Transversal family of non-autonomous conformal iterated function systems and the connectedness locus in the parameter space
(非自励的等角反復関数系の横断的族と連結性パラメータ集合)

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Abstract. We study Non-autonomous Iterated Function Systems (NIFSs) with overlaps. An NIFS $\Phi = (\{\phi_i^{(j)}\}_{i \in I^{(j)}})_{j=1}^{\infty}$ on a compact subset $X \subset \mathbb{R}^m$ consists of a sequence of finite collections of uniformly contracting maps $\phi_i^{(j)} : X \to X$, where $I^{(j)}$ is a finite set. The system Φ is an Iterated Function System (IFS) if the collections $\{\phi_i^{(j)}\}_{i \in I^{(j)}}$ are independent of j. In comparison to usual IFSs, we allow the contractions $\phi_i^{(j)}$ applied at each step j to vary as j changes.

In Chapter 1, we give an overview of the theory of IFS. In particular, we study the method of transversality and the connectedness locus for some parameterized IFSs. The method of transversality is utilized for parameterized IFSs involving some complicated overlaps. The connectedness locus arises naturally in the study of IFSs with overlaps. Finally, we give the main results in this dissertation.

In Chapter 2, we introduce transversal families of non-autonomous conformal iterated function systems on $\mathbb{R}^m$. Here, we do not assume the open set condition. We show that if a $d-$parameter family of such systems satisfies the transversality condition, then for almost every parameter value the Hausdorff dimension of the limit set is the minimum of m and the Bowen dimension. Moreover, we give an example of a family $\{\Phi_t\}_{t \in U}$ of parameterized NIFSs such that $\{\Phi_t\}_{t \in U}$ satisfies the transversality condition but Φ_t does not satisfy the open set condition for any $t \in U$.

In Chapter 3, we consider some parameterized planar sets with unbounded digits. These sets are related to some variations of NIFSs. However, we cannot apply the theory given in the previous chapter to these sets. We investigate these sets by approximating the region of transversality. We calculate the Hausdorff dimension of these sets for typical parameters in some region with respect to the 2-dimensional Lebesgue measure. In addition, we estimate the local dimension of the exceptional set of parameters.

In Chapter 4, we consider the connectedness locus $\mathcal{M}_n$ for fractal n-gons in the parameter space. The set of zeros of some families of power series is strongly related to the connectedness locus for parameterized IFSs. We prove that the sets of zeros in the unit disk are connected under some conditions. Furthermore, we apply this result to the study of the connectedness locus $\mathcal{M}_n$. We prove that for any n, $\mathcal{M}_n$ is connected.

Contents

CHAPTER 1

Introduction

1.1. Background

An Iterated Function System (IFS) $\{\phi_1, ..., \phi_n\}$ on a compact subset $X \subset \mathbb{R}^m$ consists of a collection of uniformly contracting maps $\phi_i : X \to X$. It is well-known that there uniquely exists a non-empty compact subset $A \subset X$ such that

$$A = \bigcup_{i=1}^{n} \phi_i(A),$$

called the limit set of the IFS ([**15**]). In order to analyze the fine-scale structure of the limit set, it is important to estimate the dimension of the set. If the conformal IFS satisfies some separating condition, the Hausdorff dimension of the limit set is the zero of the pressure function corresponding to the IFS (see e.g., [**10**], [**21**]). However, in general, it is difficult to estimate the Hausdorff dimension of the limit set in the overlapping case. The method of transversality is utilized for parameterized IFSs involving some complicated overlaps (see e.g., [**28**], [**32**], [**30**], [**17**], [**18**]). In particular, the application to Bernoulli convolutions is one of the recent developments in the theory of IFSs (see e.g., [**31**], [**27**]).

We now give an overview of the theory of IFSs as follows. We consider the following planar sets $A_2(\lambda)$ for $\lambda \in \mathbb{D}^*$, where $\mathbb{D}^* := \{\lambda \in \mathbb{C} \ : \ 0 < |\lambda| < 1\}$.

$$A_2(\lambda) := \left\{ \sum_{j=0}^{\infty} a_j \lambda^j \ : \ a_j \in \{0, 1\} \right\}.$$

These sets have fractal structure. Indeed, the sets $A_2(\lambda)$ are the limit sets of the iterated function systems $\{z \mapsto \lambda z, z \mapsto \lambda z + 1\}$ on the complex plane. In order to discuss these sets, we introduce a set $\mathcal{F}$ of functions and the set $\mathcal{M}_2$ of zeros in $\mathbb{D}^*$ of functions which belong to $\mathcal{F}$ as follows.

$$\mathcal{F} := \left\{ f(\lambda) = 1 + \sum_{j=1}^{\infty} a_j \lambda^j : a_j \in \{-1, 0, 1\} \right\},$$
$$\mathcal{M}_2 := \{\lambda \in \mathbb{D}^* \ : \text{there exists } f \in \mathcal{F} \text{ such that } f(\lambda) = 0\}.$$

Note that

$$\left\{ \lambda \in \mathbb{D}^* \ : \ \frac{1}{\sqrt{2}} < |\lambda| < 1 \right\} \subset \mathcal{M}_2 \subset \left\{ \lambda \in \mathbb{D}^* \ : \ \frac{1}{2} < |\lambda| < 1 \right\} \qquad (1)$$

(see [**32**, p.538 (6)]).

We set $f_1(z) = \lambda z$ and $f_2(z) = \lambda z + 1$. We say that the IFS $\{f_1, f_2\}$ satisfies *the open set condition* if there exists a non-empty bounded open set V such that $f_1(V) \cap f_2(V) = \emptyset$ and $f_i(V) \subset V$ for all $i \in \{1, 2\}$. If λ is not an element of $\mathcal{M}_2$, the corresponding IFS satisfies the open set condition, and hence we have that the Hausdorff dimension of $A_2(\lambda)$ is equal to $-\log 2/\log|\lambda|$ (see [**11**, Theorem 9.3]). However, in general, it is difficult to estimate the Hausdorff dimension of $A_2(\lambda)$ if λ is an element of $\mathcal{M}_2$. We set

$$\tilde{\mathcal{M}_2} := \{\lambda \in \mathbb{D}^* \ : \text{there exists } f \in \mathcal{F} \text{ such that } f(\lambda) = f'(\lambda) = 0\} (\subset \mathcal{M}_2).$$

For any set $A \subset \mathbb{C}$, we denote by $\dim_H(A)$ the Hausdorff dimension of A with respect to the Euclidean metric. We denote by $\mathcal{L}_2$ the 2-dimensional Lebesgue measure. Solomyak and Xu ([**31**, Theorem 2.2] and [**35**, Proposition 2.7]) proved the following theorem by using the method of transversality.

THEOREM 1.1.1.

$$\dim_H(A_2(\lambda)) = \frac{\log 2}{-\log|\lambda|} \text{ for } \mathcal{L}_2\text{-a.e. } \lambda \in \{\lambda \in \mathbb{D}^* \ : \ 0 < |\lambda| < 1/\sqrt{2}\}; \tag{2}$$

$$\mathcal{L}_2(A_2(\lambda)) > 0 \text{ for } \mathcal{L}_2\text{-a.e. } \lambda \in \{\lambda \in \mathbb{D}^* \ : \ 1/\sqrt{2} < |\lambda| < 1\} \backslash \tilde{\mathcal{M}_2}. \tag{3}$$

The local dimension of the exceptional set of parameters is estimated as the following.

THEOREM 1.1.2. [**26**, Theorem 8.2] *For any* $0 < r < R < 1/\sqrt{2}$,

$$\dim_H\left(\left\{\lambda \in \mathbb{D}^* \ : \ r < |\lambda| < R, \ \dim_H(A_2(\lambda)) < \frac{\log 2}{-\log|\lambda|}\right\}\right) \leq \frac{\log 2}{-\log R} < 2.$$

We now consider the topological property of $\mathcal{M}_2$. The set $\mathcal{M}_2$ is known as "*the Mandelbrot set for pairs of linear maps*". In 1985, Barnsley and Harrington ([**3**]) defined $\mathcal{M}_2$ as the connectedness locus for the pair of linear maps, that is,

$$\mathcal{M}_2 = \{\lambda \in \mathbb{D}^* \ : \ A_2(\lambda) \text{ is connected}\}.$$

The set $\mathcal{M}_2$ looks like a "ring" around the set of parameters λ for which $A_2(\lambda)$ is a Cantor set and has "whiskers" (see Figure 1). In fact, Barnsley and Harrington ([**3**]) proved that there is a neighborhood U of the set $\{0.5, -0.5\}$ such that $U \cap \mathcal{M}_2 \subset \mathbb{R}$. Furthermore, they conjectured that there is a non-trivial hole in $\mathcal{M}_2$.

Bousch ([**5**], [**6**]) proved that $\mathcal{M}_2$ is connected and locally connected. This is interesting since for the case of quadratic maps, the local connectedness of the Mandelbrot set is still an open problem. In [**5**] and [**6**], Bousch showed that $\mathcal{M}_2$ is equal to the set of zeros of power series with coefficients $0, 1$, and -1. He also studied the set of zeros of power series with coefficients 1 and -1, which is a subset of $\mathcal{M}_2$. At the same time, Odlyzko and Poonen ([**25**]) studied the set of zeros of power series with coefficients 1 and 0, and they proved the set of zeros is path-connected.

In 2002, Bandt ([**1**]) gave an algorithm to study geometric structure of $\mathcal{M}_2$, and managed to prove the existence of a non-trivial hole in $\mathcal{M}_2$ rigorously. Thus he positively answered the conjecture of Barnsley and Harrington ([**3**]). He also conjectured that the interior of $\mathcal{M}_2$ is dense away from $\mathcal{M}_2 \cap \mathbb{R}$, that is, $\mathrm{cl}\big(\mathrm{int}(\mathcal{M}_2)\big) \cup (\mathcal{M}_2 \cap \mathbb{R}) = \mathcal{M}_2$. Here, for a set $A \subset \mathbb{C}$, we denote by $\mathrm{cl}(A)$ and $\mathrm{int}(A)$ the closure of A and the interior of A with respect to the Euclidean topology on $\mathbb{C}$ respectively. Several researchers made partial progress on Bandt's conjecture (see [**31**], [**32**] and [**35**]).

In 2008, Bandt and Hung ([**2**]) introduced self-similar sets parameterized by $\lambda \in \mathbb{D}^*$ which are called "*fractal n-gons*", where $n \in \mathbb{N}$ with $n \geq 2$. We give the rigorous definition of "*fractal n-gons*" in the next sub-section (see Definition 1.3.6). They studied the connectedness locus $\mathcal{M}_n$ for "*fractal n-gons*", that is,

$$\mathcal{M}_n = \{\lambda \in \mathbb{D}^* \ : \ A_n(\lambda) \text{ is connected}\},$$

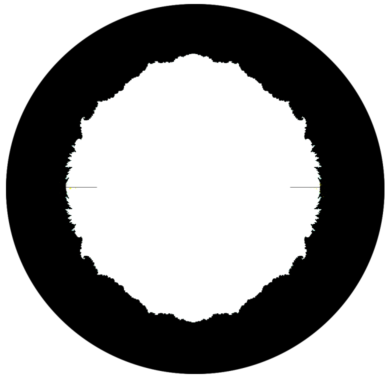

FIGURE 1. $\mathcal{M}_2$

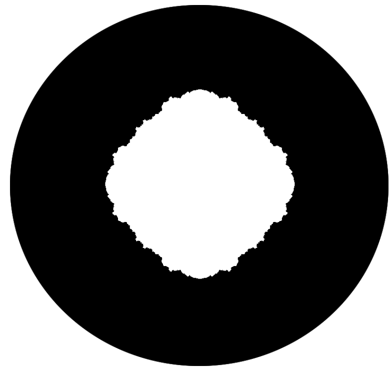

FIGURE 2. $\mathcal{M}_4$

where $A_n(\lambda)$ is the "*fractal n-gon*" corresponding to the parameter λ (see Figure 2). Note that "*fractal* 2-*gons*" are the limit sets of the iterated function systems $\{z \mapsto \lambda z, z \mapsto \lambda z + 1\}$ and $\mathcal{M}_2$ is the connectedness locus for "*fractal* 2-*gons*". Bandt and Hung ([**2**]) discovered many remarkable properties about $\mathcal{M}_n$, including the following result. For each $n \geq 3$ with $n \neq 4$, $\mathcal{M}_n$ is regular-closed, that is, $\mathrm{cl}\big(\mathrm{int}(\mathcal{M}_n)\big) = \mathcal{M}_n$.

In 2016, Calegari, Koch and Walker ([**9**]) introduced new methods for constructing interior points and positively answered Bandt's conjecture, that is, $\mathrm{cl}\big(\mathrm{int}(\mathcal{M}_2)\big) \cup (\mathcal{M}_2 \cap \mathbb{R}) = \mathcal{M}_2$. Himeki and Ishii [**13**] proved $\mathcal{M}_4$ is regular-closed. Thus the problems about the regular-closedness of $\mathcal{M}_n$ have been completely solved. Furthermore, Calegari and Walker ([**8**]) characterized the extreme points in "*fractal n-gons*" and gave an alternative proof of [**13**, Proposition 2.1], which we need to prove the regular-closedness of $\mathcal{M}_4$.

We now consider the connectedness of $\mathcal{M}_n$. Since Bandt and Hung did not study the connectedness of $\mathcal{M}_n$ (see [**2**, page. 2665]), this problem still seems to remain unsolved. However, the connectedness of $\mathcal{M}_n$ appears already in the thesis of Bousch ([**7**]). In fact, Bousch considered some parameterized iterated function systems which consist of contracting holomorphic functions on $\mathbb{C}^d$. He showed that the connectedness locus for the parameterized iterated function systems has no compact connected component under some mild conditions ([**7**, page.37 Théorème 3]). As an application of this result, he showed that $\mathcal{M}_n$ has no compact connected component ([**7**, page.42 Théorème]). This implies that $\mathcal{M}_n$ is connected since $\big\{z \in \mathbb{C} \ : \ 1/\sqrt{n} < |z| < 1\big\} \subset \mathcal{M}_n (\subset \mathbb{D}^{\times})$ (see [**2**, Proposition 3]).

In this dissertation, we approach the connectedness of $\mathcal{M}_n$ from a different aspect. Indeed, we study the connectedness of the sets of zeros of some families of power series by extending the methods of Bousch ([**5**]) and by giving a new framework (see Definition 1.3.7, Definition 1.3.8, and Main Theorem F). Furthermore, we apply this result to the study of the connectedness of $\mathcal{M}_n$ by using a characterization of $\mathcal{M}_n$ due to Bandt and Hung [**2**, remark 3](see Main Theorem E). On the other hand, Bousch ([**7**]) considered some parameterized graphs $\mathcal{G}_n(\lambda)$ associated with the parameterized iterated function systems generating the "*fractal n-gons*" $A_n(\lambda)$. Moreover, he showed that $\mathcal{M}_n = \{\lambda \in \mathbb{D}^* \ : \ \mathcal{G}_n(\lambda)$ is connected (in the sense of the graph theory)$\}$. Hence our approach in this dissertation is defferent from the approach used in [**7**]. In particular, we give some sufficient condition for the connectedness of the sets of zeros of some families of power series (see Main Theorem F). Finally we comment that this study is strongly motivated by the master thesis of Himeki [**12**].

It is natural to consider a non-autonomous version of the IFS as an application for various problems. A Non-autonomous Iterated Function System (NIFS) $\Phi = (\{\phi_i^{(j)}\}_{i\in I^{(j)}})_{j=1}^{\infty}$ on a compact subset $X \subset \mathbb{R}^m$ consists of a sequence of finite collections of uniformly contracting maps $\phi_i^{(j)} : X \to X$, where $I^{(j)}$ is a finite set. The system Φ is an Iterated Function System (for short, IFS) if the collections $\{\phi_i^{(j)}\}_{i\in I^{(j)}}$ are independent of j. In comparison to usual IFSs, we allow the contractions $\phi_i^{(j)}$ applied at each step j to vary as j changes. Rempe-Gillen and Urbański [**29**] introduced Non-autonomous Conformal Iterated Function Systems (NCIFSs). An NCIFS $\Phi = (\{\phi_i^{(j)}\}_{i\in I^{(j)}})_{j=1}^{\infty}$ on a compact subset $X \subset \mathbb{R}^m$ consists of a sequence of collections of uniformly contracting conformal maps $\phi_i^{(j)} : X \to X$ satisfying some mild conditions containing the Open Set Condition (OSC) which is defined as follows. We say that a sequence of finite collections of maps $(\{\phi_i^{(j)}\}_{i\in I^{(j)}})_{j=1}^{\infty}$ on a compact subset X with $\mathrm{int}(X) \neq \emptyset$ satisfies the OSC if for all $j \in \mathbb{N}$ and all distinct indices $a, b \in I^{(j)}$,

$$\phi_a^{(j)}(\mathrm{int}(X)) \cap \phi_b^{(j)}(\mathrm{int}(X)) = \emptyset. \tag{4}$$

Then the limit set of the NCIFS $\Phi = (\{\phi_i^{(j)}\}_{i\in I^{(j)}})_{j=1}^{\infty}$ is defined as the set of possible limit points of sequences $\phi_{\omega_1}^{(1)}(\phi_{\omega_2}^{(2)}...(\phi_{\omega_i}^{(i)}(x))...))$, $\omega_j \in I^{(j)}$ for all $j \in \{1, 2, ..., i\}$, $x \in X$. Rempe-Gillen and Urbański introduced *the lower pressure function* $\underline{P}_\Phi : [0, \infty) \to [-\infty, \infty]$ of the NCIFS Φ. Then *the Bowen dimension* s_Φ of the NCIFS Φ is defined by $s_\Phi = \sup\{s \geq 0 \ : \ \underline{P}_\Phi(s) > 0\} = \inf\{s \geq 0 \ : \ \underline{P}_\Phi(s) < 0\}$. Rempe-Gillen and Urbański proved that the Hausdorff dimension of the limit set is the Bowen dimension of the NCIFS ([**29**, 1.1 Theorem]). For related results for non-autonomous systems, see [**14**].

1.2. Notations and conventions

For the reader's convenience, we summarize our main notations and conventions as follows.

- $\mathbb{N} := \{1, 2, 3, ...\}$.
- $\mathbb{N}_0 := \{0, 1, 2, ...\}$.
- $\mathbb{R}$: the set of all real numbers.
- $\mathbb{C}$: the set of all complex numbers.
- For any $x \in \mathbb{R}^m$, we denote by $|x|$ the Euclidean norm of x.
- Usually, we identify $\mathbb{C}$ with $\mathbb{R}^2$. For $\lambda \in \mathbb{C}$, we denote by $|\lambda|$ the Euclidean norm of $\lambda \in \mathbb{R}^2$.
- $\mathbb{D} := \{\lambda \in \mathbb{C} \ : \ |\lambda| < 1\}$.
- $\mathbb{D}^* := \{\lambda \in \mathbb{C} \ : \ 0 < |\lambda| < 1\}$.
- For any $x \in \mathbb{R}^m$ and for any $a > 0$, we set $B(x, a) := \{y \in \mathbb{R}^m \ : \ |x - y| < a\}$.
- For any set $A \subset \mathbb{R}^m$, we denote by $\dim_H(A)$ the Hausdorff dimension of A with respect to the Euclidean metric.
- $\mathcal{L}_m$: the m-dimensional Lebesgue measure on $\mathbb{R}^m$.
- For any set $A \subset \mathbb{R}^m$, we denote by $\mathrm{cl}(A)$ and $\mathrm{int}(A)$ the closure of A and the interior of A with respect to the Euclidean topology on $\mathbb{R}^m$ respectively.
- For each $j \in \mathbb{N}_0$, let $G_j \subset \mathbb{R}$. Let $\lambda \in \mathbb{D}^*$. We use $\left\{\sum_{j=0}^{\infty} a_j\lambda^j \ : \ a_j \in G_j\right\}$ to denote $\left\{\sum_{j=0}^{\infty} a_j\lambda^j \ : \ \text{for each } j \in \mathbb{N}_0, a_j \in G_j\right\}$.

- If X and Y are topological spaces, and if $f : X \to Y$ is any Borel measurable map, then for any Borel measure μ on X, we define $f\mu$ as the push-forward measure $\mu \circ f^{-1}$.
- Let X be a topological space, let X_0 be a Borel measurable subspace of X and let m be a Borel measure on X_0. If we set $\tilde{m}(B) := m(B \cap X_0)$ for any Borel subset $B \subset X$, then $\tilde{m}$ is a Borel measure on X. We also denote by m the measure $\tilde{m}$.

1.3. Main results

In this section we present the main results of this dissertation.

1.3.1. Transversal family of non-autonomous conformal iterated function systems. In this subsection we present the framework of transversal families of non-autonomous conformal iterated function systems and we present the main results on them. For each $j \in \mathbb{N}$, let $I^{(j)}$ be a finite set. For any $n, k \in \mathbb{N}$ with $n \leq k$, we set

$$I_n^k := \prod_{j=n}^{k} I^{(j)}, I_n^\infty := \prod_{j=n}^{\infty} I^{(j)}, I^n := \prod_{j=1}^{n} I^{(j)}, \text{and } I^\infty := \prod_{j=1}^{\infty} I^{(j)}.$$

Let $U \subset \mathbb{R}^d$. For any $t \in U$, let $\Phi_t = (\Phi_t^{(j)})_{j=1}^\infty$ be a sequence of collections of maps on a set $X \subset \mathbb{R}^m$, where

$$\Phi_t^{(j)} = \{\phi_{i,t}^{(j)} : X \to X\}_{i \in I^{(j)}}.$$

Let $n, k \in \mathbb{N}$ with $n \leq k$. For any $\omega = \omega_n \omega_{n+1} \cdots \omega_k \in I_n^k$, we set

$$\phi_{\omega,t} := \phi_{\omega_n,t}^{(n)} \circ \cdots \circ \phi_{\omega_k,t}^{(k)}.$$

Let $n \in \mathbb{N}$. For any $\omega = \omega_n \omega_{n+1} \cdots \in I_n^\infty$ and any $j \in \mathbb{N}$, we set

$$\omega|_j := \omega_n \omega_{n+1} \cdots \omega_{n+j-1} \in I_n^{n+j-1}.$$

Let $V \subset \mathbb{R}^m$ be an open set and let $\phi : V \to \phi(V)$ be a diffeomorphism. We denote by $D\phi(x)$ the derivative of ϕ evaluated at x. We say that ϕ is *conformal* if for any $x \in V$ $D\phi(x) : \mathbb{R}^m \to \mathbb{R}^m$ is a similarity linear map, that is, $D\phi(x) = c_x \cdot A_x$, where $c_x > 0$ and A_x is an orthogonal matrix. For any conformal map $\phi : V \to \phi(V)$, we denote by $|D\phi(x)|$ its scaling factor at x, that is, if we set $D\phi(x) = c_x \cdot A_x$ we have $|D\phi(x)| = c_x$. For any set $A \subset V$, we set

$$||D\phi||_A := \sup\{|D\phi(x)| \ : \ x \in A\}.$$

We denote by $\mathcal{L}_d$ the d-dimensional Lebesgue measure on $\mathbb{R}^d$. We introduce *the transversal family of non-autonomous conformal iterated function systems* by employing the settings in [**29**] and [**30**].

Definition 1.3.1 (Transversal family of non-autonomous conformal iterated function systems). Let $m \in \mathbb{N}$ and let $X \subset \mathbb{R}^m$ be a non-empty compact convex set. Let $d \in \mathbb{N}$ and let $U \subset \mathbb{R}^d$ be an open set. For each $j \in \mathbb{N}$, let $I^{(j)}$ be a finite set. Let $t \in U$. For any $j \in \mathbb{N}$, let $\Phi_t^{(j)}$ be a collection $\{\phi_{i,t}^{(j)} : X \to X\}_{i \in I^{(j)}}$ of maps $\phi_{i,t}^{(j)}$ on X. Let $\Phi_t = (\Phi_t^{(j)})_{j=1}^\infty$. We say that $\{\Phi_t\}_{t \in U}$ is a Transversal family of Non-autonomous Conformal Iterated Function Systems (TNCIFS) if $\{\Phi_t\}_{t \in U}$ satisfies the following six conditions.

1. *Conformality* : There exists an open connected set $V \supset X$(independent of i, j and t) such that for any i, j and $t \in U$, $\phi_{i,t}^{(j)}$ extends to a C^1 conformal map on V such that $\phi_{i,t}^{(j)}(V) \subset V$.

2. *Uniform contraction* : There is a constant $0 < \gamma < 1$ such that for any $t \in U$, any $n \in \mathbb{N}$, any $\omega \in I_n^\infty$ and any $j \in \mathbb{N}$,
$$|D\phi_{\omega|_j,t}(x)| \leq \gamma^j$$
for any $x \in X$.
3. *Bounded distortion* : There exists a Borel measurable locally bounded function $K : U \to [1, \infty)$ such that for any $t \in U$, any $n \in \mathbb{N}$, any $\omega \in I_n^\infty$ and any $j \in \mathbb{N}$,
$$|D\phi_{\omega|_j,t}(x_1)| \leq K(t)|D\phi_{\omega|_j,t}(x_2)| \tag{5}$$
for any $x_1, x_2 \in V$.
4. *Distortion continuity* : For any $\eta > 0$ and $t_0 \in U$, there exists $\delta = \delta(\eta, t_0) > 0$ such that for any $t \in U$ with $|t - t_0| \leq \delta$, for any $n, j \in \mathbb{N}$ and for any $\omega \in I_n^\infty$,
$$\exp(-j\eta) \leq \frac{||D\phi_{\omega|_j,t_0}||_X}{||D\phi_{\omega|_j,t}||_X} \leq \exp(j\eta). \tag{6}$$
We define the *address map* as follows. Let $t \in U$. For all $n \in \mathbb{N}$ and all $\omega \in I_n^\infty$,
$$\bigcap_{j=1}^{\infty} \phi_{\omega|_j,t}(X)$$
is a singleton by the uniform contraction property. It is denoted by $\{y_{\omega,n,t}\}$. The map
$$\pi_{n,t} \colon I_n^\infty \to X$$
is defined by $\omega \mapsto y_{\omega,n,t}$. Then $\pi_{n,t}$ is called *the n-th address map corresponding to t*. Note that for any $t \in U$ and $n \in \mathbb{N}$ the map $\pi_{n,t}$ is continuous with respect to the product topology on I_n^∞.
5. *Continuity* : Let $n \in \mathbb{N}$. The function $I_n^\infty \times U \ni (\omega, t) \mapsto \pi_{n,t}(\omega)$ is continuous.
6. *Transversality condition* : For any compact subset $G \subset U$ there exists a sequence of positive constants $\{C_n\}_{n=1}^\infty$ with
$$\lim_{n\to\infty} \frac{\log C_n}{n} = 0$$
such that for all $\omega, \tau \in I_n^\infty$ with $\omega_n \neq \tau_n$ and for all $r > 0$,
$$\mathcal{L}_d\left(\{t \in G \ : \ |\pi_{n,t}(\omega) - \pi_{n,t}(\tau)| \leq r\}\right) \leq C_n r^m.$$

Remark 1.3.2. If $m \geq 2$, the Conformality condition implies the Bounded distortion condition. For the details, see [**29**, page. 1984 Remark].

Remark 1.3.3. Let $n \in \mathbb{N}$ and let $t \in U$. Then for any $\omega \in I_n^\infty$,
$$\pi_{n,t}(\omega) = \lim_{j\to\infty} \phi_{\omega|_j,t}(x),$$
where $x \in X$.

Remark 1.3.4. In the case of usual IFSs, the constants C_n in the transversality condition are independent of n since the n-th address maps $\pi_{n,t}$ are independent of n.

Let $\{\Phi_t\}_{t\in U}$ be a TNCIFS. For any $n \in \mathbb{N}$ and $t \in U$, *the n-th limit set $J_{n,t}$* of Φ_t is defined by

$$J_{n,t} := \pi_{n,t}(I_n^\infty).$$

For any $t \in U$, we define the lower pressure function $\underline{P}_t : [0,\infty) \to [-\infty,\infty]$ of Φ_t as the following. For any $s \geq 0$ and $n \in \mathbb{N}$, we set

$$Z_{n,t}(s) := \sum_{\omega \in I^n} (||D\phi_{\omega,t}||_X)^s,$$

and

$$\underline{P}_t(s) := \liminf_{n\to\infty} \frac{1}{n} \log Z_{n,t}(s) \in [-\infty,\infty].$$

By [**29**, Lemma 2.6], the lower pressure function has the following monotonicity. If $s_1 < s_2$, then either both $\underline{P}_t(s_1)$ and $\underline{P}_t(s_2)$ are equal to ∞, both are equal to $-\infty$, or $\underline{P}_t(s_1) > \underline{P}_t(s_2)$. Then for any $t \in U$, we set

$$s(t) := \sup\{s \geq 0 \;:\; \underline{P}_t(s) > 0\} = \inf\{s \geq 0 \;:\; \underline{P}_t(s) < 0\},$$

where we set $\sup \emptyset = 0$ and $\inf \emptyset = \infty$. The value $s(t)$ is called the Bowen dimension of Φ_t. We set $J_t := J_{1,t}$ for any $t \in U$. We now present one of the main results of this dissertation.

Main Theorem A (Theorem 2.1.8). *Let $\{\Phi_t\}_{t\in U}$ be a TNCIFS. Suppose that the function $t \mapsto s(t)$ is a real-valued and continuous function on U. Then*

$$\dim_H(J_t) = \min\{m, s(t)\}$$

for $\mathcal{L}_d$-a.e. $t \in U$.

Main Theorem A is a generalization of [**30**, Theorem 3.1 (i)]. We illustrate Main Theorem A by presenting the following important example. We set

$$X = \left\{z \in \mathbb{C} \;:\; |z| \leq \frac{1}{1 - 2 \times 5^{-5/8}}\right\}, U = \left\{t \in \mathbb{C} \;:\; |t| < 2 \times 5^{-5/8},\; t \notin \mathbb{R}\right\}.$$

Let $t \in U$. For each $j \in \mathbb{N}$, we define the maps $\phi_{1,t}^{(j)} : X \to X$ and $\phi_{2,t}^{(j)} : X \to X$ by $\phi_{1,t}^{(j)}(z) = tz$ and $\phi_{2,t}^{(j)}(z) = tz + 1/j$ respectively. For each $j \in \mathbb{N}$, we set

$$\Phi_t^{(j)} = \{\phi_{1,t}^{(j)}, \phi_{2,t}^{(j)}\} = \left\{z \mapsto tz, z \mapsto tz + \frac{1}{j}\right\}$$

and $\Phi_t = (\Phi_t^{(j)})_{j=1}^\infty$. We now present the following theorem, which is the second main result of this dissertation.

Main Theorem B (Proposition 2.2.2 and Proposition 2.2.5). *The family $\{\Phi_t\}_{t\in U}$ of parameterized systems is a TNCIFS but Φ_t does not satisfy the open set condition* (4) *for any $t \in U$.*

Since Φ_t does not satisfy the open set condition (4) for any $t \in U$, we cannot apply the framework of Rempe-Gillen and Urbański [**29**] to the study of the limit set J_t of Φ_t. We calculate the lower pressure function $\underline{P}_t$ for Φ_t as the following. For any $s \in [0,\infty)$,

$$\begin{aligned}\underline{P}_t(s) &= \liminf_{n\to\infty} \frac{1}{n} \log \sum_{\omega \in I^n} (\|D\phi_{\omega,t}\|_X)^s \\ &= \liminf_{n\to\infty} \frac{1}{n} \log \sum_{\omega \in I^n} |t|^{ns} \\ &= \liminf_{n\to\infty} \frac{1}{n} \log(2^n |t|^{ns}) \\ &= \log 2 + s \log |t|.\end{aligned}$$

Hence for each $t \in U$, $\underline{P}_t(s)$ has the zero

$$s(t) = \frac{\log 2}{-\log |t|}$$

and the function $t \mapsto s(t)$ is continuous on U. Let J_t be the (first) limit set corresponding to t. Then by Main Theorem A, we have

$$\dim_H(J_t) = \min\{2, s(t)\} = s(t)$$

for a.e. $t \in \{t \in \mathbb{C} \ : \ |t| \le 1/\sqrt{2}, t \notin \mathbb{R}\} (\subset U)$ and

$$\dim_H(J_t) = \min\{2, s(t)\} = 2$$

for a.e. $t \in \{t \in \mathbb{C} \ : \ 1/\sqrt{2} \le |t| < 2 \times 5^{-5/8}, t \notin \mathbb{R}\} (\subset U)$.

1.3.2. The Hausdorff dimension of some planar sets with unbounded digits. In this subsection we consider the following sets $L_0(\lambda)$ for $\lambda \in \mathbb{D}^*$.

$$L_0(\lambda) := \left\{ \sum_{j=0}^{\infty} a_j \lambda^j : a_j \in \{0, p_j\} \right\}, \tag{7}$$

where for all $j \in \mathbb{N}_0$, $1 \le p_j \in \mathbb{R}$, $p_j \to \infty$ as $j \to \infty$ and $\{p_j\}_{j=0}^{\infty}$ satisfies the following condition,

- $\frac{p_{j+1}}{p_j} \to 1$ as $j \to \infty$.

Note that the sets $L_0(\lambda)$ depend on the sequence $\{p_j\}_{j=0}^{\infty}$ and these sets are well-defined by the above condition (see Remark 3.1.1).

For any $\lambda \in \mathbb{D}^*$ and $j \in \mathbb{N}_0$, we define the maps $f_{0,\lambda}^{(j)} : \mathbb{C} \to \mathbb{C}$ and $f_{1,\lambda}^{(j)} : \mathbb{C} \to \mathbb{C}$ by $f_{0,\lambda}^{(j)}(z) = \lambda z$ and $f_{1,\lambda}^{(j)}(z) = \lambda z + p_j$ respectively. We can see the sets $L_0(\lambda)$ are "*the limit sets*" of the NIFSs $(\{f_{0,\lambda}^{(j)}, f_{1,\lambda}^{(j)}\})_{j=0}^{\infty}$ as the following. For any $n \in \mathbb{N}_0$ and $\lambda \in \mathbb{D}^*$, we define the address map $\Pi_{n,\lambda}$ for $(\{f_{0,\lambda}^{(j)}, f_{1,\lambda}^{(j)}\})_{j=0}^{\infty}$. We set $I^\infty := I_n^\infty = \{0,1\}^\infty$ for any $n \in \mathbb{N}_0$.

Definition 1.3.5. For each $\lambda \in \mathbb{D}^*$ and $n \in \mathbb{N}_0$, we define the address map $\Pi_{n,\lambda} : I^\infty \to \mathbb{C}$ by

$$\Pi_{n,\lambda}(\omega) := \lim_{j\to\infty} f_{\omega|_j,\lambda}(0) = \sum_{j=0}^{\infty} p_{n+j}\omega_{n+j}\lambda^j$$

$(\omega = \omega_n \omega_{n+1} \cdots \in I^\infty)$. Note that this map is well-defined.

Then we have that

$$\Pi_{n,\lambda}(I^{\infty}) = \left\{ \sum_{j=0}^{\infty} a_j \lambda^j : a_j \in \{0, p_{n+j}\} \right\}.$$

In particular, $L_0(\lambda) = \Pi_{0,\lambda}(I^{\infty})$. Moreover, Inui [**16**] gave the methods to construct *"the limit sets"* of NIFSs with some mild conditions on complete metric spaces by extending the idea of Hutchinson [**15**]. The set $L_0(\lambda)$ is also the limit set of the NIFS $(\{f_{0,\lambda}^{(j)}, f_{1,\lambda}^{(j)}\})_{j=0}^{\infty}$ in Inui's sense.

Note that there does not exist a compact subset $X \subset \mathbb{C}$ such that for each j, $f_{1,\lambda}^{(j)}(X) \subset X$ since the set of digits $\{p_j \ : \ j \in \mathbb{N}_0\}$ is not bounded. One of the aims in this subsection is to establish some methods to estimate the Hausdorff dimension of the limit sets of NIFSs on non-compact metric spaces via studying examples. We now present the main results in this subsection.

Main Theorem C (Theorem 3.3.11).

$$\dim_H(L_0(\lambda)) = \frac{\log 2}{-\log|\lambda|} \text{ for } \mathcal{L}_2\text{−a.e. } \lambda \in \{\lambda \in \mathbb{D}^* \ : \ 0 < |\lambda| < 1/\sqrt{2}\};$$

$$\mathcal{L}_2(L_0(\lambda)) > 0 \text{ for } \mathcal{L}_2\text{−a.e. } \lambda \in \{\lambda \in \mathbb{D}^* \ : \ 1/\sqrt{2} < |\lambda| < 1\} \backslash \tilde{\mathcal{M}}_2.$$

Main Theorem D (Theorem 3.3.14). *For any* $0 < R < 1/\sqrt{2}$,

$$\dim_H\left(\left\{\lambda \in \mathbb{D}^* \ : \ 0 < |\lambda| < R, \ \dim_H(L_0(\lambda)) < \frac{\log 2}{-\log|\lambda|}\right\}\right) \leq \frac{\log 2}{-\log R} < 2.$$

In order to prove our results, we use the method of transversality. Here, for a parameterized family of functions, the transversality means a condition which controls the way the functions depend on parameters. Usually, we call the set of parameters "*the region of transversality*". The method of transversality is used for self-similar sets with overlaps (e.g., [**28**], [**31**], [**17**], [**18**]), for self-similar measures (e.g., [**31**]) and for some general family of functions (e.g., [**30**], [**20**], [**36**]). Note that their setting depend on the compactness of the whole space. Hence we cannot apply their framework or methods to our setting since the set of digits $\{p_j \ : \ j \in \mathbb{N}_0\}$ is not bounded.

1.3.3. $\mathcal{M}_n$ is connected. In this subsection we consider the connectedness locus $\mathcal{M}_n$ for "*fractal n-gons*" in the parameter space. Below we fix $n \in \mathbb{N}$ with $n \geq 2$. We give the rigorous definition of fractal n-gons as the following.

DEFINITION 1.3.6 (Fractal n-gons). Let $\lambda \in \mathbb{D}^*$. We set $\xi_n = \exp(2\pi\sqrt{-1}/n)$. For each $i \in \{0, 1, ..., n-1\}$, we define $\varphi_i^{n,\lambda} : \mathbb{C} \to \mathbb{C}$ by $\varphi_i^{n,\lambda}(z) = \lambda z + {\xi_n}^i$. Then there uniquely exists a non-empty compact subset $A_n(\lambda)$ such that

$$A_n(\lambda) = \bigcup_{i=0}^{n-1} \varphi_i^{n,\lambda}(A_n(\lambda))$$

(See [**11**], [**15**]). We call $A_n(\lambda)$ a fractal n-gon corresponding to the parameter λ.

For each $n \in \mathbb{N}$ with $n \geq 2$, we define the connectedness locus $\mathcal{M}_n$ for fractal n-gons as the following.

$$\mathcal{M}_n = \{\lambda \in \mathbb{D}^* \ : \ A_n(\lambda) \text{ is connected}\}.$$

We give one of the main results in this subsection as the following.

Main Theorem E (Theorem 4.3.7 and Theorem 4.3.8). *For any $n \in \mathbb{N}$ with $n \geq 2$, $\mathcal{M}_n$ is connected.*

In [**5**], Bousch showed that $\mathcal{M}_2$ is equal to the set of zeros of power series with coefficients $0, 1$, and -1. Similarly, we can identify $\mathcal{M}_n$ with the set of zeros of some power series (see [**2**, Remark 3]). However, in the proof of the connectedness of $\mathcal{M}_n$ for general $n \in \mathbb{N}$ with $n \geq 2$, since the set Ω_n of coefficients of the power series, which corresponds to $\mathcal{M}_n$, is complicated for general $n \in \mathbb{N}$ with $n \geq 2$ (see Definition 4.3.3) in contrast to $\mathcal{M}_2$, we cannot use the methods to prove the connectedness of $\mathcal{M}_2$ and $\mathcal{M}_3$ which are given in [**5**] and [**12**]. Hence we study the connectedness of the sets of zeros of some power series by extending the methods of Bousch ([**5**]) and by using some new ideas and techniques. We need the following setting to prove Main Theorem E, which is one of the new ideas in this dissertation.

Definition 1.3.7. Let G be a subset of $\mathbb{C}$. We say that G satisfies the condition $(*)$ if G satisfies all of the following conditions (i), (ii), and (iii).

(i) $1 \in G$.
(ii) For all $a, b \in G$ with $a \neq b$, there exist $b_1, b_2, ..., b_m \in G$ with $b_1 = a$ and $b_m = b$ such that for all $c \in G$, there exist $d_1, d_2, ..., d_{m-1} \in G$ satisfying that
$$(b_2 - b_1)c + d_1 \in G, (b_3 - b_2)c + d_2 \in G, ..., (b_m - b_{m-1})c + d_{m-1} \in G.$$
(iii) G is compact.

Definition 1.3.8. Let G be a subset of $\mathbb{C}$ with $(*)$. Let $\mathbb{D}$ be the unit disk. We set
$$P^G = \{1 + \sum_{i=1}^{\infty} a_i z^i \ : \ a_i \in G\},$$
$$X^G = \{z \in \mathbb{D} \ : \ \text{there exists } f \in P^G \text{ such that } f(z) = 0\}.$$

Then the following theorem holds, which we need to prove Main Theorem E.

Main Theorem F (Theorem 4.2.3). *Let G be a subset of $\mathbb{C}$ with $(*)$. Suppose that there exists a real number R with $0 < R < 1$ such that $\{z \in \mathbb{C} \ : \ R < |z| < 1\} \subset X^G$. Then X^G is connected.*

1.4. Organization

The dissertation is organized as follows.

In Chapter 2, we study transversal families of non-autonomous conformal iterated function systems on $\mathbb{R}^m$. Section 2.1 is devoted to the proof of one of the main results. As preliminaries for the proof, we give some lemma for conformal maps on $\mathbb{R}^m$ and construct a Gibbs-like measure on the symbolic space. Finally, we give the proof by using the method of transversality. In section 2.2 we give an example of a family $\{\Phi_t\}_{t\in U}$ of parameterized NIFSs such that $\{\Phi_t\}_{t\in U}$ satisfies the transversality condition but Φ_t does not satisfy the open set condition for any $t \in U$. The contents in Chapter 2 are included in [**24**].

In Chapter 3, we study some planar sets with unbounded digits. In Section 3.1, we give the upper estimation of the Hausdorff dimension of $L_0(\lambda)$ for any $\lambda \in \mathbb{D}^*$. In Section 3.2, we give some lemmas in order to estimate the Hausdorff dimension. In addition, we give a technical lemma for the transversality (Lemma 3.2.10). In Section 3.3, we give the key lemmas (Lemmas 3.3.6 and 3.3.7), which imply the lower estimation of the Hausdorff dimension of $L_0(\lambda)$ for typical parameters λ with respect to $\mathcal{L}_2$ and the estimation of local dimension of the exceptional set of parameters. The contents in Chapter 3 are included in [**22**].

In Chapter 4, we study the connectedness of $\mathcal{M}_n$. In Sections 4.1 and 4.2, we prove Main Theorem F by extending the methods of Bousch ([**5**]) and by using some new ideas. If we set $I := \{0, 1, ..., n-1\}$ and $\Omega_n := \{({\xi_n}^j - {\xi_n}^k)/(1-\xi_n) \ : \ j, k \in I\}$, then we have that $\mathcal{M}_n = X^{\Omega_n}$ and $\{z \in \mathbb{C} \ : \ 1/\sqrt{n} < |z| < 1\} \subset \mathcal{M}_n$ (see [**2**, Remark 3] and [**2**, Proposition 3]). It is highly non-trivial that Ω_n satisfies the condition $(*)$ and in order to prove that, we need Lemmas 4.3.1 and 4.3.2, which are the key lemmas to prove Main Theorem F. In Section 4.3, by using Lemmas 4.3.1 and 4.3.2, we prove that Ω_n satisfies the condition $(*)$, and hence we get Main Theorem E as a corollary of Main Theorem F. The contents in Chapter 4 are included in [**23**].

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CHAPTER 2

Transversal family of non-autonomous conformal iterated function systems

In this chapter, we study transversal families of non-autonomous conformal iterated function systems on $\mathbb{R}^m$ and give the proofs of Main Theorems A and B.

2.1. Preliminaries and the proof of Main Theorem A

In this section we give some lemmas for conformal maps on $\mathbb{R}^m$ and give the proof of Main Theorem A. Let $\{\Phi_t\}_{t\in U} = \left\{ \left(\{\phi_{i,t}^{(j)} : X \to X\}_{i\in I^{(j)}} \right)_{j=1}^{\infty} \right\}_{t\in U}$ be a TNCIFS.

2.1.1. Lemma for conformal maps. Let $n,k \in \mathbb{N}$ with $n \le k$. Below, we set $||D\phi_{\omega,t}|| := ||D\phi_{\omega,t}||_X$ for any $\omega \in I_n^k$ and any $t \in U$. We set $I^* := \cup_{n\ge 1} I^n$. This subsection is devoted to the proof of the following lemma.

LEMMA 2.1.1. *There exists $L \ge 1$ such that for any $t \in U$, any $\omega \in I^*$ and any $x, y \in X$,*

$$|\phi_{\omega,t}(x) - \phi_{\omega,t}(y)| \ge L^{-1} K(t)^{-2} ||D\phi_{\omega,t}|| \cdot |x-y|, \tag{8}$$

where $K(t)$ comes from the bounded distortion condition (5).

In the case $X \subset \mathbb{R}^1$, we can show Lemma 2.1.1 by the Mean Value Theorem and the bounded distortion condition. In the case $X \subset \mathbb{R}^m$ for $m \ge 2$, we need some properties of conformal maps on $\mathbb{R}^m$. We prove Lemma 2.1.1 by imitating the argument in [**21**, pages.73-74] as follows. We set $|X| = \sup_{x,y\in X} |x-y| (< \infty)$. For any set $A \subset \mathbb{R}^m$, we denote by ∂A the boundary of A. Let V be an open set with $V \supset X$ in the conformality condition. We set

$$r = \min\left\{ |X|, \frac{\inf\{|x-y| \ : \ x \in X, y \in \partial V\}}{2} \right\}.$$

In order to prove Lemma 2.1.1, we give the following lemma.

LEMMA 2.1.2. *Let $t \in U$. For any $\omega \in I^*$ and $x \in X$,*

$$\phi_{\omega,t}(B(x,r)) \supset B(\phi_{\omega,t}(x), K(t)^{-1}||D\phi_{\omega,t}||r).$$

PROOF. Let $t \in U$. Fix $x \in X$. For any $\omega \in I^*$, we set

$$R_\omega = \sup\{u > 0 \ : \ B(\phi_{\omega,t}(x), u) \subset \phi_{\omega,t}(B(x,r))\}.$$

Then

$$\partial B(\phi_{\omega,t}(x), R_\omega) \cap \partial \phi_{\omega,t}(B(x,r)) \ne \emptyset. \tag{9}$$

Since $B(\phi_{\omega,t}(x), R_\omega) \subset \phi_{\omega,t}(B(x,r)) \subset \phi_{\omega,t}(V)$, by applying the Mean Value Inequality to the map $\phi_{\omega,t}^{-1}$ restricted to the convex set $B(\phi_{\omega,t}(x), R_\omega)$ and using the bounded distortion condition (5), we have

$$\phi_{\omega,t}^{-1}(B(\phi_{\omega,t}(x), R_\omega)) \subset B(x, ||D(\phi_{\omega,t}^{-1})||_{\phi_{\omega,t}(V)} R_\omega) \subset B(x, K(t)||D\phi_{\omega,t}||^{-1} R_\omega).$$

This implies

$$B(\phi_{\omega,t}(x), R_\omega) \subset \phi_{\omega,t}(B(x, K(t)||D\phi_{\omega,t}||^{-1}R_\omega)). \tag{10}$$

By (9) and (10), $K(t)||D\phi_{\omega,t}||^{-1}R_\omega \geq r$. By the definition of R_ω, we have

$$\phi_{\omega,t}(B(x,r)) \supset B(\phi_{\omega,t}(x), K(t)^{-1}||D\phi_{\omega,t}||r).$$

□

We now give a proof of Lemma 2.1.1.

(*proof of Lemma* 2.1.1*)*. Let $t \in U$. Fix $\omega \in I^*$. Take $x, y \in X$.

(Case 1: $|x-y| \leq K(t)^{-1}r$) By applying the Mean Value Inequality to the map $\phi_{\omega,t}$ restricted to the convex set $B(x, K(t)^{-1}r)$ and using the bounded distortion condition (5), we have

$$\phi_{\omega,t}(y) \in B(\phi_{\omega,t}(x), K(t)^{-1}||D\phi_{\omega,t}||r). \tag{11}$$

By Lemma 2.1.2, we have

$$B(\phi_{\omega,t}(x), K(t)^{-1}||D\phi_{\omega,t}||r) \subset \phi_{\omega,t}(B(x,r)) \subset \phi_{\omega,t}(V).$$

By (11) and applying the Mean Value Inequality to the map $\phi_{\omega,t}^{-1}$ restricted to the convex set $B(\phi_{\omega,t}(x), K(t)^{-1}||D\phi_{\omega,t}||r)$, we have

$$\begin{aligned} |x-y| &= |(\phi_{\omega,t})^{-1}(\phi_{\omega,t}(x)) - (\phi_{\omega,t})^{-1}(\phi_{\omega,t}(y))| \\ &\leq ||D(\phi_{\omega,t})^{-1}||_{\phi_{\omega,t}(V)}|\phi_{\omega,t}(x) - \phi_{\omega,t}(y)|. \end{aligned}$$

By using the bounded distortion condition (5), we have

$$|x-y| \leq K(t)||D\phi_{\omega,t}||^{-1} \cdot |\phi_{\omega,t}(x) - \phi_{\omega,t}(y)|. \tag{12}$$

Hence we obtain (8).

(Case 2: $|x-y| > K(t)^{-1}r$) Since $\phi_{\omega,t}(y) \notin \phi_{\omega,t}(B(x, K(t)^{-1}r))$, there exists $z \in \partial B(x, K(t)^{-1}r)$ such that $\phi_{\omega,t}(z)$ belongs to the straight line path from $\phi_{\omega,t}(x)$ to $\phi_{\omega,t}(y)$. Hence

$$|\phi_{\omega,t}(x) - \phi_{\omega,t}(y)| \geq |\phi_{\omega,t}(x) - \phi_{\omega,t}(z)|. \tag{13}$$

Since $|x-z| = K(t)^{-1}r$, by (12) we have

$$|\phi_{\omega,t}(x) - \phi_{\omega,t}(z)| \geq K(t)^{-1}||D\phi_{\omega,t}|| \cdot |x-z| = K(t)^{-1}||D\phi_{\omega,t}||K(t)^{-1}r. \tag{14}$$

By (13) and (14) we have

$$|\phi_{\omega,t}(x) - \phi_{\omega,t}(y)| \geq K(t)^{-1}||D\phi_{\omega,t}||K(t)^{-1}\frac{|x-y|r}{|x-y|} \geq \frac{r}{|X|}K(t)^{-2}||D\phi_{\omega,t}|| \cdot |x-y|.$$

If we set $L = |X|/r (\geq 1)$, then we obtain (8). Thus we have proved our lemma. □

2.1.2. Transversality argument. For $\omega \in I^*$, let $|\omega|$ be the length of ω. We prove the following two lemmas by imitating the proofs of Lemmas 3.2 and 3.3 in [**30**].

Lemma 2.1.3. *Let $\epsilon, a > 0$ and $t_0 \in U$. We set $\eta = \frac{-\epsilon \log \gamma}{4a+\epsilon}$ and take $\delta = \delta(\eta, t_0)$ coming from the distortion continuity* (6) *ascribed to η and t_0, where γ is the constant coming from the uniform contraction condition. Then for all $\omega \in I^*$ and $t \in U$ with $|t_0 - t| \leq \delta$, $||D\phi_{\omega,t_0}||^{a+\frac{\epsilon}{4}} \leq ||D\phi_{\omega,t}||^a$.*

PROOF. By the distortion continuity (6), we have

$$\begin{aligned}||D\phi_{\omega,t_0}||^{a+\frac{\epsilon}{4}} &\leq \exp(|\omega|\eta(a+\frac{\epsilon}{4}))\cdot||D\phi_{\omega,t}||^{a+\frac{\epsilon}{4}}\\ &\leq \exp(|\omega|\eta(a+\frac{\epsilon}{4}))\gamma^{|\omega|\frac{\epsilon}{4}}||D\phi_{\omega,t}||^{a}\\ &\text{(by the uniform contraction condition)}\\ &= \exp(|\omega|(\eta(a+\frac{\epsilon}{4})+\frac{\epsilon}{4}\log\gamma))\cdot||D\phi_{\omega,t}||^{a}.\end{aligned}$$

□

LEMMA 2.1.4. *For any compact subset $G \subset U$ and any α with $0 < \alpha < m$, there exists a sequence of positive constants $\{\tilde{C}_n\}_{n=1}^{\infty}$ such that*

$$\lim_{n\to\infty}\frac{\log\tilde{C}_n}{n} = 0$$

and for any $\omega, \tau \in I_n^{\infty}$ with $\omega_n \neq \tau_n$,

$$\int_G \frac{1}{|\pi_{n,t}(\omega)-\pi_{n,t}(\tau)|^{\alpha}}\ d\mathcal{L}_d(t) \leq \tilde{C}_n.$$

PROOF. Let $n \in \mathbb{N}$. By the transversality condition we have that

$$\begin{aligned}\int_G \frac{1}{|\pi_{n,t}(\omega)-\pi_{n,t}(\tau)|^{\alpha}}\ d\mathcal{L}_d(t) &= \int_0^{\infty}\mathcal{L}_d\left(\{t\in G\ :\ \frac{1}{|\pi_{n,t}(\omega)-\pi_{n,t}(\tau)|^{\alpha}}\geq x\}\right)\ dx\\ &= \int_0^{\infty}\mathcal{L}_d\left(\{t\in G\ :\ |\pi_{n,t}(\omega)-\pi_{n,t}(\tau)|\leq\frac{1}{x^{1/\alpha}}\}\right)\ dx\\ &= \int_0^{|X|^{-\alpha}}\mathcal{L}_d(G)\ dx+\int_{|X|^{-\alpha}}^{\infty}C_n\frac{1}{x^{m/\alpha}}\ dx\\ &= |X|^{-\alpha}\mathcal{L}_d(G)+C_n\left[\frac{1}{1-m/\alpha}x^{1-m/\alpha}\right]_{|X|^{-\alpha}}^{\infty}\\ &= |X|^{-\alpha}\mathcal{L}_d(G)+C_n\frac{1}{m/\alpha-1}|X|^{m-\alpha}=:\tilde{C}_n.\end{aligned}$$

Since $\frac{1}{n}\log C_n \to 0$ as $n \to \infty$, we have $\frac{1}{n}\log\tilde{C}_n \to 0$ as $n\to\infty$. □

For any $\omega \in I^*$, we define the cylinder set $[\omega]$ as $\{\tau \in I^{\infty}\ :\ \tau_1 = \omega_1, ..., \tau_{|\omega|} = \omega_{|\omega|}\}$. We denote by δ_ω the Dirac measure at $\omega \in I^{\infty}$. We give a Gibbs-like measure by employing the proof of *Claim* in the proof of 3.2 Theorem in [**29**] and the argument in [**14**, page. 232].

LEMMA 2.1.5 (The existence of a Gibbs-like measure). *Let $t \in U$ and let $s \geq 0$. Then there exists a Borel probability measure $\mu_{t,s}$ on I^{∞} such that for any $\omega \in I^*$,*

$$\mu_{t,s}([\omega]) \leq K(t)^{2s}\frac{||D\phi_{\omega,t}||^s}{Z_{n,t}(s)}, \tag{15}$$

where $K(t)$ is the constant comes from the bounded distortion (5) *and $Z_{n,t}(s) = \sum_{\omega\in I^n}||D\phi_{\omega,t}||^s$.*

PROOF. Let $n \in \mathbb{N}$. For any $\omega \in I^n$, take an element $\tau_\omega \in [\omega]$. For any $t \in U$, $s \geq 0$ and $n \in \mathbb{N}$, we define the Borel probability measure $\mu_{t,s,n}$ on I^{∞} as

$$\mu_{t,s,n} = \frac{1}{Z_{n,t}(s)}\sum_{\omega\in I^n}||D\phi_{\omega,t}||^s\delta_{\tau_\omega}.$$

Then

$$\mu_{t,s,n}([\omega]) = \frac{||D\phi_{\omega,t}||^s}{Z_{n,t}(s)}$$

for any $\omega \in I^n$.

If $\omega \in I^n$, $\upsilon \in I_{n+1}^{n+j}$ and $\tau = \omega\upsilon \in I^{n+j}$, then by the bounded distortion (5), $||D\phi_{\omega,t}|| \cdot ||D\phi_{\upsilon,t}|| \leq K(t)^2 ||D\phi_{\tau,t}||$. Hence

$$Z_{n+j,t}(s) \geq \frac{1}{K(t)^{2s}} Z_{n,t}(s) \sum_{\upsilon \in I_{n+1}^{n+j}} ||D\phi_{\upsilon,t}||^s. \tag{16}$$

Thus we have that for any $\omega \in I^n$,

$$\begin{aligned}
\mu_{t,s,n+j}([\omega]) &= \mu_{t,s,n+j}\Big(\bigcup_{\upsilon \in I_{n+1}^{n+j}} [\omega\upsilon]\Big) \\
&= \frac{\sum_{\upsilon \in I_{n+1}^{n+j}} ||D\phi_{\omega\upsilon,t}||^s}{Z_{n+j,t}(s)} \\
&\leq ||D\phi_{\omega,t}||^s \frac{\sum_{\upsilon \in I_{n+1}^{n+j}} ||D\phi_{\upsilon,t}||^s}{Z_{n+j,t}(s)} \\
&\leq K(t)^{2s} \frac{||D\phi_{\omega,t}||^s}{Z_{n,t}(s)} \\
&\text{(by (16))}.
\end{aligned}$$

Let $\mu_{t,s}$ be a weak$^*-$limit of a subsequence of $\{\mu_{t,s,j}\}_{j=1}^{\infty}$ in the space of Borel probability measures on I^∞ (see e.g. [**37**, Theorem 6.5]). The above inequality implies

$$\mu_{t,s}([\omega]) \leq K(t)^{2s} \frac{||D\phi_{\omega,t}||^s}{Z_{n,t}(s)}.$$

□

For any $n \in \mathbb{N}$, we define the map $\sigma^n : I^\infty \to I_{n+1}^\infty$ by $\sigma^n(\omega_1\omega_2\cdots) = \omega_{n+1}\omega_{n+2}\cdots$. This is a continuous map with respect to the product topology. We give the following simple lemma.

LEMMA 2.1.6. *Let $t \in U$. Then for any $n \in \mathbb{N}$ and $\omega \in I^\infty$,*

$$\pi_{1,t}(\omega) = \phi_{\omega|_n,t}(\pi_{n+1,t}(\sigma^n(\omega))).$$

PROOF. Let $t \in U$. For any $n \in \mathbb{N}$ and $\omega \in I^\infty$, we have

$$\begin{aligned}
\pi_{1,t}(\omega) &= \bigcap_{j=1}^{\infty} \phi_{\omega|_j,t}(X) \\
&= \bigcap_{j=n+1}^{\infty} \phi_{\omega|_j,t}(X) \\
&= \phi_{\omega|_n,t}\left(\bigcap_{j=1}^{\infty} \phi_{\sigma^n(\omega)|_j,t}(X)\right) \\
&= \phi_{\omega|_n,t}(\pi_{n+1,t}(\sigma^n(\omega))).
\end{aligned}$$

□

For any $\omega = \omega_1\omega_2\cdots, \tau = \tau_1\tau_2\cdots \in I^\infty$ with $\omega \neq \tau$ and $\omega_1 = \tau_1$, we denote by $\omega \wedge \tau (\in I^*)$ the largest common initial segment of ω and τ. In order to prove Main Theorem A, we need the following which is the key lemma for the proof.

Lemma 2.1.7. *Under the assumptions of Main Theorem A, for any $t_0 \in U$ and any $\epsilon > 0$, there exists $\delta = \delta(t_0, \epsilon) > 0$ such that*

$$\dim_H(J_t) \geq \min\{m, s(t_0)\} - \frac{\epsilon}{2}$$

for $\mathcal{L}_d$- a.e. $t \in B(t_0, \delta)$.

Proof. For any $t_0 \in U$ and $\epsilon > 0$, we set

$$\eta = \frac{-\epsilon \log \gamma}{(4(s(t_0) - \frac{\epsilon}{2}) + \epsilon)},$$

where γ is the constant coming from the uniform contraction condition. Take $\delta = \delta(\eta, t_0)$ coming from the distortion continuity (6) ascribed to η and t_0. We set $s := \min\{m, s(t_0)\}$. By Lemma 2.1.3, for any $\omega \in I^*$ and $t \in B(t_0, \delta)$,

$$||D\phi_{\omega,t}||^{s-\frac{\epsilon}{2}} \geq ||D\phi_{\omega,t}||^{s(t_0)-\frac{\epsilon}{2}} \geq ||D\phi_{\omega,t_0}||^{s(t_0)-\frac{\epsilon}{4}}. \tag{17}$$

Let $n \in \mathbb{N}$. For any $\rho \in I^n$, we set

$$\begin{aligned} F &:= \{(\omega, \tau) \in I^\infty \times I^\infty \ : \ \omega_1 \neq \tau_1\}, \\ A_\rho &:= \{(\omega, \tau) \in I^\infty \times I^\infty \ : \ \omega \wedge \tau = \rho\}, \\ H &:= \{(\omega, \tau) \in I^\infty \times I^\infty \ : \ \omega = \tau\}. \end{aligned}$$

Then we have $I^\infty \times I^\infty = H \sqcup F \sqcup \bigsqcup_{n \geq 1} \bigsqcup_{\rho \in I^n} A_\rho$ (disjoint union). Let $\rho \in I^n$. By Lemma 2.1.6 and Lemma 2.1.1, there exists $L \geq 1$ such that for any $(\omega, \tau) \in A_\rho$ and $t \in U$,

$$\begin{aligned} |\pi_{1,t}(\omega) - \pi_{1,t}(\tau)| &= |\phi_{\rho,t}(\pi_{n+1,t}(\sigma^n\omega)) - \phi_{\rho,t}(\pi_{n+1,t}(\sigma^n\tau))| \\ &\geq L^{-1}K(t)^{-2}||D\phi_{\rho,t}|| \cdot |\pi_{n+1,t}(\sigma^n\omega) - \pi_{n+1,t}(\sigma^n\tau)|. \end{aligned} \tag{18}$$

Let $\mu = \mu_{t_0, s(t_0)-\epsilon/4}$ be the Borel probability measure coming from Lemma 2.1.5 ascribed to $t_0 \in U$ and $s(t_0) - \epsilon/4 \geq 0$. Since $\liminf_{n\to\infty} \frac{1}{n} \log Z_{n,t_0}(s(t_0) - \frac{\epsilon}{4}) > 0$, there exists $b > 0$ and $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$,

$$Z_{n,t_0}\left(s(t_0) - \frac{\epsilon}{4}\right) > \exp(bn). \tag{19}$$

By (15) and (19) we have for any $\omega \in I^\infty$, $\mu(\{\omega\}) = 0$. Hence we obtain that

$$\begin{aligned} (\mu \times \mu)(H) &= \int_{I^\infty} \mu\{\omega \in I^\infty \ : \ (\omega, \tau) \in H\} \ d\mu(\tau) \\ &= \int_{I^\infty} \mu(\{\tau\}) \ d\mu(\tau) = 0. \end{aligned} \tag{20}$$

We set $\mu_2 = \mu \times \mu$ and

$$R(t) := \int\int_{I^\infty \times I^\infty} \frac{1}{|\pi_{1,t}(\omega) - \pi_{1,t}(\tau)|^{s-\epsilon/2}} \ d\mu_2.$$

Then

$$\begin{aligned}
\int_{B(t_0,\delta)} R(t)\, d\mathcal{L}_d(t) &= \sum_{n\geq 1}\sum_{\rho\in I^n}\int\int_{A_\rho}\left(\int_{B(t_0,\delta)}\frac{1}{|\pi_{1,t}(\omega)-\pi_{1,t}(\tau)|^{s-\epsilon/2}}\ dt\right)\ d\mu_2(\omega,\tau)\\
&\quad+\int\int_F\left(\int_{B(t_0,\delta)}\frac{1}{|\pi_{1,t}(\omega)-\pi_{1,t}(\tau)|^{s-\epsilon/2}}\ dt\right)\ d\mu_2(\omega,\tau)\\
&\quad\text{(by Fubini's Theorem and (20))}\\
&\leq \sum_{n\geq 1}\sum_{\rho\in I^n}\int\int_{A_\rho}\left(\int_{B(t_0,\delta)}\frac{L^{s-\epsilon/2}K(t)^{2(s-\epsilon/2)}||D\phi_{\rho,t}||^{-s+\epsilon/2}}{|\pi_{n+1,t}(\sigma^n\omega)-\pi_{n+1,t}(\sigma^n\tau)|^{s-\epsilon/2}}\ dt\right)\ d\mu_2(\omega,\tau)\\
&\quad+\int\int_F\left(\int_{B(t_0,\delta)}\frac{1}{|\pi_{1,t}(\omega)-\pi_{1,t}(\tau)|^{s-\epsilon/2}}\ dt\right)\ d\mu_2(\omega,\tau)\\
&\quad\text{(by (18))}\\
&\leq L^{s-\epsilon/2}\left(\sup_{t\in B(t_0,\delta)}K(t)^{2(s-\epsilon/2)}\right)\sum_{n\geq 1}\tilde{C}_{n+1}\sum_{\rho\in I^n}\int\int_{A_\rho}||D\phi_{\rho,t_0}||^{-s(t_0)+\epsilon/4}\ d\mu_2(\omega,\tau)\\
&\quad+\int\int_F\tilde{C}_1 d\mu_2(\omega,\tau)\\
&\quad\text{(by (17) and Lemma 2.1.4)}\\
&\leq L^{s-\epsilon/2}\left(\sup_{t\in B(t_0,\delta)}K(t)^{2(s-\epsilon/2)}\right)\sum_{n\geq 1}\tilde{C}_{n+1}\sum_{\rho\in I^n}\int\int_{A_\rho}\frac{K(t_0)^{2(s(t_0)-\epsilon/4)}}{\mu([\rho])Z_{n,t_0}(s(t_0)-\epsilon/4)}\ d\mu_2(\omega,\tau)\\
&\quad+\tilde{C}_1\\
&\quad\text{(by Lemma 2.1.5)}\\
&=\text{Const.}\sum_{n\geq 1}\frac{\tilde{C}_{n+1}}{Z_{n,t_0}(s(t_0)-\epsilon/4)}\sum_{\rho\in I^n}\frac{1}{\mu([\rho])}\int\int_{A_\rho}\ d\mu_2(\omega,\tau)+\tilde{C}_1\\
&\quad\text{(we set Const.} = L^{s-\epsilon/2}\left(\sup_{t\in B(t_0,\delta)}K(t)^{2(s-\epsilon/2)}\right)K(t_0)^{2(s(t_0)-\epsilon/4)})\\
&\leq \text{Const.}\sum_{n\geq 1}\frac{\tilde{C}_{n+1}}{Z_{n,t_0}(s(t_0)-\epsilon/4)}+\tilde{C}_1\\
&\quad\text{(since } \mu_2(A_\rho)\leq\mu([\rho])^2).
\end{aligned}$$

Since $\frac{1}{n}\log\tilde{C}_{n+1}\to 0$ as $n\to\infty$, it follows from (19) that

$$\int_{B(t_0,\delta)} R(t)\ \mathcal{L}_d(t)\leq \text{Const.}\sum_{n\geq 1}\frac{\tilde{C}_{n+1}}{Z_{n,t_0}(s(t_0)-\epsilon/4)}+\tilde{C}_1<\infty.$$

Hence we have that for $\mathcal{L}_d$-a.e. $t\in B(t_0,\delta)$,

$$R(t)=\int\int_{\mathbb{R}^m\times\mathbb{R}^m}\frac{1}{|x-y|^{s-\epsilon/2}}\ d\left(\pi_{1,t}(\mu)\times\pi_{1,t}(\mu)\right)<\infty,$$

where $\pi_{1,t}(\mu)$ is the push forward measure of μ by $\pi_{1,t}$. Since $\pi_{1,t}(\mu)\,(J_t) = 1$, by [**11**, Theorem 4.13 (a)] we have

$$\dim_H(J_t) \geq \min\{m, s(t_0)\} - \frac{\epsilon}{2}$$

for $\mathcal{L}_d$- a.e. $t \in B(t_0, \delta)$. □

2.1.3. Proof of Main Theorem A. The following is one of the main results in this dissertation.

THEOREM 2.1.8. *Let $\{\Phi_t\}_{t\in U}$ be a TNCIFS. Suppose that the function $t \mapsto s(t)$ is a real-valued and continuous function on U. Then*

$$\dim_H(J_t) = \min\{m, s(t)\}$$

for $\mathcal{L}_d$-a.e. $t \in U$.

PROOF. By [**29**, 2.8 Lemma], for any $t \in U$ we have

$$\dim_H(J_t) \leq \tilde{s}(t) := \min\{m, s(t)\}.$$

Hence it suffices to prove that

$$\dim_H(J_t) \geq \tilde{s}(t)$$

for $\mathcal{L}_d$- a.e. $t \in U$. Suppose that this is not true. Then there exist $\epsilon > 0$ and a Lebesgue density point $t_0 \in U$ of the set

$$\{t \in U \;:\; \dim_H(J_t) < \tilde{s}(t) - \epsilon\}.$$

Then there exists $\delta_0 > 0$ such that for each $0 < \delta < \delta_0$,

$$\mathcal{L}_d(\{t \in B(t_0, \delta) \;: \dim_H(J_t) < \tilde{s}(t) - \epsilon\}) > 0. \tag{21}$$

By the continuity of the function $\tilde{s}(t)$, if δ is small enough then $\tilde{s}(t) < \tilde{s}(t_0) + \epsilon/2$ for all $t \in B(t_0, \delta)$. Thus for all δ sufficiently small we obtain from (21) that

$$\mathcal{L}_d(\{t \in B(t_0, \delta) \;: \dim_H(J_t) < \tilde{s}(t_0) - \epsilon/2\}) > 0.$$

This contradicts Lemma 2.1.7 and completes the proof of our theorem. □

We consider the continuity of the map $t \mapsto s(t)$. By developing the method in the proof of Lemma 3.4 in [**30**], we give the following.

PROPOSITION 2.1.9. *Let $\{\Phi_t\}_{t\in U}$ be a TNCIFS. Suppose that for any $t \in U$ there exists $s(t) \geq 0$ such that $\underline{P}_t(s(t)) = 0$. Then the function $t \mapsto s(t)$ is continuous on U.*

PROOF. For any $t \in U$, for any $s_1 \geq 0$ with $|\underline{P}_t(s_1)| < \infty$ and for any $s_2 \in \mathbb{R}$ with $s_1 + s_2 \geq 0$ and $|\underline{P}_t(s_1 + s_2)| < \infty$, we have

$$\begin{aligned}
\underline{P}_t(s_1+s_2) - \underline{P}_t(s_1) &= \liminf_{n\to\infty} \frac{1}{n} \log Z_{n,t}(s_1+s_2) - \liminf_{n\to\infty} \frac{1}{n} \log Z_{n,t}(s_1) \\
&= \liminf_{n\to\infty} \frac{1}{n} \log Z_{n,t}(s_1+s_2) + \limsup_{n\to\infty} \frac{-1}{n} \log Z_{n,t}(s_1) \\
&\leq \limsup_{n\to\infty} \frac{1}{n} \left(\log \sum_{\omega\in I^n} ||D\phi_{\omega,t}||^{s_1+s_2} - \log Z_{n,t}(s_1) \right) \\
&\leq \limsup_{n\to\infty} \frac{1}{n} \left(\log \sum_{\omega\in I^n} ||D\phi_{\omega,t}||^{s_1} \gamma^{ns_2} - \log Z_{n,t}(s_1) \right) \\
&= \limsup_{n\to\infty} \frac{1}{n} \left(\log \sum_{\omega\in I^n} ||D\phi_{\omega,t}||^{s_1} \gamma^{ns_2} - \log Z_{n,t}(s_1) \right) \\
&= s_2 \log \gamma < 0.
\end{aligned}$$

Hence we have that for any $t \in U$, for any $s_1 \geq 0$ with $|\underline{P}_t(s_1)| < \infty$ and for any $s_2 \in \mathbb{R}$ with $s_1 + s_2 \geq 0$ and $|\underline{P}_t(s_1 + s_2)| < \infty$,

$$|\underline{P}_t(s_1+s_2) - \underline{P}_t(s_1)| \geq |s_2| \cdot |\log \gamma|. \tag{22}$$

Fix $\epsilon > 0$ and $t_0 \in U$. Take $\delta > 0$ produced by the distorsion continuity (6) with $\eta = \epsilon$ and t_0. For any $t \in U$ with $|t - t_0| < \delta$, then we have that

$$\begin{aligned}
\underline{P}_t(s(t_0)) &= \underline{P}_t(s(t_0)) - \underline{P}_{t_0}(s(t_0)) \\
&= \liminf_{n\to\infty} \frac{1}{n} \log Z_{n,t}(s(t_0)) + \limsup_{n\to\infty} \frac{-1}{n} \log Z_{n,t_0}(s(t_0)) \\
&\leq \limsup_{n\to\infty} \frac{1}{n} \left(\log \sum_{\omega\in I^n} ||D\phi_{\omega,t}||^{s(t_0)} - \log \sum_{\omega\in I^n} ||D\phi_{\omega,t_0}||^{s(t_0)} \right) \\
&= \limsup_{n\to\infty} \frac{1}{n} \left(\log \frac{\sum_{\omega\in I^n} ||D\phi_{\omega,t}||^{s(t_0)}}{\sum_{\omega\in I^n} ||D\phi_{\omega,t_0}||^{s(t_0)}} \right) \\
&\leq \limsup_{n\to\infty} \frac{1}{n} \left(\log \frac{||D\phi_{u_n,t}||^{s(t_0)}}{||D\phi_{u_n,t_0}||^{s(t_0)}} \right) \\
&\text{(we set } \frac{||D\phi_{u_n,t}||^{s(t_0)}}{||D\phi_{u_n,t_0}||^{s(t_0)}} = \max_{\omega\in I^n} \frac{||D\phi_{\omega,t}||^{s(t_0)}}{||D\phi_{\omega,t_0}||^{s(t_0)}}, \text{where } u_n \in I^n) \\
&\leq \limsup_{n\to\infty} \frac{1}{n} \log \exp(n\epsilon s(t_0)) \\
&\text{(by the distorsion continuity (6))} \\
&= \epsilon s(t_0),
\end{aligned}$$

and

$$\begin{aligned}
\underline{P}_t(s(t_0)) &= \underline{P}_t(s(t_0)) - \underline{P}_{t_0}(s(t_0)) \\
&= \liminf_{n\to\infty} \frac{1}{n} \log Z_{n,t}(s(t_0)) + \limsup_{n\to\infty} \frac{-1}{n} \log Z_{n,t_0}(s(t_0)) \\
&\geq \liminf_{n\to\infty} \frac{1}{n} \left(\log \sum_{\omega\in I^n} ||D\phi_{\omega,t}||^{s(t_0)} - \log \sum_{\omega\in I^n} ||D\phi_{\omega,t_0}||^{s(t_0)} \right) \\
&= \liminf_{n\to\infty} \frac{1}{n} \left(\log \frac{\sum_{\omega\in I^n} ||D\phi_{\omega,t}||^{s(t_0)}}{\sum_{\omega\in I^n} ||D\phi_{\omega,t_0}||^{s(t_0)}} \right) \\
&\geq \liminf_{n\to\infty} \frac{1}{n} \left(\log \frac{||D\phi_{v_n,t}||^{s(t_0)}}{||D\phi_{v_n,t_0}||^{s(t_0)}} \right) \\
&\text{(we set } \frac{||D\phi_{v_n,t}||^{s(t_0)}}{||D\phi_{v_n,t_0}||^{s(t_0)}} = \min_{\omega\in I^n} \frac{||D\phi_{\omega,t}||^{s(t_0)}}{||D\phi_{\omega,t_0}||^{s(t_0)}}, \text{where } v_n \in I^n) \\
&\geq \liminf_{n\to\infty} \frac{1}{n} \log \exp(-n\epsilon s(t_0)) \\
&\text{(by the distorsion continuity(6))} \\
&= -\epsilon s(t_0).
\end{aligned}$$

Thus we have

$$|\underline{P}_t(s(t_0))| \leq \epsilon s(t_0) < \infty. \tag{23}$$

Fix $t \in U$ with $|t_0 - t| \leq \delta$. By (22) and (23) we have

$$\begin{aligned}
|s(t_0) - s(t)| &\leq \frac{|\underline{P}_t(s(t) + s(t_0) - s(t)) - \underline{P}_t(s(t))|}{|\log\gamma|} \\
&\leq \frac{|\underline{P}_t(s(t) + s(t_0) - s(t))|}{|\log\gamma|} \leq \frac{\epsilon s(t_0)}{|\log\gamma|}.
\end{aligned}$$

Hence the function $t \mapsto s(t)$ is continuous at $t_0 \in U$. Since t_0 is arbitrary, we have proved our proposition. □

Remark 2.1.10. Let $\{\Phi_t\}_{t\in U}$ be a TNCIFS satisfying

- there exists $N \geq 1$ such that $\# I^{(j)} \leq N$ for any $j \in \mathbb{N}$;
- for any $t \in U$, $j \in \mathbb{N}$, and $i \in I^{(j)}$, the map $\phi_{i,t}^{(j)}$ has the following form

  $$\phi_{i,t}^{(j)}(x) = u(t)x + a_{i,j},$$

  where U is an open subset of $\mathbb{R}^d$, $u : U \to (0,1)$ is a continuous function on U and $a_{i,j} \in \mathbb{R}^m$.

Then we have that

$$s(t) = \frac{\liminf_{n\to\infty} \frac{1}{n} \sum_{j=1}^n \log \# I^{(j)}}{-\log|u(t)|}$$

is the zero of the lower pressure function $\underline{P}_t$ for any $t \in U$. Hence the family $\{\Phi_t\}_{t\in U}$ satisfies the assumption of Proposition 2.1.9.

We do not know any example of the family $\{\Phi_t\}_{t\in U}$ for which the map $t \mapsto s(t)$ is not continuous.

2.2. Example

In this section, we give an example of a family $\{\Phi_t\}_{t\in U}$ of parameterized NCIFSs such that $\{\Phi_t\}_{t\in U}$ satisfies the transversality condition but Φ_t does not satisfy the open set condition for any $t \in U$. We set $\mathbb{D} := \{z \in \mathbb{C} \ : \ |z| < 1\}$. For any holomorphic function f on $\mathbb{D}$, we denote by $f'(z)$ the complex derivative of f evaluated at $z \in \mathbb{D}$. For the transversality condition, we now give a slight variation of [**32**, Lemma 5.2].

LEMMA 2.2.1. *Let $\mathcal{H}$ be a compact subset of the space of holomorphic functions on $\mathbb{D}$ endowed with the compact open topology. We set*

$$\tilde{\mathcal{M}}_H := \{\lambda \in \mathbb{D} \ : \textit{there exists } f \in \mathcal{H} \text{ such that } f(\lambda) = f'(\lambda) = 0\}.$$

Let G be a compact subset of $\mathbb{D}\backslash\tilde{\mathcal{M}}_H$. Then there exists $K = K(\mathcal{H}, G) > 0$ such that for any $f \in \mathcal{H}$ and any $r > 0$,

$$\mathcal{L}_2\left(\{\lambda \in G : |f(\lambda)| \leq r\}\right) \leq Kr^2. \tag{24}$$

PROOF. We can prove the statement of our lemma by replacing $\mathcal{B}_\Gamma$ and $\tilde{\mathcal{M}}_\Gamma$ in the proof of [**32**, Lemma 5.2] by $\mathcal{H}$ and $\tilde{\mathcal{M}}_H$ respectively. □

We now give a family $\{\Phi_t\}_{t\in U}$ of parametrized systems such that $\{\Phi_t\}_{t\in U}$ is a TNCIFS but Φ_t does not satisfy the open set condition (4) for any $t \in U$. In order to do that, we set

$$U := \{t \in \mathbb{C} \ : \ |t| < 2 \times 5^{-5/8}, \ t \notin \mathbb{R}\}.$$

Note that $2 \times 5^{-5/8} \approx 0.73143 > 1/\sqrt{2}$. Let $t \in U$. For each $j \in \mathbb{N}$, we define

$$\Phi_t^{(j)} = \{z \mapsto \phi_{1,t}^{(j)}(z), z \mapsto \phi_{2,t}^{(j)}(z)\} := \left\{z \mapsto tz, z \mapsto tz + \frac{1}{j}\right\}.$$

PROPOSITION 2.2.2. *For any $t \in U$, the system $\{\Phi_t^{(j)}\}_{j=1}^\infty$ does not satisfy the open set condition.*

PROOF. Suppose that the system $\{\Phi_t^{(j)}\}_{j=1}^\infty$ satisfies the open set condition (4). Then there exists a compact subset $X \subset \mathbb{C}$ with $\mathrm{int}(X) \neq \emptyset$ such that $\phi_{1,t}^{(j)}(\mathrm{int}(X)) \cap \phi_{2,t}^{(j)}(\mathrm{int}(X)) = \emptyset$. Hence there exist $x \in X$ and $r > 0$ such that

$$\phi_{1,t}^{(j)}(B(x,r)) \cap \phi_{2,t}^{(j)}(B(x,r)) = B(tx, |t|r) \cap B(tx + 1/j, |t|r) = \emptyset.$$

In particular, we have for all $j \in \mathbb{N}$,

$$2|t|r < \frac{1}{j}.$$

This is a contradiction. □

We set

$$X := \left\{z \in \mathbb{C} \ : \ |z| \leq \frac{1}{1 - 2 \times 5^{-5/8}}\right\}.$$

Then we have that for any $t \in U$, for any $j \in \mathbb{N}$ and for any $i \in I^{(j)} := \{1, 2\}$, $\phi_{i,t}^{(j)}(X) \subset X$. We set $b_1^{(j)} = 0$ and $b_2^{(j)} = 1/j$ for each j. Let $n, j \in \mathbb{N}$. We give the following lemma.

LEMMA 2.2.3. *Let $t \in U$. For any $\omega = \omega_n \cdots \omega_{n+j-1} \in I_n^{n+j-1}$ and any $z \in X$ we have*

$$\phi_{\omega,t}(z) = \phi_{\omega_n,t}^{(n)} \circ \cdots \circ \phi_{\omega_{n+j-1},t}^{(n+j-1)}(z) = t^j z + \sum_{i=1}^{j} b_{\omega_{n+i-1}}^{(n+i-1)} t^{i-1},$$

where $b^{(n+i-1)}_{\omega_{n+i-1}} \in \{0, \frac{1}{n+i-1}\}$. *In particular, for any* $\omega = \omega_n \cdots \omega_{n+j-1} \cdots \in I_n^\infty$,

$$\pi_{n,t}(\omega) = \sum_{i=1}^{\infty} b^{(n+i-1)}_{\omega_{n+i-1}} t^{i-1}.$$

PROOF. This can be shown by induction on j. $\square$

We can show that the family of systems $\{\Phi_t\}_{t \in U}$ is a TNCIFS as follows.

1. *Conformality* : Let $t \in U$. For any $j \in \mathbb{N}$ and any $i \in I^{(j)}$, $\phi^{(j)}_{i,t}(z) = tz + b^{(j)}_i$ is a similarity map on $\mathbb{C}$.
2. *Uniform Contraction* : We set $\gamma = 1/\sqrt{2}$. Then for any $\omega \in I_n^{n+j-1}$ and $z \in X$,
$$|D\phi_{\omega,t}(z)| = |t|^j \leq \gamma^j$$
by Lemma 2.2.3.
3. *Bounded distortion* : By (2.2.3), for any $\omega = \omega_n \cdots \omega_{n+j-1} \in I_n^{n+j-1}$ and $z \in \mathbb{C}$, $|D\phi_{\omega,t}(z)| = |t|^j$. We define the Borel measurable locally bounded function $K : U \to [1, \infty)$ by $K(t) = 1$. Then for any $\omega \in I_n^{n+j-1}$,
$$|D\phi_{\omega,t}(z_1)| \leq K(t)|D\phi_{\omega,t}(z_2)|$$
for all $z_1, z_2 \in \mathbb{C}$.
4. *Distortion continuity* : Fix $t_0 \in U$. Since the map $t \mapsto \log|t|$ is continuous at $t_0 \in U$, for any $\eta > 0$ there exists $\delta = \delta(\eta, t_0) > 0$ such that for any $t \in U$ with $|t_0 - t| < \delta$,
$$|\log|t_0| - \log|t|| < \eta.$$
Hence we have
$$|\log|t_0|^j/|t|^j| < j\eta.$$
Thus we have that for any $\omega \in I_n^{n+j-1}$,
$$\exp(-j\epsilon) < \frac{||D\phi_{\omega,t_0}||}{||D\phi_{\omega,t}||} = \exp(\log|t_0|^j/|t|^j) < \exp(j\epsilon).$$
5. *Continuity* : By Lemma 2.2.3, we have for any $t \in U$ and any $\omega \in I_n^\infty$,
$$\pi_{n,t}(\omega) = \sum_{i=1}^{\infty} b^{(n+i-1)}_{\omega_{n+i-1}} t^{i-1}.$$
Hence the map $t \mapsto \pi_{n,t}(\omega)$ is continuous on U.
6. *Transversality condition* : We introduce a set $\mathcal{G}$ of holomorphic functions on $\mathbb{D}$ and the set $\tilde{\mathcal{O}_2}$ of double zeros in $\mathbb{D}$ for functions belonging to $\mathcal{G}$.
$$\mathcal{G} := \left\{ f(t) = \pm 1 + \sum_{j=1}^{\infty} a_j t^j : a_j \in [-1, 1] \right\},$$
$$\tilde{\mathcal{O}_2} := \{t \in \mathbb{D} \ : \text{there exists } f \in \mathcal{G} \text{ such that } f(t) = f'(t) = 0\}.$$
Note that $\mathcal{G}$ is a compact subset of the space of holomorphic functions on $\mathbb{D}$ endowed with the compact open topology. Let $n \in \mathbb{N}$. Then we have for any $t \in U$ and any

$\omega, \tau \in I_n^\infty$ with $\omega_n \neq \tau_n$,

$$\begin{aligned}\pi_{n,t}(\omega) - \pi_{n,t}(\tau) &= \sum_{i=1}^{\infty} b^{(n+i-1)}_{\omega_{n+i-1}} t^{i-1} - \sum_{i=1}^{\infty} b^{(n+i-1)}_{\tau_{n+i-1}} t^{i-1} \\ &= b^{(n)}_{\omega_n} - b^{(n)}_{\tau_n} + \sum_{i=2}^{\infty} \left(b^{(n+i-1)}_{\omega_{n+i-1}} - b^{(n+i-1)}_{\tau_{n+i-1}} \right) t^{i-1} \\ &= \frac{1}{n} \left(\pm 1 + \sum_{i=2}^{\infty} n \left(b^{(n+i-1)}_{\omega_{n+i-1}} - b^{(n+i-1)}_{\tau_{n+i-1}} \right) t^{i-1} \right).\end{aligned}$$

Then the function $t \mapsto \pm 1 + \sum_{i=2}^{\infty} n(b^{(n+i-1)}_{\omega_{n+i-1}} - b^{(n+i-1)}_{\tau_{n+i-1}}) t^{i-1}$ is a holomorphic function which belongs to $\mathcal{G}$. Let $G \subset \mathbb{D} \backslash \tilde{\mathcal{O}}_2$ be a compact subset. By Lemma 2.2.1, there exists $K = K(\mathcal{G}, G) > 0$ such that for any $\omega, \tau \in I_n^\infty$ with $\omega_n \neq \tau_n$ and any $r > 0$,

$$\begin{aligned}&\mathcal{L}_2(\{t \in G \ : \ |\pi_{n,t}(\omega) - \pi_{n,t}(\tau)| \leq r\}) \\ &= \mathcal{L}_2(\{t \in G \ : \ |\pm 1 + \sum_{i=2}^{\infty} n(b^{(n+i-1)}_{\omega_{n+i-1}} - b^{(n+i-1)}_{\tau_{n+i-1}}) t^{i-1}| \leq nr\}) \\ &\leq K(nr)^2.\end{aligned}$$

If we set $C_n := Kn^2$ for any $n \in \mathbb{N}$, we have

$$\mathcal{L}_2(\{t \in G \ : \ |\pi_{n,t}(\omega) - \pi_{n,t}(\tau)| \leq r\}) \leq C_n r^2$$

and

$$\frac{1}{n} \log C_n = \frac{1}{n} \log K + \frac{2}{n} \log n \to 0$$

as $n \to \infty$.

Finally, we use the following theorem.

THEOREM 2.2.4. [**35**, Proposition 2.7] *A power series of the form* $1 + \sum_{j=1}^{\infty} a_j z^j$, *with* $a_j \in [-1, 1]$, *cannot have a non-real double zero of modulus less than* $2 \times 5^{-5/8}$.

By using above theorem, we have that $U = \{t \in \mathbb{C} \ : \ |t| < 2 \times 5^{-5/8}, \ t \notin \mathbb{R}\} \subset \mathbb{D} \backslash \tilde{\mathcal{O}}$. Hence the family $\{\Phi_t\}_{t \in U}$ satisfies the transversality condition.

By the above arguments, we get the following.

PROPOSITION 2.2.5. *The family of parametrized systems* $\{\Phi_t\}_{t \in U}$ *is a TNCIFS.*

We calculate the lower pressure function $\underline{P}_t$ for Φ_t, $t \in U$ as the following. For any $s \in [0, \infty)$,

$$\begin{aligned}\underline{P}_t(s) &= \liminf_{n \to \infty} \frac{1}{n} \log \sum_{\omega \in I^n} ||D\phi_{\omega,t}||^s \\ &= \liminf_{n \to \infty} \frac{1}{n} \log \sum_{\omega \in I^n} |t|^{ns} \\ &= \liminf_{n \to \infty} \frac{1}{n} \log(2^n |t|^{ns}) \\ &= \log 2 + s \log |t|.\end{aligned}$$

Hence for each $t \in U$, $\underline{P}_t(s)$ has the zero

$$s(t) = \frac{\log 2}{-\log |t|}$$

and the function $t \mapsto s(t)$ is continuous on U. Let J_t be the (1st) limit set corresponding to t. Then by Theorem 2.1.8, we have

$$\dim_H(J_t) = \min\{2, s(t)\} = s(t)$$

for a.e. $t \in \{t \in \mathbb{C} \ : \ |t| \leq 1/\sqrt{2}, t \notin \mathbb{R}\}$ and

$$\dim_H(J_t) = \min\{2, s(t)\} = 2$$

for a.e. $t \in \{t \in \mathbb{C} \ : \ 1/\sqrt{2} \leq |t| < 2 \times 5^{-5/8}, t \notin \mathbb{R}\}$.

CHAPTER 3

The Hausdorff dimension of some planar sets with unbounded digits

In this chapter, we consider parameterized planar sets $L_0(\lambda)$ for $\lambda \in \mathbb{D}^*$ (see (7)). We investigate these sets by approximating the region of transversality. We calculate the Hausdorff dimension of these sets for typical parameters in some region with respect to the 2-dimensional Lebesgue measure. In addition, we estimate the local dimension of the exceptional set of parameters.

3.1. Preliminaries

In this section, we give the upper estimation of the Hausdorff dimension of $L_0(\lambda)$ for any $\lambda \in \mathbb{D}^*$.

3.1.1. On the symbolic space. We deal with the digits $\{p_j\}_{j=0}^{\infty}$ satisfying the following conditions.

- For each $j \in \mathbb{N}_0$, $p_j \geq 1$;
- $p_j \to \infty$ as $j \to \infty$;
- $\frac{p_{j+1}}{p_j} \to 1$ as $j \to \infty$.

The above conditions imply the following.

REMARK 3.1.1. (1) For each $n \in \mathbb{N}$, $\frac{p_{j+n}}{p_j} \to 1$ as $j \to \infty$.
(2) Let $a > 1$ and $b > 0$. For each $n \in \mathbb{N}$, $(p_{j+n})^b / a^j \to 0$ as $j \to \infty$.

We set $I := \{0, 1\}$. For each $\omega = \omega_0\omega_1 \cdots \in I^\infty$ and $k \in \mathbb{N}$, we set $\omega|_k := \omega_0\omega_1 \cdots \omega_{k-1} \in I^k$. For each $\omega = \omega_0\omega_1 \cdots \omega_{k-1} \in I^k$, we denote by $[\omega]$ the set $\{\tau \in I^\infty : \tau_0 = \omega_0, \tau_1 = \omega_1, ..., \tau_{k-1} = \omega_{k-1}\}$. For each $\omega = \omega_0\omega_1 \cdots, \tau = \tau_0\tau_1 \cdots \in I^\infty$ with $\omega \neq \tau$ we set $|\omega \wedge \tau| := \inf\{j \in \mathbb{N}_0 \ : \ \omega_j \neq \tau_j\}$. Moreover, we set $|\omega \wedge \omega| = \infty$.

PROPOSITION 3.1.2. *Let $m, n \in \mathbb{N}_0$. Then there exists minimum $j_{n,m} \in \mathbb{N}_0$ such that for all $j_1 \geq j_2 \geq j_{n,m}$, $(p_{j_1+n})^m / 2^{j_1} \leq (p_{j_2+n})^m / 2^{j_2}$.*

PROOF. Since for each $n \in \mathbb{N}_0$, $(p_{j+1+n})^m / (p_{j+n})^m \to 1$ as $j \to \infty$, there exists $k_{n,m} \in \mathbb{N}_0$ such that for each $j \geq k_{n,m}$,

$$2 \geq \frac{(p_{j+1+n})^m}{(p_{j+n})^m}.$$

Hence for any $j_1 = j_2 + l \geq j_2 \geq k_{n,m}$,

$$2 \geq \frac{(p_{j_2+1+n})^m}{(p_{j_2+n})^m}, 2 \geq \frac{(p_{j_2+2+n})^m}{(p_{j_2+1+n})^m}, \cdots, 2 \geq \frac{(p_{j_2+l+n})^m}{(p_{j_2+(l-1)+n})^m}.$$

Thus we have that

$$\frac{2^{j_1}}{2^{j_2}} = 2^l \geq \frac{(p_{j_1+n})^m}{(p_{j_2+n})^m}.$$

□

By Proposition 3.1.2, we can define the metric $\rho_{n,m}$ on I^∞ as the following.

DEFINITION 3.1.3. Let $m, n \in \mathbb{N}_0$. We define the metric $\rho_{n,m}$ on I^∞ by

$$\rho_{n,m}(\omega,\tau) := \begin{cases} K_{n,m} & (|\omega \wedge \tau| \leq j_{n,m}) \\ \frac{(p_{|\omega\wedge\tau|+n})^m}{2^{|\omega\wedge\tau|}} & (|\omega \wedge \tau| > j_{n,m}) \end{cases}$$

for each $\omega, \tau \in I^\infty$. Here, $K_{n,m} = (p_{j_{n,m}+n})^m / 2^{j_{n,m}}$.

REMARK 3.1.4. (1) The metric space $(I^\infty, \rho_{n,m})$ is a compact metric space for each $n \in \mathbb{N}_0$ and $m \in \mathbb{N}_0$.
(2) $\rho_{n,0}(\omega,\tau) = 1/2^{|\omega\wedge\tau|}$ for each $\omega, \tau \in I^\infty$.

Let X be a metric space endowed with a metric ρ. Let $A \subset X$. We define $|A|_\rho := \sup\{\rho(x,y) : x, y \in A\}$. For each $t \geq 0$ and $\delta > 0$, we set

$$\mathcal{H}^t_{\rho,\delta}(A) := \inf\left\{\sum_{i=1}^\infty |U_i|^t_\rho : A \subset \bigcup_{i=1}^\infty U_i, |U_i| \leq \delta \text{ for } U_i \subset X\right\}.$$

We define *the $t-$dimensional Hausdorff outer measure of A with respect to ρ* as

$$\mathcal{H}^t_\rho(A) := \lim_{\delta\to 0} \mathcal{H}^t_{\rho,\delta}(A) \in [0,\infty].$$

For any set $A \subset X$, we define *the Hausdorff dimension of A with respect to ρ* as

$$\dim_\rho(A) := \sup\{t \geq 0 : \mathcal{H}^t_\rho(A) = \infty\} = \inf\{t \geq 0 : \mathcal{H}^t_\rho(A) = 0\}.$$

We compute the Hausdorff dimension of I^∞ with respect to $\rho_{n,m}$ as the following.

PROPOSITION 3.1.5. *For each $n \in \mathbb{N}_0$ and $m \in \mathbb{N}_0$, $\dim_{\rho_{n,m}}(I^\infty) = 1$.*

PROOF. Let μ be a probability measure on I^∞ such that

$$\mu([\omega_0\omega_1\cdots\omega_{j-1}]) = \frac{1}{2^j}$$

for each $\omega_0\omega_1\cdots\omega_{j-1} \in I^j$ (μ is the $(1/2,1/2)-$Bernoulli measure on I^∞). Fix $m \in \mathbb{N}_0$. Then we have that for any $\omega \in I^j$ with $j > j_{n,m}$,

$$\begin{aligned} \mu(\{\tau \in I^\infty : \rho_{n,m}(\omega,\tau) \leq (p_{j+n})^m/2^j\}) &= \mu([\omega_0\omega_1\cdots\omega_{j-1}]) \\ &= \frac{1}{2^j} \\ &\leq |\{\tau \in I^\infty : \rho_{n,m}(\omega,\tau) \leq (p_{j+n})^m/2^j\}|^1_{\rho_{n,m}} (= (p_{j+n})^m/2^j) \end{aligned}$$

By the mass distribution principle (see [**11**, P.67]), we have that $1 \leq \dim_{\rho_{n,m}}(I^\infty)$.

We now prove that for each $m \in \mathbb{N}_0$, $\dim_{\rho_{n,m}}(I^\infty) \leq 1$. For any $\epsilon > 0$ and $j > j_{n,m}$, since the family of sets $\{[\omega]\}_{\omega\in I^j}$ is a covering for I^∞, we have that

$$\begin{aligned} \mathcal{H}^{1+\epsilon}_{\rho_{n,m},(p_{j+n})^m/2^j}(I^\infty) &\leq \sum_{\omega\in I^j} |[\omega]|^{1+\epsilon}_{\rho_{n,m}} \\ &= 2^j \frac{(p_{j+n})^{m(1+\epsilon)}}{2^{j(1+\epsilon)}} \to 0 \text{ as } j \to \infty. \end{aligned}$$

Hence we have that $\mathcal{H}^{1+\epsilon}_{\rho_{n,m}}(I^\infty) = 0$ and hence $\dim_{\rho_{n,m}}(I^\infty) \le 1+\epsilon$. Since $\epsilon > 0$ is arbitrary, we have that $\dim_{\rho_{n,m}}(I^\infty) \le 1$.

Hence we have proved our proposition. □

3.1.2. Address maps. We now define the address maps as follows.

Definition 3.1.6. For each $\lambda \in \mathbb{D}^*$ and $n \in \mathbb{N}_0$, we define the address map $\Pi_{n,\lambda} : I^\infty \to \mathbb{C}$ by

$$\Pi_{n,\lambda}(\omega) := \sum_{j=0}^{\infty} p_{n+j}\omega_{n+j}\lambda^j$$

($\omega = \omega_n\omega_{n+1}\cdots \in I^\infty$). Note that this map is well-defined.

Then we have that

$$\Pi_{n,\lambda}(I^\infty) = \left\{ \sum_{j=0}^{\infty} a_j\lambda^j : a_j \in \{0, p_{n+j}\} \right\}.$$

In particular, $L_0(\lambda) = \Pi_{0,\lambda}(I^\infty)$ (for the definition of $L_0(\lambda)$ see (7)). Below we set $L_n(\lambda) := \Pi_{n,\lambda}(I^\infty)$. We give the following proposition.

Proposition 3.1.7. *For each $n \in \mathbb{N}_0$, if we set $\phi_{n,\lambda}(z) := \lambda z, \varphi_{n,\lambda}(z) := \lambda z + p_n$, then*

$$L_n(\lambda) = \phi_{n,\lambda}(L_{n+1}(\lambda)) \cup \varphi_{n,\lambda}(L_{n+1}(\lambda)).$$

Proof.

$$\begin{aligned}
\phi_{n,\lambda}(L_{n+1}(\lambda)) \cup \varphi_{n,\lambda}(L_{n+1}(\lambda)) &= \left\{ \lambda\left(\sum_{j=0}^{\infty} p_{n+j+1}\omega_j\lambda^j\right) + 0 : \omega_j \in \{0,1\} \right\} \\
&\quad \cup \left\{ \lambda\left(\sum_{j=0}^{\infty} p_{n+j+1}\omega_j\lambda^j\right) + p_n : \omega_j \in \{0,1\} \right\} \\
&= \left\{ \sum_{j=0}^{\infty} p_{n+j}\omega_j\lambda^j : \omega_j \in \{0,1\} \right\} = L_n(\lambda).
\end{aligned}$$

□

Corollary 3.1.8.

$$\dim_H(L_0(\lambda)) = \dim_H(L_n(\lambda));$$
$$\mathcal{L}_2(L_0(\lambda)) \ge |\lambda|^{2n}\mathcal{L}_2(L_n(\lambda)).$$

Proof. By Proposition 3.1.7, we have that for each $n \in \mathbb{N}_0$,

$$\begin{aligned}
\dim_H(L_n(\lambda)) &= \max\{\dim_H(\phi_{n,\lambda}(L_{n+1}(\lambda))), \dim_H(\varphi_{n,\lambda}(L_{n+1}(\lambda)))\} \\
&= \max\{\dim_H(L_{n+1}(\lambda)), \dim_H(L_{n+1}(\lambda))\} = \dim_H(L_{n+1}(\lambda))
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{L}_2(L_n(\lambda)) &\ge \mathcal{L}_2(\phi_{n,\lambda}(L_{n+1}(\lambda))) \\
&= |\lambda|^2\mathcal{L}_2(L_{n+1}(\lambda)).
\end{aligned}$$

□

3.1.3. Sets of some power series. In this subsection, we introduce sets of some power series and the sets of double zeros. For each $j \in \mathbb{N}$ and $n \in \mathbb{N}_0$, we set

$$G_{n,j} := \bigcup_{m \geq n} \left\{ \frac{-p_{m+j}}{p_m}, 0, \frac{p_{m+j}}{p_m} \right\} \cup \{-1, 1\}.$$

For each $j \in \mathbb{N}$ and $n \in \mathbb{N}_0$, the set $G_{n,j}$ is compact subset in $\mathbb{R}$ since p_{m+j}/p_m tends to 1 as $m \to \infty$. If we set $b_{n,j} := \max G_{n,j} < \infty$, there exists $m_{n,j} \geq n$ such that $b_{n,j} = p_{m_{n,j}+j}/p_{m_{n,j}}$.

LEMMA 3.1.9.

$$\lim_{j \to \infty} \frac{1}{j} \log b_{n,j} = 0.$$

PROOF.

$$\begin{aligned} \log b_{n,j} &= \log \frac{p_{m_{n,j}+j}}{p_{m_{n,j}}} \\ &= \log \left(\frac{p_{m_{n,j}+1}}{p_{m_{n,j}}} \frac{p_{m_{n,j}+2}}{p_{m_{n,j}+1}} \frac{p_{m_{n,j}+3}}{p_{m_{n,j}+2}} \cdots \frac{p_{m_{n,j}+j}}{p_{m_{n,j}+(j-1)}} \right) \\ &= \sum_{k=0}^{j-1} \log \frac{p_{(m_{n,j}+k)+1}}{p_{m_{n,j}+k}}. \end{aligned}$$

For any $\epsilon > 0$, there exists $j_1 \in \mathbb{N}$ such that for any $j \geq j_1$,

$$\log \frac{p_{j+1}}{p_j} < \epsilon$$

since $p_{j+1}/p_j \to 1$ as $j \to \infty$. In addition, there exists $j_2 \in \mathbb{N}$ with $j_2 \geq j_1$ such that for any $j \geq j_2$,

$$\frac{(j_1 + 1)}{j} \log \frac{p_{m_{n,1}+1}}{p_{m_{n,1}}} < \epsilon.$$

Since $p_{m+1}/p_m \leq p_{m_{n,1}+1}/p_{m_{n,1}}$ for any $m \geq n$, we have that for any $j \geq j_2$,

$$\begin{aligned} 0 \leq \frac{1}{j} \log b_{n,j} &= \frac{1}{j} \left(\sum_{k=0}^{j_1} \log \frac{p_{(m_{n,j}+k)+1}}{p_{m_{n,j}+k}} + \sum_{k=j_1+1}^{j} \log \frac{p_{(m_{n,j}+k)+1}}{p_{m_{n,j}+k}} \right) \\ &\leq \frac{(j_1 + 1)}{j} \log \frac{p_{m_{n,1}+1}}{p_{m_{n,1}}} + \frac{(j - j_1)\epsilon}{j} < 2\epsilon. \end{aligned}$$

□

By Lemma 3.1.9, the function

$$\lambda \mapsto C_n(\lambda) := \sum_{j=0}^{\infty} b_{n,j} |\lambda|^j$$

is well-defined on $\mathbb{D}$. We define the following sets.

DEFINITION 3.1.10. For each $n \in \mathbb{N}_0$, we set

$$\mathcal{F}_n := \left\{ f(\lambda) = \pm 1 + \sum_{j=1}^{\infty} a_{n,j}\lambda^j : a_{n,j} \in G_{n,j} \right\},$$

$$\tilde{\mathcal{N}_n} := \{\lambda \in \mathbb{D}^* \ : \text{there exists } f \in \mathcal{F}_n \text{ such that } f(\lambda) = f'(\lambda) = 0\},$$

$$\mathcal{F} := \left\{ f(\lambda) = \pm 1 + \sum_{j=1}^{\infty} a_j\lambda^j : a_j \in \{-1, 0, 1\} \right\},$$

$$\tilde{\mathcal{M}_2} := \{\lambda \in \mathbb{D}^* \ : \text{there exists } f \in \mathcal{F} \text{ such that } f(\lambda) = f'(\lambda) = 0\}.$$

REMARK 3.1.11. For any $n \in \mathbb{N}_0$, the sets $\mathcal{F}_n$ and $\mathcal{F}$ are compact subsets of the space of holomorphic functions on $\mathbb{D}$ endowed with the compact open topology.

LEMMA 3.1.12.

$$\bigcap_{n \geq 0} \tilde{\mathcal{N}_n} = \tilde{\mathcal{M}_2}.$$

PROOF. Since for all $n \in \mathbb{N}_0$,

$$\mathcal{F}_n \supset \mathcal{F}$$

we have that

$$\bigcap_{n \geq 0} \tilde{\mathcal{N}_n} \supset \tilde{\mathcal{M}_2}.$$

Fix $z_0 \in \bigcap_{n \geq 0} \tilde{\mathcal{N}_n}$. Then for each $n \in \mathbb{N}_0$, there exists $f_n \in \mathcal{F}_n$ such that $f_n(z_0) = f_n'(z_0) = 0$. Here,

$$f_n(\lambda) = 1 + \sum_{j=1}^{\infty} \alpha_{n,j}\lambda^j,$$

where

$$\alpha_{n,j} = \frac{p_{m_{n,j}+j}a_{n,j}}{p_{m_{n,j}}} \text{ or } a_{n,j}$$

$(a_{n,j} \in \{-1, 0, 1\}, m_{n,j} \geq n$ for each $j \in \mathbb{N})$. For each $n \in \mathbb{N}_0$, we set

$$g_n(\lambda) := 1 + \sum_{j=1}^{\infty} a_{n,j}\lambda^j \in \mathcal{F}.$$

Then there exist a sub-sequence $\{g_{n_k}\}$ of $\{g_n\}$ and $g \in \mathcal{F}$ such that

$$g_{n_k} \to g \text{ on every compact subsets of } \mathbb{D} \text{ as } k \to \infty$$

since $\mathcal{F}$ is compact.

Then we have that

$$|f_{n_k}(z_0) - g_{n_k}(z_0)| = |(1 + \sum_{j=1}^{\infty} \alpha_{n_k,j}{z_0}^j) - (1 + \sum_{j=1}^{\infty} a_{n_k,j}{z_0}^j)|$$

$$\leq \sum_{j=1}^{\infty} |\alpha_{n_k,j} - a_{n_k,j}||z_0|^j.$$

Since $f_{n_k}(z_0) = 0$ and the last term tends to 0 as $k \to \infty$, we have that

$$g(z_0) = 0.$$

In addition,

$$|f'_{n_k}(z_0) - g'_{n_k}(z_0)| = |(\sum_{j=1}^{\infty} j\alpha_{n_k,j}{z_0}^{j-1}) - (\sum_{j=1}^{\infty} ja_{n_k,j}{z_0}^{j-1})|$$
$$\leq \sum_{j=1}^{\infty} j|\alpha_{n_k,j} - a_{n_k,j}||z_0|^{j-1}.$$

Since $f'_{n_k}(z_0) = 0$ and the last term tends to 0 as $k \to \infty$, we have that

$$g'(z_0) = 0.$$

Hence we have that $z_0 \in \tilde{\mathcal{M}}_2$. □

3.1.4. The upper estimation of the Hausdorff dimension.

PROPOSITION 3.1.13. *Let $n \in \mathbb{N}_0$. For any $(\omega, \tau) \in I^\infty \times I^\infty$ with $\omega \neq \tau$ and for any $\lambda \in \mathbb{D}^*$, there exists $f_{n,\omega,\tau} \in \mathcal{F}_n$ such that*

$$\Pi_{n,\lambda}(\omega) - \Pi_{n,\lambda}(\tau) = \lambda^{|\omega\wedge\tau|} p_{|\omega\wedge\tau|+n} f_{n,\omega,\tau}(\lambda).$$

PROOF. For any $(\omega, \tau) \in I^\infty \times I^\infty$ with $\omega \neq \tau$,

$$\begin{aligned}\Pi_{n,\lambda}(\omega) - \Pi_{n,\lambda}(\tau) &= \sum_{j=0}^{\infty} p_{n+j}\omega_j\lambda^j - \sum_{j=0}^{\infty} p_{n+j}\tau_j\lambda^j \\ &= \sum_{j=|\omega\wedge\tau|}^{\infty} p_{n+j}(\omega_j - \tau_j)\lambda^j \\ &= \lambda^{|\omega\wedge\tau|} \sum_{j=0}^{\infty} p_{|\omega\wedge\tau|+n+j}(\omega_{|\omega\wedge\tau|+j} - \tau_{|\omega\wedge\tau|+j})\lambda^j \\ &= \lambda^{|\omega\wedge\tau|} \sum_{j=0}^{\infty} p_{|\omega\wedge\tau|+n+j} a_j \lambda^j \ \ (a_0 \in \{-1, 1\}, a_j \in \{-1, 0, 1\} \text{ for } j \in \mathbb{N}) \\ &= \lambda^{|\omega\wedge\tau|} p_{|\omega\wedge\tau|+n} \sum_{j=0}^{\infty} \frac{p_{|\omega\wedge\tau|+n+j}}{p_{|\omega\wedge\tau|+n}} a_j \lambda^j\end{aligned}$$

Since $(p_{|\omega\wedge\tau|+n}/p_{|\omega\wedge\tau|+n})a_0 \in \{-1, 1\}$ and for each $j \in \mathbb{N}$, $(p_{|\omega\wedge\tau|+n+j}/p_{|\omega\wedge\tau|+n})a_j \in G_{n,j}$, we have that $f_{n,\omega,\tau}(\lambda) := \sum_{j=0}^{\infty}(p_{|\omega\wedge\tau|+n+j}/p_{|\omega\wedge\tau|+n})a_j\lambda^j \in \mathcal{F}_n$. Then we have proved our proposition. □

LEMMA 3.1.14. *Let $m \in \mathbb{N}_0$ and $n \in \mathbb{N}_0$. For any $\omega, \tau \in I^\infty$ with $|\omega \wedge \tau| > j_{n,m}$ and for any $\lambda \in \mathbb{D}^*$ with $|\lambda| \leq 1/\sqrt[m]{2}$, we have*

$$|\Pi_{n,\lambda}(\omega) - \Pi_{n,\lambda}(\tau)| \leq C_n(\lambda)\rho_{n,m}(\omega, \tau)^{\frac{-\log|\lambda|}{\log 2}},$$

where $C_n(\lambda) := \sum_{j=0}^{\infty} b_{n,j}|\lambda|^j < \infty$, $b_{n,j} := \max G_{n,j}$.

PROOF. By Proposition 3.1.13, there exists $f_{n,\omega,\tau} \in \mathcal{F}_n$ such that

$$\begin{aligned}|\Pi_{n,\lambda}(\omega) - \Pi_{n,\lambda}(\tau)| &= |\lambda|^{|\omega\wedge\tau|} p_{|\omega\wedge\tau|+n}|f_{n,\omega,\tau}(\lambda)| \\ &= (\frac{1}{2^{|\omega\wedge\tau|}})^{\frac{-\log|\lambda|}{\log 2}} p_{|\omega\wedge\tau|+n}|f_{n,\omega,\tau}(\lambda)|.\end{aligned}$$

Since $|\lambda| \leq 1/\sqrt[m]{2}$,

$$p_{|\omega\wedge\tau|+n} \leq (p_{|\omega\wedge\tau|+n})^{m\frac{-\log|\lambda|}{\log 2}}.$$

Hence we have that

$$\begin{aligned}(\frac{1}{2^{|\omega\wedge\tau|}})^{\frac{-\log|\lambda|}{\log 2}} p_{|\omega\wedge\tau|+n}|f_{n,\omega,\tau}(\lambda)| &\leq (\frac{1}{2^{|\omega\wedge\tau|}})^{\frac{-\log|\lambda|}{\log 2}} (p_{|\omega\wedge\tau|+n})^{m\frac{-\log|\lambda|}{\log 2}} |f_{n,\omega,\tau}(\lambda)| \\ &\leq C_n(\lambda)\rho_{n,m}(\omega,\tau)^{\frac{-\log|\lambda|}{\log 2}}.\end{aligned}$$

□

THEOREM 3.1.15. *Let $n \in \mathbb{N}_0$. Then for any $\lambda \in \mathbb{D}^*$,*

$$\dim_H(L_n(\lambda)) \leq \frac{\log 2}{-\log|\lambda|}.$$

PROOF. Fix $\lambda \in \mathbb{D}^*$. Since $1/\sqrt[m]{2} \to 1$ as $m \to \infty$, there exists m_0 such that $|\lambda| \leq 1/\sqrt[m_0]{2}$. By Lemma 3.1.14, for any $\omega, \tau \in I^\infty$ with $|\omega \wedge \tau| > j_{n,m_0}$,

$$|\Pi_{n,\lambda}(\omega) - \Pi_{n,\lambda}(\tau)| \leq C_n(\lambda)\rho_{n,m_0}(\omega,\tau)^{\frac{-\log|\lambda|}{\log 2}}.$$

Hence we have that

$$\dim_H(L_n(\lambda)) \leq \frac{\log 2}{-\log|\lambda|} \dim_{\rho_{n,m_0}}(I^\infty) = \frac{\log 2}{-\log|\lambda|}$$

by Proposition 3.1.5 (see the proof of [**11**, Proposition 3.3]). □

3.2. Some lemmas

3.2.1. Frostman's Lemma and an inverse Frostman's Lemma.

DEFINITION 3.2.1 (Frostman measure). Let m be a Borel measure on $\mathbb{R}^d$. Let $t \geq 0$. Let E be a Borel subset of $\mathbb{R}^d$. We say that m is a Frostman measure on E with exponent t if $0 < m(E) < \infty$ and there exists a constant $C = C_t > 0$ such that for each $x \in \mathbb{R}^d$ and for each $r > 0$, $m(B(x,r)) \leq Cr^t$.

Let $\mathcal{H}^t$ be the $t-$dimensional Hausdorff outer measure on $\mathbb{R}^d$ with respect to $|\cdot|$. We give the following lemma, which is known as Frostman's Lemma.

LEMMA 3.2.2. [**11**, Corollary 4.12] *Let E be a Borel subset of $\mathbb{R}^d$ with $\mathcal{H}^t(E) > 0$. Then there exists a Frostman measure on E with exponent t.*

COROLLARY 3.2.3. *Let $0 < t \leq 2$. For each $x \in \mathbb{R}^2$ and for each $r > 0$, there exists a Frostman measure m on $B(x,r)$ with exponent t.*

PROOF. If $0 < t < 2$, then by Lemma 3.2.2, there exists a Frostman measure m on $B(x,r)$ with exponent t since $\mathcal{H}^t(B(x,r)) = \infty$. If $t = 2$, we set $m = \mathcal{L}_2$. □

DEFINITION 3.2.4 ($s-$energy of measures). Let m be a Borel measure on $\mathbb{R}^d$. For any $s \geq 0$, we define the *$s-$energy* of m as

$$I_s(m) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{1}{|x-y|^s} dm(x)dm(y).$$

We give the following lemma, which is known as an inverse Frostman's Lemma.

LEMMA 3.2.5. [**11**, Theorem 4.13] *Let m be a finite Borel measure on $\mathbb{R}^d$. Let A be a Borel subset of $\mathbb{R}^d$ with $m(A) > 0$. If $I_s(m) < \infty$, then $\dim_H(A) \geq s$.*

3.2.2. Differentiation of measures. Let $d \in \mathbb{N}$. Let μ and m be Borel measures on $\mathbb{R}^d$ such that $\mu(G) < \infty$ and $\lambda(G) < \infty$ for any compact subset G. We say that the measure μ is absolutely continuous with respect to the measure m if $m(A) = 0$ implies $\mu(A) = 0$ for all Borel subsets A.

Definition 3.2.6. The *lower derivative* of μ with respect to m at a point $x \in \mathbb{R}^d$ is defined by

$$\underline{D}(\mu, m, x) := \liminf_{r \to 0} \frac{\mu(B(x,r))}{m(B(x,r))}.$$

Note that the function $x \mapsto \underline{D}(\mu, m, x)$ is Borel measurable. For the details of differentiation of measures, see [**19**, p. 36]. The lower derivatives of measures are related to the absolute continuity of measures by the following.

Lemma 3.2.7. [**19**, 2.12 Theorem] *Let μ and m be Borel measures on $\mathbb{R}^n$ such that $\mu(G) < \infty$ and $m(G) < \infty$ for any compact subset G. Then μ is absolutely continuous with respect to m if and only if $\underline{D}(\mu, m, x) < \infty$ for μ a.e. $x \in \mathbb{R}^n$.*

3.2.3. A technical lemma for the transversality. We give a technical lemma for the transversality condition. In order to prove it, we give some definition and remark.

Definition 3.2.8. Let G be a compact subset of $\mathbb{R}^d$. We say that a family of balls $\{B(x_i, r_i)\}_{i=1}^k$ in $\mathbb{R}^d$ is *packing for G* if for each $i \in \{1, ..., k\}$, $x_i \in G$ and for each $i, j \in \{1, ..., k\}$ with $i \neq j$, $B(x_i, r_i) \cap B(x_j, r_j) = \emptyset$.

Remark 3.2.9. Let G be a compact subset of $\mathbb{R}^d$, let $r > 0$ and let $\{B(x_i, r)\}_{i=1}^k$ be a family of balls in $\mathbb{R}^d$. If $\{B(x_i, r)\}_{i=1}^k$ is packing for G, then there exists $N \in \mathbb{N}$ which only depends on G and r such that $k \leq N$.

Proof. There exists a finite covering $\{B(y_j, r/2)\}_{j=1}^N$ for G since G is compact. Here, N only depends on G and r. Since $x_i \in G$ for each i, there exists j_i such that $x_i \in B(y_{j_i}, r/2)$. Since $\{B(x_i, r)\}_{i=1}^k$ is a disjoint family, if $i \neq l \in \{1, ..., k\}$, then $j_i \neq j_l$. Thus $k \leq N$. □

We now give a slight variation of [**32**, Lemma 5.2].

Lemma 3.2.10. *Let $\mathcal{H}$ be a compact subset of the space of holomorphic functions on $\mathbb{D}$. We set*

$$\tilde{\mathcal{M}}_H := \{\lambda \in \mathbb{D}^* \ : \textit{there exists } f \in \mathcal{H} \text{ such that } f(\lambda) = f'(\lambda) = 0\}.$$

Let G be a compact subset of $\mathbb{D}^* \backslash \tilde{\mathcal{M}}_H$. Let $t \geq 0$ and let $\mathcal{L}^t$ be a Frostman measure on G with exponent t. Then there exists $K > 0$ such that for any $f \in \mathcal{H}$ and for any $r > 0$,

$$\mathcal{L}^t(\{\lambda \in G : |f(\lambda)| \leq r\}) \leq K r^t. \tag{25}$$

Proof. Since $\mathcal{H}$ is compact and the set $\tilde{\mathcal{M}}_H$ is the set of possible double zeros, we have that there exists $\delta = \delta_G > 0$ such that for any $f \in \mathcal{H}$,

$$|f(\lambda)| < \delta \Rightarrow |f'(\lambda)| > \delta \ \text{ for } \lambda \in G. \tag{26}$$

We assume that $r < \delta$, otherwise (25) holds with $K = \mathcal{L}^t(G)/\delta^t$. Let

$$\Delta_r := \{\lambda \in G : |f(\lambda)| \leq r\}.$$

Let $\mathrm{Co}(G)$ be the convex hull of G. We set $M = M_G := \sup\{|g''(\lambda)| \in [0,\infty) : \lambda \in \mathrm{Co}(G), g \in \mathcal{H}\}$. Since $\mathrm{Co}(G)$ is compact and $\mathcal{H}$ is compact, $M < \infty$. Fix $z_0 \in \Delta_r$. By Taylor's formula, for $z \in G$,

$$|f(z) - f(z_0)| = |f'(z_0)(z - z_0) + \int_{z_0}^{z} (z - \xi) f''(\xi) d\xi|,$$

where the integration is performed along the straight line path from z_0 to z. Then $|f'(z_0)| > \delta$ by (26). Hence

$$|f(z) - f(z_0)| \geq |f'(z_0)||z - z_0| - M|z - z_0|^2 > \delta|z - z_0| - M|z - z_0|^2.$$

Now if we set

$$A_{z_0,r} := \left\{ z \in \mathbb{D}^* \ : \ \frac{4r}{\delta} < |z - z_0| < \frac{\delta}{2M} \right\},$$

then for any $z \in A_{z_0,r}$,

$$\delta|z - z_0| - M|z - z_0|^2 = |z - z_0|(\delta - M|z - z_0|) > \frac{4r}{\delta}\frac{\delta}{2} = 2r,$$

and $|f(z)| \geq |f(z) - f(z_0)| - |f(z_0)| > r$. It follows that the annulus $A_{z_0,r}$ does not intersect Δ_r.

Assume that $4r/\delta \leq \delta/4M$, otherwise (25) holds with $K = \mathcal{L}^t(G)(16M/\delta^2)^t$. Then the disc $B(z_0, \delta/4M)$ centered at z_0 with the radius $\delta/4M$ covers $\Delta_r \cap \{z : |z - z_0| < \delta/2M\}$. Then fix $z_1 \in \Delta_r \backslash \{z : |z - z_0| < \delta/2M\}$. Since the annulus $A_{z_1,r}$ does not intersect Δ_r, $B(z_1, \delta/4M)$ covers $(\Delta_r \backslash \{z : |z - z_0| < \delta/2M\}) \cap \{z : |z - z_1| < \delta/2M\}$ and $B(z_0, \delta/4M) \cap B(z_1, \delta/4M) = \emptyset$. If we repeat the procedure, we get a finite covering $\{B(z_i, \delta/4M)\}_{i=0}^k$ for Δ_r since Δ_r is compact. Then $\{B(z_i, \delta/4M)\}_{i=0}^k$ is packing for G. By Remark 3.2.9, there exists $N \in \mathbb{N}$ which only depends on $\mathcal{H}$ and G such that $k \leq N$. Since the annulus $A_{z_i,r}$ does not intersect Δ_r for each $i \in \{0, ..., k\}$, $\{B(z_i, 4r/\delta)\}_{i=0}^k$ is also a covering for Δ_r. Hence we have

$$\begin{aligned}
\mathcal{L}^t(\Delta_r) &\leq \mathcal{L}^t(\bigcup_{i=0}^{k} \{B(z_i, 4r/\delta)\}) \\
&= \sum_{i=0}^{k} \mathcal{L}^t(\{B(z_i, 4r/\delta)\}) \\
&\leq NC(\frac{4r}{\delta})^t = NC(\frac{4}{\delta})^t r^t,
\end{aligned}$$

where C denotes a constant which appears in the definition of $\mathcal{L}^t$. If we set $K := NC(4/\delta)^t$, we get the desired inequality. □

3.3. Proofs of main results

3.3.1. The lower estimation of the Hausdorff dimension for typical parameters. For each $n \in \mathbb{N}_0$, we endow I^∞ with the metric $\rho_{n,0}$ (for the definition of $\rho_{n,0}$, see Definition 3.1.3). Since the metric $\rho_{n,0}$ does not depend on n, we set $\rho_0 := \rho_{n,0}$. We consider the address maps $\Pi_{n,\lambda} : (I^\infty, \rho_0) \to \mathbb{C}$ for $\lambda \in \mathbb{D}^*$. Fix $\delta > 0$. Then for any $\lambda, \eta \in B(0, \delta) \cap \mathbb{D}^*$

and any $\omega = \omega_0\omega_1 \cdots \in I^\infty$,

$$
\begin{aligned}
|\Pi_{n,\lambda}(\omega) - \Pi_{n,\eta}(\omega)| &\le \sum_{j=0}^{\infty} p_{n+j}\omega_j |\lambda^j - \eta^j| \\
&\le \sum_{j=0}^{\infty} p_{n+j}|\lambda - \eta|(|\lambda|^{j-1} + |\lambda|^{j-2}|\eta| + \cdots + |\lambda||\eta|^{j-2} + |\eta|^{j-1}) \\
&\le \sum_{j=0}^{\infty} j p_{n+j}|\lambda - \eta|\delta^{j-1}.
\end{aligned}
$$

Hence we have the following.

Remark 3.3.1. Let $\lambda \in \mathbb{D}^*$. If $\lambda_j \to \lambda$ as $j \to \infty$, then $\Pi_{n,\lambda_j}(\cdot)$ uniformly converges to $\Pi_{n,\lambda}(\cdot)$ on I^∞. In particular, the sequence of sets $\{L_n(\lambda_j)\}_{j=1}^{\infty}$ converges to $L_n(\lambda)$ in the Hausdorff metric.

By Proposition 3.1.13, if we set $C_n(\lambda) := \sum_{j=0}^{\infty} b_{n,j}|\lambda|^j < \infty$, where $b_{n,j} := \max G_{n,j}$,

$$|\Pi_{n,\lambda}(\omega) - \Pi_{n,\lambda}(\tau)| \le |\lambda|^{|\omega\wedge\tau|} p_{|\omega\wedge\tau|+n} C_n(\lambda)$$

for any $\omega, \tau \in I^\infty$. If $\rho_0(\omega_j, \omega) = 1/2^{|\omega_j\wedge\omega|} \to 0$ as $j \to \infty$, then $|\lambda|^{|\omega_j\wedge\omega|} p_{|\omega_j\wedge\omega|+n} \to 0$. Hence for each $\lambda \in \mathbb{D}^*$, the map $\omega \mapsto \Pi_{n,\lambda}(\omega)$ is continuous on I^∞. We set $\alpha : \mathbb{D}^* \to [0, \infty)$ by

$$\alpha(\lambda) := \frac{-\log|\lambda|}{\log 2}.$$

For any compact subset $G \subset \mathbb{D}^*$, we set $\alpha_G := \sup\{\alpha(\lambda) : \lambda \in G\}$. We set $U_n := \mathbb{D}^* \setminus \tilde{\mathcal{N}}_n$ (for the definition of $\tilde{\mathcal{N}}_n$, see Definition 3.1.10).

Lemma 3.3.2. *Let G be a compact subset of U_n and let $\mathcal{L}^t$ be a Frostman measure on G with exponent t for some $t > 0$. Then there exists $K_{n,G} > 0$ such that for any $r > 0$ and any $(\omega, \tau) \in I^\infty \times I^\infty$ with $\omega \neq \tau$,*

$$\mathcal{L}^t(\{\lambda \in G : |\Pi_{n,\lambda}(\omega) - \Pi_{n,\lambda}(\tau)| \le r\}) \le K_{n,G}\,\rho_0(\omega, \tau)^{-t\alpha_G} r^t.$$

Proof. By Proposition 3.1.13, for any $(\omega, \tau) \in I^\infty \times I^\infty$ with $\omega \neq \tau$, there exists $f_{n,\omega,\tau} \in \mathcal{F}_n$ such that $\Pi_{n,\lambda}(\omega) - \Pi_{n,\lambda}(\tau) = \lambda^{|\omega\wedge\tau|} p_{|\omega\wedge\tau|+n} f_{n,\omega,\tau}(\lambda)$. Hence for any $r > 0$,

$$\{\lambda \in G : |\Pi_{n,\lambda}(\omega) - \Pi_{n,\lambda}(\tau)| \le r\} = \{\lambda \in G : |f_{n,\omega,\tau}(\lambda)| \le \rho_0(\omega, \tau)^{-\alpha(\lambda)} \frac{1}{p_{|\omega\wedge\tau|+n}} r\}.$$

Since $\mathcal{F}_n$ is a compact subset of the space of holomorphic functions on $\mathbb{D}$, by Lemma 3.2.10 we have that for any compact subset $G \subset \mathbb{D}^* \setminus \tilde{\mathcal{N}}_n$, there exists $K_{n,G} > 0$ such that for any $r > 0$,

$$
\begin{aligned}
\mathcal{L}^t(\{\lambda \in G : |\Pi_{n,\lambda}(\omega) - \Pi_{n,\lambda}(\tau)| \le r\}) &= \mathcal{L}^t(\{\lambda \in G : |f_{n,\omega,\tau}(\lambda)| \le \rho_0(\omega, \tau)^{-\alpha(\lambda)} \frac{1}{p_{|\omega\wedge\tau|+n}} r\}) \\
&\le K_{n,G}\,\rho_0(\omega, \tau)^{-t\alpha(\lambda)} \frac{1}{(p_{|\omega\wedge\tau|+n})^t} r^t \\
&\le K_{n,G}\,\rho_0(\omega, \tau)^{-t\alpha_G} r^t.
\end{aligned}
$$

$\square$

Let μ be the $(1/2, 1/2)-$Bernoulli measure on I^∞. Let $\nu_{n,\lambda} := \Pi_{n,\lambda}\mu$, where $\Pi_{n,\lambda}\mu$ denotes the push-forward measure of μ under $\Pi_{n,\lambda}$. This is a Borel probability measure on $\Pi_{n,\lambda}(I^\infty) = L_n(\lambda)$, since the map $\omega \mapsto \Pi_{n,\lambda}(\omega)$ is continuous on I^∞.

LEMMA 3.3.3. *Let* $0 \leq s < 1$. *Then*

$$\int_{I^\infty}\int_{I^\infty} \rho_0(\omega,\tau)^{-s}\ d\mu(\omega)d\mu(\tau) < \infty.$$

PROOF. For any $i \in I$, we set

$$\tilde{i} := \begin{cases} 1 & (i = 0) \\ 0 & (i = 1). \end{cases}$$

Then

$$\begin{aligned}
\int_{I^\infty}\int_{I^\infty} \rho_0(\omega,\tau)^{-s}\ d\mu(\omega)d\mu(\tau) &= \int_{I^\infty}\int_{I^\infty} 2^{s|\omega\wedge\tau|}\ d\mu(\omega)d\mu(\tau) \\
&= \int_{I^\infty}\sum_{j=0}^{\infty}\int_{\{\omega\ :\ |\omega\wedge\tau|=j\}} 2^{s|\omega\wedge\tau|}\ d\mu(\omega)d\mu(\tau) \\
&= \int_{I^\infty}\sum_{j=0}^{\infty} 2^{sj}\mu([\tau_0\tau_1\cdots\tau_{j-1}\tilde{\tau_j}])d\mu(\tau) \\
&= \frac{1}{2}\int_{I^\infty}\sum_{j=0}^{\infty} 2^{(s-1)j}d\mu(\tau) \\
&= \frac{1}{2}\int_{I^\infty}\frac{1}{1-2^{(s-1)}}d\mu(\tau) \\
&= \frac{1}{2}\frac{1}{1-2^{(s-1)}}.
\end{aligned}$$

$\square$

LEMMA 3.3.4. *Let* $\lambda \in \mathbb{D}^*$. *Let* $s_1 \geq s_2 \geq 0$. *If*

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |u-v|^{-s_2}\ d\nu_{n,\lambda}(u)d\nu_{n,\lambda}(v) = \infty,$$

then

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |u-v|^{-s_1}\ d\nu_{n,\lambda}(u)d\nu_{n,\lambda}(v) = \infty.$$

PROOF. Since for any Borel subset $B \subset \mathbb{R}^2$ with $B \cap L_n(\lambda) = \emptyset$, $\nu_{n,\lambda}(B) = 0$, we have

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |u-v|^{-s_1}\ d\nu_{n,\lambda}(u)d\nu_{n,\lambda}(v) = \int_{L_n(\lambda)}\int_{L_n(\lambda)} |u-v|^{-s_1}\ d\nu_{n,\lambda}(u)d\nu_{n,\lambda}(v).$$

If we set $D := \sup_{u,v \in L_n(\lambda)} |u-v| < \infty$, then we have

$$\begin{aligned}\int_{L_n(\lambda)}\int_{L_n(\lambda)} |u-v|^{-s_1}\ d\nu_{n,\lambda}(u)d\nu_{n,\lambda}(v) &= \int_{L_n(\lambda)}\int_{L_n(\lambda)} D^{-s_1}(\frac{|u-v|}{D})^{-s_1}\ d\nu_{n,\lambda}(u)d\nu_{n,\lambda}(v)\\ &\geq \int_{L_n(\lambda)}\int_{L_n(\lambda)} D^{-s_1}(\frac{|u-v|}{D})^{-s_2}\ d\nu_{n,\lambda}(u)d\nu_{n,\lambda}(v)\\ &= D^{-s_1+s_2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |u-v|^{-s_2}\ d\nu_{n,\lambda}(u)d\nu_{n,\lambda}(v)\\ &= \infty.\end{aligned}$$

□

LEMMA 3.3.5. *Let $\beta > 0$. Then the function*

$$\lambda \mapsto \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |u-v|^{-\beta}\ d\nu_{n,\lambda}(u)d\nu_{n,\lambda}(v)$$

is Borel measurable on $\mathbb{D}^*$.

PROOF. For any $\lambda \in \mathbb{D}^*$,

$$\Phi(\lambda) := \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |u-v|^{-\beta}\ d\nu_{n,\lambda}(u)d\nu_{n,\lambda}(v) = \int_{I^\infty}\int_{I^\infty} |\Pi_{n,\lambda}(\omega)-\Pi_{n,\lambda}(\tau)|^{-\beta}\ d\mu(\omega)d\mu(\tau).$$

Fix a sequence $\{\lambda_j\}_{j=1}^{\infty} \to \lambda$ as $j \to \infty$. Then

$$|\Pi_{n,\lambda_j}(\omega)-\Pi_{n,\lambda_j}(\tau)|^{-\beta} \to |\Pi_{n,\lambda}(\omega)-\Pi_{n,\lambda}(\tau)|^{-1/\beta} \in (0,\infty]$$

as $j \to \infty$ for each $\omega, \tau \in I^\infty$ by Remark 3.3.1. By Fatou's Lemma,

$$\begin{aligned}&\int_{I^\infty}\int_{I^\infty} |\Pi_{n,\lambda}(\omega)-\Pi_{n,\lambda}(\tau)|^{-\beta}\ d\mu(\omega)d\mu(\tau)\\ &= \int_{I^\infty}\int_{I^\infty} \liminf_{j\to\infty} |\Pi_{n,\lambda_j}(\omega)-\Pi_{n,\lambda_j}(\tau)|^{-\beta}\ d\mu(\omega)d\mu(\tau)\\ &\leq \liminf_{j\to\infty}\int_{I^\infty}\int_{I^\infty} |\Pi_{n,\lambda_j}(\omega)-\Pi_{n,\lambda_j}(\tau)|^{-\beta}\ d\mu(\omega)d\mu(\tau).\end{aligned}$$

Hence the function $\lambda \mapsto \Phi(\lambda)$ is lower semi-continuous, and hence Borel measurable. □

We give key lemmas as the following.

LEMMA 3.3.6. *Let $0 < t \leq 2$. For any $\lambda_0 \in U_n \cap \{\lambda \in \mathbb{D}^* : 1/\alpha(\lambda) \leq t\}$ and any $\epsilon > 0$, there exists $\delta > 0$ such that for any Frostman measure $\mathcal{L}^t$ on $B(\lambda_0, \delta)$ with exponent t,*

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |u-v|^{-(1/\alpha(\lambda_0)-\epsilon)}\ d\nu_{n,\lambda}(u)d\nu_{n,\lambda}(v) < \infty$$

for $\mathcal{L}^t$−a.e. λ in $B(\lambda_0, \delta)$.

PROOF. Fix $\lambda_0 \in U_n \cap \{\lambda \in \mathbb{D}^* : 1/\alpha(\lambda) \leq t\}$ and any $\epsilon > 0$. There exists $\delta > 0$ such that $1/\alpha(\lambda_0) - \epsilon < 1/\alpha_{\mathrm{cl}(B(\lambda_0,\delta))}$ since α is continuous. Below, we set $s = 1/\alpha(\lambda_0) - \epsilon$ and $G := \mathrm{cl}(B(\lambda_0, \delta))$. Then

$$\int_{I^\infty}\int_{I^\infty} \rho_0(\omega,\tau)^{-s\alpha_G}\ d\mu(\omega)d\mu(\tau) < \infty$$

by Lemma 3.3.3 since $s\alpha_G < 1$. If we prove

$$\mathcal{S} := \int_G \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} |u-v|^{-s} \ d\nu_{n,\lambda}(u) d\nu_{n,\lambda}(v) d\mathcal{L}^t(\lambda) < \infty,$$

we get the desired result. By changing variables and Fubini's Theorem,

$$\mathcal{S} = \int_{I^\infty} \int_{I^\infty} \int_G |\Pi_{n,\lambda}(\omega) - \Pi_{n,\lambda}(\tau)|^{-s} \ d\mathcal{L}^t(\lambda) d\mu(\omega) d\mu(\tau).$$

By using Lemma 3.3.2 and $\mathcal{L}^t(G) < \infty$, we have that for any $r > 0$ and any $\omega, \tau \in I^\infty$,

$$\mathcal{L}^t(\{\lambda \in G : |\Pi_{n,\lambda}(\omega) - \Pi_{n,\lambda}(\tau)| \le r\}) \le \text{Const.} \min\{1, \rho_0(\omega,\tau)^{-t\alpha_G} r^t\}.$$

Here, we set Const. $:= \max\{1, \mathcal{L}^t(G)\} K_{n,G}$, where $K_{n,G}$ comes from Lemma 3.3.2. Then by using that $s < t$, we obtain

$$\begin{aligned}
\int_G |\Pi_{n,\lambda}(\omega) - \Pi_{n,\lambda}(\tau)|^{-s} \ d\mathcal{L}^t(\lambda) &= \int_0^\infty \mathcal{L}^t(\{\lambda \in G : |\Pi_{n,\lambda}(\omega) - \Pi_{n,\lambda}(\tau)|^{-s} \ge x\}) \ dx \\
&\le \text{Const.} \int_0^\infty \min\{1, \rho_0(\omega,\tau)^{-t\alpha_G} x^{-t/s}\} \ dx \\
&= \text{Const.} \Big(\int_0^{\rho_0(\omega,\tau)^{-s\alpha_G}} 1 \ dx \\
&+ \rho_0(\omega,\tau)^{-t\alpha_G} \int_{\rho_0(\omega,\tau)^{-s\alpha_G}}^\infty x^{-t/s} \ dx \Big) \\
&= \text{Const.}' \rho_0(\omega,\tau)^{-s\alpha_G}.
\end{aligned}$$

Here, we set Const.$' := \left(\text{Const.} + \frac{1}{t/s-1} \right)$. Hence we have $\mathcal{S} < \infty$. □

LEMMA 3.3.7. *For any $\lambda_0 \in U_n \cap \{\lambda \in \mathbb{D}^* \ : \ 1/\alpha(\lambda) > 2\}$, there exists $\delta > 0$ such that*

$$\mathcal{L}_2(L_n(\lambda)) > 0$$

for $\mathcal{L}_2$-a.e. λ in $B(\lambda_0, \delta)$.

PROOF. Fix any $\lambda_0 \in U_n \cap \{\lambda \in \mathbb{D}^* \ : \ 1/\alpha(\lambda) > 2\}$ and any $\epsilon > 0$ with $(1-\epsilon)/\alpha(\lambda_0) > 2$. Then by Lemma 3.3.3,

$$\int_{I^\infty} \int_{I^\infty} \rho_0(\omega,\tau)^{-(1-\epsilon)} \ d\mu(\omega) d\mu(\tau) < \infty.$$

There exists $\delta > 0$ such that $(1-\epsilon)/\alpha_{\mathrm{cl}(B(\lambda_0,\delta))} > 2$ since α is continuous. It suffices to prove that $\nu_{n,\lambda}$ is absolutely continuous with respect to $\mathcal{L}_2$ for $\mathcal{L}_2$-a.e. λ in $B(\lambda_0, \delta)$. We set $G = \mathrm{cl}(B(\lambda_0,\delta))$. Let

$$\underline{D}(\nu_{n,\lambda}, u) := \liminf_{r \to 0} \frac{\nu_{n,\lambda}(B(u,r))}{\mathcal{L}_2(B(u,r))}$$

be the lower derivative of $\nu_{n,\lambda}$ with respect to $\mathcal{L}_2$ at the point u. If we show that

$$\mathcal{S} := \int_G \int_{\mathbb{R}^2} \underline{D}(\nu_{n,\lambda}, u) \ d\nu_{n,\lambda} d\mathcal{L}_2(\lambda) < \infty,$$

then for $\mathcal{L}_2$- a.e. $\lambda \in G$ we have $\underline{D}(\nu_{n,\lambda}, u) < \infty$ for $\nu_{n,\lambda}$-a.e. u and hence $\nu_{n,\lambda}$ is absolutely continuous with respect to $\mathcal{L}_2$ by Lemma 3.2.7. By Fatou's Lemma,

$$\mathcal{S} \le \text{Const.} \liminf_{r \to 0} r^{-2} \int_G \int_{\mathbb{R}^2} \nu_{n,\lambda}(B(u,r)) \ d\nu_{n,\lambda}(u) d\mathcal{L}_2(\lambda).$$

Then

$$\int_{\mathbb{R}^2} \nu_{n,\lambda}(B(u,r))\ d\nu_{n,\lambda}(u) = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \chi_{B(u,r)}(v)\ d\nu_{n,\lambda}(v)d\nu_{n,\lambda}(u)$$
$$= \int_{I^\infty}\int_{I^\infty} \chi_{\{\tau\in I^\infty : |\Pi_{n,\lambda}(\omega)-\Pi_{n,\lambda}(\tau)|\le r\}}\ d\mu(\tau)d\mu(\omega),$$

where χ_A is the characteristic function with respect to the set A. By Fubini's Theorem, integrating with respect to λ,

$$\mathcal{S} \le \text{Const.} \liminf_{r\to 0} r^{-2} \int_{I^\infty}\int_{I^\infty} \mathcal{L}_2\{\lambda \in G : |\Pi_{n,\lambda}(\omega) - \Pi_{n,\lambda}(\tau)| \le r\}\ d\mu(\omega)\mu(\tau).$$

By using Lemma 3.3.2, we have that

$$\mathcal{S} \le \text{Const.}' \int_{I^\infty}\int_{I^\infty} \rho_0(\omega,\tau)^{-2\alpha_G}\ d\mu(\omega)d\mu(\tau),$$

which is finite by the inequality $2\alpha_G < 1-\epsilon$ and Lemma 3.3.3. □

THEOREM 3.3.8. *Let $n \in \mathbb{N}_0$. Then we have the following.*

(i)

$$\dim_H(L_n(\lambda)) \ge \frac{\log 2}{-\log|\lambda|} \text{ for } \mathcal{L}_2\text{-a.e. } \lambda \in \{\lambda \in \mathbb{D}^* \ :\ 0 < |\lambda| < 1/\sqrt{2}\}\backslash\tilde{\mathcal{N}_n}.$$

(ii)

$$\mathcal{L}_2(L_n(\lambda)) > 0 \text{ for } \mathcal{L}_2\text{-a.e. } \lambda \in \{\lambda \in \mathbb{D}^* \ :\ 1/\sqrt{2} < |\lambda| < 1\}\backslash\tilde{\mathcal{N}_n}. \tag{27}$$

PROOF. We first prove (i). We set $V_n := \{\lambda \in \mathbb{D}^* \ :\ 0 < |\lambda| < 1/\sqrt{2}\}\backslash\tilde{\mathcal{N}_n}$. Fix $k \in \mathbb{N}$ with $k \ge 2$. We prove

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |u-v|^{-(1/\alpha(\lambda)-1/k)}\ d\nu_{n,\lambda}(u)d\nu_{n,\lambda}(v) < \infty \tag{28}$$

for $\mathcal{L}_2$-a.e. λ in V_n.

Suppose that (28) does not hold. Then there exists a Lebesgue density point $\lambda_0 \in V_n$ of the set

$$\{\lambda \in V_n \ :\ \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |u-v|^{-(1/\alpha(\lambda)-1/k)}\ d\nu_{n,\lambda}(u)d\nu_{n,\lambda}(v) = \infty\}.$$

Then there exists $\delta_0 > 0$ such that for each $\delta \in (0,\delta_0)$,

$$\mathcal{L}_2\left(\{\lambda \in B(\lambda_0,\delta) \ :\ \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |u-v|^{-(1/\alpha(\lambda)-1/k)}\ d\nu_{n,\lambda}(u)d\nu_{n,\lambda}(v) = \infty\}\right) > 0.$$

By the continuity of the function $\lambda \mapsto 1/\alpha(\lambda)$, if δ is small enough, then $1/\alpha(\lambda) - 1/k < 1/\alpha(\lambda_0) - 1/2k$ for each $\lambda \in B(\lambda_0,\delta)$. Hence for all sufficiently small δ , by Lemma 3.3.4, we have that

$$\mathcal{L}_2\left(\{\lambda \in B(\lambda_0,\delta) \ :\ \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |u-v|^{-(1/\alpha(\lambda_0)-1/2k)}\ d\nu_{n,\lambda}(u)d\nu_{n,\lambda}(v) = \infty\}\right) > 0.$$

This however contradicts Lemma 3.3.6 since $\mathcal{L}_2$ is a Frostman measure on $B(\lambda_0,\delta)$ with exponent 2. Thus we have proved (28). By Lemma 3.2.5, we have that

$$\dim_H(L_n(\lambda)) \ge \frac{\log 2}{-\log|\lambda|} - \frac{1}{k} \text{ for } \mathcal{L}_2\text{-a.e. } \lambda \in \{\lambda \in \mathbb{D}^* \ :\ 0 < |\lambda| < 1/\sqrt{2}\}\backslash\tilde{\mathcal{N}_n}.$$

By letting $k \to \infty$, we prove (i).

Statement (ii) follows from Lemma 3.3.7 in a similar way.

□

COROLLARY 3.3.9.

$$\dim_H(L_0(\lambda)) \geq \frac{\log 2}{-\log|\lambda|} \text{ for } \mathcal{L}_2\text{−a.e. } \lambda \in \{\lambda \in \mathbb{D}^* \ : \ 0 < |\lambda| < 1/\sqrt{2}\}\backslash\tilde{\mathcal{M}}_2;$$

$$\mathcal{L}_2(L_0(\lambda)) > 0 \text{ for } \mathcal{L}_2\text{−a.e. } \lambda \in \{\lambda \in \mathbb{D}^* \ : \ 1/\sqrt{2} < |\lambda| < 1\}\backslash\tilde{\mathcal{M}}_2.$$

PROOF. By Theorem 3.3.8 and Corollary 3.1.8, we have that

$$\dim_H(L_0(\lambda)) \geq \frac{\log 2}{-\log|\lambda|} \text{ for } \mathcal{L}_2\text{−a.e. } \lambda \in \{\lambda \in \mathbb{D}^* \ : \ 0 < |\lambda| < 1/\sqrt{2}\}\backslash\tilde{\mathcal{N}}_n;$$

$$\mathcal{L}_2(L_0(\lambda)) > 0 \text{ for } \mathcal{L}_2\text{−a.e. } \lambda \in \{\lambda \in \mathbb{D}^* \ : \ 1/\sqrt{2} < |\lambda| < 1\}\backslash\tilde{\mathcal{N}}_n.$$

By Lemma 3.1.12, letting $n \to \infty$, we get our corollary. □

We use the following theorem in order to prove one of our main results of this dissertation.

THEOREM 3.3.10. [**35**, Proposition 2.7] *A power series of the form* $1 + \sum_{j=1}^{\infty} a_j z^j$, *with* $a_j \in [-1, 1]$, *cannot have a non-real double zero of modulus less than* $2 \times 5^{-5/8} \approx 0.73143 (> 1/\sqrt{2})$.

Finally, we get the following theorem by using Theorem 3.1.15, Corollary 3.3.9 and Theorem 3.3.10.

THEOREM 3.3.11.

$$\dim_H(L_0(\lambda)) = \frac{\log 2}{-\log|\lambda|} \text{ for } \mathcal{L}_2\text{−a.e. } \lambda \in \{\lambda \in \mathbb{D}^* \ : \ 0 < |\lambda| < 1/\sqrt{2}\};$$

$$\mathcal{L}_2(L_0(\lambda)) > 0 \text{ for } \mathcal{L}_2\text{−a.e. } \lambda \in \{\lambda \in \mathbb{D}^* \ : \ 1/\sqrt{2} < |\lambda| < 1\}\backslash\tilde{\mathcal{M}}_2.$$

3.3.2. The estimation of local dimension of the exceptional set of parameters. In this subsection we give the estimation of local dimension of the exceptional set of parameters. Recall that $U_n = \mathbb{D}^*\backslash\tilde{\mathcal{N}}_n$ and $\alpha(\lambda) = -\log|\lambda|/\log 2$ for $\lambda \in \mathbb{D}^*$. Note that $\bigcup_{n\in\mathbb{N}_0} U_n = \mathbb{D}^*\backslash\tilde{\mathcal{M}}_2$ by Lemma 3.1.12.

LEMMA 3.3.12. *Let* G *be a compact subset of* U_n. *Then we have*

$$\dim_H\left(\left\{\lambda \in G \ : \ \dim_H(L_n(\lambda)) < \frac{\log 2}{-\log|\lambda|}\right\}\right) \leq \sup_{\lambda\in G} \frac{\log 2}{-\log|\lambda|}.$$

PROOF. We may assume that

$$G \subset \left\{\lambda \in \mathbb{D}^* \ : \ |\lambda| \leq \frac{1}{\sqrt{2}}\right\}.$$

We set $s_G := \sup_{\lambda\in G} \log 2/-\log|\lambda|$. By the countable stability of the Hausdorff dimension, it suffices to prove that for each $k \in \mathbb{N}$,

$$\dim_H\left(\left\{\lambda \in G \ : \ \dim_H(L_n(\lambda)) < \frac{\log 2}{-\log|\lambda|} - \frac{1}{k}\right\}\right) \leq s_G.$$

Since G is compact, it is enough to prove that for each $\lambda \in G$, there exists $\delta > 0$ such that

$$\dim_H\left(\left\{\lambda \in B(\lambda, \delta) \ : \ \dim_H(L_n(\lambda)) < \frac{\log 2}{-\log|\lambda|} - \frac{1}{k}\right\}\right) \leq s_G.$$

Suppose that this is false, that is, there exists $\lambda_0 \in G$ such that for any $\delta > 0$,

$$\dim_H \left(\left\{ \lambda \in B(\lambda_0, \delta) \ : \ \dim_H(L_n(\lambda)) < \frac{\log 2}{-\log|\lambda|} - \frac{1}{k} \right\} \right) > s_G.$$

Then by the continuity of the function $\lambda \mapsto \log 2/-\log|\lambda|$, there exists $\delta_0 > 0$ such that for any $0 < \delta < \delta_0$,

$$\dim_H \left(\left\{ \lambda \in B(\lambda_0, \delta) \ : \ \dim_H(L_n(\lambda)) < \frac{\log 2}{-\log|\lambda_0|} - \frac{1}{2k} \right\} \right) > s_G.$$

Take $\delta_1 > 0$ with $\delta_1 < \delta_0$ so that Lemma 3.3.6 holds with $t = s_G$ and $\epsilon = 1/2k$. By Lemma 3.2.5, we have

$$\begin{aligned} &\left\{ \lambda \in B(\lambda_0, \delta_1) \ : \ \dim_H(L_n(\lambda)) < \frac{\log 2}{-\log|\lambda_0|} - \frac{1}{2k} \right\} \\ &\subset \left\{ \lambda \in B(\lambda_0, \delta_1) \ : \ \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} |u - v|^{-(1/\alpha(\lambda_0) - 1/2k)} \ d\nu_{n,\lambda}(u) d\nu_{n,\lambda}(v) = \infty \right\} =: E. \end{aligned}$$

By Lemma 3.3.5, the set E is a Borel subset of $\mathbb{D}^*$. Since $\mathcal{H}^{s_G}(E) > 0$, by Lemma 3.2.2, there exists a Frostman measure $\mathcal{L}^{s_G}$ on E with exponent s_G. However this contradicts Lemma 3.3.6 since $\mathcal{L}^{s_G}$ is also a Frostman measure on $B(\lambda_0, \delta_1)$ with exponent s_G. $\square$

THEOREM 3.3.13. *Let G be a compact subset of $\mathbb{D}^* \backslash \tilde{\mathcal{M}}_2$. Then we have*

$$\dim_H \left(\left\{ \lambda \in G \ : \ \dim_H(L_0(\lambda)) < \frac{\log 2}{-\log|\lambda|} \right\} \right) \leq \sup_{\lambda \in G} \frac{\log 2}{-\log|\lambda|}.$$

PROOF. Since $\bigcup_{n \in \mathbb{N}_0} U_n = \mathbb{D}^* \backslash \tilde{\mathcal{M}}_2$, there exists $n_0 \in \mathbb{N}_0$ such that $G \subset U_n$. By Lemma 3.3.12, we have

$$\dim_H \left(\left\{ \lambda \in G \ : \ \dim_H(L_n(\lambda)) < \frac{\log 2}{-\log|\lambda|} \right\} \right) \leq \sup_{\lambda \in G} \frac{\log 2}{-\log|\lambda|}.$$

By Corollary 3.1.8, we have that

$$\dim_H \left(\left\{ \lambda \in G \ : \ \dim_H(L_0(\lambda)) < \frac{\log 2}{-\log|\lambda|} \right\} \right) \leq \sup_{\lambda \in G} \frac{\log 2}{-\log|\lambda|}.$$

$\square$

THEOREM 3.3.14. *For any $0 < R < 1/\sqrt{2}$,*

$$\dim_H \left(\left\{ \lambda \in \mathbb{D}^* \ : \ 0 < |\lambda| < R, \ \dim_H(L_0(\lambda)) < \frac{\log 2}{-\log|\lambda|} \right\} \right) \leq \frac{\log 2}{-\log R} < 2.$$

PROOF. Let $0 < r < R < 1/\sqrt{2}$. If $R \leq 1/2$, by (1) and since $\tilde{\mathcal{M}}_2 \subset \mathcal{M}_2$,

$$\{\lambda \in \mathbb{D}^* \ : \ r < |\lambda| < R\} \backslash \tilde{\mathcal{M}}_2 = \{\lambda \in \mathbb{D}^* \ : \ r < |\lambda| < R\}.$$

For each $k \in \mathbb{N}$, we set $G_k := \{\lambda \in \mathbb{D}^* \ : \ r + 1/k \leq |\lambda| \leq R - 1/k\}$. Then G_k is a compact subset of $\mathbb{D}^* \backslash \tilde{\mathcal{M}}_2$ and $\bigcup_{k \in \mathbb{N}} G_k = \{\lambda \in \mathbb{D}^* \ : \ r < |\lambda| < R\}$. By Theorem 3.3.13 and the countable stability of the Hausdorff dimension, we have that

$$\dim_H \left(\left\{ \lambda \in \mathbb{D}^* \ : \ r < |\lambda| < R, \ \dim_H(L_0(\lambda)) < \frac{\log 2}{-\log|\lambda|} \right\} \right) \leq \frac{\log 2}{-\log R}.$$

If $1/2 < R \leq 1/\sqrt{2}$, then by Theorem 3.3.10,

$$\{\lambda \in \mathbb{D}^* \ : \ r < |\lambda| < R\} \backslash \tilde{\mathcal{M}}_2 = \{\lambda \in \mathbb{D}^* \backslash \mathbb{R} \ : \ r < |\lambda| < R\} \cup \{\lambda \in \mathbb{R} \ : \ r < |\lambda| < R\} \backslash \tilde{\mathcal{M}}_2.$$

For each $k \in \mathbb{N}$, we set

$$\begin{aligned} G_k := &\{\lambda \in \mathbb{D}^* \ : \ r + 1/k \leq |\lambda| \leq R - 1/k, \mathrm{Im}(\lambda) \geq 1/k\} \\ &\cup \{\lambda \in \mathbb{D}^* \ : \ r + 1/k \leq |\lambda| \leq R - 1/k, \mathrm{Im}(\lambda) \leq -1/k\}, \end{aligned}$$

where $\mathrm{Im}(\lambda)$ denotes the imaginary part of λ. Then G_k is a compact subset of $\mathbb{D}^* \backslash \tilde{\mathcal{M}}_2$ and $\bigcup_{k \in \mathbb{N}} G_k = \{\lambda \in \mathbb{D}^* \backslash \mathbb{R} \ : \ r < |\lambda| < R\}$. By Theorem 3.3.13 and the countable stability of the Hausdorff dimension, we have that

$$\dim_H \left(\left\{ \lambda \in \mathbb{D}^* \backslash \mathbb{R} \ : \ r < |\lambda| < R, \ \dim_H(L_0(\lambda)) < \frac{\log 2}{-\log |\lambda|} \right\} \right) \leq \frac{\log 2}{-\log R}.$$

Since $\dim_H(\mathbb{R}) = 1 < \log 2 / -\log R$, we have that

$$\dim_H \left(\left\{ \lambda \in \mathbb{D}^* \ : \ r < |\lambda| < R, \ \dim_H(L_0(\lambda)) < \frac{\log 2}{-\log |\lambda|} \right\} \right) \leq \frac{\log 2}{-\log R}.$$

By the countable stability of the Hausdorff dimension, we have that

$$\dim_H \left(\left\{ \lambda \in \mathbb{D}^* \ : \ 0 < |\lambda| < R, \ \dim_H(L_0(\lambda)) < \frac{\log 2}{-\log |\lambda|} \right\} \right) \leq \frac{\log 2}{-\log R}.$$

□

CHAPTER 4

$\mathcal{M}_n$ is connected

In this chapter, we consider the connectedness locus $\mathcal{M}_n$ for fractal n-gons in the parameter space.

4.1. Preliminaries

In this section we extend the method of Bousch [**5**]. Let G be a subset of $\mathbb{C}$ with $(*)$ in Definition 1.3.7. Let $N \in \mathbb{N}$ with $N \geq 2$. Let $\mathbb{D}$ be the unit disk. We set

$$\begin{aligned}
P^G &= \{1 + \sum_{i=1}^{\infty} a_i z^i \ : \ a_i \in G\}, \\
X^G &= \{z \in \mathbb{D} \ : \ \text{there exists } f \in P^G \text{ such that } f(z) = 0\}, \\
Q_N^G &= \{1 + \sum_{i=1}^{N-1} a_i z^i \ : \ a_i \in G\}, \\
Y_N^G &= \{z \in \mathbb{C} \ : \ \text{there exists } f \in Q_N^G \text{ such that } f(z) = 0\}, \\
Y^G &= \bigcup_{N \geq 2} Y_N^G.
\end{aligned}$$

Let $\mathcal{O}(\mathbb{D})$ be the set of holomorphic functions on $\mathbb{D}$.

Lemma 4.1.1. *Let G be a subset of $\mathbb{C}$ with $(*)$. Then $X^G = \mathrm{cl}(Y^G) \cap \mathbb{D}$.*

Proof. $(\subset)$Take $z \in X^G$. Then there exists $\{a_i\}_{i=1}^{\infty} \subset G$ such that $1 + \sum_{i=1}^{\infty} a_i z^i = 0$. Fix $\epsilon > 0$ with $B(z, \epsilon) \subset \mathbb{D}$. Then there exist $N \in \mathbb{N}$ and $z' \in B(z, \epsilon)$ such that $1 + \sum_{i=1}^{N-1} a_i z' = 0$ by theorem of Rouché. Hence $z \in \mathrm{cl}(Y^G) \cap \mathbb{D}$.

$(\supset)$Since P^G is a compact subset of $\mathcal{O}(\mathbb{D})$ endowed with the compact open topolgy, the set X^G is relatively closed in $\mathbb{D}$. Hence it suffices to prove that $X^G \supset Y^G \cap \mathbb{D}$. Take $z_0 \in Y^G \cap \mathbb{D}$. Then there exist $N \geq 2$ and $a_i \in G$ for any $i \in \{1, 2, ..., N-1\}$ such that

$$f(z_0) = 1 + \sum_{i=1}^{N-1} a_i {z_0}^i = 0.$$

We set $\tilde{f}(z) := f(z) \times \sum_{i=0}^{\infty} z^{iN} \in P^G$. Then $\tilde{f}(z_0) = 0$. Thus $z_0 \in X^G$.

Thus we have proved our lemma. $\square$

Below we fix a set $G \subset \mathbb{C}$ with $(*)$. We set $\mathbb{N}_{\geq 2} := \{n \in \mathbb{N} \ : \ n \geq 2\}$.

DEFINITION 4.1.2. Let $N \in \mathbb{N}_{\geq 2}$. We set $L := \sup\{|a|, |ab|, |(a-b)c| \ : \ a, b, c \in G\}(< \infty)$. Then we define the sets of functions W and W_N as the following.

$$W := \{1 + \sum_{i=1}^{\infty} a_i z^i \ : \ |a_i| \leq L\},$$

$$W_N := \{1 + \sum_{i=1}^{N-1} a_i z^i \ : \ |a_i| \leq L\}.$$

REMARK 4.1.3. $Q_N^G \subset W_N \subset W$ and $P^G \subset W$.

Let $N \in \mathbb{N}_{\geq 2}$. We identify $(1, a_1, a_2, ...)$ with the power series $1 + \sum_{i=1}^{\infty} a_i z^i$. We identify $(1, a_1, ..., a_{N-1})$ or $(1, a_1, ..., a_{N-1}, 0, 0, ...)$ with the polynomial $1 + \sum_{i=1}^{N-1} a_i z^i$. Let $f = (1, a_1, a_2, ...)$ and $g = (1, b_1, b_2, ...)$. We set $\mathrm{Val}(f, g) := \inf\{i \in \mathbb{N} \mid a_i - b_i \neq 0\}$. If $f = g$, we set $\mathrm{Val}(f, g) = \infty$. Let $N \in \mathbb{N}_{\geq 2}$. We define the map $C_N : W \to W_N$ by

$$C_N\big((1, a_1, a_2, ...)\big) := (1, a_1, ..., a_{N-1}).$$

We now give a slight variation of [**5**, Lemme 2].

LEMMA 4.1.4. *Let $R > 0$ and $\epsilon > 0$ with $R + \epsilon < 1$. Then there exists $N_{R,\epsilon} \in \mathbb{N}_{\geq 2}$ such that for all $(f, s) \in F := \{(f, s) \in W \times \mathrm{cl}(B(0, R)) \ : \ f(s) = 0\}$ and for all $g \in W$ with $\mathrm{Val}(f, g) \geq N_{R,\epsilon}$, there exists $s' \in B(s, \epsilon)$ such that $g(s') = 0$.*

PROOF. Since W is a compact subset of $\mathcal{O}(\mathbb{D})$ endowed with compact open topology, F is a compact subset of $\mathcal{O}(\mathbb{D}) \times \mathbb{D}$.

Fix $(f, s) \in F$. Let $\delta_{f,s}$ be a positive real number which satisfies that

- $\delta_{f,s} < \epsilon/2$, and
- f has the unique root s in $\mathrm{cl}\big(B(s, \delta_{f,s})\big)$.

Let $\eta_{f,s} = \min\{|f(z)| \ : \ z \in \partial B(s, \delta_{f,s})\} > 0$. Let $g \in W$. Let $N \in \mathbb{N}_{\geq 2}$. If $\mathrm{Val}(f, g) \geq N$, then for all $z \in \mathbb{D}$ with $|z| \leq R + \epsilon$,

$$|f(z) - g(z)| \leq \sum_{i=N}^{\infty} 2L(R + \epsilon)^i.$$

Recall that $L := \sup\{|a|, |ab|, |(a - b)c| \mid a, b, c \in G\}(< \infty)$. Let $N_{f,s}$ be a natural number which satisfies

$$\sum_{i=N_{f,s}}^{\infty} 2L(R + \epsilon)^i \leq \eta_{f,s}/2.$$

Let

$$V_{f,s} := \{(g, s') \in F \ : \ s' \in B(s, \delta_{f,s}) \text{ and } \max_{z \in \mathrm{cl}\big(B(0,R+\epsilon)\big)} |f(z) - g(z)| < \eta_{f,s}/2\}$$

Then the set $V_{f,s}$ is open in F. Since F is compact, there exist $(f_{i_1}, s_{i_1}), ..., (f_{i_k}, s_{i_k}) \in F$ such that $F \subset \bigcup_{j=1}^{k} V_{f_{i_j}, s_{i_j}}$.

We set for each $j \in \{1, ..., k\}$, $f_j := f_{i_j}, s_j := s_{i_j}, \delta_j := \delta_{f_{i_j}, s_{i_j}}, \eta_j := \eta_{f_{i_j}, s_{i_j}}, N_j := N_{f_{i_j}, s_{i_j}}$, and $V_j := V_{f_{i_j}, s_{i_j}}$. We set $N_{R,\epsilon} := \max\{N_1, ..., N_k\}$.

We now prove that $N_{R,\epsilon}$ satisfies the statement of our lemma.

Fix $(f,s) \in F$ and $g \in W$ with $\mathrm{Val}(f,g) \geq N_{R,\epsilon}$. Since $F \subset \bigcup_{j=1}^{k} V_j$, there exists $j \in \{1,...,k\}$ such that $(f,s) \in V_j$. Hence $s \in B(s_j, \delta_j)$ and $\max_{z \in \mathrm{cl}\left(B(0,R+\epsilon)\right)} |f_j(z) - f(z)| < \eta_j/2$. For each $z \in \partial B(s_j, \delta_j)$,

$$|f_j(z) - g(z)| \leq |f_j(z) - f(z)| + |f(z) - g(z)|.$$

Since $s_j \in \mathrm{cl}(B(0,R))$ and $\delta_j < \epsilon/2$, we have $z \in \mathrm{cl}\big(B(0, R+\epsilon)\big)$, and hence

$$|f_j(z) - f(z)| < \eta_j/2.$$

Moreover, since $\mathrm{Val}(f,g) \geq N_{R,\epsilon} \geq N_j$,

$$|f(z) - g(z)| < \eta_j/2.$$

Thus we have that

$$|f_j(z) - g(z)| < \eta_j (= \min_{z \in \partial B(s_j,\delta_j)} |f_j(z)|)$$

for each $z \in \partial B(s_j, \delta_j)$. By theorem of Rouché, there exists $s' \in B(s_j, \delta_j)$ such that $g(s') = 0$. Since $s \in B(s_j, \delta_j)$, we have that

$$\begin{aligned} |s' - s| &\leq |s' - s_j| + |s_j - s| \\ &< \delta_j + \delta_j \\ &< \epsilon. \end{aligned}$$

Hence we have proved our lemma.

□

DEFINITION 4.1.5. Let $N \in \mathbb{N}_{\geq 2}$. Let $A, B \in Q_N^G$ with $A \neq B$.

Let $R := \{p_0, q_0, p_1, q_1, ..., p_{m-1}, q_{m-1}, p_m\}$ be a sequence of functions on $\mathbb{D}$. We say that R is a sequence of functions which joins A to B with respect to N if R satisfies the following:

(1) for each i, $p_i \in Q_N^G$;
(2) for each i, $q_i \in W$;
(3) for each i, there exists a holomorphic function f on $\mathbb{D}$ such that $q_i(z) = f(z) \cdot p_i(z)$ for all $z \in \mathbb{D}$;
(4) for each i, $C_N(q_i) = p_{i+1}$;
(5) $p_0 = A, p_m = B$.

We prove the following lemma by extending the methods in the proof of [**5**, Lemme 3] and adding new ideas.

LEMMA 4.1.6. *Let $N \in \mathbb{N}_{\geq 2}$. Let $A, B \in Q_N^G$ with $A \neq B$. Then there exists a sequence of functions $p_0, q_0, p_1, q_1, ..., p_{m-1}, q_{m-1}, p_m$ which joins A to B.*

PROOF. This is done by induction with respect to $\mathrm{Val}(A,B) \in \{1, ..., N-1\}$. We first prove that the statement holds in the case $\mathrm{Val}(A,B) = N-1$. We set

$$\begin{aligned} A &:= (1, a_1, ..., a_{N-2}, a), \\ B &:= (1, a_1, ..., a_{N-2}, b), \end{aligned}$$

where $a \neq b$. Since G satisfies the condition $(*)$, there exist elements $(a =)b_1, b_2, ..., b_m(= b) \in G$ which satisfy Definition 1.3.7 (ii). We set

$$\begin{aligned} q_0^0 :=& \{1 + (b-a)z^{N-1}\}A \\ =& (1, a_1, ..., a_{N-2}, a) + \\ & (\underbrace{0, 0, ..., 0}_{N-1}, (b-a), (b-a)a_1, ..., (b-a)a_{N-2}, (b-a)a) \\ =& (1, a_1, ..., a_{N-2}, b, (b-a)a_1, ..., (b-a)a_{N-2}, (b-a)a) \in W, \\ p_1^0 :=& C_N(q_0^0) \\ =& (1, a_1, ..., a_{N-2}, b) = B \in Q_N^G. \end{aligned}$$

Hence we find a sequence $\{A, q_0^0, B\}$ of functions which joins A to B.

Fix $j \in \{1, ..., N-2\}$. Suppose that the statement holds in the case $\mathrm{Val}(A, B) > j$. We prove that the statement holds in the case $\mathrm{Val}(A, B) = j$. We set

$$A := (1, a_1, ..., a_{j-1}, a, * \cdots *),$$
$$B := (1, a_1, ..., a_{j-1}, b, * \cdots *),$$

where $a \neq b$. Since G satisfies the condition $(*)$, there exist $(a =)b_1, b_2, ..., b_m(= b) \in G$ which satisfies Definition 1.3.7 (ii). Let k, l be natural numbers such that $N - 1 = jk + l$ and $0 \leq l \leq j - 1$. By the condition (ii) in Definition 1.3.7 for $a_1 \in G$ there exists $c_1^1 \in G$ such that

$$(b_2 - b_1)a_1 + c_1^1 \in G.$$

Similarly, for $a_i \in G$ there exists $c_i^1 \in G$ such that

$$(b_2 - b_1)a_i + c_i^1 \in G,$$

where $i \in \{1, 2, ..., j\}$ and we set $a_j = a$. As in the same manner, for $c_i^m \in G$ there exists $c_i^{m+1} \in G$ such that

$$(b_2 - b_1)c_i^m + c_i^{m+1} \in G,$$

where $i \in \{1, 2, ..., j\}$ and $m \in \{1, 2, ..., k-1\}$. We set

$$A_1 := (1, a_1, ..., a_{j-1}, a, c_1^1, ..., c_j^1, c_1^2, ..., c_j^2, ..., c_1^k, ..., c_l^k) \in Q_N^G.$$

Since $\mathrm{Val}(A, A_1) > j$, by induction hypothesis, there exists a sequence R_1 of functions which joins A to A_1. We set

$$\begin{aligned} q_1 :=& \{1 + (b_2 - b_1)z^j\}A_1 \\ =& (1, a_1, ..., a_{j-1}, a, c_1^1, ..., c_j^1, c_1^2, ..., c_j^2, ..., c_1^k, ..., c_l^k) + \\ & (\underbrace{0, 0, ..., 0}_{j}, (b_2 - b_1), (b_2 - b_1)a_1, ..., (b_2 - b_1)a, (b_2 - b_1)c_1^1, ..., (b_2 - b_1)c_j^1, ...) \\ =& (1, a_1, ..., a_{j-1}, b_2, (b_2 - b_1)a_1 + c_1^1, ..., (b_2 - b_1)c_l^{k-1} + c_l^k, (b_2 - b_1)c_{l+1}^{k-1}, ..., (b_2 - b_1)c_l^k) \\ & \in W. \end{aligned}$$

Here, recall that $b_1 = a$. We set

$$\begin{aligned} p_2 :=& C_N(q_1) \\ =& (1, a_1, ..., a_{j-1}, b_2, (b_2 - b_1)a_1 + c_1^1, ..., (b_2 - b_1)c_l^{k-1} + c_l^k) \in Q_N^G. \end{aligned}$$

By the condition (ii) in Definition 1.3.7 for $a_1 \in G$ there exists $d_1^1 \in G$ such that

$$(b_2 - b_1)a_1 + d_1^1 \in G.$$

We set $a_i' = a_i$ and $a_j' = a_j$ for any $i \in \{2, 3, ..., j-1\}$. Similarly, for any $i \in \{2, 3, ..., j\}$ and $a_i' \in G$ there exists $d_i^1 \in G$ such that

$$(b_2 - b_1)a_i' + d_i^1 \in G.$$

As in the same manner, for $d_i^m \in G$ there exists $d_i^{m+1} \in G$ such that

$$(b_2 - b_1)d_i^m + d_i^{m+1} \in G,$$

where $i \in \{1, 2, ..., j\}$ and $m \in \{1, 2, ..., k-1\}$. We set

$$A_2 := (1, a_1, ..., a_{j-1}, b_2, d_1^1, ..., d_j^1, d_1^2, ..., d_j^2, ..., d_1^k, ..., d_l^k) \in Q_N^G.$$

Since $\mathrm{Val}(p_2, A_2) > j$, by induction hypothesis, there exists a sequence R_2 of functions which joins p_2 to A_2. We set

$$\begin{aligned} q_2 &:= \{1 + (b_3 - b_2)z^j\}A_2 \in W, \\ p_3 &:= C_N(q_2) \in Q_N^G. \end{aligned}$$

If we continue this process, we find sequences $R_1, R_2, ..., R_{r-1}$ of functions, functions $q_1, q_2, ..., q_{r-1} \in W$ and a function $p_r \in Q_N^G$ such that R_1 joins A to A_1, R_i joins p_i to A_i for each $i \in \{2, ..., r-1\}$ and such that $p_i = C_N(q_{i-1})$ for each $i \in \{2, ..., r\}$. Here,

$$\begin{aligned} &R_1 := \{A, q_0^1, p_1^1, q_1^1, ..., A_1\}, \\ &R_2 := \{p_2, q_0^2, p_1^2, q_1^2, ..., A_2\}, \\ &\cdots, \\ &R_{r-1} := \{p_{r-1}, q_0^{r-1}, p_1^{r-1}, q_1^{r-1}, ..., A_{r-1}\}. \end{aligned}$$

Then we find a sequence $\{A, q_0^1, p_1^1, q_1^1, ..., A_1, q_1, p_2, q_0^2, p_1^2, q_1^2, ..., A_2, ..., p_{r-1}, q_0^{r-1}, p_1^{r-1}, q_1^{r-1}, ..., A_{r-1}, q_{r-1}, p_r\}$ of functions which joins A to p_r, where p_r has the following form.

$$p_r = (1, a_1, ..., a_{j-1}, b, * \cdots *).$$

Since $\mathrm{Var}(p_r, B) > j$, by induction hypothesis, there exists a sequence of functions $R_r = \{p_r, q_0^r, p_1^r, q_1^r, ..., B\}$ which joins p_r to B. Hence we find a sequence $\{A, q_0^1, p_1^1, q_1^1, ..., A_1, q_1, p_2, q_0^2, p_1^2, q_1^2, ..., A_2, ..., p_{r-1}, q_0^{r-1}, p_1^{r-1}, q_1^{r-1}, ..., A_{r-1}, q_{r-1}, p_r, q_0^r, p_1^r, q_1^r, ..., B\}$ of functions which joins A to B. Thus we have proved our lemma.

□

4.2. Proof of Main Theorem F

Definition 4.2.1 (ϵ-connected). Let $A \subset \mathbb{C}$. Let $\epsilon > 0$. Let $x, y \in A$ and $\{e_0, ..., e_k\} \subset A$. We say that $\{e_0, ..., e_k\}$ is an ϵ-chain for (x, y) if $x = e_0, y = e_k$ and for each $i \in \{0, ..., k-1\}, |e_i - e_{i+1}| \leq \epsilon$.

We say that A is ϵ-connected if for all $x, y \in A$, there exists an ϵ-chain for (x, y).

Remark 4.2.2. If $A \subset \mathbb{C}$ is compact, A is connected if and only if for an arbitrary small $\epsilon > 0$, A is ϵ-connected.

PROOF. Suppose that A is not connected. Then there exist non-empty compact subsets A_1 and A_2 of A such that $A_1 \cup A_2 = A$ and $A_1 \cap A_2 = \emptyset$. If we set $\epsilon_0 = \inf_{a_1 \in A_1, a_2 \in A_2} |a_1 - a_2|/2$, then A is not ϵ_0-connected.

Suppose that A is connected. Since A is compact, for an arbitrary small $\epsilon > 0$ there exist $a_1, a_2, ..., a_k \in A$ such that $A \subset \cup_{i=1}^k \mathrm{cl}(B(a_i, \epsilon))$. Since A is connected, for any $i, j \in \{1, ..., k\}$ with $i \neq j$ there exist $i_1, ..., i_m \in \{1, 2, ..., k\}$ such that $a_i = a_{i_1}, a_j = a_{i_m}$, and $\mathrm{cl}(B(a_{i_l}, \epsilon)) \cap \mathrm{cl}(B(a_{i_{l+1}}, \epsilon)) \neq \emptyset$ for any $l \in \{1, ..., m-1\}$. Hence A is ϵ-connected. □

The following theorem is Main Theorem F.

THEOREM 4.2.3. *Let G be a subset of $\mathbb{C}$ with $(*)$. Suppose that there exists a real number R with $0 < R < 1$ such that $\{z \in \mathbb{C} \ : \ R < |z| < 1\} \subset X^G$. Then X^G is connected.*

PROOF. We set $M_R := \{z \in \mathbb{C} \ : \ R < |z| < 1\}$. Since $M_R \subset X^G$, it suffices to prove that $X^G \cup \partial\mathbb{D}$ is connected. By Lemma 4.1.1, $X^G \cup \partial\mathbb{D}$ is compact. Hence it suffices to prove that $X^G \cup \partial\mathbb{D}$ is ϵ-connected for an arbitrary small $\epsilon > 0$.

Fix $\epsilon > 0$ with $R + \epsilon < 1$. Take $s \in X^G$. We prove that there exist $s' \in M_R$ and an ϵ-chain for (s, s'). We may assume that $s \in \mathrm{cl}(B(0, R))$. Since $s \in X^G$, there exists $f \in P^G$ such that $f(z) = 0$. Let $N_{R,\epsilon}$ be a natural number defined by Lemma 4.1.4. We set $A := C_{N_{R,\epsilon}}(f) \in Q^G_{N_{R,\epsilon}}$. Since $\mathrm{Val}(f, A) \geq N_{R,\epsilon}$, there exists $s_0 \in B(s, \epsilon)$ such that $A(s_0) = 0$. If $s_0 \in M_R$, our theorem holds. If $s_0 \notin M_R$, that is, $s_0 \in \mathrm{cl}(B(0, R))$, we set

$$B(z) := 1 + z + z^2 + \cdots + z^{N_{R,\epsilon}-1} = \frac{1 - z^{N_{R,\epsilon}}}{1 - z} \in Q^G_{N_{R,\epsilon}}.$$

By Lemma 4.1.6, there exists a sequence of functions $p_0, q_0, p_1, q_1, ..., p_{m-1}, q_{m-1}, p_m$ which joins A to B. Since $q_0(s_0) = 0$ and $\mathrm{Val}(q_0, p_1) \geq N_{R,\epsilon}$, there exists $s_1 \in B(s_0, \epsilon)$ such that $p_1(s_1) = 0$ by Lemma 4.1.4. If $s_1 \in M_R$, our theorem holds. If $s_1 \in \mathrm{cl}(B(0, R))$, since $q_1(s_1) = 0$ and $\mathrm{Val}(q_1, p_2) \geq N_{R,\epsilon}$, there exists $s_2 \in B(s_1, \epsilon)$ such that $p_2(s_2) = 0$ by Lemma 4.1.4. If we continue this process, there exists $i \in \{1, ..., m-1\}$ such that $s_i \in M_R$ and $p_i(s_i) = 0$.

For, if this is not true, there exists $s_m \in \mathbb{D}$ such that $p_m(s_m) = B(s_m) = 0$. But this contradicts that B does not have any roots in $\mathbb{D}$.

Since $A, p_j \in Q^G_{N_{R,\epsilon}}$ for each $j \in \{1, ..., i\}$, we have that $s_0, s_j \in X^G$ by Lemma 4.1.1. We set $s' := s_i$. Then $\{s, s_0, s_1, ..., s_i(= s')\}$ is an ϵ-chain for (s, s').

Hence we have proved our theorem. □

4.3. Application (proof of Main Theorem E)

We use the following lemmas, which are key lemmas to prove Main Theorem E.

LEMMA 4.3.1. *Let n be an odd number. Let q, r be integers such that $2 \leq q \leq (n-1)/2$ and $0 \leq r \leq (n-1)/2$. We set*

$$j = \begin{cases} q + r - 1 & (1 \leq q + r - 1 \leq (n-1)/2) \\ n - (q + r - 1) & ((n+1)/2 \leq q + r - 1 \leq n - 2) \end{cases}$$

and

$$k = r - q + 1.$$

Then

$$\Big(\sin\frac{q\pi}{n} - \sin\frac{(q-2)\pi}{n}\Big)\sin\frac{r\pi}{n} - \Big(\sin\frac{j\pi}{n}\Big)\Big(\sin\frac{\pi}{n}\Big) - \Big(\sin\frac{k\pi}{n}\Big)\Big(\sin\frac{\pi}{n}\Big) = 0.$$

PROOF. (Case 1: $j = q + r - 1$ and $k = r - q + 1$)

$$\begin{aligned}
\Big(\sin\frac{j\pi}{n}\Big)\Big(\sin\frac{\pi}{n}\Big)+\Big(\sin\frac{k\pi}{n}\Big)\Big(\sin\frac{\pi}{n}\Big)=&-\frac{1}{2}\Big(\cos\frac{(j+1)\pi}{n}-\cos\frac{(j-1)\pi}{n}\Big)\\
&-\frac{1}{2}\Big(\cos\frac{(k+1)\pi}{n}-\cos\frac{(k-1)\pi}{n}\Big)\\
=&-\frac{1}{2}\Big(\cos\frac{(q+r)\pi}{n}-\cos\frac{(q+r-2)\pi}{n}\Big)\\
&-\frac{1}{2}\Big(\cos\frac{(r-q+2)\pi}{n}-\cos\frac{(r-q)\pi}{n}\Big)\\
=&-\frac{1}{2}\Big(\cos\frac{(q+r)\pi}{n}-\cos\frac{(q+r-2)\pi}{n}\Big)\\
&-\frac{1}{2}\Big(\cos\frac{(q-r-2)\pi}{n}-\cos\frac{(q-r)\pi}{n}\Big)\\
=&-\frac{1}{2}\Big(\cos\frac{(q+r)\pi}{n}-\cos\frac{(q-r)\pi}{n}\Big)\\
&+\frac{1}{2}\Big(\cos\frac{(q+r-2)\pi}{n}-\cos\frac{(q-r-2)\pi}{n}\Big)\\
=&\Big(\sin\frac{q\pi}{n}-\sin\frac{(q-2)\pi}{n}\Big)\sin\frac{r\pi}{n}.
\end{aligned}$$

(Case 2: $j = n - (q + r - 1)$ and $k = r - q + 1$)

$$\begin{aligned}
\Big(\sin\frac{j\pi}{n}\Big)\Big(\sin\frac{\pi}{n}\Big)+\Big(\sin\frac{k\pi}{n}\Big)\Big(\sin\frac{\pi}{n}\Big)=&\Big(\sin\frac{(n-(q+r-1))\pi}{n}\Big)\Big(\sin\frac{\pi}{n}\Big)\\
&+\Big(\sin\frac{(r-q+1)\pi}{n}\Big)\Big(\sin\frac{\pi}{n}\Big)\\
=&\Big(\sin\frac{(q+r-1)\pi}{n}\Big)\Big(\sin\frac{\pi}{n}\Big)\\
&+\Big(\sin\frac{(r-q+1)\pi}{n}\Big)\Big(\sin\frac{\pi}{n}\Big).
\end{aligned}$$

By Case 1,

$$\Big(\sin\frac{(q+r-1)\pi}{n}\Big)\Big(\sin\frac{\pi}{n}\Big)+\Big(\sin\frac{(r-q+1)\pi}{n}\Big)\Big(\sin\frac{\pi}{n}\Big)=\Big(\sin\frac{q\pi}{n}-\sin\frac{(q-2)\pi}{n}\Big)\sin\frac{r\pi}{n}.$$

□

LEMMA 4.3.2. *Let n be an even number. Let q, r be integers such that $2 \le q \le n/2$ and $0 \le r \le n/2$. We set*

$$j = \begin{cases} q+r-1 & (1 \le q+r-1 \le n/2-1) \\ n-(q+r-1) & (n/2 \le q+r-1 \le n-1) \end{cases}$$

and

$$k = r - q + 1.$$

Then

$$\Big(\sin\frac{q\pi}{n}-\sin\frac{(q-2)\pi}{n}\Big)\sin\frac{r\pi}{n}-\Big(\sin\frac{j\pi}{n}\Big)\Big(\sin\frac{\pi}{n}\Big)-\Big(\sin\frac{k\pi}{n}\Big)\Big(\sin\frac{\pi}{n}\Big)=0.$$

PROOF. We can prove Lemma 4.3.2 as in the proof of Lemma 4.3.1. □

We define the set of coefficients Ω_n which corresponds to $\mathcal{M}_n$ as the following.

DEFINITION 4.3.3. We set $I := \{0, 1, ..., n-1\}$. Also we set

$$\Omega_n := \{({\xi_n}^j - {\xi_n}^k)/(1-\xi_n) \ : \ j, k \in I\}.$$

Here, recall that $\xi_n = \exp(2\pi\sqrt{-1}/n)$.

REMARK 4.3.4. For each $a \in \Omega_n$, we have that $-a \in \Omega_n$.

The following two lemmas can be found in [**2**].

LEMMA 4.3.5. [**2**, Remark 3]

$$\mathcal{M}_n = X^{\Omega_n}.$$

LEMMA 4.3.6. [**2**, Proposition 3]

$$\Big\{z \in \mathbb{C} \ : \ \frac{1}{\sqrt{n}} < |z| < 1\Big\} \subset \mathcal{M}_n.$$

The proof of Main Theorem E is divided into the following two theorems (Theorems 4.3.7 and 4.3.8).

THEOREM 4.3.7. *If n is odd, $\mathcal{M}_n$ is connected.*

PROOF. By Theorem 4.2.3, Lemmas 4.3.5 and 4.3.6, it suffices to prove that Ω_n satisfies the condition $(*)$.

If $n = 2p+1$, where p is a natural number, Ω_n has the following form (see [**2**, p.2662]).

$$\Omega_n = \left\{ \frac{{\xi_n}^{l/2} \sin \frac{r\pi}{n}}{\sin \frac{\pi}{n}} \ : \ l = 0, ..., 2n-1, r = 0, 1, ..., p \right\}. \tag{29}$$

Since Ω_n contains 1 and Ω_n is finite, it suffices to prove that Ω_n satisfies the condition (ii) in Definition 1.3.7.

In order to prove that, suppose that for each $a \in \Omega_n$ with $a \neq 0$, there exist $b_1, b_2, ..., b_m \in \Omega_n$ with $b_1 = 0$ and $b_m = a$ such that for all $c \in \Omega_n$, there exist $d_1, d_2, ..., d_{m-1} \in \Omega_n$ such that

$$(b_2 - b_1)c + d_1 \in \Omega_n, (b_3 - b_2)c + d_2 \in \Omega_n, ..., (b_m - b_{m-1})c + d_{m-1} \in \Omega_n.$$

Then for each $a, b \in \Omega_n$ with $a, b \neq 0$ and $a \neq b$, there exist $b_1, b_2, ..., b_m, b'_1, b'_2 ..., b'_k \in \Omega_n$ with $b_1 = 0, b_m = a, b'_1 = 0$ and $b'_k = b$ such that for all $c \in \Omega_n$, there exist $d_1, d_2, ..., d_{m-1}, d'_1, d'_2, ..., d'_{k-1} \in \Omega_n$ such that

$$(b_2 - b_1)c + d_1 \in \Omega_n, (b_3 - b_2)c + d_2 \in \Omega_n, ..., (b_m - b_{m-1})c + d_{m-1} \in \Omega_n$$

and

$$(b'_2 - b'_1)c + d'_1 \in \Omega_n, (b'_3 - b'_2)c + d'_2 \in \Omega_n, ..., (b'_k - b'_{k-1})c + d'_{k-1} \in \Omega_n.$$

We set $\tilde{b}_1 = b_m, \tilde{b}_2 = b_{m-1}, ..., \tilde{b}_m = b_1, \tilde{b}_{m+1} = b'_2, \tilde{b}_{m+2} = b'_3, ..., \tilde{b}_{m+k-1} = b'_k$. Since for each $e \in \Omega_n$, $-e \in \Omega_n$, we have that

$$(\tilde{b}_2 - \tilde{b}_1)c - d_{m-1} \in \Omega_n, (\tilde{b}_3 - \tilde{b}_2)c - d_{m-2} \in \Omega_n, ..., (\tilde{b}_m - \tilde{b}_{m-1})c - d_1 \in \Omega_n,$$

$$(\tilde{b}_{m+1} - \tilde{b}_m)c + d'_1 \in \Omega_n, (\tilde{b}_{m+2} - \tilde{b}_{m+1})c + d'_2 \in \Omega_n, ..., (\tilde{b}_{m+k-1} - \tilde{b}_{m+k-2})c + d'_{k-1} \in \Omega_n.$$

Hence it suffices to prove that for each $a \in \Omega_n$ with $a \neq 0$, there exist $b_1, b_2, ..., b_m \in \Omega_n$ with $b_1 = 0$ and $b_m = a$ such that for all $c \in \Omega_n$, there exist $d_1, d_2, ..., d_{m-1} \in \Omega_n$ such that

$$(b_2 - b_1)c + d_1 \in \Omega_n, (b_3 - b_2)c + d_2 \in \Omega_n, ..., (b_m - b_{m-1})c + d_{m-1} \in \Omega_n.$$

In order to prove that, fix $a \in \Omega_n$ with $a \neq 0$.

(Case 1 : $a = {\xi_n}^{l/2} \sin \frac{q\pi}{n} / \sin \frac{\pi}{n}$, where $q \in \{0, ..., p\}$ is even and $l \in \{0, ..., 2n-1\}$)

We set

$$b_1 = 0, b_2 = \frac{{\xi_n}^{l/2} \sin \frac{2\pi}{n}}{\sin \frac{\pi}{n}}, ..., b_{q/2} = \frac{{\xi_n}^{l/2} \sin \frac{(q-2)\pi}{n}}{\sin \frac{\pi}{n}}, b_{q/2+1} = \frac{{\xi_n}^{l/2} \sin \frac{q\pi}{n}}{\sin \frac{\pi}{n}}.$$

Fix $c \in \Omega_n$. We set $c = {\xi_n}^{l_1/2} \sin \frac{r\pi}{n} / \sin \frac{\pi}{n}$, where $r \in \{0, 1, ..., p\}$ and $l_1 \in \{0, ..., 2n-1\}$.

For each $i \in \{1, ..., q/2\}$,

$$(b_{i+1} - b_i)c = {\xi_n}^{(l+l_1)/2} \left(\frac{\sin \frac{2i\pi}{n}}{\sin \frac{\pi}{n}} - \frac{\sin \frac{(2i-2)\pi}{n}}{\sin \frac{\pi}{n}} \right) \frac{\sin \frac{r\pi}{n}}{\sin \frac{\pi}{n}}.$$

By Lemma 4.3.1, if we set

$$j_i = \begin{cases} 2i + r - 1 & (1 \leq 2i + r - 1 \leq (n-1)/2) \\ n - (2i + r - 1) & ((n+1)/2 \leq 2i + r - 1 \leq n - 2) \end{cases}$$

and

$$k_i = r - 2i + 1,$$

we have that

$$(b_{i+1} - b_i)c - \frac{{\xi_n}^{(l+l_1)/2} \sin \frac{j_i\pi}{n}}{\sin \frac{\pi}{n}} - \frac{{\xi_n}^{(l+l_1)/2} \sin \frac{k_i\pi}{n}}{\sin \frac{\pi}{n}} = 0.$$

Here, $-{\xi_n}^{(l+l_1)/2} \sin \frac{j_i\pi}{n} / \sin \frac{\pi}{n} \in \Omega_n$ and ${\xi_n}^{(l+l_1)/2} \sin \frac{k_i\pi}{n} / \sin \frac{\pi}{n} \in \Omega_n$ by (29) and Remark 4.3.4. Hence we have that

$$(b_{i+1} - b_i)c + \left(-\frac{{\xi_n}^{(l+l_1)/2} \sin \frac{j_i\pi}{n}}{\sin \frac{\pi}{n}}\right) = \frac{{\xi_n}^{(l+l_1)/2} \sin \frac{k_i\pi}{n}}{\sin \frac{\pi}{n}} \in \Omega_n.$$

(Case 2 : $a = {\xi_n}^{l/2} \sin \frac{q\pi}{n} / \sin \frac{\pi}{n}$, where $q \in \{0, ..., p\}$ is odd and $l \in \{0, ..., 2n-1\}$)

We set

$$b_1 = 0, b_2 = {\xi_n}^{l/2}, b_3 = \frac{{\xi_n}^{l/2} \sin \frac{3\pi}{n}}{\sin \frac{\pi}{n}}, ..., b_{(q+1)/2} = \frac{{\xi_n}^{l/2} \sin \frac{(q-2)\pi}{n}}{\sin \frac{\pi}{n}}, b_{(q+1)/2+1} = \frac{{\xi_n}^{l/2} \sin \frac{q\pi}{n}}{\sin \frac{\pi}{n}}.$$

Fix $c \in \Omega_n$. We set $c = {\xi_n}^{l_1/2} \sin \frac{r\pi}{n} / \sin \frac{\pi}{n}$, where $r \in \{0, 1, ..., p\}$ and $l_1 \in \{0, ..., 2n-1\}$.

$$(b_2 - b_1)c = {\xi_n}^{(l+l_1)/2} \frac{\sin \frac{r\pi}{n}}{\sin \frac{\pi}{n}}.$$

Here, $-{\xi_n}^{(l+l_1)/2} \sin \frac{r\pi}{n} / \sin \frac{\pi}{n} \in \Omega_n$ by (29) and Remark 4.3.4, and hence

$$(b_2 - b_1)c + \left(-{\xi_n}^{(l+l_1)/2} \frac{\sin \frac{r\pi}{n}}{\sin \frac{\pi}{n}}\right) = 0 \in \Omega_n.$$

For each $i \in \{2, ..., (q+1)/2\}$,

$$(b_{i+1} - b_i)c = {\xi_n}^{(l+l_1)/2} \left(\frac{\sin \frac{(2i-1)\pi}{n}}{\sin \frac{\pi}{n}} - \frac{\sin \frac{(2i-3)\pi}{n}}{\sin \frac{\pi}{n}} \right) \frac{\sin \frac{r\pi}{n}}{\sin \frac{\pi}{n}}.$$

By Lemma 4.3.1, if we set

$$j_i = \begin{cases} 2i-1+r-1 & (1 \leq 2i-1+r-1 \leq (n-1)/2) \\ n-(2i-1+r-1) & ((n+1)/2 \leq 2i-1+r-1 \leq n-2) \end{cases}$$

and

$$k_i = r-(2i-1)+1,$$

we have that

$$(b_{i+1}-b_i)c - \frac{{\xi_n}^{(l+l_1)/2} \sin \frac{j_i \pi}{n}}{\sin \frac{\pi}{n}} - \frac{{\xi_n}^{(l+l_1)/2} \sin \frac{k_i \pi}{n}}{\sin \frac{\pi}{n}} = 0.$$

Here, $-{\xi_n}^{(l+l_1)/2} \sin \frac{j_i \pi}{n} / \sin \frac{\pi}{n} \in \Omega_n$ and ${\xi_n}^{(l+l_1)/2} \sin \frac{k_i \pi}{n} / \sin \frac{\pi}{n} \in \Omega_n$ by (29) and Remark 4.3.4. Hence we have that

$$(b_{i+1}-b_i)c + (-\frac{{\xi_n}^{(l+l_1)/2} \sin \frac{j_i \pi}{n}}{\sin \frac{\pi}{n}}) = \frac{{\xi_n}^{(l+l_1)/2} \sin \frac{k_i \pi}{n}}{\sin \frac{\pi}{n}} \in \Omega_n.$$

Hence we have proved our theorem. □

THEOREM 4.3.8. *If n is even, $\mathcal{M}_n$ is connected.*

PROOF. By Theorem 4.2.3, Lemmas 4.3.5 and 4.3.6, it suffices to prove that Ω_n satisfies the condition $(*)$.

If $n = 4p$, where p is a natural number, Ω_n has the following form (See [**2**, p.2662]).

$$\begin{aligned} \Omega_n = & \left\{ \frac{{\xi_n}^{l+1/2} \sin \frac{r\pi}{n}}{\sin \frac{\pi}{n}} \ : \ l = 0, ..., n-1, r = 0, 2, ..., 2p \right\} \\ & \bigcup \left\{ \frac{{\xi_n}^{l} \sin \frac{r\pi}{n}}{\sin \frac{\pi}{n}} \ : \ l = 0, ..., n-1, r = 1, 3, ..., 2p-1 \right\}. \end{aligned} \tag{30}$$

If $n = 4p+2$, where p is a natural number, Ω_n has the following form (See [**2**, p.2662]).

$$\begin{aligned} \Omega_n = & \left\{ \frac{{\xi_n}^{l+1/2} \sin \frac{r\pi}{n}}{\sin \frac{\pi}{n}} \ : \ l = 0, ..., n-1, r = 0, 2, ..., 2p \right\} \\ & \bigcup \left\{ \frac{{\xi_n}^{l} \sin \frac{r\pi}{n}}{\sin \frac{\pi}{n}} \ : \ l = 0, ..., n-1, r = 1, 3, ..., 2p+1 \right\}. \end{aligned} \tag{31}$$

Since Ω_n contains 1 and Ω_n is finite, it suffices to prove that Ω_n satisfies the condition (ii) in Definition 1.3.7.

As in the proof of Theorem 4.3.7, it suffices to prove that for each $a \in \Omega_n$ with $a \neq 0$, there exist $b_1, b_2, ..., b_m \in \Omega_n$ with $b_1 = 0$ and $b_m = a$ such that for all $c \in \Omega_n$, there exist $d_1, d_2, ..., d_{m-1} \in \Omega_n$ such that

$$(b_2-b_1)c + d_1 \in \Omega_n, (b_3-b_2)c + d_2 \in \Omega_n, ..., (b_m - b_{m-1})c + d_{m-1} \in \Omega_n.$$

In order to prove that, fix $a \in \Omega_n$ with $a \neq 0$.

(Case 1 : $a = {\xi_n}^{l+1/2} \sin \frac{q\pi}{n} / \sin \frac{\pi}{n}$, where $q \in \{0, 2, ..., 2p\}$ and $l \in \{0, ..., n-1\}$)

We set

$$b_1 = 0, b_2 = \frac{{\xi_n}^{l+1/2} \sin \frac{2\pi}{n}}{\sin \frac{\pi}{n}}, ..., b_{q/2} = \frac{{\xi_n}^{l+1/2} \sin \frac{(q-2)\pi}{n}}{\sin \frac{\pi}{n}}, b_{q/2+1} = \frac{{\xi_n}^{l+1/2} \sin \frac{q\pi}{n}}{\sin \frac{\pi}{n}}.$$

Fix $c \in \Omega_n$.

(Case 1-1 : $c = {\xi_n}^{l_1+1/2} \sin\frac{r\pi}{n}/\sin\frac{\pi}{n}$, where $r \in \{0, 2, ..., 2p\}$ and $l_1 \in \{0, ..., n-1\}$)

For each $i \in \{1, ..., q/2\}$,

$$(b_{i+1} - b_i)c = {\xi_n}^{l+l_1+1}\left(\frac{\sin\frac{2i\pi}{n}}{\sin\frac{\pi}{n}} - \frac{\sin\frac{(2i-2)\pi}{n}}{\sin\frac{\pi}{n}}\right)\frac{\sin\frac{r\pi}{n}}{\sin\frac{\pi}{n}}.$$

By Lemma 4.3.2, if we set

$$j_i = \begin{cases} 2i + r - 1 & (1 \le 2i + r - 1 \le n/2 - 1) \\ n - (2i + r - 1) & (n/2 \le 2i + r - 1 \le n - 1) \end{cases}$$

and

$$k_i = r - 2i + 1,$$

then we have that

$$(b_{i+1} - b_i)c - \frac{{\xi_n}^{l+l_1+1}\sin\frac{j_i\pi}{n}}{\sin\frac{\pi}{n}} - \frac{{\xi_n}^{l+l_1+1}\sin\frac{k_i\pi}{n}}{\sin\frac{\pi}{n}} = 0.$$

Since $2i$ and r are even, j_i and $|k_i|$ are odd. Hence $-{\xi_n}^{l+l_1+1}\sin\frac{j_i\pi}{n}/\sin\frac{\pi}{n} \in \Omega_n$ and ${\xi_n}^{l+l_1+1}\sin\frac{k_i\pi}{n}/\sin\frac{\pi}{n} \in \Omega_n$ by (30), (31), and Remark 4.3.4. Hence we have that

$$(b_{i+1} - b_i)c + (-\frac{{\xi_n}^{l+l_1+1}\sin\frac{j_i\pi}{n}}{\sin\frac{\pi}{n}}) = \frac{{\xi_n}^{l+l_1+1}\sin\frac{k_i\pi}{n}}{\sin\frac{\pi}{n}} \in \Omega_n.$$

(Case 1-2 : $c = {\xi_n}^{l_1}\sin\frac{r\pi}{n}/\sin\frac{\pi}{n}$, where

$$r \in \begin{cases} \{1, 3, ..., 2p-1\} & (\text{if } n = 4p) \\ \{1, 3, ..., 2p+1\} & (\text{if } n = 4p+2) \end{cases}$$

and $l_1 \in \{0, ..., n-1\}$)

For each $i \in \{1, ..., q/2\}$,

$$(b_{i+1} - b_i)c = {\xi_n}^{l+l_1+1/2}\left(\frac{\sin\frac{2i\pi}{n}}{\sin\frac{\pi}{n}} - \frac{\sin\frac{(2i-2)\pi}{n}}{\sin\frac{\pi}{n}}\right)\frac{\sin\frac{r\pi}{n}}{\sin\frac{\pi}{n}}.$$

By Lemma 4.3.2, if we set

$$j_i = \begin{cases} 2i + r - 1 & (1 \le 2i + r - 1 \le n/2 - 1) \\ n - (2i + r - 1) & (n/2 \le 2i + r - 1 \le n - 1) \end{cases}$$

and

$$k_i = r - 2i + 1,$$

then we have that

$$(b_{i+1} - b_i)c - \frac{{\xi_n}^{l+l_1+1/2}\sin\frac{j_i\pi}{n}}{\sin\frac{\pi}{n}} - \frac{{\xi_n}^{l+l_1+1/2}\sin\frac{k_i\pi}{n}}{\sin\frac{\pi}{n}} = 0.$$

Since $2i$ is even and r is odd, j_i and $|k_i|$ are even. Hence $-{\xi_n}^{l+l_1+1/2}\sin\frac{j_i\pi}{n}/\sin\frac{\pi}{n} \in \Omega_n$ and ${\xi_n}^{l+l_1+1/2}\sin\frac{k_i\pi}{n}/\sin\frac{\pi}{n} \in \Omega_n$ by (30), (31), and Remark 4.3.4. Hence we have that

$$(b_{i+1} - b_i)c + (-\frac{{\xi_n}^{l+l_1+1/2}\sin\frac{j_i\pi}{n}}{\sin\frac{\pi}{n}}) = \frac{{\xi_n}^{l+l_1+1/2}\sin\frac{k_i\pi}{n}}{\sin\frac{\pi}{n}} \in \Omega_n.$$

(Case 2 : $a = {\xi_n}^l \sin\frac{q\pi}{n}/\sin\frac{\pi}{n}$, where

$$q \in \begin{cases} \{1,3,...,2p-1\} & (\text{if } n = 4p) \\ \{1,3,...,2p+1\} & (\text{if } n = 4p+2) \end{cases}$$

and $l \in \{0,...,n-1\}$)

We set

$$b_1 = 0, b_2 = {\xi_n}^l, b_3 = \frac{{\xi_n}^l \sin\frac{3\pi}{n}}{\sin\frac{\pi}{n}}, ..., b_{(q+1)/2} = \frac{{\xi_n}^l \sin\frac{(q-2)\pi}{n}}{\sin\frac{\pi}{n}}, b_{(q+1)/2+1} = \frac{{\xi_n}^l \sin\frac{q\pi}{n}}{\sin\frac{\pi}{n}}.$$

Fix $c \in \Omega_n$.

(Case 2-1 : $c = {\xi_n}^{l_1+1/2} \sin\frac{r\pi}{n}/\sin\frac{\pi}{n}$, where $r \in \{0,2,...,2p\}$ and $l_1 \in \{0,...,n-1\}$)

$$(b_2 - b_1)c = {\xi_n}^{l+l_1+1/2}\frac{\sin\frac{r\pi}{n}}{\sin\frac{\pi}{n}}.$$

Since r is even, by (30), (31), and Remark 4.3.4, we have $-{\xi_n}^{l+l_1+1/2}\sin\frac{r\pi}{n}/\sin\frac{\pi}{n} \in \Omega_n$. Hence

$$(b_2 - b_1)c + (-{\xi_n}^{l+l_1+1/2}\frac{\sin\frac{r\pi}{n}}{\sin\frac{\pi}{n}}) = 0 \in \Omega_n.$$

For each $i \in \{2,...,(q+1)/2\}$,

$$(b_{i+1} - b_i)c = {\xi_n}^{l+l_1+1/2}\left(\frac{\sin\frac{(2i-1)\pi}{n}}{\sin\frac{\pi}{n}} - \frac{\sin\frac{(2i-3)\pi}{n}}{\sin\frac{\pi}{n}}\right)\frac{\sin\frac{r\pi}{n}}{\sin\frac{\pi}{n}}.$$

By Lemma 4.3.2, if we set

$$j_i = \begin{cases} 2i-1+r-1 & (1 \le 2i-1+r-1 \le n/2-1) \\ n-(2i-1+r-1) & (n/2 \le 2i-1+r-1 \le n-1) \end{cases}$$

and

$$k_i = r - (2i-1) + 1,$$

then we have that

$$(b_{i+1} - b_i)c - \frac{{\xi_n}^{l+l_1+1/2}\sin\frac{j_i\pi}{n}}{\sin\frac{\pi}{n}} - \frac{{\xi_n}^{l+l_1+1/2}\sin\frac{k_i\pi}{n}}{\sin\frac{\pi}{n}} = 0.$$

Since $2i-1$ is odd and r is even, j_i and $|k_i|$ are even. Hence $-{\xi_n}^{l+l_1+1/2}\sin\frac{j_i\pi}{n}/\sin\frac{\pi}{n} \in \Omega_n$ and ${\xi_n}^{l+l_1+1/2}\sin\frac{k_i\pi}{n}/\sin\frac{\pi}{n} \in \Omega_n$ by (30), (31), and Remark 4.3.4. Hence we have that

$$(b_{i+1} - b_i)c + (-\frac{{\xi_n}^{l+l_1+1/2}\sin\frac{j_i\pi}{n}}{\sin\frac{\pi}{n}}) = \frac{{\xi_n}^{l+l_1+1/2}\sin\frac{k_i\pi}{n}}{\sin\frac{\pi}{n}} \in \Omega_n.$$

(Case 2-2 : $c = {\xi_n}^{l_1}\sin\frac{r\pi}{n}/\sin\frac{\pi}{n}$, where

$$r \in \begin{cases} \{1,3,...,2p-1\} & (\text{if } n = 4p) \\ \{1,3,...,2p+1\} & (\text{if } n = 4p+2) \end{cases}$$

and $l_1 \in \{0,...,n-1\}$)

$$(b_2 - b_1)c = {\xi_n}^{l+l_1}\frac{\sin\frac{r\pi}{n}}{\sin\frac{\pi}{n}}.$$

Since r is odd, by (30), (31), and Remark 4.3.4, we have $-{\xi_n}^{l+l_1}\sin\frac{r\pi}{n}/\sin\frac{\pi}{n}\in\Omega_n$. Hence

$$(b_2-b_1)c+(-{\xi_n}^{l+l_1}\frac{\sin\frac{r\pi}{n}}{\sin\frac{\pi}{n}})=0\in\Omega_n.$$

For each $i\in\{2,...,(q+1)/2\}$,

$$(b_{i+1}-b_i)c={\xi_n}^{l+l_1}\left(\frac{\sin\frac{(2i-1)\pi}{n}}{\sin\frac{\pi}{n}}-\frac{\sin\frac{(2i-3)\pi}{n}}{\sin\frac{\pi}{n}}\right)\frac{\sin\frac{r\pi}{n}}{\sin\frac{\pi}{n}}.$$

By Lemma 4.3.2, if we set

$$j_i=\begin{cases}2i-1+r-1 & (1\leq 2i-1+r-1\leq n/2-1)\\ n-(2i-1+r-1) & (n/2\leq 2i-1+r-1\leq n-1)\end{cases}$$

and

$$k_i=r-(2i-1)+1,$$

then we have that

$$(b_{i+1}-b_i)c-\frac{{\xi_n}^{l+l_1}\sin\frac{j_i\pi}{n}}{\sin\frac{\pi}{n}}-\frac{{\xi_n}^{l+l_1}\sin\frac{k_i\pi}{n}}{\sin\frac{\pi}{n}}=0.$$

Since $2i-1$ and r are odd, j_i and $|k_i|$ are odd. Hence $-{\xi_n}^{l+l_1}\sin\frac{j_i\pi}{n}/\sin\frac{\pi}{n}\in\Omega_n$ and ${\xi_n}^{l+l_1}\sin\frac{k_i\pi}{n}/\sin\frac{\pi}{n}\in\Omega_n$ by (30), (31), and Remark 4.3.4. Hence we have that

$$(b_{i+1}-b_i)c+(-\frac{{\xi_n}^{l+l_1}\sin\frac{j_i\pi}{n}}{\sin\frac{\pi}{n}})=\frac{{\xi_n}^{l+l_1}\sin\frac{k_i\pi}{n}}{\sin\frac{\pi}{n}}\in\Omega_n.$$

Hence we have proved our theorem.

$\square$

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