^n))) \to U(\widehat{\mathfrak{gl}}(n))^{\otimes l}_{\mathrm{comp}},$$

where $U(\widehat{\mathfrak{gl}}(n))^{\otimes l}_{\mathrm{comp}}$ is the standard degreewise completion of $U(\widehat{\mathfrak{gl}}(n))^{\otimes l}$ in the sense of [33]. By the definition of μ_D, we have

$$\widetilde{\mu}_D(\widetilde{W}^{(1)}_{i,j}t^s) = \sum_{1\leq r\leq l'} (e^{(r)}_{j,i} - (-1)^{\widehat{i}+\widehat{j}} e^{(r)}_{-i,-j})t^s$$

and

$$
\begin{aligned}
\widetilde{\mu}_D(\widetilde{W}^{(2)}_{i,i}t) = &\sum_{\substack{1\leq r_1<r_2\leq l'\\ s\in\mathbb{Z}}} e^{(r_1)}_{u,i}t^{-s}e^{(r_2)}_{i,u}t^s + \sum_{\substack{1\leq r_1<r_2\leq l'\\ s\in\mathbb{Z}}} e^{(r_1)}_{u,-i}t^{-s}e^{(r_2)}_{-i,u}t^s \\
&- \sum_{\substack{1\leq r_1,r_2\leq l'\\ u<i,s\geq 0}} (-1)^{\widehat{u}+\widehat{i}} e^{(r_1)}_{-i,-u}t^{-s}e^{(r_2)}_{i,u}t^s - \sum_{\substack{r_1,r_2\\ u<i\\ s\geq 1}} (-1)^{\widehat{u}+\widehat{i}} e^{(r_1)}_{i,u}t^{-s}e^{(r_2)}_{-i,-u}t^s \\
&- \sum_{\substack{1\leq r_1,r_2\leq l'\\ u>i,s\geq 0}} (-1)^{\widehat{u}+\widehat{i}} e^{(r_1)}_{-i,-u}t^{-s}e^{(r_2)}_{i,u}t^s - \sum_{\substack{1\leq r_1,r_2\leq l'\\ u>i,s\geq 1}} (-1)^{\widehat{u}+\widehat{i}} e^{(r_1)}_{i,u}t^{-s}e^{(r_2)}_{-i,-u}t^s \\
&- \sum_{\substack{1\leq r_1,r_2\leq l'\\ s\geq 0}} e^{(r_1)}_{-i,-i}t^{-s}e^{(r_2)}_{i,i}t^s - \sum_{\substack{1\leq r_1,r_2\leq l'\\ s\geq 1}} e^{(r_1)}_{i,i}t^{-s}e^{(r_2)}_{-i,-i}t^s \\
&- \sum_{1\leq r\leq l'} \frac{(2r+1)\alpha+1}{2} e^{(r)}_{i,i} - \sum_{1\leq r\leq l'} \frac{(2r+1)\alpha+1}{2} e^{(r)}_{-i,-i}.
\end{aligned} \tag{13.4}
$$

Theorem 13.5. *For $n\geq 4$ and $l\geq 2$, there exists a homomorphism*
$$\Phi\colon TY(\widehat{so}(n))\to\mathcal{U}(\mathcal{W}^k(\mathfrak{so}(nl),(l^n)))$$
defined by $\tilde{\mu}_D\circ\Phi=(\bigotimes_{r=1}^{l}\mathrm{ev}_{\xi_r})\circ\widetilde{\Delta}^l$, *where* $\xi_r=\dfrac{(2r+1)\alpha+1}{2}\hbar$.

Proof. It is enough to show that $((\bigotimes_{r=1}^{l}\mathrm{ev}_{\xi_r})\circ\widetilde{\Delta}^l)(B(h_i-h_{-i-2}))$ can be written by the sum of the image of $\widetilde{\mu}_D$.

By the proof of Proposition 12.6, we obtain
$$\begin{aligned}&((\bigotimes_{r=1}^{l}\mathrm{ev}_{\xi_r})\circ\widetilde{\Delta}^l)(B(h_i-h_{-i-2}))\\&=\square^l(\mathrm{ev}_0(B(h_i-h_{-i-2})))+C^l_i+C^l_{-i}-C^l_{i+2}-C^l_{-i-2}+F^l_i+F^l_{-i}-F^l_{i+2}-F^l_{-i-2}+G_i-G_{i+2},\end{aligned}$$
where
$$C^l_i=-\frac{\hbar}{2}\sum_{\substack{1\leq r_1<r_2\leq l'\\u\neq i,s\in\mathbb{Z}}}e^{(r_1)}_{u,i}t^{-s}e^{(r_2)}_{i,u}t^s+\frac{\hbar}{2}\sum_{\substack{1\leq r_1<r_2\leq l'\\u\neq i,s\in\mathbb{Z}}}e^{(r_1)}_{i,u}t^se^{(r_2)}_{u,i}t^{-s},\tag{13.6}$$
$$F^l_i=-\frac{\hbar}{2}\sum_{\substack{1\leq r_1<r_2\leq l'\\v\neq i,s\in\mathbb{Z}}}(-1)^{\widehat{v}+\widehat{i}}e^{(r_1)}_{-v,-i}t^{-s}e^{(r_2)}_{v,i}t^s+\frac{\hbar}{2}\sum_{\substack{1\leq r_1<r_2\leq l'\\u\neq i,s\in\mathbb{Z}}}(-1)^{\widehat{u}+\widehat{i}}e^{(r_1)}_{-i,-u}t^{-s}e^{(r_2)}_{i,u}t^s,\tag{13.7}$$
$$G_i=\hbar\sum_{1\leq r\leq l'}\frac{(2r+1)\alpha+1}{2}e^{(r)}_{i,i}+\hbar\sum_{1\leq r\leq l'}\frac{(2r+1)\alpha+1}{2}e^{(r)}_{-i,-i}.\tag{13.8}$$
By the definition of C^l_i and F^l_i, we have
$$C^l_{-i}=-\frac{\hbar}{2}\sum_{\substack{1\leq r_1<r_2\leq l'\\u\neq i,s\in\mathbb{Z}}}e^{(r_1)}_{-u,-i}t^{-s}e^{(r_2)}_{-i,-u}t^s+\frac{\hbar}{2}\sum_{\substack{1\leq r_1<r_2\leq l'\\u\neq i,s\in\mathbb{Z}}}e^{(r_1)}_{-i,-u}t^se^{(r_2)}_{-u,-i}t^{-s},\tag{13.9}$$
$$F^l_{-i}=-\frac{\hbar}{2}\sum_{\substack{1\leq r_1<r_2\leq l'\\v\neq i,s\in\mathbb{Z}}}(-1)^{\widehat{v}+\widehat{i}}e^{(r_1)}_{v,i}t^{-s}\otimes e^{(r_2)}_{v,i}t^s+\frac{\hbar}{2}\sum_{\substack{1\leq r_1<r_2\leq l'\\u\neq i,s\in\mathbb{Z}}}(-1)^{\widehat{u}+\widehat{i}}e^{(r_1)}_{i,u}t^{-s}\otimes e^{(r_2)}_{-i,-u}t^s.\tag{13.10}$$
Thus, it is enough to prove that $C^l_i+C^l_{-i}+F^l_i+F^l_{-i}+G_i+\hbar(\widetilde{\mu}_D(\widetilde{W}^{(2)}_{i,i}t+\widetilde{W}^{(2)}_{-i,-i}t))$ can be written by the sum of the image of $\widetilde{\mu}_D$. We compute $C^l_i+C^l_{-i}+F^l_i+F^l_{-i}+G_i+\hbar(\widetilde{\mu}_D(\widetilde{W}^{(2)}_{i,i}t+\widetilde{W}^{(2)}_{-i,-i}t))$. We denote the i-th term of (equation number) by (equation number)$_i$. By a direct computation, we obtain
$$(13.6)_1+\hbar(13.4)_1=\hbar\sum_{\substack{1\leq r_1<r_2\leq l'\\s\in\mathbb{Z}}}e^{(r_1)}_{i,i}t^{-s}e^{(r_2)}_{i,i}t^s+\frac{\hbar}{2}\sum_{\substack{1\leq r_1<r_2\leq l'\\s\in\mathbb{Z},u\neq i}}e^{(r_1)}_{u,i}t^{-s}e^{(r_2)}_{i,u}t^s,\tag{13.11}$$
$$(13.9)_1+\hbar(13.4)_2=\hbar\sum_{\substack{1\leq r_1<r_2\leq l\\s\in\mathbb{Z}}}e^{(r_1)}_{-i,-i}t^{-s}e^{(r_2)}_{-i,-i}t^s+\frac{\hbar}{2}\sum_{\substack{1\leq r_1<r_2\leq l'\\s\in\mathbb{Z},u\neq -i}}e^{(r_1)}_{u,-i}t^{-s}e^{(r_2)}_{-i,u}t^s,\tag{13.12}$$
$$\begin{aligned}&(13.7)_2+(13.10)_2+\hbar(13.4)_3+\hbar(13.4)_4+\hbar(13.4)_5+\hbar(13.4)_6\\&=-\hbar\sum_{\substack{1\leq r\leq l',\\u<i,s\geq 0}}(-1)^{\widehat{u}+\widehat{i}}e^{(r)}_{-i,-u}t^{-s}e^{(r)}_{i,u}t^s-\hbar\sum_{\substack{1\leq r\leq l',\\u<i,s\geq 1}}(-1)^{\widehat{u}+\widehat{i}}e^{(r)}_{i,u}t^{-s}e^{(r)}_{-i,-u}t^s\\&\quad-\hbar\sum_{\substack{1\leq r\leq l'\\u>i,s\geq 0}}(-1)^{\widehat{u}+\widehat{i}}e^{(r)}_{-i,-u}t^{-s}e^{(r)}_{i,u}t^s-\hbar\sum_{\substack{1\leq r\leq l'\\u>i,s\geq 1}}(-1)^{\widehat{u}+\widehat{i}}e^{(r_1)}_{i,u}t^{-s}e^{(r_1)}_{-i,-u}t^s\end{aligned}$$

$$-\frac{\hbar}{2}\sum_{\substack{1\leq r_1<r_2\leq l'\\ u\neq i,s\in\mathbb{Z}}}(-1)^{\widehat{u}+\widehat{i}}e^{(r_1)}_{-i,-u}t^{-s}e^{(r_2)}_{i,u}t^s-\frac{\hbar}{2}\sum_{\substack{1\leq r_1<r_2\leq l'\\ u\neq i,s\in\mathbb{Z}}}(-1)^{\widehat{u}+\widehat{i}}e^{(r_1)}_{i,u}t^{-s}e^{(r_2)}_{-i,-u}t^s, \tag{13.13}$$

$$\hbar(13.4)_9+\hbar(13.4)_{10}+G_i=0. \tag{13.14}$$

By a direct computation, we obtain

$$\begin{aligned}
&(13.6)_2+(13.9)_2+(13.7)_1+(13.10)_1+(13.11)_2+(13.12)_2+(13.13)_5+(13.13)_6\\
&=(\widetilde{\Delta}^l-\square^l)\big(\frac{\hbar}{2}\sum_{\substack{v\neq i\\ s\geq 0}}(e_{i,v}-e_{-v,-i})t^{-s}(e_{v,i}-e_{-i,-v})t^s\big)\\
&\quad+(\widetilde{\Delta}^l-\square^l)\big(\frac{\hbar}{2}\sum_{\substack{v\neq i\\ s\geq 1}}(e_{i,v}-e_{-v,-i})t^{-s}(e_{v,i}-e_{-i,-v})t^s\big),
\end{aligned} \tag{13.15}$$

$$\begin{aligned}
&\hbar(13.4)_7+\hbar(13.4)_8+(13.11)_1+(13.12)_1\\
&=(\widetilde{\Delta}^l-\square^l)\big(-\frac{\hbar}{2}(e_{i,i}-e_{-i,-i})^2+\hbar\sum_{s\geq 0}(e_{i,i}-e_{-i,-i})t^{-s}(e_{i,i}-e_{-i,-i})t^s\big)\\
&\quad-\square^l(\sum_{s\geq 0}e_{-i,-i}t^{-s}e_{i,i}t^s-\sum_{s\geq 1}e_{i,i}t^{-s}e_{-i,-i}t^s).
\end{aligned} \tag{13.16}$$

Then, we obtain

$$\begin{aligned}
&((\bigotimes_{r=1}^{l}\mathrm{ev}_{\xi_r})\circ\widetilde{\Delta}^l)(B(h_i-h_{-i-2}))+\hbar(\widetilde{\mu}_D(\widetilde{W}^{(2)}_{i,i}t+\widetilde{W}^{(2)}_{-i,-i}t))\\
&=\square^l(\mathrm{ev}_0(B(h_i-h_{-i-2}))-\square^l(\hbar\sum_{u<i,s\geq 0}(-1)^{\widehat{u}+\widehat{i}}e_{-i,-u}t^{-s}e^{(r)}_{i,u}t^s)\\
&\quad-\square^l(\hbar\sum_{u<i,s\geq 1}(-1)^{\widehat{u}+\widehat{i}}e_{i,u}t^{-s}e_{-i,-u}t^s)\\
&\quad-\square^l(\hbar\sum_{u>i,s\geq 0}(-1)^{\widehat{u}+\widehat{i}}e_{-i,-u}t^{-s}e_{i,u}t^s)-\square^l(\hbar\sum_{u>i,s\geq 1}(-1)^{\widehat{u}+\widehat{i}}e_{i,u}t^{-s}e_{-i,-u}t^s)\\
&\quad+(\widetilde{\Delta}^l-\square^l)\big(\frac{\hbar}{2}\sum_{\substack{v\neq i\\ s\geq 0}}(e_{i,v}-e_{-v,-i})t^{-s}(e_{v,i}-e_{-i,-v})t^s\big)\\
&\quad+(\widetilde{\Delta}^l-\square^l)\big(\frac{\hbar}{2}\sum_{\substack{v\neq i\\ s\geq 1}}(e_{i,v}-e_{-v,-i})t^{-s}(e_{v,i}-e_{-i,-v})t^s\big)\\
&\quad+(\widetilde{\Delta}^l-\square^l)\big(-\frac{\hbar}{2}(e_{i,i}-e_{-i,-i})^2+\hbar\sum_{s\geq 0}(e_{i,i}-e_{-i,-i})t^{-s}(e_{i,i}-e_{-i,-i})t^s\big)\\
&\quad-\square^l(\hbar\sum_{s\geq 0}e_{-i,-i}t^{-s}e_{i,i}t^s+\hbar\sum_{s\geq 1}e_{i,i}t^{-s}e_{-i,-i}t^s).
\end{aligned}$$

Thus, it is enough to show that

$$\begin{aligned}
&\mathrm{ev}_0(B(h_i-h_{-i-2}))-\hbar\sum_{u<i,s\geq 0}(-1)^{\widehat{u}+\widehat{i}}e_{-i,-u}t^{-s}e^{(r)}_{i,u}t^s-\hbar\sum_{u<i,s\geq 1}(-1)^{\widehat{u}+\widehat{i}}e_{i,u}t^{-s}e_{-i,-u}t^s\\
&-\hbar\sum_{u>i,s\geq 0}(-1)^{\widehat{u}+\widehat{i}}e_{-i,-u}t^{-s}e_{i,u}t^s-\hbar\sum_{u>i,s\geq 1}(-1)^{\widehat{u}+\widehat{i}}e_{i,u}t^{-s}e_{-i,-u}t^s\\
&-\hbar\sum_{s\geq 0}e_{-i,-i}t^{-s}e_{i,i}t^s+\hbar\sum_{s\geq 1}e_{i,i}t^{-s}e_{-i,-i}t^s
\end{aligned} \tag{13.17}$$

is equal to

$$\frac{\hbar}{2}\sum_{\substack{v\neq i\\ s\geq 0}}(e_{i,v}-e_{-v,-i})t^{-s}(e_{v,i}-e_{-i,-v})t^s+\frac{\hbar}{2}\sum_{\substack{v\neq i\\ s\geq 1}}(e_{i,v}-e_{-v,-i})t^{-s}(e_{v,i}-e_{-i,-v})t^s$$
$$-\frac{\hbar}{2}(e_{i,i}-e_{-i,-i})^2+\hbar\sum_{s\geq 0}(e_{i,i}-e_{-i,-i})t^{-s}(e_{i,i}-e_{-i,-i})t^s.$$

Here after, we prove this statement. By a direct computation, we obtain

$$\mathrm{ev}_0(J(h_i))=A_i-A_{i+2},$$

where

$$A_i=-\frac{\hbar}{2}e_{i,i}^2+\frac{\hbar}{2}\sum_{\substack{u>i\\ s\geq 0}}e_{u,i}t^{-s}e_{i,u}t^s+\frac{\hbar}{2}\sum_{\substack{i>v\\ s\geq 0}}e_{i,v}t^{-s}e_{v,i}t^s$$
$$+\frac{\hbar}{2}\sum_{\substack{u<i\\ s\geq 1}}e_{u,i}t^{-s}e_{i,u}t^s+\frac{\hbar}{2}\sum_{\substack{i<v\\ s\geq 1}}e_{i,v}t^{-s}e_{v,i}t^s+\hbar\sum_{s\geq 0}e_{i,i}t^{-s}e_{i,i}t^s. \tag{13.18}$$

By changing i to $-i$, we have

$$A_{-i}=-\frac{\hbar}{2}e_{-i,-i}^2+\frac{\hbar}{2}\sum_{\substack{u<i\\ s\geq 0}}e_{-u,-i}t^{-s}e_{-i,-u}t^s+\frac{\hbar}{2}\sum_{\substack{i<v\\ s\geq 0}}e_{-i,-v}t^{-s}e_{-v,-i}t^s$$
$$+\frac{\hbar}{2}\sum_{\substack{u>i\\ s\geq 1}}e_{-u,-i}t^{-s}e_{-i,-u}t^s+\frac{\hbar}{2}\sum_{\substack{i>v\\ s\geq 1}}e_{-i,-v}t^{-s}e_{-v,-i}t^s+\hbar\sum_{s\geq 0}e_{-i,-i}t^{-s}e_{-i,-i}t^s. \tag{13.19}$$

By a direct computation, we also obtain

$$\frac{\hbar}{8}[\sum_{\substack{u<v\\ s\geq 1}}(e_{u,v}-(-1)^{\widehat{u}+\widehat{v}}e_{-v,-u})t^{-s}(e_{v,u}-(-1)^{\widehat{u}+\widehat{v}}e_{-u,-v})t^s,e_{i,i}+e_{-i,-i}]$$
$$=\frac{\hbar}{2}\sum_{\substack{u<i\\ s\geq 1}}(-1)^{\widehat{u}+\widehat{i}}e_{-i,-u}t^{-s}e_{i,u}t^s-\frac{\hbar}{2}\sum_{\substack{i<v\\ s\geq 1}}(-1)^{\widehat{i}+\widehat{v}}e_{-v,-i}t^{-s}e_{v,i}t^s$$
$$+\frac{\hbar}{2}\sum_{\substack{u<-i\\ s\geq 1}}(-1)^{\widehat{u}+\widehat{i}+1}e_{i,-u}t^{-s}e_{-i,u}t^s-\frac{\hbar}{2}\sum_{\substack{-i<v\\ s\geq 1}}(-1)^{\widehat{i}+\widehat{v}+1}e_{-v,i}t^{-s}e_{v,-i}t^s, \tag{13.20}$$

$$\frac{\hbar}{8}[\sum_{\substack{u>v\\ s\geq 0}}(e_{u,v}-(-1)^{\widehat{u}+\widehat{v}}e_{-v,-u})t^{-s}(e_{v,u}-(-1)^{\widehat{u}+\widehat{v}}e_{-u,-v})t^s,e_{i,i}+e_{-i,-i}]$$
$$=\frac{\hbar}{2}\sum_{\substack{u>i\\ s\geq 0}}(-1)^{\widehat{u}+\widehat{i}}e_{-i,-u}t^{-s}e_{i,u}t^s-\frac{\hbar}{2}\sum_{\substack{i>v\\ s\geq 0}}(-1)^{\widehat{i}+\widehat{v}}e_{-v,-i}t^{-s}e_{v,i}t^s$$
$$+\frac{\hbar}{2}\sum_{\substack{u>-i\\ s\geq 0}}(-1)^{\widehat{u}+\widehat{i}+1}e_{i,-u}t^{-s}e_{-i,u}t^s-\frac{\hbar}{2}\sum_{\substack{-i>v\\ s\geq 0}}(-1)^{\widehat{i}+\widehat{v}+1}e_{-v,i}t^{-s}e_{v,-i}t^s, \tag{13.21}$$

Then, we have

$$(13.17)_4+(13.21)_1+(13.21)_4+(13.18)_2+(13.19)_3$$
$$=\frac{\hbar}{2}\sum_{\substack{u>i\\ s\geq 0}}(e_{u,i}-(-1)^{\widehat{i}+\widehat{u}}e_{-i,-u})t^{-s}(e_{i,u}-(-1)^{\widehat{i}+\widehat{u}}e_{-u,-i})^s,$$

$$
\begin{aligned}
&(13.17)_2+(13.21)_2+(13.21)_3+(13.18)_3+(13.19)_4\\
&=\frac{\hbar}{2}\sum_{\substack{i>v\\ s\geq 0}}(e_{i,v}-(-1)^{\widehat{i}+\widehat{u}}e_{-v,-i})t^{-s}(e_{v,i}-(-1)^{\widehat{i}+\widehat{u}}e_{-i,-v})t^s,\\
&(13.17)_3+(13.20)_1+(13.20)_4+(13.18)_4+(13.19)_5\\
&=\frac{\hbar}{2}\sum_{\substack{u<i\\ s\geq 1}}(e_{u,i}-(-1)^{\widehat{i}+\widehat{u}}e_{-i,-u})t^{-s}(e_{i,u}-(-1)^{\widehat{i}+\widehat{u}}e_{-u,-i})t^s,\\
&(13.17)_5+(13.20)_2+(13.20)_3+(13.18)_5+(13.19)_2\\
&=\frac{\hbar}{2}\sum_{\substack{i<v\\ s\geq 1}}(e_{i,v}-(-1)^{\widehat{i}+\widehat{u}}e_{-v,-i})t^{-s}(e_{v,i}-(-1)^{\widehat{i}+\widehat{u}}e_{-i,-v})t^s,\\
&(13.11)_1+(13.12)_1+(13.4)_7+(13.4)_8+(13.18)_1+(13.18)_6+(13.19)_1+(13.19)_6\\
&=-\frac{\hbar}{2}(e_{i,i}-e_{-i,-i})^2+\hbar\sum_{s\geq 0}(e_{i,i}-e_{-i,-i})t^{-s}(e_{i,i}-e_{-i,-i})t^s.
\end{aligned}
$$

Adding the above five relations, we complete the proof of Theorem 13.5. □

Theorem 13.22. *Provided that $\alpha\geq 0$, the homomorphism Φ is surjective.*

Proof. We denote the image of $TY(\widehat{so}(n))$ via Φ by $\mathrm{Im}\Phi$. By Theorem 11.20, it is enough to show that $\{\widetilde{W}^{(r)}_{i,j}t^s\mid 1\leq i,j\leq n, r=1,2, s\in\mathbb{Z}\}$ is contained in $\mathrm{Im}\Phi$. By the definition of $\Phi(U(\widehat{\mathfrak{k}}))$, $\mathrm{Im}\Phi$ contains $\widetilde{W}^{(1)}_{j,i}t^s$ for all $i\neq j$. Take (i,j) such that $i\neq\pm j,-i-2$. By the definition of $\Phi(B(h_i-h_{i+2}))$, we find that

$$
\begin{aligned}
\gamma_i=(\widetilde{W}^{(2)}_{i,i}-\widetilde{W}^{(2)}_{i+2,i+2})t-\sum_{m\geq 1}\widetilde{W}^{(1)}_{i,i}t^{-m}\widetilde{W}^{(1)}_{i,i}t^m-\frac{1}{2}(\widetilde{W}^{(1)}_{i,i})^2\\
+\sum_{m\geq 1}\widetilde{W}^{(1)}_{i+2,i+2}t^{-m}\widetilde{W}^{(1)}_{i+2,i+2}t^m+\frac{1}{2}(\widetilde{W}^{(1)}_{i+2,i+2})^2
\end{aligned}
$$

is contained in $\mathrm{Im}\Phi$. By Lemma 11.21, we find that $[\gamma_i,\widetilde{W}^{(1)}_{j,i}t^s]$ is equal to

$$
\begin{aligned}
\gamma_{i,s}&=(1+\delta_{j,i+2}-(-1)^{p(i)+p(j)}\delta_{-j,i+2})\widetilde{W}^{(2)}_{j,i}t^{s+1}+\frac{l-1}{2}s\alpha(1+\delta_{j,i+2}+\delta_{-i,j}+\delta_{-j,i+2})\widetilde{W}^{(1)}_{j,i}t^s\\
&\quad+\frac{1}{2}sW^{(1)}_{-i,-j}t^s+\frac{1}{2}\delta_{j,-i-2}W^{(1)}_{-i-2,i}-\frac{1}{2}\delta_{j,i+2}W^{(1)}_{i+2,i}\\
&\quad-\sum_{m\geq 1}\widetilde{W}^{(1)}_{j,i}t^{-m+s}\widetilde{W}^{(1)}_{i,i}t^m-\sum_{m\geq 1}\widetilde{W}^{(1)}_{i,i}t^{-m}\widetilde{W}^{(1)}_{j,i}t^{m+s}\\
&\quad-\frac{1}{2}(1+\delta_{-i,j})\widetilde{W}^{(1)}_{j,i}t^s\widetilde{W}^{(1)}_{i,i}-\frac{1}{2}(1+\delta_{-i,j})\widetilde{W}^{(1)}_{i,i}\widetilde{W}^{(1)}_{j,i}t^s\\
&\quad-(\delta_{j,i+2}-\delta_{j,-i-2})\sum_{m\geq 1}\widetilde{W}^{(1)}_{i+2,j}t^{-m+s}\widetilde{W}^{(1)}_{i+2,i+2}t^m\\
&\quad-(\delta_{j,i+2}-\delta_{j,-i-2})\sum_{m\geq 1}\widetilde{W}^{(1)}_{i+2,i+2}t^{-m}\widetilde{W}^{(1)}_{i+2,j}t^{m+s}\\
&\quad-\frac{1}{2}(\delta_{j,i+2}-\delta_{j,-i-2})(\widetilde{W}^{(1)}_{j,i+2}t^s\widetilde{W}^{(1)}_{i+2,i+2}+\widetilde{W}^{(1)}_{i+2,i+2}\widetilde{W}^{(1)}_{j,i+2}t^s)
\end{aligned}
\tag{13.23}
$$

for all $i\neq j$. Then, by Lemma 11.21, we obtain

$$
[\widetilde{W}^{(1)}_{i,j}t,\gamma_{i,s}]-[\widetilde{W}^{(1)}_{i,j},\gamma_{i,s+1}]
$$

$$= \frac{l-1}{2}\alpha(1+\delta_{j,i+2}-(-1)^{p(i)+p(j)}\delta_{-j,i+2})(\widetilde{W}^{(1)}_{i,i}+\widetilde{W}^{(1)}_{j,j})t^{s+1}-\frac{l}{2}\alpha\widetilde{W}^{(1)}_{i,i}t^{s+1}$$
$$+ \text{completion of sum of terms of } U(\widehat{\mathfrak{k}}).$$

Then, we find that $\widetilde{W}^{(1)}_{i,i}t^{s+1}$ is contained in $\mathrm{Im}\Phi$. Since $(\widetilde{W}^{(1)}_{p,p}-\widetilde{W}^{(1)}_{q,q})t^{s+1}$ is contained in $\mathrm{Im}\Phi$ for all p,q, we find that $\widetilde{W}^{(1)}_{p,p}t^{s+1}$ is contained in $\mathrm{Im}\Phi$ for all p.

Since $\widetilde{W}^{(1)}_{i,j}t^s$ is contained in $\mathrm{Im}\Phi$ for all i,j, $(\widetilde{W}^{(2)}_{i,i}-\widetilde{W}^{(2)}_{j,j})t$ is contained in $\mathrm{Im}\Phi$. By using Lemma 11.21 (1), we obtain

$$\begin{aligned}&[\widetilde{W}^{(1)}_{j,i}t^s,(\widetilde{W}^{(2)}_{i,i}-\widetilde{W}^{(2)}_{j,j})t]\\&=(2+\delta_{i,-j}(-1)^{\hat{i}+\hat{j}})\widetilde{W}^{(2)}_{j,i}t^{s+1}+\frac{l-1}{2}s\alpha(2+\delta_{i,-j})\widetilde{W}^{(1)}_{j,i}t^s.\end{aligned}\tag{13.24}$$

By (13.24) and (13.23), we find that $\widetilde{W}^{(2)}_{i,j}t^s$ $(i\neq j)$ is contained in $\mathrm{Im}\Phi$. By using Lemma 11.21 (1), we have

$$[\widetilde{W}^{(1)}_{i,i+2},\widetilde{W}^{(2)}_{i+2,i}t^s]=(1+\delta_{2i+2,0})(\widetilde{W}^{(2)}_{i+2,i+2}-\widetilde{W}^{(2)}_{i,i})t^s.$$

Since $\widetilde{W}^{(1)}_{i,i+2}$ and $\widetilde{W}^{(2)}_{i+2,i}t^s$ are contained in $\mathrm{Im}\Phi$, $(\widetilde{W}^{(2)}_{i+2,i+2}-\widetilde{W}^{(2)}_{i,i})t^s$ is contained in $\mathrm{Im}\Phi$. By using Lemma 11.21 (1), we obtain

$$\begin{aligned}&[(\widetilde{W}^{(2)}_{i,i}-\widetilde{W}^{(2)}_{i+2,i+2})t,(\widetilde{W}^{(2)}_{i,i}-\widetilde{W}^{(2)}_{i+2,i+2})t^s]-[(\widetilde{W}^{(2)}_{i,i}-\widetilde{W}^{(2)}_{i+2,i+2}),(\widetilde{W}^{(2)}_{i,i}-\widetilde{W}^{(2)}_{i+2,i+2})t^{s+1}]\\&=\alpha(\widetilde{W}^{(2)}_{i,i}+\widetilde{W}^{(2)}_{i+2,i+2})t^s\end{aligned}$$

for all $\hat{i}=\widehat{i+2}$. By the assumption $\alpha\geq 0$, $(\widetilde{W}^{(2)}_{i,i}+\widetilde{W}^{(2)}_{i+2,i+2})t^s$ is contained in $\mathrm{Im}\Phi$. Since we have already shown that $(\widetilde{W}^{(2)}_{i,i}-\widetilde{W}^{(2)}_{i+2,i+2})t^s$ is contained in $\mathrm{Im}\Phi$, $\widetilde{W}^{(2)}_{i,i}t^s$ is contained in $\mathrm{Im}\Phi$. This completes the proof. □

A The proof of Theorem 8.17

In this section, we prove Theorem 8.17. We define a grading on $\mathfrak{b}$ by setting $\deg(x)=j$ if $x\in\mathfrak{b}\cap\mathfrak{g}_j$. Since

$$\{\sum_{s=1}^{l-r}e_{(r+s-1)(m+n)+j,(s-1)(m+n)+i}\mid 0\leq r\leq l-1,\ 1\leq i,j\leq m+n\}$$

forms a basis of $\mathfrak{gl}(ml|nl)^f=\{g\in\mathfrak{gl}(ml|nl)|[f,g]=0\}$, it is enough to show that $W^{(1)}_{i,j}$ and $W^{(2)}_{i,j}$ generate the term whose form is

$$\sum_{s=1}^{l-r}e_{(r+s-1)(m+n)+j,(s-1)(m+n)+i}[-1]+\text{higher terms}$$

for all $0\leq r\leq l-1,\ 1\leq i,j\leq m+n$ by Theorem 4.1 of [27]. We show that $W^{(1)}_{i,j}$ and $W^{(2)}_{i,j}$ generate these terms by two claims, that is, Claim A.4 and Claim A.5. In Claim A.4 below, we show that $W^{(1)}_{i,j}$ and $W^{(2)}_{i,j}$ generate the term whose form is

$$\begin{aligned}&(-1)^{p(i)}\sum_{s=1}^{l-r}e_{(r+s-1)(m+n)+i,(s-1)(m+n)+i}[-1]\\&\qquad-(-1)^{p(i+1)}\sum_{s=1}^{l-r}e_{(r+s-1)(m+n)+i+1,(s-1)(m+n)+i+1}[-1]+\text{higher terms}\end{aligned}$$

or
$$\sum_{s=1}^{l-r} e_{(r+s-1)(m+n)+j,(s-1)(m+n)+i}[-1] + \text{higher terms } (i \neq j)$$

for all $0 \leq r \leq l-1$. In Claim A.5 below, we prove that $W^{(1)}_{i,j}$ and $W^{(2)}_{i,j}$ generate the term whose form is
$$\sum_{s=1}^{l-r} e_{(r+s-1)(m+n)+i,(s-1)(m+n)+i}[-1] + \text{higher terms}$$

for all $1 \leq r \leq l-1$. Since $\sum_{s=1}^{l-0} e_{(0+s-1)(m+n)+i,(s-1)(m+n)+i}[-1]$ is nothing but $W^{(1)}_{i,i}$, Theorem 8.17 is derived from Claim A.4 and Claim A.5.

In order to prove Claims A.4 and A.5, we prepare the following claim.

Claim A.1. (1) The following equation holds for all $0 \leq w \leq l-1,\ 1 \leq i,j,u,v \leq m+n$;
$$\begin{aligned}
&(\sum_{s=1}^{l-1} e_{s(m+n)+j,(s-1)(m+n)+i}[-1])_{(0)} \sum_{t=1}^{l-w} e_{(w+t-1)(m+n)+u,(t-1)(m+n)+v}[-1]\\
&= \delta_{i,u} \sum_{t=1}^{l-w-1} e_{(w+t)(m+n)+j,(t-1)(m+n)+v}[-1]\\
&\quad - \delta_{j,v}(-1)^{p(e_{i,j})p(e_{u,v})} \sum_{t=1}^{l-w-1} e_{(w+t)(m+n)+u,(t-1)(m+n)+i}[-1].
\end{aligned} \tag{A.2}$$

(2) We obtain
$$\begin{aligned}
&(W^{(1)}_{i,j})_{(0)}(\sum_{s=1}^{l-r} e_{(r+s-1)(m+n)+x,(s-1)(m+n)+y}[-1])\\
&= \delta_{i,x} \sum_{s=1}^{l-r} e_{(r+s-1)(m+n)+j,(s-1)(m+n)+y}[-1]\\
&\quad - \delta_{j,y}(-1)^{p(e_{i,j})p(e_{x,y})} \sum_{s=1}^{l-r} e_{(r+s-1)(m+n)+x,(s-1)(m+n)+i}[-1]
\end{aligned} \tag{A.3}$$

for all $0 \leq r \leq l-1,\ 1 \leq i,j,x,y \leq m+n$.

Claim A.1 is proven by direct computation. We omit the proof. By (A.2) and (A.3), it is easy to obtain the following claim.

Claim A.4. (1) For all $0 \leq r \leq l-1$, the elements $W^{(1)}_{i,j}$ and $W^{(2)}_{i,j}$ generate the term whose form is
$$\sum_{s=1}^{l-r} e_{(r+s-1)(m+n)+i,(s-1)(m+n)+j}[-1] + \text{higher terms } (i \neq j).$$

(2) For all $0 \leq r \leq l-1$, the elements $W^{(1)}_{i,j}$ and $W^{(2)}_{i,j}$ generate the term whose form is
$$\begin{aligned}
&(-1)^{p(i)} \sum_{s=1}^{l-r} e_{(r+s-1)(m+n)+i,(s-1)(m+n)+i}[-1]\\
&\quad - (-1)^{p(i+1)} \sum_{s=1}^{l-r} e_{(r+s-1)(m+n)+i+1,(s-1)(m+n)+i+1}[-1] + \text{higher terms}.
\end{aligned}$$

Proof. First, let us show (1). Since $W^{(2)}_{i,j}$ has the form such that

$$\sum_{s=1}^{l-1} e_{s(m+n)+j,(s-1)(m+n)+i}[-1] + \text{degree 0 terms},$$

we obtain

$$((W^{(2)}_{i,i})_{(0)})^r W^{(1)}_{j,i} = \left((\sum_{s=1}^{l-1} e_{s(m+n)+i,(s-1)(m+n)+i}[-1])_{(0)} \right)^r W^{(1)}_{j,i} + \text{higher terms}$$

for all $i \neq j,\ 0 \leq r \leq l-1$. By (A.2), we have

$$((W^{(2)}_{i,i})_{(0)})^r W^{(1)}_{j,i} = \sum_{s=1}^{l-r} e_{(r+s-1)(m+n)+i,(s-1)(m+n)+j}[-1] + \text{higher terms}.$$

Thus, we have proved (1).

Next, let us prove (2). By (1), the element whose form is

$$\sum_{s=1}^{l-r} e_{(r+s-1)(m+n)+i,(s-1)(m+n)+i+1}[-1] + \text{higher terms}$$

is generated by $W^{(1)}_{i,j}$ and $W^{(2)}_{i,j}$. By (A.3), we have

$$\begin{aligned}
&(W^{(1)}_{i,i+1})_{(0)}\Big(\sum_{s=1}^{l-r} e_{(r+s-1)(m+n)+i,(s-1)(m+n)+i+1}[-1] + \text{higher terms}\Big)\\
&= \sum_{s=1}^{l-r} e_{(r+s-1)(m+n)+i+1,(s-1)(m+n)+i+1}[-1]\\
&\quad - (-1)^{p(e_{i,i+1})} \sum_{s=1}^{l-r} e_{(r+s-1)(m+n)+i,(s-1)(m+n)+i}[-1] + \text{higher terms}.
\end{aligned}$$

Thus, we have proved (2). □

Claim A.5. The elements $W^{(1)}_{i,j}$ and $W^{(2)}_{i,j}$ generate the term whose form is

$$\sum_{1\leq t\leq l-r} e_{(t+r-1)(m+n)+i,(t-1)(m+n)+i}[-1] + \text{higher terms}$$

for all $1 \leq r \leq l-1$.

Proof. It is enough to show that

$$\begin{aligned}
&(W^{(2)}_{i,i})_{(1)}(W^{(1)}_{i,i+1})_{(0)}\{(W^{(2)}_{i,i})_{(0)}\}^r W^{(1)}_{i+1,i}\\
&= (-1)^{p(e_{i,i+1})}\alpha \sum_{1\leq t\leq l} e_{(t-1)(m+n)+i,(t-r-1)(m+n)+i}[-1]\\
&\quad + (-1)^{p(i+1)} r \sum_{1\leq t\leq l} e_{(t-1)(m+n)+i,(t-r-1)(m+n)+i}[-1]\\
&\quad - (-1)^{p(i)} r \sum_{1\leq t\leq l} e_{(t-1)(m+n)+i+1,(t-r-1)(m+n)+i+1}[-1] + \text{higher terms}
\end{aligned} \tag{A.6}$$

since we have already shown that

$$\sum_{1\leq t\leq l} (e_{(t-1)(m+n)+i,(t-r-1)(m+n)+i}[-1] - (-1)^{p(e_{i,i+1})} e_{(t-1)(m+n)+i+1,(t-r-1)(m+n)+i+1}[-1])$$

is generated by $W^{(1)}_{i,j}$ and $W^{(2)}_{i,j}$. Let us set

$$Z=\sum_{1\leq s\leq l-1}e_{s(m+n)+i,(s-1)(m+n)+i}[-1],\quad W=W^{(2)}_{i,i}-Z.$$

The element $W^{(2)}_{i,i}$ is the sum of degree -1 element Z and degree 0 element W. We can rewrite the left hand side of (A.6) as

$$\begin{aligned}&Z_{(1)}(W^{(1)}_{i,i+1})_{(0)}(Z_{(0)})^rW^{(1)}_{i+1,i}+W_{(1)}(W^{(1)}_{i,i+1})_{(0)}(Z_{(0)})^rW^{(1)}_{i+1,i}\\&+\sum_{1\leq d\leq r}Z_{(1)}(W^{(1)}_{i,i+1})_{(0)}(Z_{(0)})^{r-d}W_{(0)}(Z_{(0)})^{d-1}W^{(1)}_{i+1,i}+\text{higher terms.}\end{aligned}\tag{A.7}$$

In order to simplify the notation, here after, we denote $\sum\limits_{a\leq s\leq l-b}e_{(b+s-1)(m+n)+i,(s-a)(m+n)+j}[-u]$ by $\sum\limits_{1\leq s\leq l}e_{(b+s-1)(m+n)+i,(s-a)(m+n)+j}[-u]$. Let us compute the each terms of (A.7). First, we compute the first term of (A.7). By (A.2) and (A.3), we have

$$\begin{aligned}&(W^{(1)}_{i,i+1})_{(0)}(Z_{(0)})^rW^{(1)}_{i+1,i}\\&=\sum_{1\leq t\leq l}e_{(t-1)(m+n)+i+1,(t-r-1)(m+n)+i+1}[-1]\\&\qquad-(-1)^{p(e_{i,i+1})}\sum_{1\leq t\leq l}e_{(t-1)(m+n)+i,(t-r-1)(m+n)+i}[-1].\end{aligned}\tag{A.8}$$

Applying (A.8) to the first term of (A.7), we obtain

$$\text{the first term of (A.7)}=(\sum_{1\leq t\leq l}e_{s(m+n)+i,(s-1)(m+n)+i}[-1])_{(1)}(\text{the right hand side of (A.8)})=0$$

since $\kappa(e_{s(m+n)+j,(s-1)n+j},e_{(t-1)(m+n)+i,(t-r-1)(m+n)+i})=0$. Next, let us compute the second term of (A.7). By (A.8), it is the sum of

$$\begin{aligned}&(-1)^{p(i)}(\sum_{\substack{r_1<r_2\\1\leq u\leq m+n}}e^{(r_1)}_{u,i}[-1]e^{(r_2)}_{i,u}[-1])_{(1)}(\sum_{1\leq t\leq l}e_{(t-1)(m+n)+i+1,(t-r-1)(m+n)+i+1}[-1])\\&\quad-(-1)^{p(i+1)}(\sum_{\substack{r_1<r_2\\1\leq u\leq m+n}}e^{(r_1)}_{u,i}[-1]e^{(r_2)}_{i,u}[-1])_{(1)}(\sum_{1\leq t\leq l}e_{(t-1)(m+n)+i,(t-r-1)(m+n)+i}[-1])\end{aligned}\tag{A.9}$$

and

$$\begin{aligned}&(\alpha\sum_{2\leq s\leq l}(s-1)e^{(s)}_{i,i}[-2])_{(1)}\sum_{1\leq t\leq l}e_{(t-1)(m+n)+i+1,(t-r-1)(m+n)+i+1}[-1]\\&\quad-(-1)^{p(e_{i,i+1})}(\alpha\sum_{2\leq s\leq l}(s-1)e^{(s)}_{i,i}[-2])_{(1)}\sum_{1\leq t\leq l}e_{(t-1)(m+n)+i,(t-r-1)(m+n)+i}[-1]).\end{aligned}\tag{A.10}$$

Let us compute (A.9) and (A.10). By direct computation, the second term of (A.9) is equal to

$$\begin{aligned}&-(-1)^{p(i+1)}\sum_{\substack{r_1<r_2\\1\leq u\leq m+n\\1\leq t\leq l}}[e^{(r_1)}_{u,i},[e^{(r_2)}_{i,u},e_{(t-1)(m+n)+i,(t-r-1)(m+n)+i}]][-1]\\&\quad-(-1)^{p(i+1)}\sum_{\substack{r_1<r_2\\1\leq u\leq m+n\\1\leq t\leq l}}\kappa(e^{(r_2)}_{i,u},e_{(t-1)(m+n)+i,(t-r-1)(m+n)+i})e^{(r_1)}_{u,i}[-1]\end{aligned}$$

$$- (-1)^{p(i+1)} \sum_{\substack{r_1<r_2\\ 1\leq u\leq m+n\\ 1\leq t\leq l}} (-1)^{p(e_{i,u})}\kappa(e^{(r_1)}_{u,i}, e_{(t-1)(m+n)+i,(t-r-1)(m+n)+i})e^{(r_2)}_{i,u}[-1]$$
$$= -(-1)^{p(i+1)} \sum_{1\leq t\leq l} e_{(t-1)(m+n)+i,(t-r-1)(m+n)+i}[-1] + 0 + 0.$$

By the similar computation, the first term of (A.9) is equal to

$$(-1)^{p(i+1)} \sum_{1\leq t\leq l} e_{(t-1)(m+n)+i,(t-r-1)(m+n)+i}[-1].$$

By direct computation, we rewrite the second term of (A.10) as

$$(-1)^{p(e_{i,i+1})}\alpha \sum_{\substack{1\leq s\leq l\\ 1\leq t\leq l}} (s-1)[e^{(s)}_{i,i}, e_{(t-1)(m+n)+i,(t-r-1)(m+n)+i}][-1]$$
$$= (-1)^{p(e_{i,i+1})} r\alpha \sum_{1\leq t\leq l} e_{(t-1)(m+n)+i,(t-r-1)(m+n)+i}[-1].$$

By the similar computation, we find that the first term of (A.10) is zero. Thus, we obtain

$$\text{the sum of first two terms of (A.7)} = (-1)^{p(e_{i,i+1})} r\alpha \sum_{1\leq t\leq l} e_{(t-1)(m+n)+i,(t-r-1)(m+n)+i}[-1]. \tag{A.11}$$

Finally, we compute the third term of (A.7). Since the relation $(\sum_{1\leq s\leq l}(s-1)e^{(s)}_{i,i}[-2])_{(0)} = 0$ holds, we can rewrite the third term of (A.7) as

$$\sum_{1\leq d\leq r} Z_{(1)}(W^{(1)}_{i,i+1})_{(0)}(Z_{(0)})^{r-d}\cdot((-1)^{p(i)} \sum_{\substack{r_1<r_2\\ 1\leq t\leq m+n}} e^{(r_1)}_{t,i}[-1]e^{(r_2)}_{i,t}[-1])_{(0)}(Z_{(0)})^{d-1}W^{(1)}_{i+1,i}.$$

Let us set

$$T_d = Z_{(1)}(W^{(1)}_{i,i+1})_{(0)}(Z_{(0)})^{r-d}, \quad B_d = ((-1)^{p(i)} \sum_{\substack{r_1<r_2\\ 1\leq t\leq m+n}} e^{(r_1)}_{t,i}[-1]e^{(r_2)}_{i,t}[-1])_{(0)}(Z_{(0)})^{d-1}W^{(1)}_{i+1,i}$$

Then, the third term of (A.7) is equal to $\sum_{1\leq d\leq r} T_d(B_d)$.

We rewrite B_d and T_d. By (A.2) and (A.3), T_d is the sum of T^1_d and T^2_d such that

$$T^1_d = -\sum_{g=0}^{r-d}\binom{r-d}{g}(Z_{(0)})^{r-d-g}(\sum_{1\leq s\leq l} e_{(s+g)(m+n)+i+1,(s-1)(m+n)+i}[-1])_{(1)}, \tag{A.12}$$
$$T^2_d = (W^{(1)}_{i,i+1})_{(0)}(Z_{(0)})^{r-d}Z_{(1)}. \tag{A.13}$$

Since

$$(Z_{(0)})^{d-1}W^{(1)}_{i+1,i} = \sum_{d\leq t\leq l} e_{(t-1)(m+n)+i,(t-d)(m+n)+i+1}[-1]. \tag{A.14}$$

by (A.2) and (A.3), B_d is equal to

$$((-1)^{p(i)} \sum_{\substack{1\leq r_1<r_2\leq l\\ 1\leq u\leq m+n}} e^{(r_1)}_{u,i}[-1]e^{(r_2)}_{i,u}[-1])_{(0)} \sum_{d\leq t\leq l} e_{(t-1)(m+n)+i,(t-d)(m+n)+i+1}[-1]$$

$$
\begin{aligned}
&= (-1)^{p(i)} \sum_{\substack{1\leq r_1<r_2\leq l\\ 1\leq u\leq m+n}} \sum_{d\leq t\leq l} e^{(r_1)}_{u,i}[-1][e^{(r_2)}_{i,u}, e_{(t-1)(m+n)+i,(t-d)(m+n)+i+1}][-1]\\
&\quad+\sum_{\substack{1\leq r_1<r_2\leq l\\ 1\leq u\leq m+n}} \sum_{d\leq t\leq l} (-1)^{p(u)} e^{(r_2)}_{i,u}[-1][e^{(r_1)}_{u,i}, e_{(t-1)(m+n)+i,(t-d)(m+n)+i+1}][-1]\\
&\quad+\sum_{\substack{1\leq r_1<r_2\leq l\\ 1\leq u\leq m+n}} \sum_{d\leq t\leq l} (-1)^{p(u)} \kappa(e^{(r_1)}_{u,i}, e_{(t-1)(m+n)+i,(t-d)(m+n)+i+1}) e^{(r_2)}_{i,u}[-2]. \qquad (A.15)
\end{aligned}
$$

By direct computation, we find that the first term of the right hand side of (A.15) is equal to

$$
(-1)^{p(i)} \sum_{d\leq r_1<t\leq l} e^{(r_1)}_{i,i}[-1] e_{(t-1)(m+n)+i,(t-d)(m+n)+i+1}[-1] \qquad (A.16)
$$

and the second term of the right hand side of (A.15) is equal to

$$
\begin{aligned}
&\sum_{\substack{d\leq t<r_2\leq l\\ 1\leq u\leq m+n}} (-1)^{p(u)} e^{(r_2)}_{i,u}[-1] e_{(t-1)(m+n)+u,(t-d)(m+n)+i+1}[-1]\\
&\qquad -(-1)^{p(i+1)} \sum_{t-d+1<r_2} e^{(r_2)}_{i,i+1}[-1] e_{(t-1)(m+n)+i,(t-d)(m+n)+i}[-1]. \qquad (A.17)
\end{aligned}
$$

By the definition of κ, the third term of the right hand side of (A.15) is equal to

$$
\delta_{d,1}\alpha \sum_{1\leq r_2\leq l} (r_2-1) e^{(r_2)}_{i,i+1}[-2]. \qquad (A.18)
$$

Adding (A.16), (A.17), and (A.18), we obtain

$$
\begin{aligned}
B_d &= (-1)^{p(i)} \sum_{d\leq r_1<t\leq l} e^{(r_1)}_{i,i}[-1] e_{(t-1)(m+n)+i,(t-d)(m+n)+i+1}[-1]\\
&\quad+\sum_{\substack{d\leq t<r_2\leq l\\ 1\leq u\leq m+n}} (-1)^{p(u)} e^{(r_2)}_{i,u}[-1] e_{(t-1)(m+n)+u,(t-d)(m+n)+i+1}[-1]\\
&\quad-(-1)^{p(i)} \sum_{t-d+1<r_2\leq l} e^{(r_2)}_{i,i+1}[-1] e_{(t-1)(m+n)+i,(t-d)(m+n)+i}[-1]\\
&\quad+\delta_{d,1}\alpha \sum_{1\leq r_2\leq l} (r_2-1) e^{(r_2)}_{i,i+1}[-2]\\
&= (-1)^{p(i)} \sum_{r_1\neq t} e^{(r_1)}_{i,i}[-1] e_{(t-1)(m+n)+i,(t-d)(m+n)+i+1}[-1])\\
&\quad+\sum_{\substack{r_2>t\\ u\neq i}} (-1)^{p(u)} e^{(r_2)}_{i,u}[-1] e_{(t-1)(m+n)+u,(t-d)(m+n)+i+1}[-1]\\
&\quad-(-1)^{p(i)} \sum_{t-d+1<r_2\leq l} e^{(r_2)}_{i,i+1}[-1] e_{(t-1)(m+n)+i,(t-d)(m+n)+i}[-1]\\
&\quad+\delta_{d,1}\alpha \sum_{1\leq r_2\leq l} (r_2-1) e^{(r_2)}_{i,i+1}[-2]. \qquad (A.19)
\end{aligned}
$$

Now, we compute $T_d(B_d)$. We divide B_d into two parts such that

$$
\begin{aligned}
B^1_d &= (-1)^{p(i)} \sum_{r_1\neq t} e^{(r_1)}_{i,i}[-1] e_{(t-1)(m+n)+i,(t-d)(m+n)+i+1}[-1]\\
&\quad+\sum_{\substack{r_2>t\\ u\neq i}} (-1)^{p(u)} e^{(r_2)}_{i,u}[-1] e_{(t-1)(m+n)+u,(t-d)(m+n)+i+1}[-1]
\end{aligned}
$$

$$-(-1)^{p(i)}\sum_{t-d+1<r_2\leq l}e^{(r_2)}_{i,i+1}[-1]e_{(t-1)(m+n)+i,(t-d)(m+n)+i}[-1],$$

$$B^2_d=\delta_{d,1}\alpha\sum_{1\leq r_2\leq l}(r_2-1)e^{(r_2)}_{i,i+1}[-2].$$

First, let us compute $T_d(B^2_d)$. By (A.2) and (A.3), we obtain

$$T_d(B^2_d)=-\delta_{d,1}(-1)^{p(e_{i,i+1})}(r-1)\alpha\sum_{1\leq t\leq l}e_{(t-1)(m+n)+i,(t-r-1)(m+n)+i}[-1]. \tag{A.20}$$

Next, let us compute $T_d(B^1_d)=T^1_d(B^1_d)+T^2_d(B^1_d)$. We compute $T^1_d(B^1_d)$ and $T^2_d(B^1_d)$ respectively. In order to compute $T^1_d(B^1_d)$, we prepare the following three relations;

$$\begin{aligned}&\sum_{1\leq s\leq l}(e_{(s+g)(m+n)+i+1,(s-1)(m+n)+i}[-1])_{(1)}\\&\qquad\cdot((-1)^{p(i)}\sum_{r_1\neq t}e^{(r_1)}_{i,i}[-1]e_{(t-1)(m+n)+i,(t-d)(m+n)+i+1}[-1])\\&=-(-1)^{p(i+1)}\sum_{1\leq t\leq l}e_{(t-1)(m+n)+i,(t-g-d-1)(m+n)+i}[-1],\end{aligned} \tag{A.21}$$

$$\begin{aligned}&\sum_{1\leq s\leq l}(e_{(s+g)(m+n)+i+1,(s-1)(m+n)+i}[-1])_{(1)}\\&\qquad\cdot(\sum_{\substack{r_2>t\\u\neq i}}(-1)^{p(u)}e^{(r_2)}_{i,u}[-1]e_{(t-1)(m+n)+u,(t-d)(m+n)+i+1}[-1])=0,\end{aligned} \tag{A.22}$$

$$\begin{aligned}&\sum_{1\leq s\leq l}(e_{(s+g)(m+n)+i+1,(s-1)(m+n)+i}[-1])_{(1)}\\&\qquad\cdot((-1)^{p(i)}\sum_{t-d+1<r_2\leq l}e^{(r_2)}_{i,i+1}[-1]e_{(t-1)(m+n)+i,(t-d)(m+n)+i}[-1])\\&=-(-1)^{p(i+1)}\sum_{1\leq t\leq l}e_{(t-1)(m+n)+i,(t-g-d-1)(m+n)+i}[-1].\end{aligned} \tag{A.23}$$

We only show the relation (A.23) holds. The other relations are proven similarly. By direct computation, (A.23) is equal to

$$\begin{aligned}&(-1)^{p(i)}\sum_{1\leq s\leq l}\sum_{t-d+1<r_2\leq l}[[e_{(s+g)(m+n)+i+1,(s-1)(m+n)+i},e^{(r_2)}_{i,i+1}],e_{(t-1)n+i,(t-d)n+i})][-1]\\&=-(-1)^{p(i+1)}\sum_{1\leq t\leq l}e_{(t-1)(m+n)+i,(t-g-d-1)(m+n)+i}[-1].\end{aligned}$$

Thus, we have obtained (A.23). By (A.21)-(A.23) and (A.12), we find the relation

$$T^1_d(B^1_d)=0. \tag{A.24}$$

Similarly to (A.21)-(A.23), we obtain the following three equations;

$$\begin{aligned}&\sum_{1\leq s\leq l}(e_{s(m+n)+i,(s-1)(m+n)+i}[-1])_{(1)}\\&\qquad\cdot((-1)^{p(i)}\sum_{r_1\neq t}e^{(r_1)}_{i,i}[-1]e_{(t-1)(m+n)+i,(t-d)(m+n)+i+1}[-1])\\&=-(-1)^{p(i)}\sum_{1\leq t\leq l}e_{(t-1)(m+n)+i,(t-d-1)(m+n)+i+1}[-1],\end{aligned} \tag{A.25}$$

$$\sum_{1\le s\le l}(e_{s(m+n)+i,(s-1)(m+n)+i}[-1])_{(1)}$$

$$\cdot(\sum_{\substack{r_2>t\\ u\neq i}}(-1)^{p(u)}e^{(r_2)}_{i,u}[-1]e_{(t-1)(m+n)+u,(t-d)(m+n)+i+1}[-1])=0, \tag{A.26}$$

$$\sum_{1\le s\le l}(e_{s(m+n)+i,(s-1)(m+n)+i}[-1])_{(1)}$$

$$\cdot((-1)^{p(i)}\sum_{t-d+1<r_2\le l}e^{(r_2)}_{i,i+1}[-1]e_{(t-1)(m+n)+i,(t-d)(m+n)+i}[-1])=0. \tag{A.27}$$

By (A.25)-(A.27) and (A.13), we obtain

$$\begin{aligned}T^2_d(B^1_d)&=-(-1)^{p(i)}(W^{(1)}_{i,i+1})_{(0)}(Z_{(0)})^{r-d}\sum_{1\le t\le l}e_{(t-1)(m+n)+i,(t-d-1)(m+n)+i+1}[-1]\\ &=-(-1)^{p(i)}\sum_{1\le t\le l}e_{(t-1)(m+n)+i+1,(t-r-1)(m+n)+i+1}[-1]\\ &\quad+(-1)^{p(i+1)}\sum_{1\le t\le l}e_{(t-1)(m+n)+i,(t-r-1)(m+n)+i}[-1],\end{aligned} \tag{A.28}$$

where the second equality is due to (A.2) and (A.3). By (A.20), (A.24) and (A.28), we have

$$\begin{aligned}\sum_{1\le d\le r}T_d(B_d)&=-(-1)^{p(e_{i,i+1})}(r-1)\alpha\sum_{1\le t\le l}e_{(t-1)(m+n)+i,(t-r-1)(m+n)+i}[-1]\\ &\quad-(-1)^{p(i)}r\sum_{1\le t\le l}e_{(t-1)(m+n)+i+1,(t-r-1)(m+n)+i+1}[-1]\\ &\quad+(-1)^{p(i+1)}r\sum_{1\le t\le l}e_{(t-1)(m+n)+i,(t-r-1)(m+n)+i}[-1].\end{aligned} \tag{A.29}$$

Adding (A.11) and (A.29), (A.7) is equal to

$$\begin{aligned}&(-1)^{p(e_{i,i+1})}\alpha\sum_{1\le t\le l}e_{(t-1)(m+n)+i,(t-r-1)(m+n)+i}[-1]\\ &-(-1)^{p(i+1)}r\sum_{1\le t\le l}e_{(t-1)(m+n)+i,(t-r-1)(m+n)+i}[-1]\\ &+(-1)^{p(i)}r\sum_{1\le t\le l}e_{(t-1)(m+n)+i+1,(t-r-1)(m+n)+i+1}[-1]+\text{higher terms}.\end{aligned}$$

We have obtained (A.6). □

Since we complete the proof of Claims A.4 and A.5, we have proved Theorem 8.17.

B Generators of rectangular W-algebras of type D

This section is devoted to the proof of Theorem 11.20. We define a grading on $\mathfrak{b}$ by setting $\deg(x)=j$ if $x\in\mathfrak{b}\cap\mathfrak{g}_j$. For $a,b\in I_{nl}$, let $\gamma_{a,b}$ be $\sum\limits_{0<2u\le q-p}\widehat{q+2u}+\widehat{p}\cdot\widehat{j}+\widehat{q}\cdot\widehat{i}$, where $p=\mathrm{col}(a), q=\mathrm{col}(b), j=\mathrm{row}(a), i=\mathrm{row}(b)$. Since

$$\{\sum_{\substack{\mathrm{row}(a)=j,\mathrm{row}(b)=i,\\ \mathrm{col}(a)=\mathrm{col}(b)+2s}}(-1)^{\gamma_{a,b}}f_{a,b}\mid 0\le s\le l-1, 1\le i,j\le n\}$$

forms a basis of $\mathfrak{so}(nl)^f = \{g \in \mathfrak{so}(nl) | [f,g] = 0\}$, it is enough to show that $\widetilde{W}^{(1)}_{i,j}$ and $\widetilde{W}^{(2)}_{i,j}$ generate the term whose form is

$$\sum_{\substack{\mathrm{row}(a)=j,\mathrm{row}(b)=i,\\ \mathrm{col}(a)=\mathrm{col}(b)+2s}} (-1)^{\gamma_{a,b}} f_{a,b}[-1] + \text{higher terms}$$

for all $0 \le s \le l-1,\ 1 \le i,j \le n$ by Theorem 4.1 of [27]. The proof is completed by two claims, that is, Lemma B.1 and Lemma B.3. In order to simplify computations, we prepare the following notations. Let us set

$$Z_{i,i} = \sum_{\substack{\mathrm{row}(a)=i,\mathrm{row}(b)=i,\\ \mathrm{col}(a)=\mathrm{col}(b)+2=p}} (-1)^{\widehat{p}+\widehat{p}\cdot\widehat{i}+\widehat{p-2}\cdot\widehat{i}} f_{a,b}[-1], \quad V_{i,i} = \widetilde{W}^{(2)}_{i,i} - Z_{i,i}.$$

Then, $Z_{i,i}$ is a degree -2 term and $V_{i,i}$ is a degree -1 term. We also denote the condition that $\mathrm{row}(a) = i, \mathrm{row}(b) = j, \mathrm{col}(a) = \mathrm{col}(b) + 2$ by $(A)_{i,j}$, the condition that

$$\mathrm{row}(a_2) = i, \mathrm{row}(b_1) = j, p = \mathrm{col}(a_1) = \mathrm{col}(b_1) < \mathrm{col}(a_2) = \mathrm{col}(b_2) = q, \mathrm{row}(a_1) = \mathrm{row}(b_2) = r$$

by $(B)_{i,j}$, the condition that $\mathrm{row}(a) = i, \mathrm{row}(b) = j, \mathrm{col}(a) = \mathrm{col}(b)+2s$ by $(C)^s_{i,j}$, and the condition that $\mathrm{row}(c) = i, \mathrm{row}(d) = j, \mathrm{col}(c) = \mathrm{col}(d) + 2s$ by $(D)^s_{i,j}$. Moreover, for all $a_i \in V^\kappa(\mathfrak{gl}(n))^{\otimes l}$ and $s_i \in \mathbb{Z}$, we set

$$(a_1)_{(s_1)}(a_2)_{(s_2)} \cdots (a_{u-1})_{(s_{u-1})} a_u = (a_1)_{(s_1)}\Big((a_2)_{(s_2)}\big(\cdots((a_{u-1})_{(s_{u-1})} a_u)\cdots\big)\Big).$$

Lemma B.1. (1) *For all* $i \neq j$, $\{\widetilde{W}^{(r)}_{p,q} \mid 1 \le p.q \le n, r = 1,2\}$ *generate*

$$\sum_{(C)^s_{j,i}} (-1)^{\gamma_{a,b}} f_{a,b}[-1] + \textit{higher terms.}$$

(2) *For all* $i \neq j$, $\{\widetilde{W}^{(r)}_{p,q} \mid 1 \le p,q \le n, r = 1,2\}$ *generate*

$$\sum_{(C)^s_{i,i}} (-1)^{\gamma_{a,b}} f_{a,b}[-1] - \sum_{(C)^s_{j,j}} (-1)^{\gamma_{a,b}} f_{a,b}[-1] + \textit{higher terms.}$$

Proof. (1) By a direct computation, we obtain

$$\begin{aligned}
&\Big(\sum_{(C)^s_{j,i}} (-1)^{\gamma_{a,b}} f_{a,b}[-1]\Big)_{(0)} \sum_{(C)^s_{v,u}} (-1)^{\gamma_{a,b}} f_{a,b}[-1]\\
&= \delta_{i,v} \sum_{(C)^{s+t}_{j,u}} (-1)^{\gamma_{a,b}} f_{a,b}[-1] - \delta_{j,u} \sum_{(C)^{s+t}_{v,i}} (-1)^{\gamma_{a,b}} f_{a,b}[-1]\\
&\quad - \delta_{-j,v} \sum_{(C)^{s+t}_{-i,u}} (-1)^{s+\hat{i}+\hat{j}+\gamma_{a,b}} f_{a,b}[-1] + \delta_{i,-u} \sum_{(C)^{s+t}_{v,-j}} (-1)^{s+\hat{i}+\hat{j}+\gamma_{a,b}} f_{a,b}[-1]. \qquad \text{(B.2)}
\end{aligned}$$

By (B.2), we have the following equation;

$$\begin{aligned}
(\widetilde{W}^{(2)}_{j,j})^s \widetilde{W}^{(1)}_{i,j} &= (Z_{j,j})^s \widetilde{W}^{(1)}_{i,j} + \text{higher terms}\\
&= \sum_{\substack{\mathrm{row}(a)=j,\mathrm{row}(b)=i,\\ \mathrm{col}(a)=\mathrm{col}(b)+2s}} (-1)^{\gamma_{a,b}} f_{a,b}[-1] + \text{higher terms}
\end{aligned}$$

for all $i \neq -j, j$. Then, we have proven (1) in the case that $j \neq i, -i$. Taking $p \in I_n$ such that $i \neq \pm p$, we obtain

$$(\widetilde{W}^{(1)}_{p,i})_{(0)} \sum_{\substack{\mathrm{row}(a)=p,\mathrm{row}(b)=-i,\\ \mathrm{col}(a)=\mathrm{col}(b)+s}} (-1)^{\gamma_{a,b}} f_{a,b}[-1] = \sum_{\substack{\mathrm{row}(a)=p,\mathrm{row}(b)=-i,\\ \mathrm{col}(a)=\mathrm{col}(b)+s}} (-1)^{\gamma_{a,b}} f_{a,b}[-1]$$

by (B.2). We have shown (1) in the case that $j=-i$. This completes the proof of (1).

(2) It is enough to show the case when $i\neq\pm j$ since the case that $i=-j$ is naturally derived from other cases. By (B.2), we obtain

$$\begin{aligned}&(\widetilde{W}^{(1)}_{i,j})_{(0)}\sum_{\substack{\mathrm{row}(a)=i,\mathrm{row}(b)=j,\\ \mathrm{col}(a)=\mathrm{col}(b)+s}}(-1)^{\gamma_{a,b}}f_{a,b}[-1]\\ =&\sum_{\substack{\mathrm{row}(a)=j,\mathrm{row}(b)=j,\\ \mathrm{col}(a)=\mathrm{col}(b)+s}}(-1)^{\hat{i}+\hat{j}+\gamma_{a,b}}f_{a,b}[-1]-\sum_{\substack{\mathrm{row}(a)=i,\mathrm{row}(b)=i,\\ \mathrm{col}(a)=\mathrm{col}(b)+s}}(-1)^{\hat{i}+\hat{j}+\gamma_{a,b}}f_{a,b}[-1]\end{aligned}$$

for all $i\neq\pm j$. TWe have shown (2). □

Lemma B.3. *Suppose that $j\neq\pm i$. We obtain*

$$\begin{aligned}&(\widetilde{W}^{(2)}_{i,i})_{(1)}(\widetilde{W}^{(1)}_{i,j})_{(0)}((\widetilde{W}^{(2)}_{i,i})_{(0)})^s\widetilde{W}^{(1)}_{j,i}\\ =&-s\sum_{\substack{\mathrm{row}(a)=i,\mathrm{row}(b)=i,\\ \mathrm{col}(a)=\mathrm{col}(b)+2s}}(-1)^{\gamma_{a,b}}f_{a,b}[-1]+s\sum_{\substack{\mathrm{row}(a)=j,\mathrm{row}(b)=j,\\ \mathrm{col}(a)=\mathrm{col}(b)+2s}}(-1)^{\gamma_{a,b}}f_{a,b}[-1]\\ &+\alpha\sum_{\substack{\mathrm{row}(a)=i,\mathrm{row}(b)=i,\\ \mathrm{col}(a)=\mathrm{col}(b)+2s}}(-1)^{\gamma_{a,b}}f_{a,b}[-1].\end{aligned}$$

Proof. By the degree of $Z_{i,i}$ and $V_{i,i}$, we obtain

$$\begin{aligned}&(\widetilde{W}^{(2)}_{i,i})_{(1)}(\widetilde{W}^{(1)}_{i,j})_{(0)}((\widetilde{W}^{(2)}_{i,i})_{(0)})^s\widetilde{W}^{(1)}_{j,i}\\ =&(Z_{i,i})_{(1)}(\widetilde{W}^{(1)}_{i,j})_{(0)}((Z_{i,i})_{(0)})^s\widetilde{W}^{(1)}_{j,i}+(V_{i,i})_{(1)}(\widetilde{W}^{(1)}_{i,j})_{(0)}((Z_{i,i})_{(0)})^s\widetilde{W}^{(1)}_{j,i}\\ &+\sum_{1\leq t\leq s}(Z_{i,i})_{(1)}(\widetilde{W}^{(1)}_{i,j})_{(0)}((Z_{i,i})_{(0)})^{s-t}(V_{i,i})_{(0)}((Z_{i,i})_{(0)})^{t-1}\widetilde{W}^{(1)}_{j,i}+\text{higher terms.}\end{aligned}\tag{B.4}$$

We compute each terms of the right hand side of (B.4). Let us compute the first term of the right hand side of (B.4). By (B.2), we obtain

$$(\widetilde{W}^{(1)}_{i,j})_{(0)}((Z_{i,i})_{(0)})^s\widetilde{W}^{(1)}_{j,i}=\sum_{(C)^s_{j,j}}(-1)^{\gamma_{a,b}}f_{a,b}[-1]-\sum_{(C)^s_{i,i}}(-1)^{\gamma_{a,b}}f_{a,b}[-1].\tag{B.5}$$

By (B.2) and (B.5), we have

$$(Z_{i,i})_{(1)}(\widetilde{W}^{(1)}_{i,j})_{(0)}((Z_{i,i})_{(0)})^s\widetilde{W}^{(1)}_{j,i}=0.\tag{B.6}$$

Next, let us compute the second term of the right hand side of (B.4). By (B.5), we obtain

$$\begin{aligned}&(V_{i,i})_{(1)}(\widetilde{W}^{(1)}_{i,j})_{(0)}((Z_{i,i})_{(0)})^s\widetilde{W}^{(1)}_{j,i}\\ =&\sum_{(A)_{i,i},(C)^s_{j,j}}((-1)^{(\widehat{r}+\widehat{i})\cdot(\widehat{p}+\widehat{q})+\gamma_{a,b}}f_{a_1,b_1}[-1]f_{a_2,b_2}[-1])_{(1)}f_{a,b}[-1]\\ &-\sum_{(A)_{i,i},(C)^s_{i,i}}((-1)^{(\widehat{r}+\widehat{i})\cdot(\widehat{p}+\widehat{q})+\gamma_{a,b}}f_{a_1,b_1}[-1]f_{a_2,b_2}[-1])_{(1)}f_{a,b}[-1]\\ &+(\alpha\sum_{(C)^p_{i,i}}\frac{p}{2}f_{a,b}[-2])_{(1)}\sum_{(C)^s_{i,i}}(-1)^{\gamma_{a,b}}f_{a,b}[-1].\end{aligned}\tag{B.7}$$

By a direct computation, we obtain

the first term of (B.7) = the second term of (B.7)

$$= \sum_{\substack{\mathrm{row}(a)=i,\mathrm{row}(b)=i,\\ \mathrm{col}(a)=\mathrm{col}(b)+s=p}} (-1)^{(\widehat{j}+\widehat{i})\cdot(\widehat{p}+\widehat{p-s})+\gamma_{a,b}} f_{a,b}[-1],$$

$$\text{the third term of (B.7)} = \alpha \sum_{(C)^s_{i,i}} (-1)^{\gamma_{a,b}} s f_{a,b}[-1].$$

Thus, we obtain

$$(V_{i,i})_{(1)}(\widetilde{W}^{(1)}_{i,j})_{(0)}((Z_{i,i})_{(0)})^s \widetilde{W}^{(1)}_{j,i} = \alpha \sum_{(C)^s_{i,i}} (-1)^{\gamma_{a,b}} s f_{a,b}[-1]. \tag{B.8}$$

Next, let us compute the third term of (B.4). By (B.2), we obtain

$$((Z_{i,i})_{(0)})^{t-1}\widetilde{W}^{(1)}_{j,i} = \sum_{(C)^{t-1}_{i,i}} (-1)^{\gamma_{a,b}} f_{a,b}[-1].$$

Since

$$(V_{i,i})_{(0)} = (\sum_{(A)_{i,i}} (-1)^{(\widehat{r}+\widehat{i})\cdot\widehat{p}+(\widehat{i}+\widehat{r})\cdot\widehat{q}} f_{a_1,b_1}[-1] f_{a_2,b_2}[-1])_{(0)}$$

holds, we can rewrite $(V_{i,i})_{(0)}((Z_{i,i})_{(0)})^{t-1}\widetilde{W}^{(1)}_{j,i}$ as

$$\sum_{(A)_{i,i},(C)^{t-1}_{i,j}} (-1)^{\beta_1} f_{a_1,b_1}[-1][f_{a_2,b_2}, f_{a,b}][-1] + \sum_{(A)_{i,i},(C)^{t-1}_{i,j}} (-1)^{\beta_1} f_{a_2,b_2}[-1][f_{a_1,b_1}, f_{a,b}][-1],$$

where $\beta_1 = \gamma_{a,b} + (\widehat{r}+\widehat{i})\cdot(\widehat{p}+\widehat{q})$ such that $\mathrm{row}(a_1) = r, \mathrm{col}(a_1) = p, \mathrm{col}(a_2) = q$. By a direct computation, we can rewrite $(V_{i,i})_{(0)}((Z_{i,i})_{(0)})^{t-1}\widetilde{W}^{(1)}_{j,i}$ as

$$\begin{aligned}
&\sum_{(A)_{i,i},(C)^{t-1}_{i,j}} (-1)^{\beta_1} \delta_{b_2,a} f_{a_1,b_1}[-1] f_{a_2,b}[-1] + \sum_{(A)_{i,i},(C)^{t-1}_{i,j}} (-1)^{\beta_1} \delta_{b_2,-b} f_{a_1,b_1}[-1] f_{a,-a_2}[-1]\\
&\quad + \sum_{(A)_{i,i},(C)^{t-1}_{i,j}} (-1)^{\beta_1} \delta_{b_1,a} f_{a_2,b_2}[-1] f_{a_1,b}[-1] - \sum_{(A)_{i,i},(C)^{t-1}_{i,j}} (-1)^{\beta_1} \delta_{a_1,b} f_{a_2,b_2}[-1] f_{a,b_1}[-1]\\
&\quad - \sum_{(A)_{i,i},(C)^{t-1}_{i,j}} (-1)^{\beta_1} \delta_{a_1,-a} f_{a_2,b_2}[-1] f_{-b_1,b}[-1] + \delta_{t,1} \sum_{(C)^{t-1}_{i,j}} \alpha (-1)^{\gamma_{a,b}} \frac{(\mathrm{col}(b)-1+n)}{2} f_{a,b}[-2].
\end{aligned} \tag{B.9}$$

Let us denote the sum of the first five terms of (B.9) by B_t. We can rewrite

$$(Z_{i,i})_{(1)}(\widetilde{W}^{(1)}_{i,j})_{(0)}((Z_{i,i})_{(0)})^{s-t}(V_{i,i})_{(0)}((Z_{i,i})_{(0)})^{t-1}\widetilde{W}^{(1)}_{j,i}$$

as

$$\begin{aligned}
&-\sum_{g=0}^{s-t} \binom{r-t}{g} ((Z_{i,i})_{(0)})^{s-t-g} (\sum_{(D)^g_{j,i}} (-1)^{\gamma_{c,d}} f_{c,d}[-1])_{(1)} B_t + (\widetilde{W}^{(1)}_{i,j})_{(0)}((Z_{i,i})_{(0)})^{s-t}((Z_{i,i})_{(1)}) B_t\\
&\quad + (Z_{i,i})_{(1)}(\widetilde{W}^{(1)}_{i,j})_{(0)}((Z_{i,i})_{(0)})^s \delta_{t,1} \sum_{(C)^{t-1}_{i,j}} \alpha (-1)^{\gamma_{a,b}} \frac{(\mathrm{col}(b)-1+n)}{2} f_{a,b}[-2].
\end{aligned} \tag{B.10}$$

Let us compute each terms of (B.10). By a direct computation, we obtain

$$\text{the third term of (B.10)} = -\delta_{t,1}\alpha \sum_{\substack{\mathrm{row}(a)=i,\mathrm{row}(b)=i,\\ \mathrm{col}(a)=\mathrm{col}(b)+2s}} (-1)^{\gamma_{a,b}} (s-1) f_{a,b}[-1]. \tag{B.11}$$

Next, we compute the first term of (B.10). By (B.9), we can rewrite $(\sum_{(D)^g_{j,i}}(-1)^{\gamma_{c,d}}f_{c,d}[-1])_{(1)}B_t$ as

$$\begin{aligned}
&\sum_{(A)_{i,i},(C)^{t-1}_{i,j},(D)^g_{i,i}}(-1)^{\beta_1+\gamma_{c,d}}\delta_{b_2,a}[[f_{c,d},f_{a_1,b_1}],f_{a_2,b}][-1]\\
&\quad+\sum_{(A)_{i,i},(C)^{t-1}_{i,j},(D)^g_{i,i}}(-1)^{\beta_1+\gamma_{c,d}}\delta_{b_2,-b}[[f_{c,d},f_{a_1,b_1}],f_{a,-a_2}][-1]\\
&\quad+\sum_{(A)_{i,i},(C)^{t-1}_{i,j},(D)^g_{i,i}}(-1)^{\beta_1+\gamma_{c,d}}\delta_{b_1,a}[[f_{c,d},f_{a_2,b_2}],f_{a_1,b}][-1]\\
&\quad-\sum_{(A)_{i,i},(C)^{t-1}_{i,j},(D)^g_{i,i}}(-1)^{\beta_1+\gamma_{c,d}}\delta_{a_1,b}[[f_{c,d},f_{a_2,b_2}],f_{a,b_1}][-1]\\
&\quad-\sum_{(A)_{i,i},(C)^{t-1}_{i,j},(D)^g_{i,i}}(-1)^{\beta_1+\gamma_{c,d}}\delta_{a_1,-a}[[f_{c,d},f_{a_2,b_2}]f_{-b_1,b}][-1]. \qquad (B.12)
\end{aligned}$$

We compute each terms of the right hand side of (B.12). By a direct computation, we obtain

$$\text{the first term of (B.12)}=-\sum_{(A)_{i,i},(C)^{t-1}_{i,j},(D)^g_{i,i}}(-1)^{\beta_1+\gamma_{c,d}}\delta_{b_2,a}\delta_{d,a_1}\delta_{c,b}f_{a_2,b_1}[-1], \qquad (B.13)$$

$$\text{the second term of (B.12)}=0, \qquad (B.14)$$

$$\begin{aligned}
\text{the third term of (B.12)}&=-\sum_{(A)_{i,i},(C)^{t-1}_{i,j},(D)^g_{i,i}}(-1)^{\beta_1+\gamma_{c,d}}\delta_{b_1,a}\delta_{d,-b_2}\delta_{a_2,-a_1}f_{c,b}[-1]\\
&\quad-\sum_{(A)_{i,i},(C)^{t-1}_{i,j},(D)^g_{i,i}}(-1)^{\beta_1+\gamma_{c,d}}\delta_{b_1,a}\delta_{d,-b_2}\delta_{c,b}f_{a_1,-a_2}[-1], \qquad (B.15)
\end{aligned}$$

$$\text{the 4-th term of (B.12)}=\sum_{(A)_{i,i},(C)^{t-1}_{i,j},(D)^g_{i,i}}(-1)^{\beta_1+\gamma_{c,d}}\delta_{a_1,b}\delta_{b_2,c}\delta_{d,a}f_{a_2,b_1}[-1], \qquad (B.16)$$

$$\begin{aligned}
\text{the 5-th term of (B.12)}&=\sum_{(A)_{i,i},(C)^{t-1}_{i,j},(D)^g_{i,i}}(-1)^{\beta_1+\gamma_{c,d}}\delta_{a_1,-a}\delta_{d,a_2}\delta_{b_2,-b_1}f_{c,b}[-1]\\
&\quad+\sum_{(A)_{i,i},(C)^{t-1}_{i,j},(D)^g_{i,i}}(-1)^{\beta_1+\gamma_{c,d}}\delta_{a_1,-a}\delta_{d,a_2}\delta_{b,c}f_{-b_1,b_2}[-1]. \qquad (B.17)
\end{aligned}$$

Since

$$\begin{gathered}
\text{(B.13)}=-\text{(B.16)},\quad \text{the first term of (B.15)}=-\text{the first term of (B.17)},\\
\text{the second term of (B.15)}=-\text{the second term of (B.17)}
\end{gathered}$$

hold, we obtain

$$\text{the first term of (B.10)}=0 \qquad (B.18)$$

by adding (B.13)-(B.17).

Next, let us compute the second term of (B.10). By a direct computation, we also obtain

$$\begin{aligned}
&((Z_{i,i})_{(0)})^{s-t}((Z_{i,i})_{(1)})B_t\\
&=\sum_{(A)_{i,i},(C)^{t-1}_{i,j},(D)^1_{i,i}}(-1)^{\beta_1+\gamma}\delta_{b_2,a}[[f_{c,d},f_{a_1,b_1}],f_{a_2,b}][-1]\\
&\quad+\sum_{(A)_{i,i},(C)^{t-1}_{i,j},(D)^1_{i,i}}(-1)^{\beta_1+\gamma_{c,d}}\delta_{b_2,-b}[[f_{c,d},f_{a_1,b_1}],f_{a,-a_2}][-1]
\end{aligned}$$

$$
\begin{aligned}
&+ \sum_{(A)_{i,i},(C)^{t-1}_{i,j},(D)^1_{i,i}} (-1)^{\beta_1+\gamma_{c,d}} \delta_{b_1,a}[[f_{c,d}, f_{a_2,b_2}], f_{a_1,b}][-1] \\
&- \sum_{(A)_{i,i},(C)^{t-1}_{i,j},(D)^1_{i,i}} (-1)^{\beta_1+\gamma_{c,d}} \delta_{a_1,b}[[f_{c,d}, f_{a_2,b_2}], f_{a,b_1}][-1] \\
&- \sum_{(A)_{i,i},(C)^{t-1}_{i,j},(D)^1_{i,i}} (-1)^{\beta_1+\gamma_{c,d}} \delta_{a_1,-a}[[f_{c,d}, f_{a_2,b_2}], f_{-b_1,b}][-1].
\end{aligned} \tag{B.19}
$$

Let us compute each terms of (B.19). By a direct computation, we obtain

$$\text{the first term of (B.19)} = 0, \tag{B.20}$$

$$\text{the second term of (B.19)} = \sum_{(A)_{i,i},(C)^{t-1}_{i,j},(D)^1_{i,i}} (-1)^{\beta_1+\gamma_{c,d}} \delta_{b_2,-b}\delta_{b_1,c}\delta_{d,a} f_{a_1,-a_2}[-1], \tag{B.21}$$

$$
\begin{aligned}
\text{the third term of (B.19)} &= \sum_{(A)_{i,i},(C)^{t-1}_{i,j},(D)^1_{i,i}} (-1)^{\beta_1+\gamma_{c,d}} \delta_{b_1,a}\delta_{b_2,c}\delta_{d,a_1} f_{a_2,b}[-1] \\
&\quad - \sum_{(A)_{i,i},(C)^{t-1}_{i,j},(D)^1_{i,i}} (-1)^{\beta_1+\gamma_{c,d}} \delta_{d,-b_2}\delta_{-c,a_1} f_{a_2,b}[-1],
\end{aligned} \tag{B.22}
$$

$$\text{the 4-th term of (B.19)} = 0, \tag{B.23}$$

$$
\begin{aligned}
\text{the 5-th term of (B.19)} &= \sum_{(A)_{i,i},(C)^{t-1}_{i,j},(D)^1_{i,i}} (-1)^{\beta_1+\gamma_{c,d}} \delta_{a_1,-a}\delta_{d,a_2}\delta_{b_2,-b_1} f_{c,b}[-1] \\
&\quad - \sum_{(A)_{i,i},(C)^{t-1}_{i,j},(D)^2_{i,i}} (-1)^{\beta_1+\gamma} \delta_{d,-b_2}\delta_{c,b_1} f_{a_2,b}[-1].
\end{aligned} \tag{B.24}
$$

Since

$$
\begin{gathered}
\text{(B.21)} = -\text{the first term of (B.24)}, \quad \text{(B.22)}_2 = -\text{the second term of (B.24)}, \\
\text{(B.22)}_1 = \sum_{\substack{\text{row}(a)=i,\text{row}(b)=j, \\ \text{col}(a)=\text{col}(b)+s}} (-1)^{\gamma_{a,b}} f_{a,b}[-1]
\end{gathered}
$$

hold, we obtain

$$\text{the second term of (B.10)} = \sum_{\substack{\text{row}(a)=i,\text{row}(b)=j, \\ \text{col}(a)=\text{col}(b)+s}} (-1)^{\gamma_{a,b}} f_{a,b}[-1]. \tag{B.25}$$

by adding (B.20)-(B.24). Adding (B.11), (B.18) and (B.25), we obtain

$$
\begin{aligned}
&\text{the third term of (B.4)} \\
&= -s \sum_{\substack{\text{row}(a)=i,\text{row}(b)=i, \\ \text{col}(a)=\text{col}(b)+2s}} (-1)^{\gamma_{a,b}} f_{a,b}[-1] + s \sum_{\substack{\text{row}(a)=j,\text{row}(b)=j, \\ \text{col}(a)=\text{col}(b)+2s}} (-1)^{\gamma_{a,b}} f_{a,b}[-1] \\
&\quad - \alpha \sum_{\substack{\text{row}(a)=i,\text{row}(b)=i, \\ \text{col}(a)=\text{col}(b)+2s}} (-1)^{\gamma_{a,b}} (s-1) f_{a,b}[-1]
\end{aligned} \tag{B.26}
$$

by (B.2). Adding (B.6), (B.8) and (B.26), we obtain the proof. □

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