A CONJECTURE TO FIND FOUR-TERM ARITHMETIC PROGRESSIONS OF PIATETSKI-SHAPIRO SEQUENCES

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Abstract. For every non-integral $\alpha > 1$, the sequence of the integer parts of n^α ($n = 1, 2, \dots$) is called the Piatetski-Shapiro sequence with exponent α. Let $\mathrm{PS}(\alpha)$ be the set of all terms of this sequence. The aim of this article is to propose a conjecture to find infinitely many four-term arithmetic progressions of $\mathrm{PS}(\alpha)$.

1. Introduction

We let $\lfloor x \rfloor$ denote the integer part of $x \in \mathbb{R}$. For every non-integral $\alpha > 1$, the sequence $(\lfloor n^\alpha \rfloor)_{n=1}^\infty$ is called the Piatetski-Shapiro sequence with exponent α, and we let $\mathrm{PS}(\alpha)$ be the set of all terms of this sequence. A real sequence $(a_j)_{j=0}^{k-1}$ is called a *k-term arithmetic progression (k-AP)* if there exists $\ell > 0$ such that

$$a_j = a_0 + j\ell$$

for all $j = 0, 1, \dots, k-1$. In this article, we discuss APs of $\mathrm{PS}(\alpha)$. By the result of Frantzikinakis and Wierdl [FW09], $\mathrm{PS}(\alpha)$ contains arbitrarily long APs for all $1 < \alpha < 2$. Further, Matsusaka and the author recently showed that for all $2 < \beta < \gamma$, there are uncountably many $\alpha \in [\beta, \gamma]$ such that $\mathrm{PS}(\alpha)$ contains infinitely many 3-APs [MS21]. More precisely, for any fixed $a, b, c \in \mathbb{N}$, they showed that the Hausdorff dimension of

$$\{\alpha \in [\beta,\gamma] \colon ax + by = cz \text{ has infinitely many solutions } (x,y,z) \in \mathrm{PS}(\alpha)^3 \text{ with } \#\{x,y,z\} = 3\}$$

is greater than or equal to $1/s^3$. By substituting $a = b = 1$ and $c = 2$, they obtained the result on 3-APs of $\mathrm{PS}(\alpha)$. However, there is no research to find 4-APs of $\mathrm{PS}(\alpha)$ when $\alpha > 2$ is non-integral. The aim of this article is to propose a sufficient condition to find infinitely many 4-APs of $\mathrm{PS}(\alpha)$. Let $\alpha > 1$, and we define the following condition which depends on α.

Condition 1.1. *There exists $\xi > \alpha^2$ such that for infinitely many tuples $(p, q, r, s) \in \mathbb{N}^4$ with $p < q < r < s$, one has*

$$|p^\alpha + r^\alpha - 2q^\alpha| \le q^{-\xi}, \quad |q^\alpha + s^\alpha - 2r^\alpha| \le q^{-\xi}. \tag{1.1}$$

Theorem 1.2. *Assume that there exists $\alpha > 1$ satisfying Condition 1.1. Then $\mathrm{PS}(\alpha)$ contains infinitely many four-term arithmetic progressions.*

In view of this theorem, we would expect to find infinitely many 4-APs of $\mathrm{PS}(\alpha)$ by using simultaneous Diophantine approximations. However, we do not find any α which satisfies Condition 1.1.

Question 1.3. *Let $1 < \beta < \gamma$. What is a lower bound for the Hausdorff dimension of*

$$\{\alpha \in (\beta,\gamma) \colon \text{for infinitely many } (p,q,r,s) \in \mathbb{N}^2 \text{ with } p<q<r<s,$$
$$|p^\alpha + r^\alpha - 2q^\alpha| \le q^{-\gamma^2}, \quad |q^\alpha + s^\alpha - 2r^\alpha| \le q^{-\gamma^2}\}?$$

If the set had positive Hausdorff dimension, then by Theorem 1.2, we would find uncountably many $\alpha \in (\beta,\gamma)$ such that $\mathrm{PS}(\alpha)$ contains infinitely many four-term arithmetic progressions.

Notation 1.4. Let $\mathbb{N}$ be the set of all positive integers. For $x \in \mathbb{R}$, let $\{x\}$ denote the fractional part of x. For all $\ell \in \mathbb{N}$, we define $[\ell] = \mathbb{N} \cap [1,\ell]$. Let $\sqrt{-1}$ denote the imaginary unit, and define $\mathrm{e}(x)$ by $e^{2\pi\sqrt{-1}x}$ for all $x \in \mathbb{R}$.

2. Preparation

Let $d \in \mathbb{N}$. We mainly discuss the case when $d = 1$ or 2. For all $\mathbf{x} = (x^{(1)}, \dots, x^{(d)}) \in \mathbb{R}^d$, we define $\{\mathbf{x}\} = (\{x^{(1)}\}, \dots, \{x^{(d)}\})$. Let $(\mathbf{x}_n)_{n=1}^N$ be a sequence composed of $\mathbf{x}_n \in \mathbb{R}^d$ for all $1 \le n \le N$. We define the *discrepancy* of $(\mathbf{x}_n)_{n=1}^N$ by

$$\mathcal{D}_{1\le n\le N}(\mathbf{x}_n) = \sup_{\substack{0\le a_i<b_i\le 1\\ \forall i\in[d]}} \left| \frac{\#\left\{n \in \mathbb{N}\cap[1,N] \colon \{\mathbf{x}_n\} \in \prod_{i=1}^d [a_i,b_i)\right\}}{N} - \prod_{i=1}^d (b_i - a_i) \right|.$$

We can find upper bounds of the discrepancy from evaluating exponential sums by the following inequality. This is shown by Koksma [Kok50] and Szüsz [Szü52] independently: there exists $C_d > 0$ which depends only on d such that for all $H \in \mathbb{N}$, we have

$$\mathcal{D}_{1\le n\le N}(\mathbf{x}_n) \le C_d \left(\frac{1}{H} + \sum_{\substack{0<\|\mathbf{h}\|_\infty \le H\\ \mathbf{h}\in\mathbb{Z}^d}} \frac{1}{\nu(\mathbf{h})} \left| \frac{1}{N} \sum_{n=1}^N \mathrm{e}(\langle \mathbf{h}, \mathbf{x}_n\rangle) \right| \right), \tag{2.1}$$

where we let $\langle\cdot,\cdot\rangle$ denote the standard inner product and define

$$\|\mathbf{h}\|_\infty = \max\{|h^{(1)}|, \dots, |h^{(d)}|\}, \quad \nu(\mathbf{h}) = \prod_{i=1}^d \max\{1, |h^{(i)}|\}$$

for all $\mathbf{h} = (h^{(1)}, \dots, h^{(d)}) \in \mathbb{R}^d$. This inequality is sometimes reffered as the Erdős-Turán-Koksma inequality. We refer [DT97, Theorem 1.21] to the readers for more details. In order to evaluate upper bounds for the right-hand side on (2.1), we will use the following lemma which is called van der Corput's k-th derivative test.

Lemma 2.1. *Let V_1, V_2 be real numbers with $V_2 - V_1 \ge 1$. Let $f : [V_1,V_2] \to \mathbb{R}$ be a function which has continuous derivatives up to the k-th order, where $k \ge 4$. Let λ_k and T be positive real numbers. Suppose that*

$$\lambda_k \le |f^{(k)}(x)| \le T\lambda_k$$

for all $x \in [V_1,V_2]$. Then there exists $C(T,k) > 0$ such that

$$\left| \sum_{V_1<n\le V_2} \mathrm{e}(f(n)) \right| \le C(T,k)\left((V_2-V_1)\lambda_k^{1/(2^k-2)} + (V_2-V_1)^{1-2^{2-k}} \lambda_k^{-1/(2^k-2)} \right).$$

Proof. See the book written by Titchmarsh [Tit86, Theorem 5.13]. $\square$

3. Lemma

We write $O(1)$ for a bounded quantity. If this bound depends only on some parameters $a_1,\dots,a_n$, then for instance we write $O_{a_1,a_2,\dots,a_n}(1)$. As is customary, we often abbreviate $O(1)X$ and $O_{a_1,\dots,a_n}(1)X$ to $O(X)$ and $O_{a_1,\dots,a_n}(X)$ respectively for a non-negative quantity X. We also state $f(X)\ll g(X)$ and $f(X)\ll_{a_1,\dots,a_n} g(X)$ as $f(X)=O(g(X))$ and $f(X)=O_{a_1,\dots,a_n}(g(X))$ respectively, where $g(X)$ is non-negative.

Lemma 3.1. *Let $\alpha>1$, $\xi>\alpha^2$, $0<\epsilon<(\xi-\alpha^2)/\alpha$, and $\delta>0$. Then there exists $\theta=\theta(\alpha,\xi,\epsilon)>0$ such that for all $p,q\in\mathbb{N}$ with $p<q$, by setting $V=(\delta q^\xi)^{1/\alpha}$ and $U=q^{(\xi-\alpha^2)/\alpha^2-\epsilon/\alpha}$, either one has*

$$\mathcal{D}_{V<v\le 2V}((vp)^\alpha,(vq)^\alpha)\ll_{\alpha,\xi,\epsilon,\delta} q^{-\theta},$$

or there exist $h_1,h_2\in\mathbb{N}$ with $h_2<h_1\le q^\theta$ such that $|h_1(p/q)^\alpha-h_2|\le q^{-\xi/\alpha+\epsilon}$ and

$$\mathcal{D}_{U<u\le 2U}((uq)^\alpha/h_1)\ll_{\alpha,\xi,\epsilon,\delta} q^{-2\theta}.$$

Proof. Take any $p,q\in\mathbb{N}$ with $p<q$. Take a small parameter $\theta=\theta(\alpha,\xi,\epsilon)>0$, and large parameter $q_0=q_0(\alpha,\xi,\epsilon)$ which satisfies $q_0^\theta\ge 2$. Let $\eta=\xi/\alpha-\epsilon+2\theta$. We may assume that $q\ge q_0$. Let $H=\lfloor q^\theta\rfloor$, and let $L(h_1,h_2)=h_1(p/q)^\alpha+h_2$ for all $h_1,h_2\in\mathbb{Z}$. By (2.1), we have

$$\mathcal{D}_{V<v\le 2V}((vp)^\alpha,(vq)^\alpha)\ll\frac{1}{H}+\sum_{0<\|(h_1,h_2)\|_\infty\le H}\frac{1}{\nu(h_1,h_2)}\left|S(h_1,h_2)\right|,$$

where $S(h_1,h_2)=\frac{1}{V}\sum_{V<v\le 2V}\mathrm{e}(L(h_1,h_2)q^\alpha v^\alpha)$. Firstly, we discuss the case when

$$|L(h_1,h_2)|\ge q^{-\eta} \tag{3.1}$$

for all $h_1,h_2\in\mathbb{Z}$ with $0<\|(h_1,h_2)\|_\infty\le H$. Let $k=\lfloor\alpha(\alpha+\xi)/\xi\rfloor+1$. Here $\alpha(\alpha+\xi)/\xi>\alpha>1$, which implies that $k\ge 2$. In addition, $k=\lfloor\alpha(\alpha+\xi)/\xi\rfloor+1>\alpha(\alpha+\xi)/\xi>\alpha$. Fix any $h_1,h_2\in\mathbb{Z}$ with $0<\|(h_1,h_2)\|_\infty\le H$. Define $f(x)=L(h_1,h_2)q^\alpha x^\alpha$. Then

$$|L(h_1,h_2)|q^\alpha V^{\alpha-k}\ll_{\alpha,\xi}|f^{(k)}(x)|\ll_{\alpha,\xi}|L(h_1,h_2)|q^\alpha V^{\alpha-k}$$

for all real numbers $x\in(V,2V]$. Therefore by Lemma 2.1, we obtain

$$|S(h_1,h_2)|\ll_{\alpha,\xi}(|L(h_1,h_2)|q^\alpha V^{\alpha-k})^{1/(2^k-2)}+V^{-2^{2-k}}(|L(h_1,h_2)|q^\alpha V^{\alpha-k})^{-1/(2^k-2)}. \tag{3.2}$$

Let S_1 and S_2 be the first and second term on the right-hand side of (3.2), respectively. Then we have

$$S_1^{2^k-2}\le|L(h_1,h_2)|q^\alpha V^{\alpha-k}\ll_{\alpha,\xi,\delta} q^\theta q^\alpha q^{(\alpha-k)\xi/\alpha}.$$

Further, we observe that

$$\alpha+\frac{(\alpha-k)\xi}{\alpha}<\alpha+\frac{(\alpha-\alpha(\alpha+\xi)/\xi)\xi}{\alpha}=0.$$

Therefore, by taking small $\theta>0$, one has $S_1\ll_{\alpha,\xi,\delta} q^{-2\theta}$.

Let us next evaluate S_2. By (3.1), it follows that

$$S_2^{\alpha(2^k-2)}\ll_{\alpha,\xi,\delta} q^{-2^{2-k}(2^k-2)\xi}q^{\eta\alpha}q^{-\alpha^2}q^{\xi(k-\alpha)}.$$

By $2\le k\le\alpha(\alpha+\xi)/\xi+1$, the exponent of q is

$$\begin{aligned}
&-2^{2-k}(2^k-2)\xi+\eta\alpha-\alpha^2+\xi(k-\alpha)\\
&\le(-4+2^{3-k})\xi+\xi-\epsilon\alpha+2\theta\alpha-\alpha^2+\xi(\alpha(\alpha+\xi)/\xi+1-\alpha)\\
&\le-\epsilon\alpha+2\theta\alpha<0
\end{aligned}$$

if θ is sufficiently small. Therefore, by taking small $\theta > 0$, one has $S_2 \ll_{\alpha,\xi} q^{-2\theta}$. Hence

$$\mathcal{D}_{V<v\le 2V}((vp)^\alpha, (vq)^\alpha) \ll_{\alpha,\xi,\delta} q^{-\theta} + q^{-2\theta}(\log H)^2.$$

This implies that

$$\mathcal{D}_{V<v\le 2V}((vp)^\alpha, (vq)^\alpha) \ll_{\alpha,\xi,\epsilon,\delta} q^{-\theta}.$$

Let us next discuss the case when there exist $h_1, h_2 \in \mathbb{Z}$ with $0 < \|(h_1,h_2)\|_\infty \le H$ such that $|L(h_1,h_2)| \le q^{-\eta}$. In this case, it follows that either $h_1 < 0 < h_2$ or $h_2 < 0 < h_1$. Indeed, if $h_1, h_2 \ge 0$ or $h_1, h_2 \le 0$ holds, then by $0 < \|(h_1,h_2)\|_\infty \le H$, one has

$$q^{-\eta} \ge |L(h_1,h_2)| = |h_1|(p/q)^\alpha + |h_2| \ge (1/q)^\alpha = q^{-\alpha}.$$

Therefore, $\xi/\alpha - \epsilon \le \eta \le \alpha$ which contradicts $\epsilon < (\xi - \alpha^2)/\alpha$. Hence

$$|L(h_1,h_2)| = ||h_1|(p/q)^\alpha - |h_2|| \le q^{-\eta}.$$

In addition, this implies that $|h_1| \ge (q/p)^\alpha |h_2| - q^{\alpha-\eta}/p^\alpha > |h_2|$ since $\eta > \alpha$ and $q > p$. We replace $|h_1|$ and $|h_2|$ with h_1 and h_2, respectively. Let $\psi = \eta - 2\theta = \xi/\alpha - \epsilon$. Let $U = q^{(\psi-\alpha)/\alpha}$. Let $K = \lfloor q^{2\theta} \rfloor$. By (2.1),

$$\mathcal{D}_{U<u\le 2U}((uq)^\alpha/h_1) \ll \frac{1}{K} + \sum_{1\le h\le K} \frac{|T(h)|}{h},$$

where $T(h) = \frac{1}{U}\sum_{U<u\le 2U} \mathrm{e}(h(uq)^\alpha/h_1)$. Let $\ell = \lfloor \alpha\psi/(\psi-\alpha) \rfloor + 1$. We define $g(x) = (xq)^\alpha/h_1$. Then for all real numbers $x \in (U, 2U]$

$$q^\alpha U^{\alpha-\ell}/h_1 \ll_{\ell,\alpha} |g^{(\ell)}(x)| \ll_{\ell,\alpha} q^\alpha U^{\alpha-\ell}/h_1.$$

Hence, by Lemma 2.1, one has

$$T(h) \ll_{\ell,\alpha} (q^\alpha U^{\alpha-\ell}/h_1)^{1/(2^\ell-2)} + U^{-2^{2-\ell}}(q^\alpha U^{\alpha-\ell}/h_1)^{-1/(2^\ell-2)}.$$

Let T_1 and T_2 be the first and second term on the right-hand side of this equation. Then

$$T_1^{2^\ell-2} = q^\alpha U^{\alpha-\ell}/h_1 \le q^\alpha q^{(\alpha-\ell)(\psi-\alpha)/\alpha}.$$

The exponent of q is

$$\alpha + (\alpha-\ell)(\psi-\alpha)/\alpha < \alpha + (\alpha - \psi\alpha/(\psi-\alpha))(\psi-\alpha)/\alpha = 0.$$

Therefore, by taking small $\theta > 0$, we have $T_1 \ll_{\alpha,\ell} q^{-3\theta}$. Let us evaluate T_2. We have

$$T_2^{\alpha(2^\ell-2)} \le U^{-2^{2-\ell}(2^\ell-2)\alpha} H^\alpha q^{-\alpha^2} U^{(\ell-\alpha)\alpha} \ll_{\alpha,\ell} q^{-2^{2-\ell}(2^\ell-2)(\psi-\alpha)} q^{-\alpha^2} q^{(\ell-\alpha)(\psi-\alpha)} H^\alpha.$$

Let $\xi' = \psi - \alpha$. Note that $\xi' > 0$ holds since $\xi > \alpha^2 + \alpha\epsilon$ and $\psi = \xi/\alpha - \epsilon$. From $2 \le \ell \le \alpha(\alpha+\xi')/\xi' + 1$, the exponent of q is

$$\begin{aligned}
&-2^{2-\ell}(2^\ell-2)\xi' - \alpha^2 + (\ell-\alpha)\xi' \\
&\le -4\xi' + 2^{3-2}\xi' - \alpha^2 + (\alpha(\alpha+\xi')/\xi' + 1 - \alpha)\xi' \\
&\le -2\xi' - \alpha^2 + (\alpha^2/\xi' + 1)\xi' = -\xi'.
\end{aligned}$$

Therefore $T_2^{\alpha(2^\ell-2)} \ll_{\alpha,\ell} q^{-\xi'+\alpha\theta}$. By taking sufficiently small $\theta = \theta(\alpha,\xi,\epsilon)$, we have $T_2 \ll_{\alpha,\ell} q^{-3\theta}$. Hence we obtain

$$\mathcal{D}_{U<u\le 2U}((uq)^\alpha/h_1') \ll_{\alpha,\ell} q^{-2\theta} + q^{-3\theta}\log q \ll_{\alpha,\xi,\epsilon} q^{-2\theta}.$$

□

4. Proof of Theorem 1.2

Let $\alpha > 1$. Suppose that α satisfies Condition 1.1. Then there exists $\xi > \alpha^2$ such that infinitely many (p, q, r, s)'s with $p < q < r < s$ satisfy (1.1). Let $((p_n, q_n, r_n, s_n))_{n=1}^{\infty}$ be a sequence of which each term satisfies (1.1). We may assume that $q_1 < q_2 < \cdots \to \infty$. If not, then there are at most finitely many (p, q, r, s)'s satisfy (1.1). This is a contradiction. Let $0 < \epsilon < (\xi - \alpha^2)/\alpha$, and $\delta = 2^{-6}$. Take any large $n \in \mathbb{N}$. By Lemma 3.1, by setting $V_n = (\delta q_n^{\xi})^{1/\alpha}$ and $U_n = q_n^{(\xi-\alpha^2)/\alpha^2-\epsilon/\alpha}$, either we have

$$\mathcal{D}_{V_n < v \leq 2V_n}((vp)^{\alpha}, (vq)^{\alpha}) \ll_{\alpha,\epsilon,\xi,\delta} q_n^{-\theta},$$

or there exist $h_1, h_2 \in \mathbb{N}$ with $h_2 < h_1 \leq q_n^{\theta}$ such that $|h_1(p_n/q_n)^{\alpha} - h_2| \leq q_n^{-\xi/\alpha+\epsilon}$ and

$$\mathcal{D}_{U_n < u \leq 2U_n}((uq_n)^{\alpha}/h_1) \ll_{\alpha,\epsilon,\xi,\delta} q_n^{-2\theta}.$$

In the farmer case, let

$$A_n = \{v \in (V_n, 2V_n] \cap \mathbb{N} \colon (\{(vp_n)^{\alpha}\}, \{(vq_n)^{\alpha}\}) \in [1/8, 1/4) \times [0, 1/8)\}.$$

By the definition of the discrepancy, it follows that

$$\# A_n = V_n/64 + O(V_n q_n^{-\theta}).$$

Therefore $A_n \neq \emptyset$ if n is sufficiently large. Fix any $v_n \in A$. Then we have

$$\begin{aligned}(v_n r_n)^{\alpha} &= 2(v_n q_n)^{\alpha} - (v_n p_n)^{\alpha} + E_n^{(1)}\\ &= 2\lfloor (v_n q_n)^{\alpha} \rfloor - \lfloor (v_n p_n)^{\alpha} \rfloor + 2\{(v_n q_n)^{\alpha}\} - \{(v_n p_n)^{\alpha}\} + E_n^{(1)}\end{aligned}$$

where we let $E_n^{(1)} = (v_n r_n)^{\alpha} - 2(v_n q_n)^{\alpha} + (v_n p_n)^{\alpha}$. By $v_n \in A_n$, it follows that

$$1/8 = 1/4 - 1/8 \leq 2\{(v_n q_n)^{\alpha}\} - \{(v_n p_n)^{\alpha}\} \leq 1/2.$$

Since $|E_n^{(1)}| \leq \delta = 2^{-6}$, we find that $2\{(v_n q_n)^{\alpha}\} - \{(v_n p_n)^{\alpha}\} + E_n^{(1)} \in [0, 1)$. Thus

$$\{(v_n r_n)^{\alpha}\} = 2\{(v_n q_n)^{\alpha}\} - \{(v_n p_n)^{\alpha}\} + E_n^{(1)}.$$

Let $E_n^{(2)} = (v_n q_n)^{\alpha} + (v_n s_n)^{\alpha} - 2(v_n r_n)^{\alpha}$. Then we have

$$\begin{aligned}(v_n s_n)^{\alpha} &= 2(v_n r_n)^{\alpha} - (v_n q_n)^{\alpha} + E_n^{(2)}\\ &= 2\lfloor (v_n r_n)^{\alpha} \rfloor - \lfloor (v_n q_n)^{\alpha} \rfloor + 2\{(v_n r_n)^{\alpha}\} - \{(v_n q_n)^{\alpha}\} + E_n^{(2)}.\end{aligned}$$

By $v_n \in A_n$, we obtain that

$$\begin{gathered}2\{(v_n r_n)^{\alpha}\} - \{(v_n q_n)^{\alpha}\} + F_n = 3\{(v_n q_n)^{\alpha}\} - 2\{(v_n p_n)^{\alpha}\} + 2E_n^{(1)} + E_n^{(2)},\\ 1/8 = 3/8 - 1/4 \leq 3\{(v_n q_n)^{\alpha}\} - 2\{(v_n p_n)^{\alpha}\} \leq 3/4,\\ |2E_n^{(1)} + E_n^{(2)}| \leq 3\delta \leq 2^{-4}.\end{gathered}$$

These imply that $\{(v_n s_n)^{\alpha}\} = 2\{(v_n r_n)^{\alpha}\} - \{(v_n q_n)^{\alpha}\} + E_n^{(2)}$. Therefore

$$\begin{aligned}&|\lfloor (v_n p_n)^{\alpha} \rfloor + \lfloor (v_n r_n)^{\alpha} \rfloor - 2\lfloor (v_n q_n)^{\alpha} \rfloor|\\ &\leq |E_n^{(1)}| + |\{(v_n p_n)^{\alpha}\} + \{(v_n r_n)^{\alpha}\} - 2\{(v_n q_n)^{\alpha}\}|\\ &\leq \delta + \delta = 2\delta = 2^{-5},\end{aligned}$$

and

$$\begin{aligned}&|\lfloor (v_nq_n)^\alpha\rfloor + \lfloor (v_ns_n)^\alpha\rfloor - 2\lfloor (v_nr_n)^\alpha\rfloor|\\ &\le |E_n^{(2)}| + |\{(v_nq_n)^\alpha\} + \{(v_ns_n)^\alpha\} - 2\{(v_nr_n)^\alpha\}|\\ &\le \delta+\delta = 2\delta = 2^{-5}.\end{aligned}$$

Hence we conclude that

$$\lfloor (v_np_n)^\alpha\rfloor + \lfloor (v_nr_n)^\alpha\rfloor = 2\lfloor (v_nq_n)^\alpha\rfloor,\quad \lfloor (v_nq_n)^\alpha\rfloor + \lfloor (v_ns_n)^\alpha\rfloor = 2\lfloor (v_nr_n)^\alpha\rfloor.$$

In the latter case when there exist $h_1, h_2 \in \mathbb{N}$ with $h_2 < h_1 \le q_n^\theta$ such that

$$|h_1(p_n/q_n)^\alpha - h_2| \le q_n^{-\xi/\alpha+\epsilon-2\theta},\quad \mathcal{D}_{U_n<u\le 2U_n}((uq_n)^\alpha/h_1) \ll_{\alpha,\epsilon,\xi} q_n^{-2\theta}$$

where $U_n = q_n^{(\xi-\alpha^2)/\alpha^2-\epsilon/\alpha}$. Let

$$B_n = \{u\in (U_n, 2U_n]\colon h_1^{-1}2^{-6} \le \{(u_nq_n)^\alpha/h_1\} < h_1^{-1}2^{-5}\}$$

By the definition of the discrepancy, it follows that

$$\#B_n = U_nh_1^{-1}2^{-6} + O(U_nq_n^{-2\theta}) \ge U_nq_n^{-\theta}2^{-6} + O(U_nq_n^{-2\theta}) > 0.$$

Therefore $B_n \ne \emptyset$ if n is sufficiently large. Fix any $u_n\in B_n$. Then we observe that

$$(u_nq_n)^\alpha = h_1\lfloor (u_nq_n)^\alpha/h_1\rfloor + h_1\{(u_nq_n)^\alpha/h_1\}.$$

Since $u_n\in B_n$, we also observe that $2^{-6}\le h_1\{(u_nq_n)^\alpha/h_1\}\le 2^{-5}$. Thus

$$\{(u_nq_n)^\alpha\} = h_1\{(u_nq_n)^\alpha/h_1\}. \tag{4.1}$$

Let $E_n^{(3)} = (u_np_n)^\alpha - (h_2/h_1)(u_nq_n)^\alpha$. It follows that

$$\begin{aligned}(u_np_n)^\alpha &= (h_2/h_1)(u_nq_n)^\alpha + E_n^{(3)}\\ &= h_2\lfloor (u_nq_n)^\alpha/h_1\rfloor + h_2\{(u_nq_n)^\alpha/h_1\} + E_n^{(3)}.\end{aligned}$$

By $u_n\in B_n$, $1\le h_2 < h_1 \le q_n^\theta$, and $u_n \le U_n = q_n^{(\xi-\alpha^2)/\alpha^2-\epsilon/\alpha}$, we have

$$q_n^{-\theta}2^{-6} \le h_2\cdot h_1^{-1}2^{-6} \le h_2\{(u_nq_n)^\alpha/h_1\} \le h_2\cdot h_1^{-1}2^{-5} \le 2^{-5} - q_n^{-\theta}2^{-5},$$

$$|E_n^{(3)}| = (u_nq_n)^\alpha|h_1(p_n/q_n)^\alpha - h_2|/h_1 \le U_n^\alpha q_n^\alpha q_n^{-\xi/\alpha+\epsilon-2\theta} = q_n^{(\xi-\alpha^2)/\alpha-\epsilon} q_n^\alpha q_n^{-\xi/\alpha+\epsilon-2\theta} = q_n^{-2\theta}.$$

Therefore, we obtain $\{(u_np_n)^\alpha\} = h_2\{(u_nq_n)^\alpha/h_1\} + E_n^{(3)}$ by taking large n. Further, let $E_n^{(4)} = (u_nr_n)^\alpha - 2(u_nq_n)^\alpha + (u_np_n)^\alpha$. We observe that

$$\begin{aligned}(u_nr_n)^\alpha &= 2(u_nq_n)^\alpha - (u_np_n)^\alpha + E_n^{(4)}\\ &= 2h_1\lfloor (u_nq_n)^\alpha\rfloor - \lfloor (u_np_n)^\alpha\rfloor + 2\{(v_nq_n)^\alpha\} - \{(v_np_n)^\alpha\} + E_n^{(4)}.\end{aligned}$$

In addition, we have $|E_n^{(4)}| \le U_n^\alpha q_n^{-\xi} = q_n^{(\xi-\alpha^2)/\alpha-\epsilon}q_n^{-\xi} = q_n^{-(1-1/\alpha)\xi-\alpha-\epsilon}$. Further,

$$\begin{aligned}&2\{(u_nq_n)^\alpha\} - \{(u_np_n)^\alpha\} + E_n^{(4)}\\ &= 2h_1\{(u_nq_n)^\alpha/h_1\} - \{(u_np_n)^\alpha\} + E_n^{(4)}\\ &= 2h_1\{(u_nq_n)^\alpha/h_1\} - h_2\{(u_nq_n)^\alpha/h_1\} - E_n^{(3)} + E_n^{(4)}\\ &= (2h_1-h_2)\{(u_nq_n)^\alpha/h_1\} - E_n^{(3)} + E_n^{(4)}\end{aligned}$$

Note that $h_1 > h_2$ and $u_n\in B_n$ imply

$$2^{-6} = h_1h_1^{-1}2^{-6} \le (2h_1-h_2)\{(u_nq_n)^\alpha/h_1\} \le 2h_1\cdot h_1^{-1}2^{-5} = 2^{-4}.$$

Therefore we have $\{(u_nr_n)^\alpha\} = 2\{(u_nq_n)^\alpha\} - \{(u_np_n)^\alpha\} + E_n^{(4)}$ for sufficiently large n. We next let $E_n^{(5)} = (u_nq_n)^\alpha + (u_ns_n)^\alpha - 2(u_nr_n)^\alpha$. Similarly to the evaluation of $E^{(4)}$, it follows that $|E_n^{(5)}| \leq q_n^{-(1-1/\alpha)\xi-\alpha-\epsilon}$. In addition, we observe that

$$(u_ns_n)^\alpha = 2\lfloor (u_nr_n)^\alpha \rfloor - \lfloor (u_nq_n)^\alpha \rfloor + 2\{(u_nr_n)^\alpha\} - \{(u_nq_n)^\alpha\} + E_n^{(5)},$$

and

$$\begin{aligned}&2\{(u_nr_n)^\alpha\} - \{(u_nq_n)^\alpha\}\\ &= 2(2h_1 - h_2)\{(u_nq_n)^\alpha/h_1\} - h_1\{(u_nq_n)^\alpha/h_1\} - E_n^{(3)} + E_n^{(4)}\\ &= (3h_1 - 2h_2)\{(u_nq_n)^\alpha/h_1\} - E_n^{(3)} + E_n^{(4)}.\end{aligned}$$

Note that $h_1 > h_2$ and $u_n \in B_n$ imply

$$2^{-6} = h_1 \cdot h_1^{-1}2^{-6} \leq (3h_1 - 2h_2)\{(u_nq_n)^\alpha/h_1\} \leq 3h_1 \cdot h_1^{-1}2^{-5} \leq 2^{-3}.$$

Therefore $\{(u_ns_n)^\alpha\} = 2\{(u_nr_n)^\alpha\} - \{(u_nq_n)^\alpha\} + E_n^{(5)}$. Similarly to the farmer case, by taking sufficiently large $n \in \mathbb{N}$, we conclude that

$$\lfloor (u_np_n)^\alpha \rfloor + \lfloor (u_nr_n)^\alpha \rfloor = 2\lfloor (u_nq_n)^\alpha \rfloor, \quad \lfloor (u_nq_n)^\alpha \rfloor + \lfloor (u_ns_n)^\alpha \rfloor = 2\lfloor (u_nr_n)^\alpha \rfloor.$$

Hence PS(α) contains infinitely many 4-APs assuming that α satisfies Condition 1.1

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