TILINGS OF THE PLANE ARISING FROM ITERATED FUNCTION SYSTEMS GENERATED BY THREE SIMILARITY TRANSFORMATIONS

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ABSTRACT. A similarity tiling is a partition of the plane by tiles all similar to each other with pairwise disjoint interiors. We investigate similarity tilings arising from iterated function systems generated by two or three similarity transformations. To find parameters of the generating similarity transformations, we execute search programs to find candidates of the complex ratios, and find locations of the centers by using Fractal Gazer, a computer program developed by the author.

INTRODUCTION

A similarity tiling is a partition of the plane by tiles all similar to each other with pairwise disjoint interiors,

$$\mathbf{R}^2 = \sum_{i=0}^{\infty} g_i(\Lambda), \qquad \mathrm{Int}(g_i(\Lambda)) \cap \mathrm{Int}(g_j(\Lambda)) = \emptyset \ (i \neq j).$$

The fundamental tile Λ is a compact subset of $\mathbf{R}^2$ satisfying $\overline{\mathrm{Int}(\Lambda)} = \Lambda$, and $g_0(= id), g_1, g_2, \ldots$ are similarity transformations.

Such tilings naturally arise from certain type of iterated function systems, which are indeed the subject of our study. We investigate iterated functions systems generated by two or three contractive similarity transformations producing similarity tilings.

Our main goal is not investigating properties of such tilings, but rather finding examples. To find parameters of the generating similarity transformations, we first execute search programs to find candidates of complex ratios. Finding location of the center of similarity transformation is done by hand using Fractal Gazer [6]. Fractal Gazer is a computer program developed by the author to visualize iterated function systems generated by contractive similarity transformations. All the figures in this paper have been produced using Fractal Gazer.

Euclidean tilings of the plane have been subject of numerous studies in history of mathematics. Similarity tilings may be thought of as a variation of Euclidean tilings. Connection between newly found tilings and aperiodic tilings (*cf.* [2]) should be investigated elsewhere. Relationship to Pisot numbers (*cf.* [1]) also needs to be addressed. We hope that the examples in this paper will motivate further studies.

1. NOTATIONS AND DEFINITIONS

Consider an iterated function system

$$\mathcal{F} = \langle f_0, \ldots, f_{m-1} \rangle$$

generated by orientation-preserving contractive similarity transformations of the complex plane

$$f_i(z) = a_i z + (1 - a_i)c_i \quad (z \in \mathbf{C}),$$

which we call a similarity iterated function system. The complex ratio a_i satisfies $0 < |a_i| < 1$. We write $|f_i|$ to mean the scale factor $|a_i|$. The center c_i is the fixed point of f_i. We assume that the centers $c_0, \ldots, c_{m-1}$ are all distinct.

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1.1. **The minimal invariant disk.**

Proposition 1. *Given a similarity iterated function system $\mathcal{F} = \langle f_0, \ldots, f_{m-1} \rangle$, there exists a unique disk D of minimal radius r satisfying*

$$f_i(D) \subset D \quad (i = 0, \ldots, m-1).$$

Proof. Denote the disk of radius $r > 0$ centered at $w \in \mathbf{C}$ by

$$D_r(w) = \{z \in \mathbf{C} \mid |z - w| \leq r\}.$$

The following holds in general.

$$D_{r'}(w') \subset D_r(w) \quad \Leftrightarrow \quad |w - w'| \leq r - r'$$

For the similarity transformation $f(z) = az + (1-a)c$, since

$$f(D_r(w)) = D_{|a|r}(aw + (1-a)c),$$

we have

$$f(D_r(w)) \subset D_r(w) \quad \Leftrightarrow \quad |w - c| \leq \frac{1 - |a|}{|1 - a|} r.$$

For $D = D_r(w)$, the condition is therefore equivalent to

$$|w - c_i| \leq \alpha_i r \quad (i = 0, \ldots, m-1),$$

where $\alpha_i = \frac{1 - |a_i|}{|1 - a_i|}$. Thus the smallest r such that

$$\bigcap_{i=0}^{m-1} D_{\alpha_i r}(c_i) \neq \emptyset$$

gives the solution. □

1.2. **The code map and the limit set.**

Denote by $\Sigma = \{0, \ldots, m-1\}$ the set of indices. A sequence of indices

$$\omega = \omega_1 \cdots \omega_n \in \Sigma^n$$

is called a code of length n.

For an infinite code

$$\omega = \omega_1 \omega_2 \cdots \in \Sigma^\infty,$$

we denote the code consisting of the first n indices by

$$\omega|_n = \omega_1 \cdots \omega_n.$$

The code obtained by concatenating α and β is denoted by $\alpha\beta$.

Suppose that we have a similarity iterated function system $\mathcal{F} = \langle f_0, \ldots, f_{m-1} \rangle$. Let $D = D_r(w)$ be the minimal invariant disk (or any invariant disk), and we have

$$D \supset f_{\omega_1}(D) \supset f_{\omega_1} f_{\omega_2}(D) \supset \cdots \supset f_{\omega|_n}(D) \supset \cdots,$$

where

$$f_{\omega|_n} = f_{\omega_1} \cdots f_{\omega_n}.$$

Note that the disk $f_{\omega|_n}(D)$ shrinks to a point uniformly as $n \to \infty$, for its radius $|f_{\omega_1}| \cdots |f_{\omega_n}| r$ converges to 0 uniformly.

Definition 2. The code map

$$\pi : \Sigma^\infty \to \mathbf{C}$$

is defined by

$$\pi(\omega) = \bigcap_{n=0}^{\infty} f_{\omega|_n}(D) \quad (\omega \in \Sigma^\infty).$$

Definition 3. The image of the code map,

$$\Lambda = \Lambda(\mathcal{F}) = \pi(\Sigma^\infty) \subset \mathbf{C},$$

is called the limit set of $\mathcal{F}$.

Equip Σ with the discrete topology, and the product space Σ^∞ is compact due to Tychonoff's theorem. The code map $\pi : \Sigma^\infty \to \mathbf{C}$ is uniformly continuous since the disk $f_{\omega|_n}(D)$ shrinks to a point uniformly as $n \to \infty$. Therefore, as a continuous image of a compact set, the limit set Λ is compact.

For $i \in \Sigma$, define

$$\sigma_i : \Sigma^\infty \to \Sigma^\infty, \qquad \sigma_i(\omega) = i\omega \quad (\omega \in \Sigma^\infty).$$

Then the diagram

$$\begin{array}{ccc} \Sigma^\infty & \xrightarrow{\sigma_i} & \Sigma^\infty \\ \pi \big\downarrow & \circlearrowright & \big\downarrow \pi \\ \Lambda & \xrightarrow[f_i|_\Lambda]{} & \Lambda \end{array}$$

commutes since

$$f_i(\pi(\omega)) = \bigcap_{n=0}^{\infty} f_i(f_{\omega|_n}(D)) = \bigcap_{n=0}^{\infty} f_{(i\omega)|_{n+1}}(D) = \pi(i\omega).$$

Thus the limit set $\Lambda = \pi(\Sigma^\infty)$ is invariant under the iterated function system. Since

$$\Sigma^\infty = \bigcup_{i=0}^{m-1} \sigma_i(\Sigma^\infty),$$

we have

$$\Lambda = \bigcup_{i=0}^{m-1} f_i(\Lambda). \tag{1}$$

In fact, the limit set is the unique nonempty compact subset of $\mathbf{C}$ satisfying this property.

1.3. **The Hutchinson operator.**

Definition 4. The Hutchinson operator associated with an iterated function system $\mathcal{F}$, also denoted $\mathcal{F}$, is defined for any subset $S \subset \mathbf{C}$ by

$$\mathcal{F}(S) = \bigcup_{i=0}^{m-1} f_i(S).$$

The Hutchinson operator is a contraction mapping on the set of non-empty compact subsets with respect to the Hausdorff metric, and the limit set Λ is the unique fixed point([3]). For any non-empty compact set $K \subset \mathbf{C}$, the sequence of subsets $\mathcal{F}^n(K)$ $(n = 0, 1, \dots)$ converges to the limit set Λ.

Note that a similarity transformation f_i is a homeomorphism and commutes with topological operators like closure and interior; $f_i(\overline{S}) = \overline{f_i(S)}$ and $f_i(\mathrm{Int}(S)) = \mathrm{Int}(f_i(S))$. Hence we have

$$\mathcal{F}(\overline{S}) = \overline{\mathcal{F}(S)}$$

and

$$\mathcal{F}(\mathrm{Int}(S)) \subset \mathrm{Int}(\mathcal{F}(S))$$

in general.

1.4. **Equivalence of similarity iterated function systems.**

Definition 5. Similarity iterated function systems $\mathcal{F} = \langle f_0, \dots, f_{m-1} \rangle$ and $\mathcal{F}' = \langle f'_0, \dots, f'_{m-1} \rangle$ are said to be equivalent and denoted $\mathcal{F} \simeq \mathcal{F}'$ if there exists an (either orientation-preserving or orientation-reversing) similarity transformation $g : \mathbf{C} \to \mathbf{C}$ such that

$$g f_i \, g^{-1} = f'_i \quad (i = 0, \dots, m-1)$$

under appropriate permutation of the indices.

If $\mathcal{F}$ and $\mathcal{F}'$ are equivalent via g, the centers, the minimal invariant disk and the limit set of $\mathcal{F}$ are mapped to those of $\mathcal{F}'$ by g.

2. Similarity tilings

2.1. **m-similarity tiling.** We are interested in similarity iterated function systems satisfying the following condition.

Definition 6. A similarity iterated function system $\mathcal{F} = \langle f_0, \dots, f_{m-1} \rangle$ is called a similarity tiling, or m-similarity tiling, if

- $\mathrm{Int}(\Lambda) \neq \emptyset$, and
- $f_i(\mathrm{Int}(\Lambda)) \cap f_j(\mathrm{Int}(\Lambda)) = \emptyset$ if $i \neq j$.

In view of (1), a similarity tiling may seem more appropriate to be called a similarity self-tiling. However, as we will see later, a similarity tiling actually gives rise to genuine tilings of the plane, and we choose the shorter name for the concept.

Proposition 7. *If the limit set Λ has non-empty interior, then $\overline{\mathrm{Int}(\Lambda)} = \Lambda$.*

Proof. Since Λ is closed, obviously we have

$$\overline{\mathrm{Int}(\Lambda)} \subset \Lambda.$$

On the other hand, we have

$$\mathcal{F}(\mathrm{Int}(\Lambda)) \subset \mathrm{Int}(\mathcal{F}(\Lambda)) = \mathrm{Int}(\Lambda)$$

and

$$\mathcal{F}(\overline{\mathrm{Int}(\Lambda)}) = \overline{\mathcal{F}(\mathrm{Int}(\Lambda))} \subset \overline{\mathrm{Int}(\Lambda)},$$

hence by induction

$$\overline{\mathrm{Int}(\Lambda)} \supset \mathcal{F}(\overline{\mathrm{Int}(\Lambda)}) \supset \mathcal{F}^2(\overline{\mathrm{Int}(\Lambda)}) \supset \cdots \supset \bigcap_{k=0}^{\infty} \mathcal{F}^k(\overline{\mathrm{Int}(\Lambda)}) = \Lambda.$$

This completes the proof. □

Definition 8. An iterated function system $\mathcal{F}$ is said to satisfy the open set condition if there exists a non-empty open set $O \subset \mathbf{C}$ such that

- $\mathcal{F}(O) \subset O$, and
- $f_i(O) \cap f_j(O) = \emptyset$ if $i \neq j$.

Proposition 9. *A similarity iterated function system whose limit set has non-empty interior is a similarity tiling if and only if it satisfies the open set condition.*

Proof. A similarity tiling satisfies the open set condition with $O = \mathrm{Int}(\Lambda)$, for we have $\mathcal{F}(\mathrm{Int}(\Lambda)) \subset \mathrm{Int}(\mathcal{F}(\Lambda)) = \mathrm{Int}(\Lambda)$.

Conversely, let O be an open set satisfying the open set condition. From $\mathcal{F}(O) \subset O$ we obtain

$$\overline{O} \supset \mathcal{F}(\overline{O}) \supset \mathcal{F}^2(\overline{O}) \supset \cdots \supset \bigcap_{k=0}^{\infty} \mathcal{F}^k(\overline{O}) = \Lambda,$$

hence

$$\mathrm{Int}(\Lambda) \subset \mathrm{Int}(\overline{O}). \tag{2}$$

On the other hand, for $i \neq j$ we see, in turn,

$$\begin{aligned} f_i(O) \cap f_j(O) &= \emptyset, \\ f_i(O) \cap f_j(\overline{O}) &= \emptyset, \\ f_i(O) \cap f_j(\mathrm{Int}(\overline{O})) &= \emptyset, \\ f_i(\overline{O}) \cap f_j(\mathrm{Int}(\overline{O})) &= \emptyset, \\ f_i(\mathrm{Int}(\overline{O})) \cap f_j(\mathrm{Int}(\overline{O})) &= \emptyset. \end{aligned}$$

Therefore by (2), we have

$$f_i(\mathrm{Int}(\Lambda)) \cap f_j(\mathrm{Int}(\Lambda)) = \emptyset.$$

This completes the proof. □

Proposition 10. *The scale factors of a similarity tiling* $\mathcal{F} = \langle f_0, \ldots, f_{m-1}\rangle$ *must satisfy*

$$\sum_{i=0}^{m-1} |f_i|^2 = 1.$$

Proof. The area (2-dimensional Lebesgue measure) of the limit set must satisfy

$$\begin{aligned}\text{Area}(\Lambda) &= \sum_{i=0}^{m-1} \text{Area}(f_i(\Lambda)) \\ &= \sum_{i=0}^{m-1} |f_i|^2 \text{Area}(\Lambda).\end{aligned}$$

From this we obtain

$$1 = \sum_{i=0}^{m-1} |f_i|^2,$$

since Λ has non-empty interior and $\text{Area}(\Lambda) \neq 0$. □

2.2. **Tilings of the plane.** A similarity tiling $\mathcal{F} = \langle f_0, \ldots, f_{m-1}\rangle$ gives rise to tilings of the plane $\mathbf{C}$, with subsets similar to the limit set Λ.

Proposition 11. *There exist uncountably many tilings of* $\boldsymbol{C}$ *such that*

$$\boldsymbol{C} = \bigcup_{i=0}^{\infty} g_i(\Lambda),$$

where $g_0 = id$ *and* $g_1, g_2, \ldots$ *are similarity transformations, and*

$$g_i(\text{Int}(\Lambda)) \cap g_j(\text{Int}(\Lambda)) = \emptyset \qquad \text{if } i \neq j.$$

Proof. For a finite code $\omega = \omega_1 \cdots \omega_n$, we use the notation

$$\check{f}_\omega = f_{\omega_1}^{-1} \cdots f_{\omega_n}^{-1}.$$

Lemma 12. *Given an infinite code* $\omega \in \Sigma^\infty$, *we have a partition for each* $n = 1, 2, \ldots$,

$$P_n(\omega): \quad \check{f}_{\omega|_n}(\Lambda) = \bigcup_{i=0}^{(m-1)n} g_i(\Lambda),$$

where $g_0 = id$ *and* $g_1, \ldots, g_{(m-1)n}$ *are similarity transformations, and*

$$g_i(\text{Int}(\Lambda)) \cap g_j(\text{Int}(\Lambda)) = \emptyset \qquad \text{if } i \neq j.$$

Furthermore, the partition $P_{n+1}(\omega)$ *naturally extends* $P_n(\omega)$.

Fig. 1 shows partitions $P_1(\omega)$ and $P_2(\omega)$ corresponding to some codes.

Proof. Recall that we have

$$\Lambda = \bigcup_{i=0}^{m-1} f_i(\Lambda), \qquad f_i(\text{Int}(\Lambda)) \cap f_j(\text{Int}(\Lambda)) = \emptyset \text{ if } i \neq j. \tag{3}$$

First, for $n = 1$, we have

$$f_{\omega_1}^{-1}(\Lambda) = \Lambda \cup \bigcup_{i=0,\ldots,m-1,\ i\neq\omega_1} f_{\omega_1}^{-1} f_i(\Lambda),$$

and this is $P_1(\omega)$.

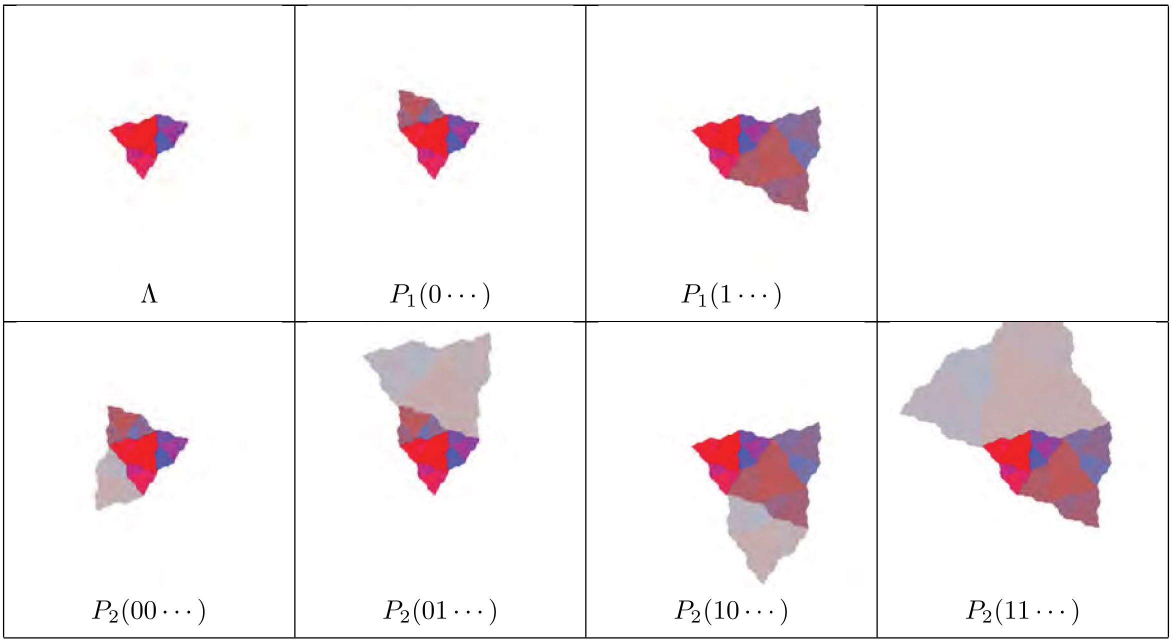

FIGURE 1. Partitions corresponding to some codes.

Next, assume that we already have $P_n(\omega)$. Then,

$$\begin{aligned}\check{f}_{\omega|_{n+1}}(\Lambda) &= \check{f}_{\omega|_n} f^{-1}_{\omega_{n+1}}(\Lambda)\\ &= \check{f}_{\omega|_n}(\Lambda \cup \bigcup_{i=0,\ldots,m-1,\ i\neq\omega_{n+1}} f^{-1}_{\omega_{n+1}} f_i(\Lambda))\\ &= \check{f}_{\omega|_n}(\Lambda) \cup \bigcup_{i=0,\ldots,m-1,\ i\neq\omega_{n+1}} \check{f}_{\omega|_n} f^{-1}_{\omega_{n+1}} f_i(\Lambda))\\ &= \bigcup_{i=0}^{(m-1)n} g_i(\Lambda) \cup \bigcup_{i=0,\ldots,m-1,\ i\neq\omega_{n+1}} \check{f}_{\omega|_{n+1}} f_i(\Lambda)\end{aligned}$$

Thus we have a partition $P_{n+1}(\omega)$ naturally extending $P_n(\omega)$. This completes the proof by induction. □

Lemma 13. *Let $D_r(w)$ be the minimal invariant disk for $\mathcal{F}$. Then, for any point $z \in \boldsymbol{C}$, we have*

$$f_\tau(z) \in D_{2r}(w)$$

as long as the length of τ is sufficiently large.

Proof. Write

$$\mu = \max_{i=0,\ldots,m-1} |f_i|,$$

and we have

$$|f_\tau(z) - f_\tau(w)| \leq \mu^{|\tau|}|z - w| \leq r$$

as long as the length $|\tau|$ is sufficiently large. Since $f_\tau(w) \in D_r(w)$, we have $f_\tau(z) \in D_{2r}(w)$. □

Lemma 14. *There exists a code $\sigma = \sigma_1 \cdots \sigma_k$ such that*

$$f_\sigma(D_{2r}(w)) \subset \Lambda.$$

Proof. Let $\alpha \in \Sigma^\infty$ be a code corresponding to an interior point of Λ. Then, there exists a number k such that

$$f_{\alpha|_k}(D_{2r}(w)) \subset \mathrm{Int}(\Lambda).$$

It suffices to put $\sigma = \alpha|_k$. □

Now we show:

Lemma 15. *We have*

$$\bigcup_{n=1}^{\infty} \check{f}_{\omega|_n}(\Lambda) = \boldsymbol{C} \tag{4}$$

for uncountably many codes $\omega \in \Sigma^\infty$.

Proof. The condition (4) holds if for any $z \in \mathbf{C}$ there exists a number n such that

$$f_{\omega_n} \cdots f_{\omega_1}(z) \in \Lambda.$$

But for any $z \in \mathbf{C}$ we know by Lemma 13 and Lemma 14 that

$$f_{\sigma_1} \cdots f_{\sigma_k} f_\tau(z) \in \Lambda$$

as long as the length of τ is sufficiently large. Therefore, (4) holds if the sequence $\sigma_k \cdots \sigma_1$ appears in ω infinitely many times. Obviously there are uncountably many such codes $\omega \in \Sigma^\infty$. □

The tilings of Lemma 15 are different from each other because if $\omega \neq \omega'$, there is a number k such that $\omega|_k \neq \omega'|_k$ and we have $\check{f}_{\omega|_k} \neq \check{f}_{\omega'|_k}$, since

$$\check{f}^{-1}_{\omega|_k}(O) \cap \check{f}^{-1}_{\omega'|_k}(O) = \emptyset$$

by the open set condition. This complete the proof of Proposition 11. □

3. 2-similarity tilings

Let us consider a 2-similarity tiling $\mathcal{F} = \langle f_0, f_1 \rangle$. Since there always exists a similarity transformation g mapping the center of f_0 to 0 and the center of f_1 to 1, we may assume that the centers of f_0 and f_1 are 0 and 1 respectively up to equivalence. Thus, we have

$$f_0(z) = a_0 z \qquad \text{and} \qquad f_1(z) = a_1 z + 1 - a_1.$$

The pair of complex numbers (a_0, a_1) satisfying $0 < |a_0| < 1$ and $0 < |a_1| < 1$ determine the similarity iterated function system, which we denote by $\mathcal{F}(a_0, a_1)$. Further considering the reflection in the real line, and the π-rotation about the point 1/2, we see that

$$\mathcal{F}(a_0, a_1) \simeq \mathcal{F}(\bar{a}_0, \bar{a}_1) \simeq \mathcal{F}(a_1, a_0) \simeq \mathcal{F}(\bar{a}_1, \bar{a}_0).$$

Given a complex number z, we use the notation $\tilde{z}$ for either z or $\bar{z}$, just like $\pm z$ means either z or $-z$. Thus, $\pm\tilde{z}$ means z, $-z$, $\bar{z}$, or $-\bar{z}$. In general, the 16 classes of similarity iterated function systems $\mathcal{F}(\pm\tilde{u}, \pm\tilde{v})$ are grouped into eight conjugate pairs $\mathcal{F}(\pm u, \pm\tilde{v})$. If $u = v$ up to sign and conjugation, they are further reduced to six; a complete system of representatives is

$$F(u,u), \quad F(u,-u), \quad F(u,\bar{u}), \quad F(u,-\bar{u}), \quad F(-u,-u), \quad F(-u,-\bar{u}).$$

Example 16. Consider the case $(a_0, a_1) = (\frac{i}{\sqrt{2}}, -\frac{i}{\sqrt{2}})$. The limit set is the rectangle,

$$\Lambda = \{z \in \mathbf{C} \,|\, -1 \leq \Re z \leq 2, \ -\frac{1}{\sqrt{2}} \leq \Im z \leq \sqrt{2}\}$$

shown in Fig. 2(right). One can easily verify that

$$f_0(\Lambda) = \{x + yi \in \mathbf{C} \,|\, -1 \leq \Re z \leq \frac{1}{2}, \ -\frac{1}{\sqrt{2}} \leq \Im z \leq \sqrt{2}\},$$
$$f_1(\Lambda) = \{x + yi \in \mathbf{C} \,|\, \frac{1}{2} \leq \Re z \leq 2, \ -\frac{1}{\sqrt{2}} \leq \Im z \leq \sqrt{2}\},$$

and thus $\Lambda = f_0(\Lambda) \cup f_1(\Lambda)$. This proves that $\mathcal{F}(\frac{i}{\sqrt{2}}, -\frac{i}{\sqrt{2}})$ is a similarity tiling.

Example 17. Consider the case $(a_0, a_1) \sim (-0.662359 - 0.562280i, 0.460202 + 0.182582i)$. See Fig. 5 $\mathcal{F}(-\alpha, -\alpha^5)$ for the limit set. This 2-similarity tiling has been studied by Thurston [5], Akiyama [1] and several others in relation to Pisot numbers. The similarity iterated function system $\mathcal{F} = \langle f_0, f_1 \rangle$ satisfies

$$f_{000}(1) = f_{10}(1), \qquad \text{and} \qquad f_{0010}(1) = f_{100}(1),$$

which imply

$$1 + a_0 + a_0^2 = a_1, \qquad \text{and} \qquad 1 + a_0 - a_1 - a_0 a_1 + a_0^2 a_1 = 0. \tag{5}$$

Substituting a_1 of the first equation in the second, we see that a_0 is one of the complex root of the polynomial

$$p(z) = z^3 - z - 1.$$

We also see that $a_1 = a_0^5$ by checking $z^5 \equiv 1 + z + z^2 \mod p(z)$.

Let λ be the real root, and $\alpha, \bar{\alpha}$ be the complex roots of the polynomial $p(z)$ so that $a_0 = \alpha$ and $a_1 = \alpha^5$. From $p(z) = (z - \lambda)(z - \alpha)(z - \bar{\alpha})$, we easily deduce that $r = |\alpha|^2 = \alpha\bar{\alpha}$ satisfies

$$r^2 + r^3 = 1,$$

and then

$$r + r^5 = 1,$$

for $r^5 + r - 1$ is divisible by $r^3 + r^2 - 1$. Hence we have $|a_0|^2 + |a_1|^2 = 1$. Namely, a_0, a_1 satisfy the necessary condition of Proposition 10. Proof that this actually gives a similarity tiling is non-trivial ([4]).

Besides the necessary condition of Proposition 10, the complex ratios a_0 and a_1 of a 2-similarity tiling would also satisfy some sort of algebraic condition like (5). We impose the following ad hoc assumption. We do not have a rational explanation for the assumption, but all 2-similarity tilings known so far satisfy this.

Assumption 18. There exists a complex root α of some polynomial $p(z)$ with integer coefficients such that

$$a_0 = \pm\tilde{\alpha}^{e_0} \qquad \text{and} \qquad a_1 = \pm\tilde{\alpha}^{e_1}$$

for some positive integers e_0, e_1.

We may also assume that $\Re\alpha \geq 0$ and $\Im\alpha > 0$ since, after all, we will take conjugates and change signs of α^e to look for similarity tilings. Let us express the polynomial $p(z)$ as

$$p(z) = \sum_{k=0}^{d} p_k z^k \quad (p_k \in \mathbf{Z}).$$

A computer search using Mathematica for (a_0, a_1) satisfying Assumption 18 and

$$|a_0|^2 + |a_1|^2 = 1$$

restricting to $d \leq 3$, $|p_k| \leq 2$ and $e_0, e_1 \leq 20$ yields the following five polynomials and six cases.

(i) $p(z) = 1 + 2z^2$, $\alpha = \frac{i}{\sqrt{2}} \sim 0.707107i$, $(e_0, e_1) = (1, 1)$,
(ii) $p(z) = 1 - z + 2z^2$, $\alpha = \frac{1+\sqrt{7}i}{4} \sim 0.25 + 0.661438i$, $(e_0, e_1) = (1, 1)$,
(iii) $p(z) = 1 - 2z + 2z^2$, $\alpha = \frac{1+i}{2}$, $(e_0, e_1) = (1, 1)$,
(iv) $p(z) = 1 - z + z^3$, $\alpha \sim 0.662359 + 0.562280i$, $(e_0, e_1) = (1, 5)$, $(2, 3)$,
(v) $p(z) = 1 + z^2 + z^3$, $\alpha \sim 0.232786 + 0.792552i$, $(e_0, e_1) = (1, 3)$.

Extending the range of search to $d \leq 3$, $|p_k| \leq 7$ adds eight more cases, none of which leads to a 2-similarity tiling. Another search where $d \leq 6$, $|p_k| \leq 2$ finds 33 additional cases none of which leads to a new 2-similarity tiling. In fact we have loosen the assumption slightly and tried all possible combinations of

$$a_0 = \pm\alpha^{e_0 - i}\bar{\alpha}^i \quad (0 \leq i \leq e_0), \qquad a_1 = \pm\alpha^{e_1 - j}\bar{\alpha}^j \quad (0 \leq j \leq e_1),$$

in the range $d \leq 4$, $|p_k| \leq 6$, $e_0, e_1 \leq 20$, and still have no new 2-similarity tiling.

It should be stressed that we judge whether a given similarity iterated function system is a similarity tiling or not solely by looking at the picture of the limit set produced by Fractal Gazer. In fact, we do not know how to prove most of what we claim to be similarity tilings are actually similarity tilings. For instance, is $\mathcal{F}(-\alpha^2, -\alpha^3)$ of Fig. 5 really a similarity tiling?

Anyhow, Figs. 2, 3, 4, 5 and 6 show 2-similarity tilings associated with $1 + 2z^2$, $1 - z + 2z^2$, $1 - 2z + 2z^2$, $1 - z + z^3$ and $1 + z^2 + z^3$, respectively. These are all the 2-similarity tilings we have found.

There are only two similarity tilings associated with $1 + 2z^2$ shown in Fig. 2, for $\bar{\alpha} = -\alpha$. The two similarity tilings are not equivalent, but their limit sets are similar. This follows from the symmetry of the limit set. The limit set $\Lambda = \Lambda(\mathcal{F}(\alpha, \alpha))$ is invariant under $g(z) = 1 - z$. It follows that

$$\Lambda = f_0(\Lambda) \cup f_1 g(\Lambda),$$

and Λ is also the limit set of the 2-similarity tiling $\langle f_0, f_1 g \rangle$, which is equivalent to $\mathcal{F}(\alpha, -\alpha)$. The limit sets of $\mathcal{F}(\alpha, \alpha)$, $\mathcal{F}(\alpha, -\alpha)$, $\mathcal{F}(-\alpha, -\alpha)$, associated with $1 - z + 2z^2$ shown in Fig. 3, as well as those associated with $1 - 2z + 2z^2$ shown in Fig. 4 are similar by the same reason.

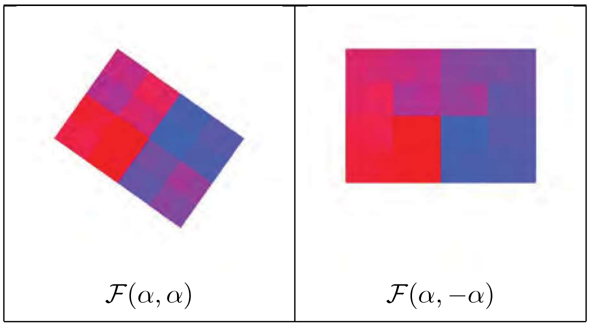

FIGURE 2. 2-similarity tilings associated with $1 + 2z^2$, $\alpha = \frac{i}{\sqrt{2}}$.

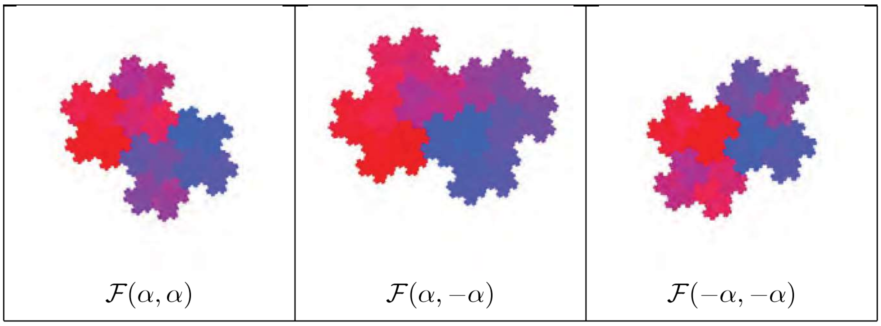

FIGURE 3. 2-similarity tilings associated with $1 - z + 2z^2$, $\alpha = \frac{1+\sqrt{7}i}{4}$.

4. 3-SIMILARITY TILINGS

Let us search for 3-similarity tilings $\mathcal{F} = \langle f_0, f_1, f_2 \rangle$. We do not care about where the centers of similarity transformations are when searching for 2-similarity tilings, but for 3-similarity tilings we need to take location of the centers into account. We assume that the centers of f_0 and f_1 are 0 and 1 respectively, and the similarity iterated function system $\mathcal{F}$ is determined by the complex ratios a_0, a_1, a_2 and the center $c = c_2$ of f_2. We denote the similarity iterated function system by $\mathcal{F}(a_0, a_1, a_2|_c)$. It is obvious that

$$\mathcal{F}(a_0, a_1, a_2|_c) \simeq \mathcal{F}(\bar{a}_0, \bar{a}_1, \bar{a}_2|_{\bar{c}_2}).$$

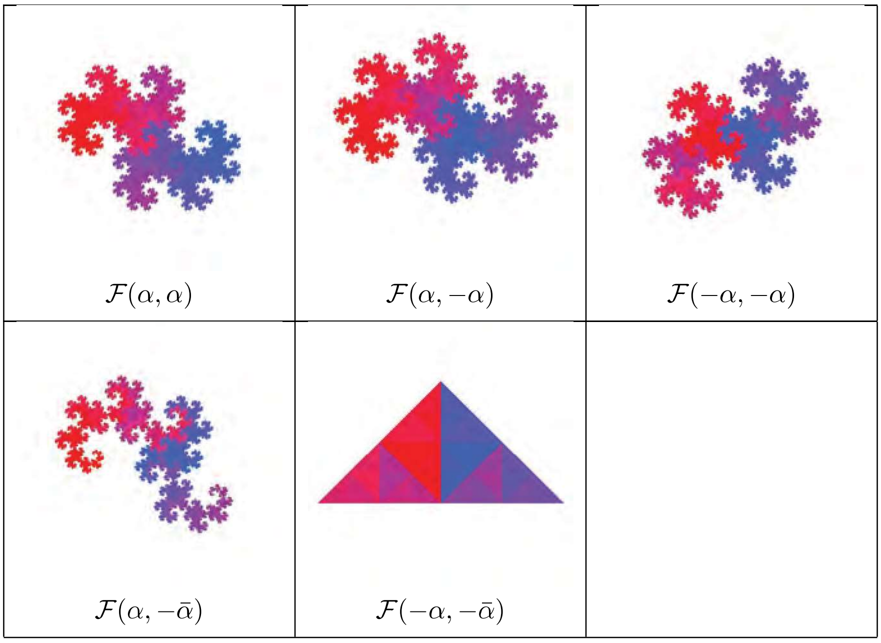

FIGURE 4. 2-similarity tilings associated with $1 - 2z + 2z^2$, $\alpha = \frac{1+i}{2}$.

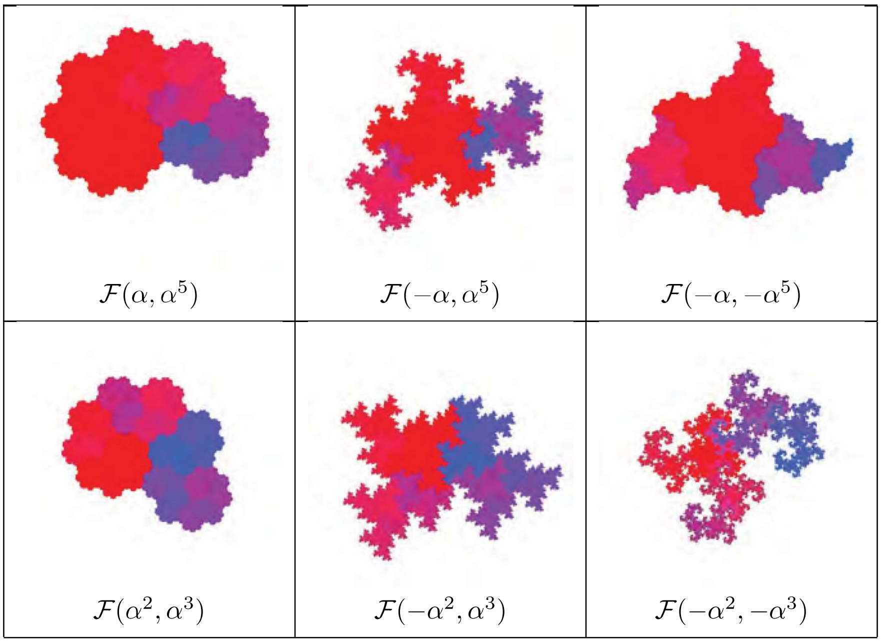

FIGURE 5. 2-similarity tilings associated with $1 - z + z^3$, $(\alpha, \alpha^5) \sim (0.662359 + 0.562280i, -0.460202 - 0.182582i)$, $(\alpha^2, \alpha^3) \sim (0.122561 + 0.744862i, -0.337641 + .562280i)$.

Under the permutations of the generators, we have

$$\begin{aligned}\mathcal{F}\big(a_0, a_1, a_2|_c\big) &\simeq \mathcal{F}\big(a_1, a_2, a_0|_{\frac{1}{1-c}}\big) \simeq \mathcal{F}\big(a_2, a_0, a_1|_{(1-\frac{1}{c})}\big) \\ &\simeq \mathcal{F}\big(a_0, a_2, a_1|_{\frac{1}{c}}\big) \simeq \mathcal{F}\big(a_1, a_0, a_2|_{(1-c)}\big) \simeq \mathcal{F}\big(a_2, a_1, a_0|_{\frac{-c}{1-c}}\big).\end{aligned}$$

In general, the 64 families of similarity iterated function systems $\mathcal{F}(\pm\tilde{u}, \pm\tilde{v}, \pm\tilde{w}|.)$ are grouped into 32 conjugate pairs $\mathcal{F}(\pm u, \pm\tilde{v}, \pm\tilde{w}|.)$. If two of u, v, w are equal up to sign and conjugation,

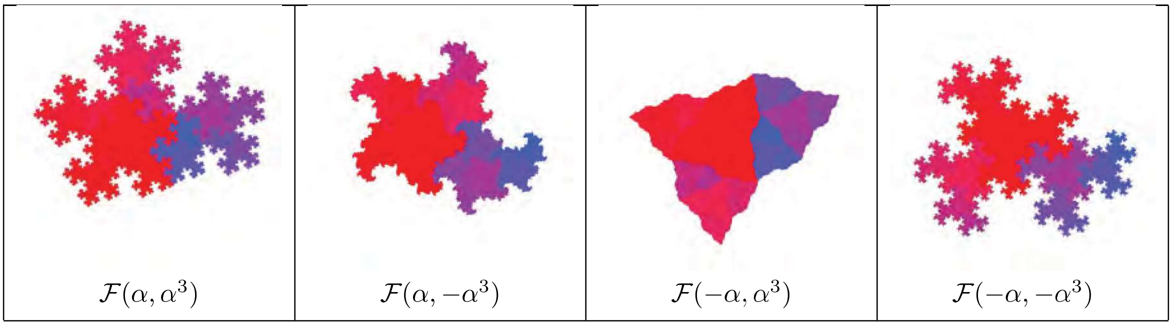

FIGURE 6. 2-similarity tilings associated with $1+z^2+z^3$, $(\alpha,\alpha^3) \sim (0.232786+0.792552i, -0.426050-0.368989i)$.

the number reduces to 24:

$$\mathcal{F}(u,\tilde{u},\pm\tilde{v}|.),\quad \mathcal{F}(u,-\tilde{u},\pm\tilde{v}|.),\quad \mathcal{F}(-u,-\tilde{u},\pm\tilde{v}|.).$$

If $u=v=w$ up to sign and conjugation, the number further reduces to ten:

$$\begin{gathered}\mathcal{F}(u,u,u|.),\quad \mathcal{F}(u,u,-u|.),\quad \mathcal{F}(u,-u,-u|.),\quad \mathcal{F}(-u,-u,-u|.),\\ \mathcal{F}(u,u,\bar{u}|.),\quad \mathcal{F}(u,-u,\bar{u}|.),\quad \mathcal{F}(-u,-u,\bar{u}|.),\\ \mathcal{F}(u,u,-\bar{u}|.),\quad \mathcal{F}(u,-u,-\bar{u}|.),\quad \mathcal{F}(-u,-u,-\bar{u}|.).\end{gathered}$$

4.1. **Derived 3-similarity tilings.** Suppose that we have two (same or different) 2-similarity tilings

$$\mathcal{F}=\langle f_0,f_1\rangle \qquad\text{and}\qquad \mathcal{F}'=\langle f_0',f_1'\rangle$$

having the same limit set so that

$$\Lambda = f_0(\Lambda)\cup f_1(\Lambda) = f_0'(\Lambda)\cup f_1'(\Lambda).$$

It is obvious from

$$\begin{aligned}\Lambda &= f_0(\Lambda)\cup f_1(f_0'(\Lambda)\cup f_1'(\Lambda))\\ &= f_0(\Lambda)\cup f_1f_0'(\Lambda)\cup f_1f_1'(\Lambda),\end{aligned}$$

that $\mathcal{G}_0=\langle f_0, f_1f_0', f_1f_1'\rangle$ is a 3-similarity tiling having the limit set Λ. By a similar reason, $\mathcal{G}_1=\langle f_0f_0', f_0f_1', f_1\rangle$ is also a 3-similarity tiling having the limit set Λ.

Definition 19. We say that 3-similarity tilings $\mathcal{G}_0$ and $\mathcal{G}_1$ are derived from 2-similarity tilings $\mathcal{F}$ and $\mathcal{F}'$.

If α is a complex root of a polynomial $p(z)$ with integer coefficients and the complex ratios of f_0, f_1, f_0', f_1' are expressed as

$$a_0=\pm\tilde{\alpha}^{e_0},\qquad a_1=\pm\tilde{\alpha}^{e_1},\qquad a_0'=\pm\tilde{\alpha}^{e_0'},\qquad a_1'=\pm\tilde{\alpha}^{e_1'},$$

$\mathcal{G}_0$ and $\mathcal{G}_1$ are equivalent to

$$\mathcal{F}(\pm\tilde{\alpha}^{e_0},\pm\tilde{\alpha}^{e_1}\tilde{\alpha}^{e_0'},\pm\tilde{\alpha}^{e_1}\tilde{\alpha}^{e_1'}|_c)\qquad\text{and}\qquad \mathcal{F}(\pm\tilde{\alpha}^{e_0}\tilde{\alpha}^{e_0'},\pm\tilde{\alpha}^{e_0}\tilde{\alpha}^{e_1'},\pm\tilde{\alpha}^{e_1}|_{c'})$$

for some c and c' respectively. Note that we have

$$|\alpha^{e_0}|^2+|\alpha^{e_1+e_0'}|^2+|\alpha^{e_1+e_1'}|^2 = |\alpha^{e_0+e_0'}|^2+|\alpha^{e_0+e_1'}|^2+|\alpha^{e_1}|^2=1.$$

Derived 3-similarity tilings $\mathcal{G}_0$ and $\mathcal{G}_1$ are nothing interesting for the limit sets are the same as that of the original. However this does not exclude possibility of our finding new 3-similarity tilings by varying the centers c and c'.

4.2. **3-similarity tilings associated with a polynomial.** More generally:

Definition 20. Let α be a root of a polynomial $p(z)$ with integer coefficients satisfying

$$|\alpha^{e_0}|^2 + |\alpha^{e_1}|^2 + |\alpha^{e_2}|^2 = 1$$

for some positive integers e_0, e_1, e_2. Then a 3-similarity tiling equivalent to one of the form

$$\mathcal{F}(\pm\alpha^{e_0-i}\bar{\alpha}^i, \pm\alpha^{e_1-j}\bar{\alpha}^j, \pm\alpha^{e_2-k}\bar{\alpha}^k|_c) \quad (0 \le i \le e_0,\ 0 \le j \le e_1,\ 0 \le k \le e_2,\ c \ne 0, 1)$$

is said to be associated with $p(z)$.

The method we use to find 3-similarity tilings is primitive. We input each candidate of the complex ratios (a_0, a_1, a_2) in Fractal Gazer, and vary the center c of the third similarity transformation by hand using the mouse and literally search for tilings by looking at the displayed limit set. While some of the 3-similarity tilings are easy to find, there are also 3-similarity tilings quite difficult to find. We have tried to be careful but it is quite possible that we have missed a few 3-similarity tilings. Anyhow, given complex ratios (a_0, a_1, a_2), the number of c yielding 3-similarity tilings seems to be finite.

5. 3-similarity tilings associated with polynomials of 2-similarity tilings

We first search for 3-similarity tilings associated with polynomials of 2-similarity tilings.

Fig. 7 shows 3-similarity tilings associated with $1+2z^2$. The first in each family is the derived 3-similarity tiling. The results show that we can vary the center of derived 3-similarity tilings $\mathcal{F}(a_0, a_1, a_2|_c)$ and find non-trivial 3-similarity tilings.

The 3-similarity tilings associated with $1 - z + 2z^2$ and $1 - 2z + 2z^2$ that we have found are listed in Figs. 8, 9 and Figs. 10, 11, respectively. We have not found any tiling of types other than the listed.

The complex root α of $1 - z + z^3$ satisfies

$$|\alpha|^2 + |\alpha^5|^2 = |\alpha^2|^2 + |\alpha^3|^2 = 1.$$

We can combine them in various ways to obtain six triples $(e_0, e_1, e_2) = (1, 6, 10)$, $(1, 7, 8)$, $(2, 4, 8)$, $(2, 5, 6)$, $(3, 3, 7)$, $(3, 4, 5)$ all satisfying

$$|\alpha^{e_0}|^2 + |\alpha^{e_1}|^2 + |\alpha^{e_2}|^2 = 1.$$

Figs. 12–17 show 3-similarity tilings of respective types that we have found.

The complex root α of $1 + z^2 + z^3$ satisfies $|\alpha|^2 + |\alpha^3|^2 = 1$, from which we obtain triples $(e_0, e_1, e_2) = (1, 4, 6)$, $(2, 2, 7)$, $(2, 3, 4)$ satisfying $|\alpha^{e_0}|^2 + |\alpha^{e_1}|^2 + |\alpha^{e_2}|^2 = 1$. Figs. 18–20 show all the 3-similarity tilings of respective types that we have found.

6. 3-similarity tilings not related to 2-similarity tilings

Table 6 shows all polynomials $p(z)$ of degree $d \le 4$ with integer coefficients $|p_k| \le 6$, having a root α ($\Re\alpha \ge 0$, $\Im\alpha > 0$) satisfying

$$|\alpha^{e_0}|^2 + |\alpha^{e_1}|^2 + |\alpha^{e_2}|^2 = 1$$

for some positive integers e_0, e_1, e_2. The mark 2 in the first column indicates that the parameters are related to 2-similarity tilings. The mark 3 indicates that the parameters give 3-similarity tilings but not related to 2-similarity tilings. Parentheses indicate that the given parameters reduce to a simpler case, like $(\alpha^2, \alpha^4, \alpha^4)$ for a root of $1 + 2z^4$ is actually $(\alpha, \alpha^2, \alpha^2)$ for a root of $1 + 2z^2$. For the parameters without mark, we have not found any 3-similarity tiling, though we have tried.

Figs. 21–31 show all our findings on 3-similarity tilings associated with the polynomials marked 3.

The limit set Λ of the similarity tiling $\mathcal{F}(\alpha, \alpha, \alpha|_c) = \langle f_0, f_1, f_2 \rangle$ ($\alpha = \frac{i}{\sqrt{3}}$, $c = \frac{1+\sqrt{3}i}{2}$) in Fig. 21 is unique in that it has 3-fold symmetry. Let $g : \mathbf{C} \to \mathbf{C}$ be the $2\pi/3$-rotation around the point $\frac{3+\sqrt{3}i}{6}$, and we have $g(\Lambda) = \Lambda$. It follows that Λ is the limit set of any of the

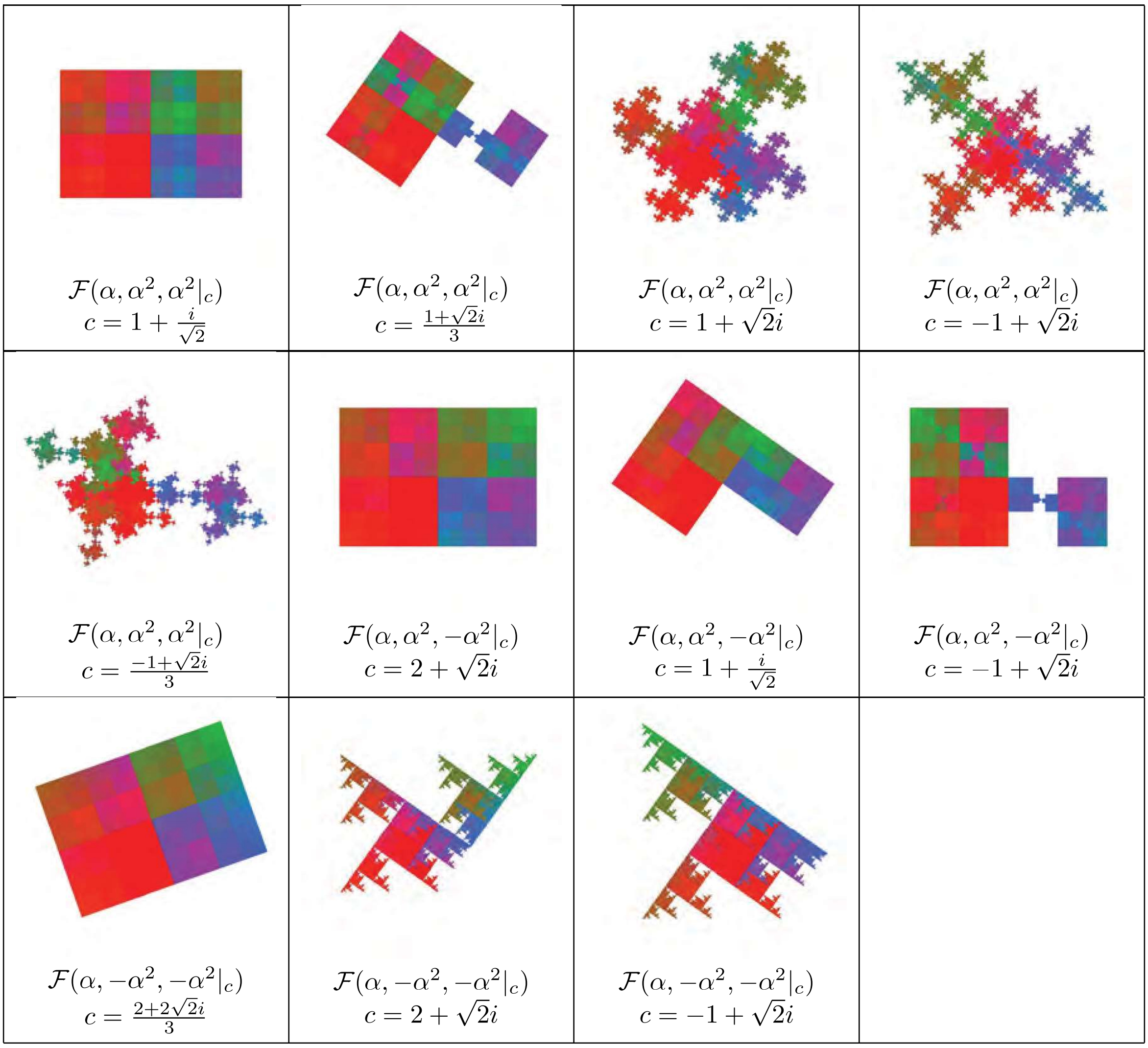

FIGURE 7. 3-similarity tilings associated with $1 + 2z^2$, $\alpha = \frac{i}{\sqrt{2}}$.

similarity iterated function systems $\langle f_0 g^i, f_1 g^j, f_2 g^k \rangle \quad (i, j, k \in \{0, \pm 1\})$. Therefore, each family of similarity iterated function systems

$$\mathcal{F}(\alpha\omega^i, \alpha\omega^j, \alpha\omega^k|_c) \quad (\omega = \tfrac{1+\sqrt{2}i}{2})$$

contains at least one 3-similarity tiling whose limit set is Λ, but perhaps more. We have loosened the condition slightly and searched for 3-similarity tiling of the form

$$\mathcal{F}(\pm\alpha\omega^i, \pm\alpha\omega^j, \pm\alpha\omega^k|_c).$$

Note that these include all the 3-similarity tilings associated with $1 + 3z^2$ as well as those associated with $1 - 3z + 3z^2$. Indeed $\pm\alpha\omega^i$ $(i \in \{0, \pm 1\})$ are the roots of

$$1 + 27z^6 = (1 + 3z^2)(1 - 3z + 3z^2)(1 + 3z + 3z^2).$$

Our choice of the complex root of $1 - 3z + 3z^2$ is $\frac{3+\sqrt{3}i}{6} = -\alpha\omega$ in terms of the root α of $1 + 3z^2$. The 3-similarity tilings associated with $1 + 3z^2$ and those associated with $1 - 3z + 3z^2$ are shown in Fig. 21 and Fig. 22 respectively, while the 3-similarity tilings related to both $1 + 3z^2$ and $1 - 3z + 3z^2$ are shown in Figs. 23 and 24.

ACKNOWLEDGMENT

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	$p(z)$	e_0	e_1	e_2	α^{e_0}	α^{e_1}	α^{e_2}
2	$1+2z^2$	1	2	2	$0.707107i$	-0.5	-0.5
2	$1-z+2z^2$	1	2	2	$0.25+0.661438i$	$-0.375+0.330719i$	$-0.375+0.330719i$
2	$1-2z+2z^2$	1	2	2	$0.5+0.5i$	$0.5i$	$0.5i$
2	$1-z+z^3$	1	6	10	$0.662359+0.562280i$	$-0.202157-0.379697i$	$0.178450+0.168050i$
2	$1-z+z^3$	1	7	8	$0.662359+0.562280i$	$0.079596-0.365165i$	$0.258045-0.197115i$
2	$1-z+z^3$	2	4	8	$0.122561+0.744862i$	$-0.539798+0.182582i$	$0.258045-0.197115i$
2	$1-z+z^3$	2	5	6	$0.122561+0.744862i$	$-0.460202-0.182582i$	$-0.202157-0.379697i$
2	$1-z+z^3$	3	3	7	$-0.337641+0.562280i$	$-0.337641+0.562280i$	$0.079596-0.365165i$
2	$1-z+z^3$	3	4	5	$-0.337641+0.562280i$	$-0.539798+0.182582i$	$-0.460202-0.182582i$
2	$1+z^2+z^3$	1	4	6	$0.232786+0.792552i$	$0.193265-0.423563i$	$0.045366+0.314416i$
2	$1+z^2+z^3$	2	2	7	$-0.573950+0.368989i$	$-0.573950+0.368989i$	$-0.238631+0.109146i$
2	$1+z^2+z^3$	2	3	4	$-0.573950+0.368989i$	$-0.426050-0.368989i$	$0.193265-0.423563i$
3	$1+3z^2$	1	1	1	$0.577350i$	$0.577350i$	$0.577350i$
3	$1-3z+3z^2$	1	1	1	$0.5+0.288675i$	$0.5+0.288675i$	$0.5+0.288675i$
3	$1-z+3z^2$	1	1	1	$0.166667+0.552771i$	$0.166667+0.552771i$	$0.166667+0.552771i$
3	$1-2z+3z^2$	1	1	1	$0.333333+0.471405i$	$0.333333+0.471405i$	$0.333333+0.471405i$
3	$1+2z^2+z^3$	1	1	3	$0.102785+0.665457i$	$0.102785+0.665457i$	$-0.135463-0.273595i$
3	$1-z+z^2+z^3$	1	2	3	$0.419643+0.606291i$	$-0.191488+0.508852i$	$-0.388869+0.097439i$
3	$1-z+2z^2-z^3$	1	2	4	$0.122561+0.744862i$	$-0.539798+0.182582i$	$0.258045-0.197115i$
(2)	$1+2z^4$	2	4	4	$0.707107i$	-0.5	-0.5
(2)	$1-2z^4$	2	4	4	-0.707107	0.5	0.5
(3)	$1+3z^4$	2	2	2	$0.577350i$	$0.577350i$	$0.577350i$
3	$1+z^2-z^4$	1	3	4	$0.786151i$	$-0.485868i$	0.381966
	$1+z^2-z^4$	2	2	3	-0.618034	-0.618034	$-0.485868i$
(2)	$1+z^2+2z^4$	2	4	4	$-0.25+0.661438i$	$-0.375-0.330719i$	$-0.375-0.330719i$
(3)	$1+z^2+3z^4$	2	2	2	$-0.166667+0.552771i$	$-0.166667+0.552771i$	$-0.166667+0.552771i$
(2)	$1-z^2+2z^4$	2	4	4	$0.25+0.661438i$	$-0.375+0.330719i$	$-0.375+0.330719i$
	$1+2z^2-z^4$	1	1	2	$0.643594i$	$0.643594i$	-0.414214
(2)	$1+2z^2+2z^4$	2	4	4	$-0.5+0.5i$	$-0.5i$	$-0.5i$
(3)	$1+2z^2+3z^4$	2	2	2	$-0.333333+0.471405i$	$-0.333333+0.471405i$	$-0.333333+0.471405i$
(2)	$1-2z^2+2z^4$	2	4	4	$0.5+0.5i$	$0.5i$	$0.5i$
	$1-2z+z^2-2z^4$	2	4	4	$-0.457107+0.539494i$	$-0.082107-0.493212i$	$-0.082107-0.493212i$
	$1+z+3z^3-z^4$	1	3	4	$0.190983+0.762600i$	$-0.326238-0.360051i$	$0.212269-0.317553i$
	$1+z+3z^3-z^4$	2	2	3	$-0.545085+0.291287i$	$-0.545085+0.291287i$	$-0.326238-0.360051i$
	$1-z+2z^3-z^4$	1	3	4	$0.5+0.606658i$	$-0.427051+0.231723i$	$-0.354102-0.143213i$
	$1-z+2z^3-z^4$	2	2	3	$-0.118034+0.606658i$	$-0.118034+0.606658i$	$-0.427051+0.231723i$
	$1-2z+z^2+2z^3-z^4$	1	1	2	$0.5+0.405233i$	$0.5+0.405233i$	$0.085786+0.405233i$
	$1-2z+2z^2-z^3-z^4$	1	3	4	$0.309017+0.722871i$	$-0.454915-0.170647i$	$-0.017221-0.381578i$
	$1-2z+2z^2-z^3-z^4$	2	2	3	$-0.427051+0.446759i$	$-0.427051+0.446759i$	$-0.454915-0.170647i$
	$1-2z+3z^2-2z^3-z^4$	1	1	2	$0.207107+0.609361i$	$0.207107+0.609361i$	$-0.328427+0.252405i$

TABLE 1. Parameters of 3-similarity tilings.

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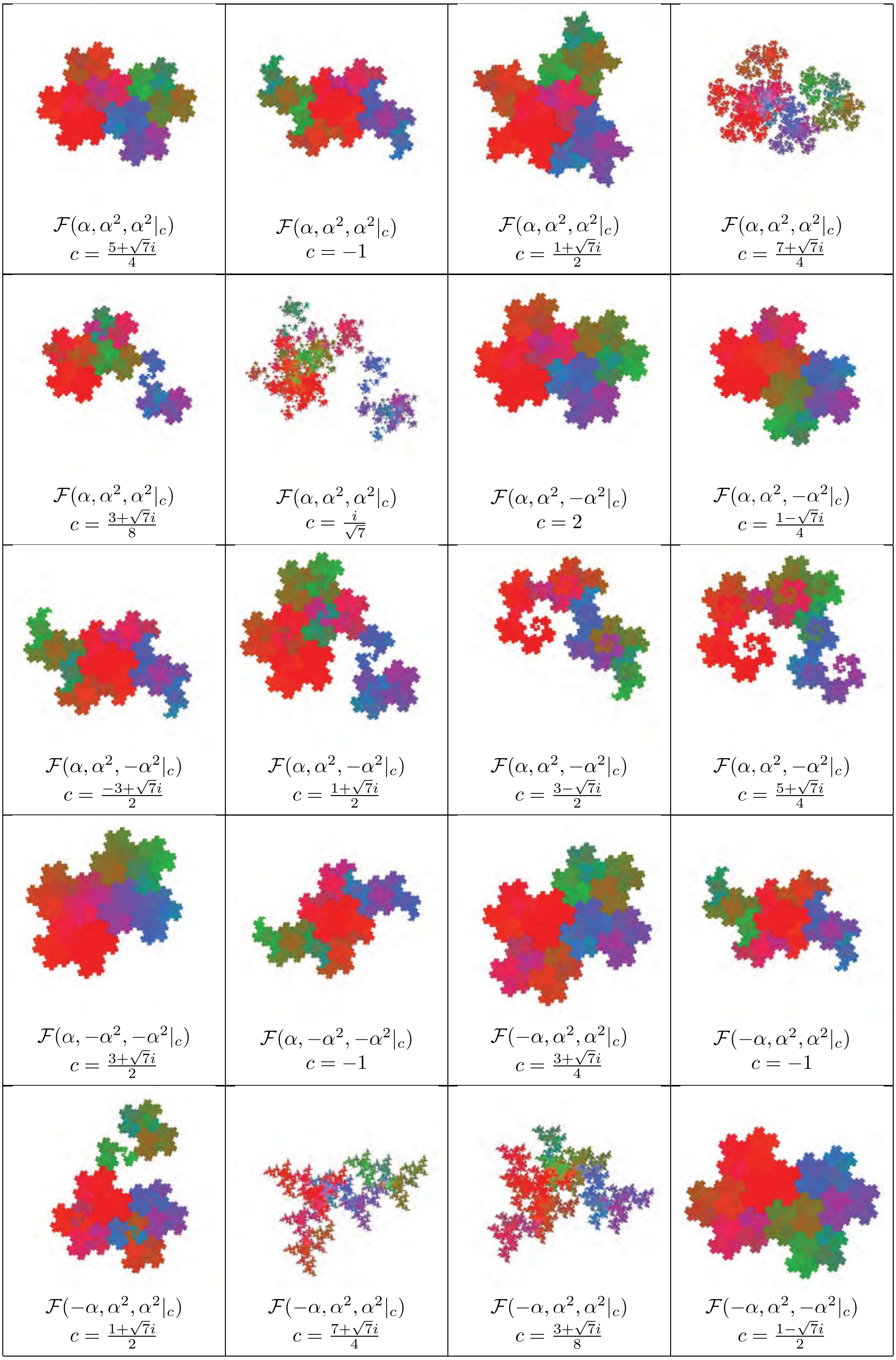


FIGURE 8. 3-similarity tilings associated with $1 - z + 2z^2$, $\alpha = \frac{1+\sqrt{7}i}{4}$. (1)

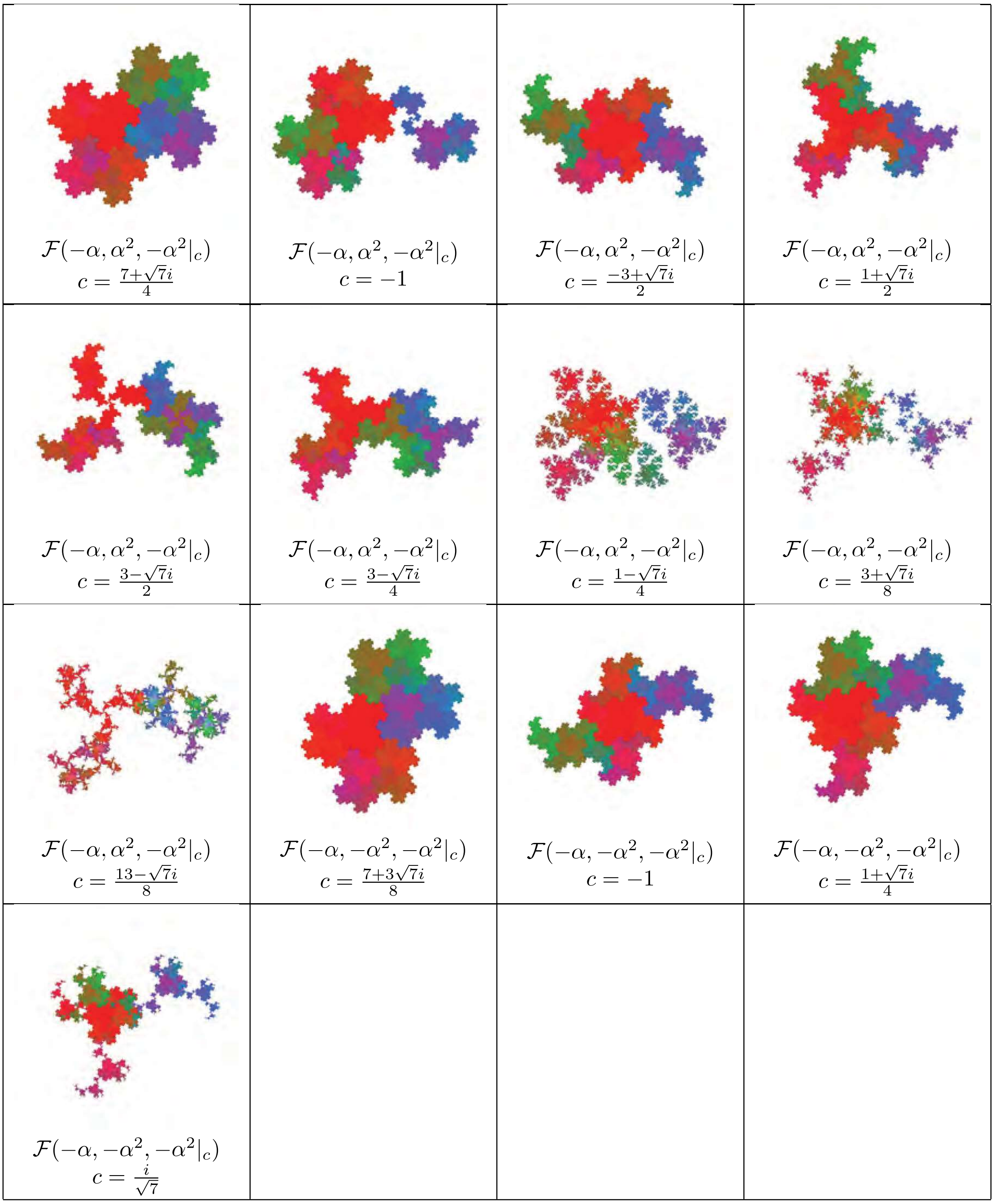


FIGURE 9. 3-similarity tilings associated with $1 - z + 2z^2$, $\alpha = \frac{1+\sqrt{7}i}{4}$. (2)

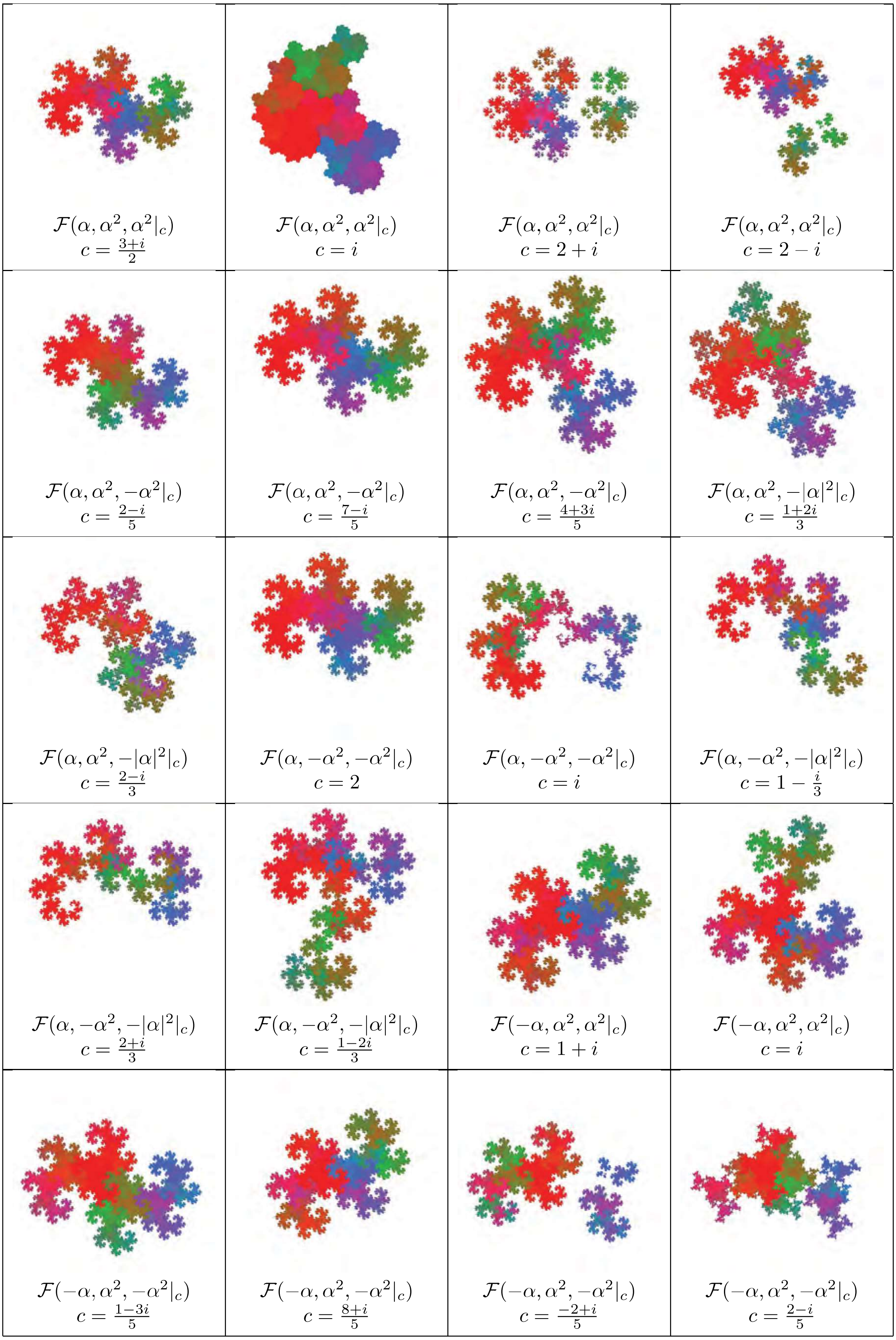


Figure 10. 3-similarity tilings associated with $1-2z+2z^2$, $\alpha=\frac{1+i}{2}$. (1)

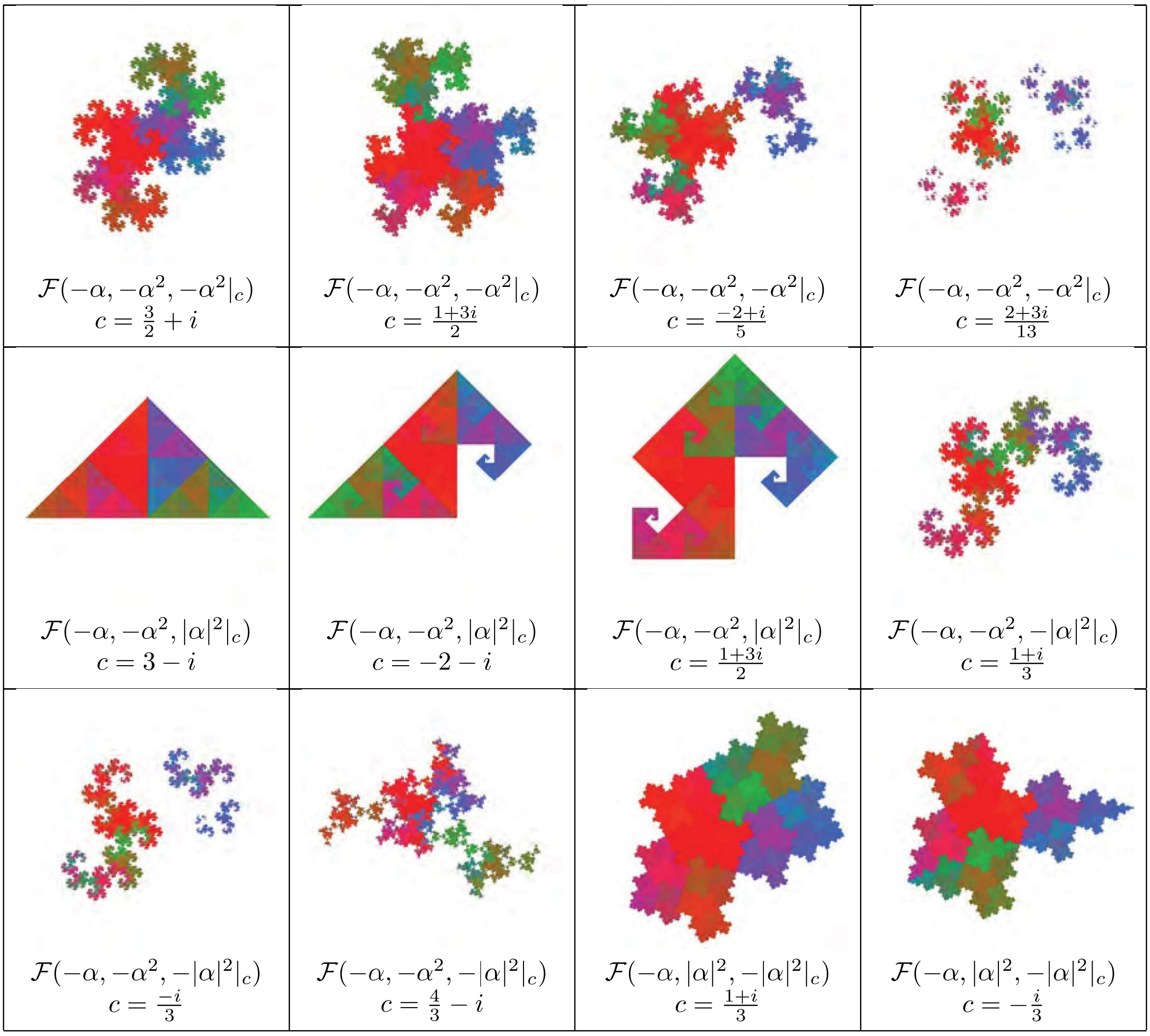


FIGURE 11. 3-similarity tilings associated with $1 - 2z + 2z^2$, $\alpha = \frac{1+i}{2}$. (2)

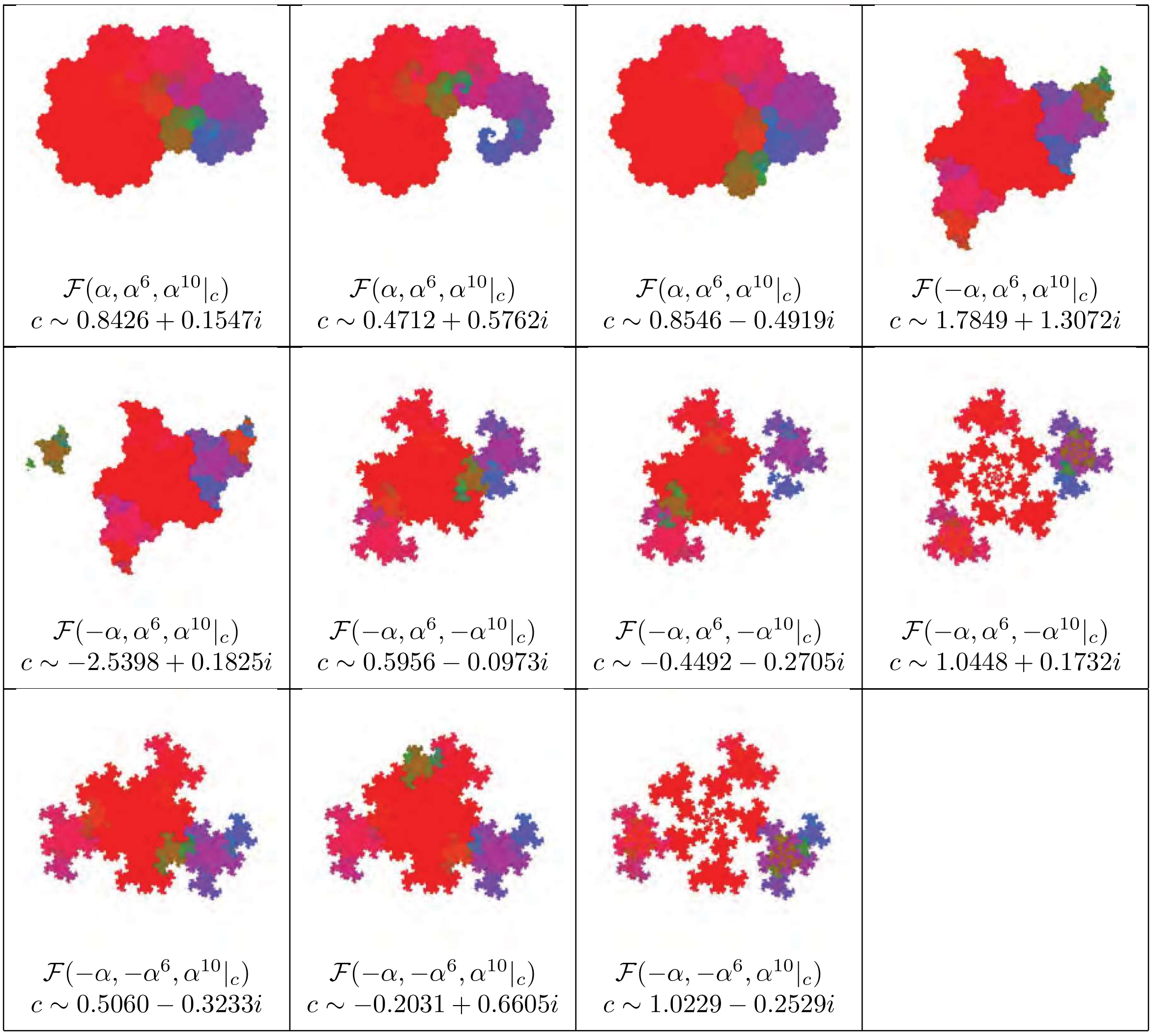


FIGURE 12. 3-similarity tilings of type $(1, 6, 10)$ associated with $1 - z + z^3$, $(\alpha, \alpha^6, \alpha^{10}) \sim (0.662359{+}0.562280i, -0.202157{-}0.379697i, 0.178450{+}0.168050i)$.

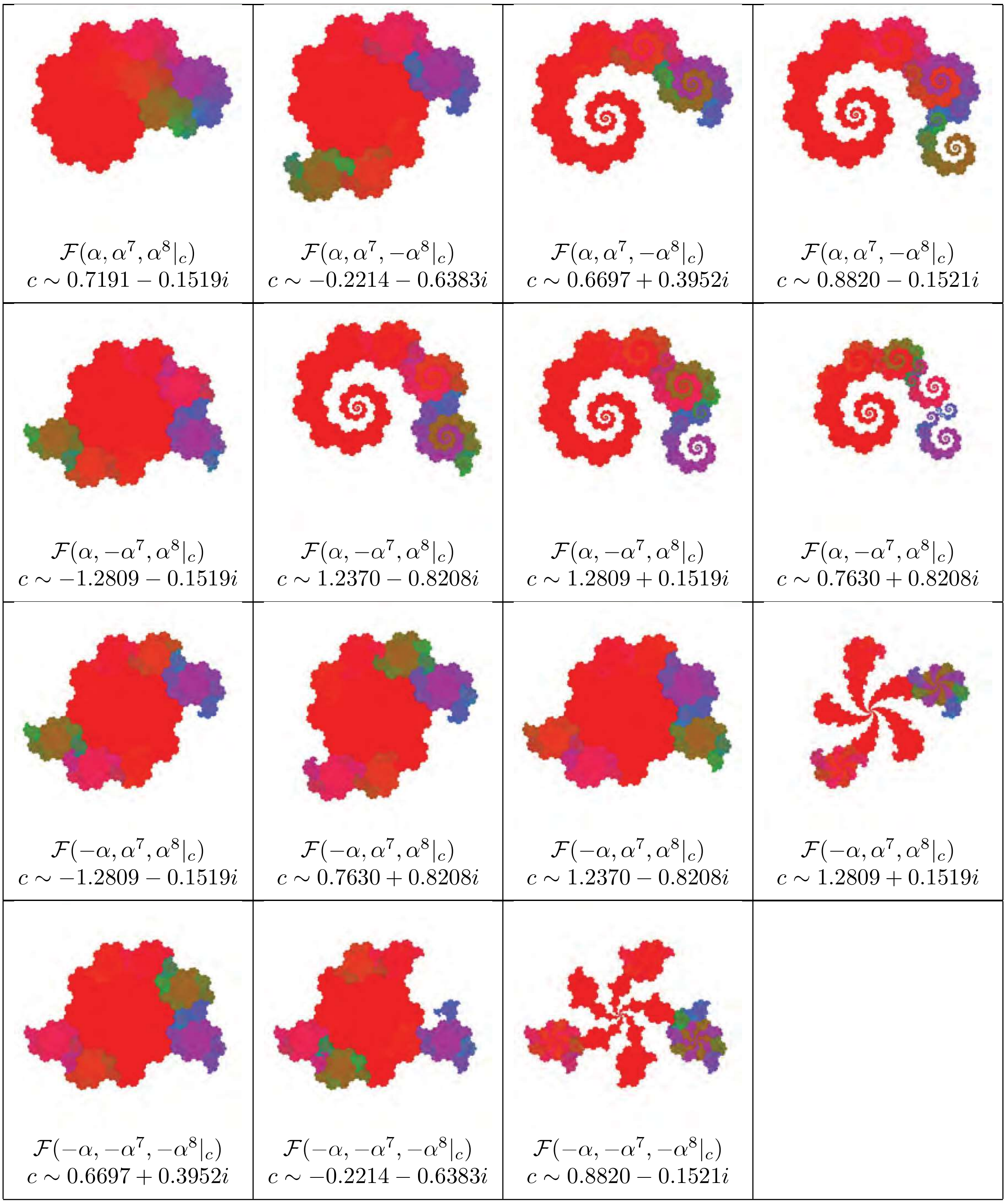


FIGURE 13. 3-similarity tilings of type $(1,7,8)$ associated with $1-z+z^3$, $(\alpha,\alpha^7,\alpha^8)\sim(0.662359+0.562280i, 0.079596-0.365165i, 0.258045-0.197115i)$.

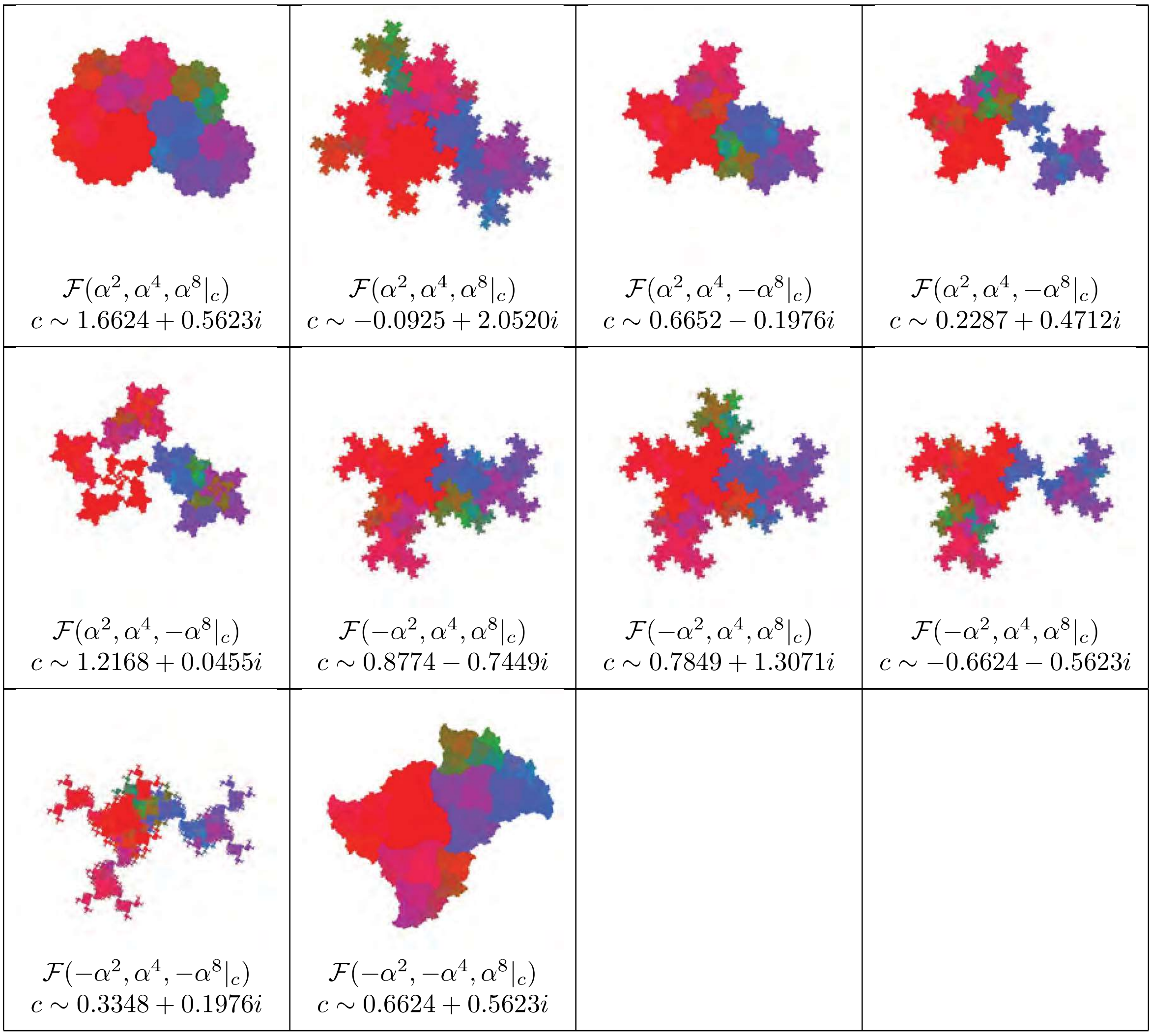


FIGURE 14. 3-similarity tilings of type $(2, 4, 8)$ associated with $1 - z + z^3$, $(\alpha^2, \alpha^4, \alpha^8) \sim (0.122561+0.744862i, -0.539798+0.182582i, 0.258045-0.197115i)$.

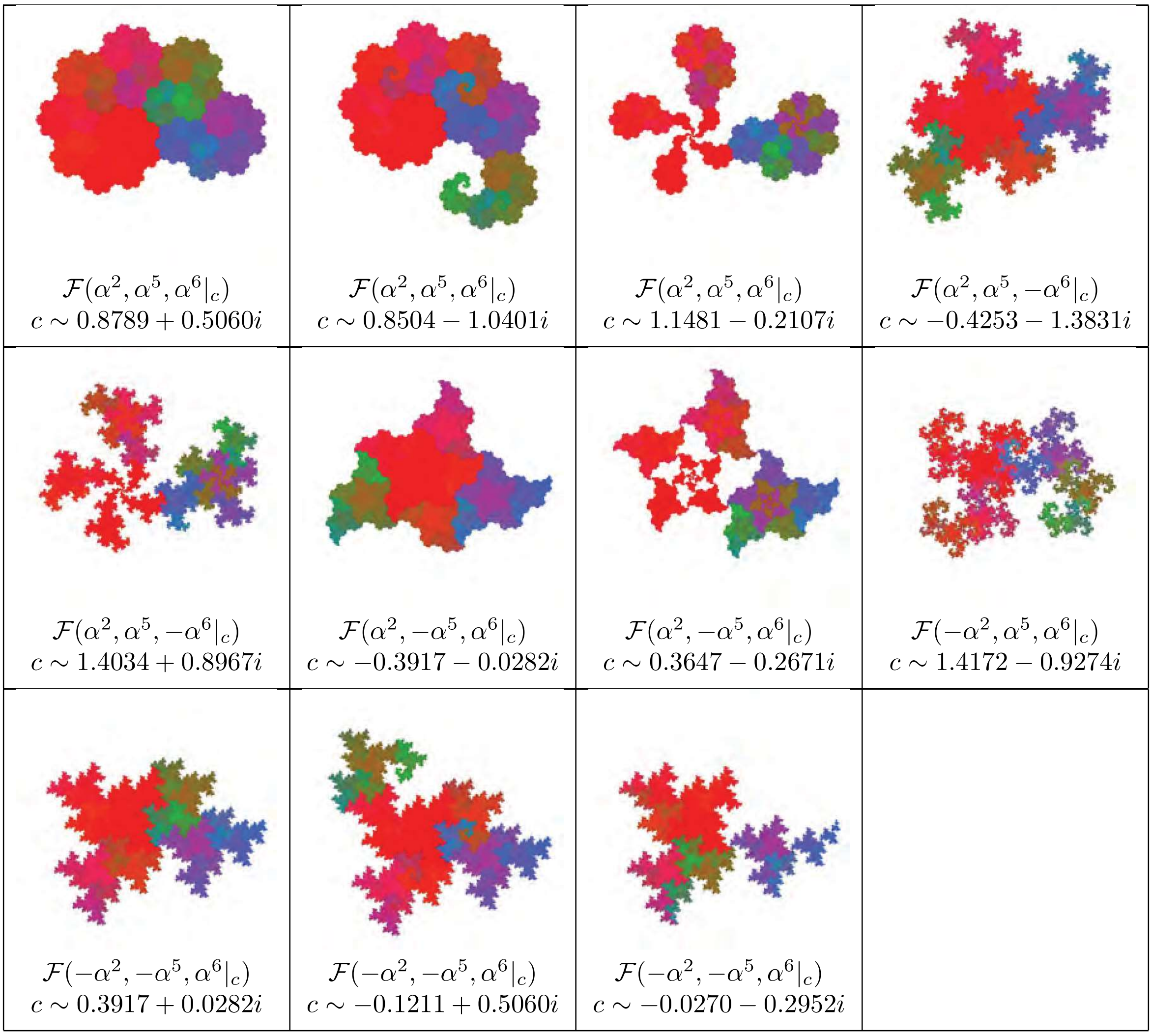


FIGURE 15. 3-similarity tilings of type $(2,5,6)$ associated with $1 - z + z^3$, $(\alpha^2, \alpha^5, \alpha^6) \sim (0.122561 + 0.744862i, -0.460202 - 0.182582i, -0.202157 - 0.379697i)$.

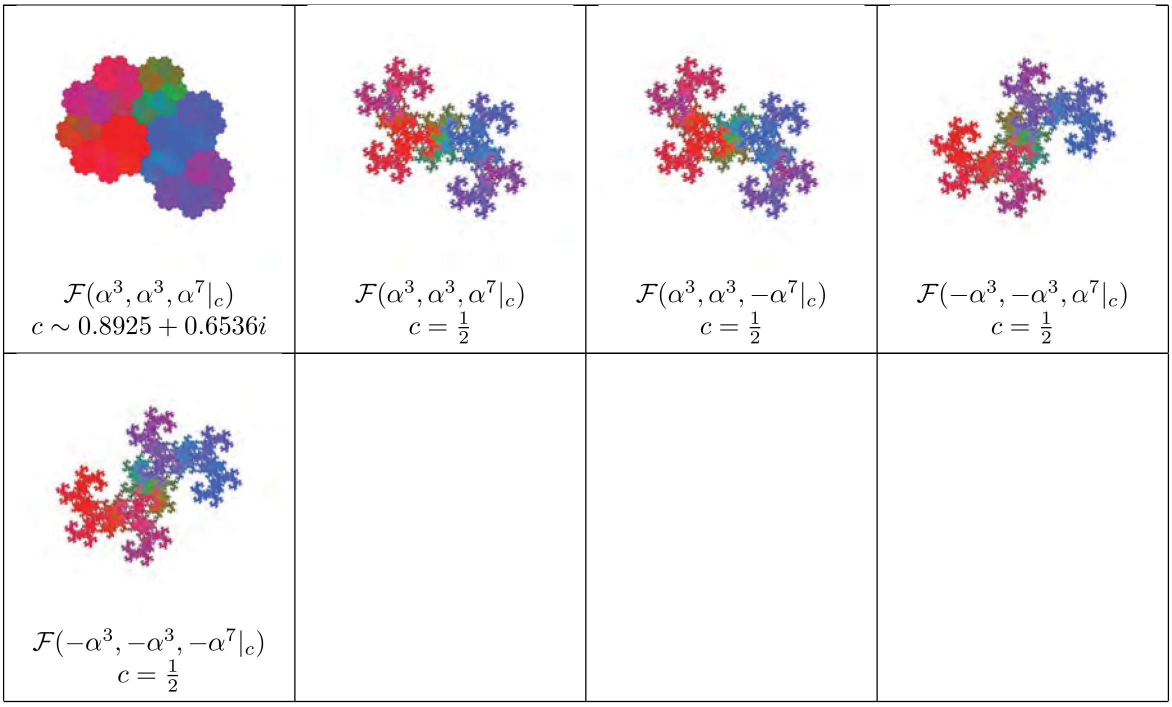


FIGURE 16. 3-similarity tilings of type $(3, 3, 7)$ associated with $1 - z + z^3$, $(\alpha^3, \alpha^3, \alpha^7) \sim (-0.337641 + 0.562280i, -0.337641 + 0.562280i, 0.079596 - 0.365165i)$.

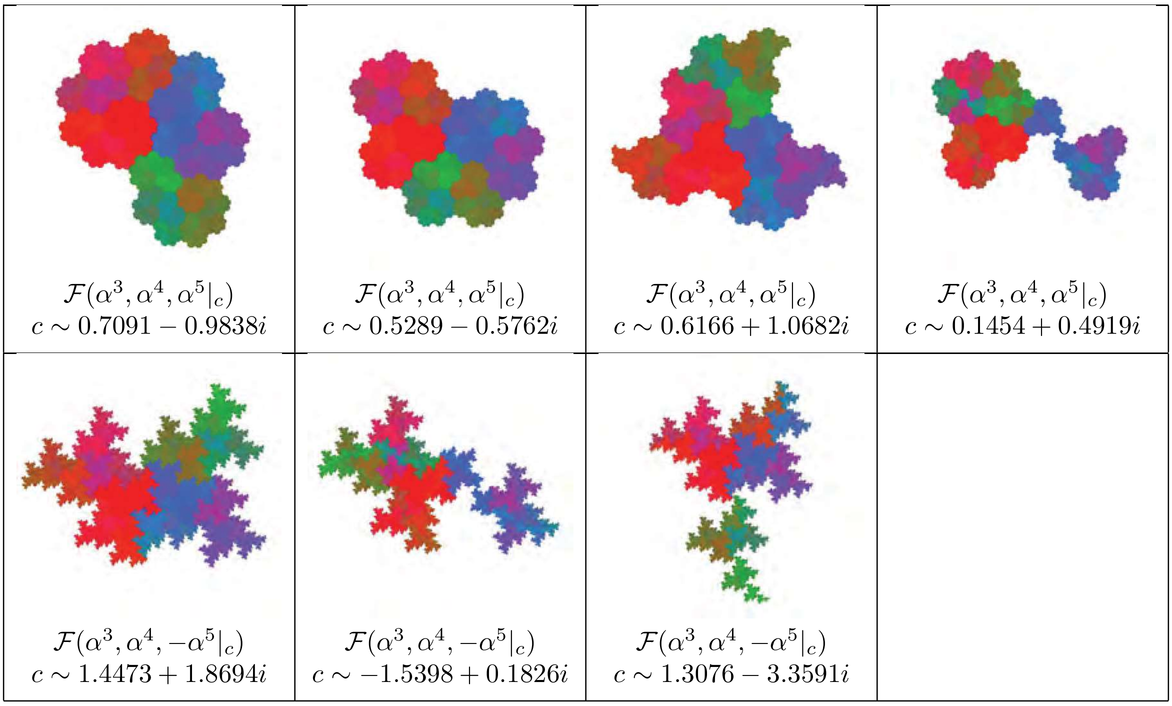
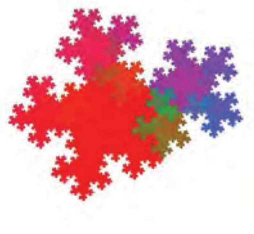
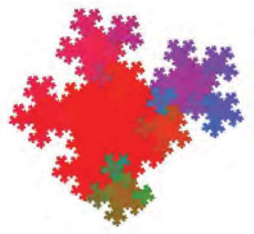


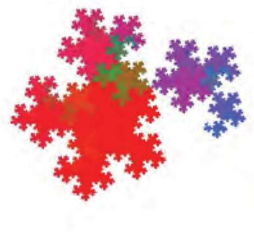
FIGURE 17. 3-similarity tilings of type $(3, 4, 5)$ associated with $1 - z + z^3$, $(\alpha^3, \alpha^4, \alpha^5) \sim (-0.337641 + 0.562280i, -0.539798 + 0.182582i, -0.460202 - 0.182582i)$.



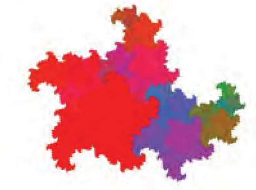
$\mathcal{F}(\alpha, \alpha^4, \alpha^6|_c)$
$c \sim 0.6022 + 0.1412i$



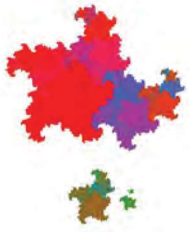
$\mathcal{F}(\alpha, \alpha^4, \alpha^6|_c)$
$c \sim 0.3695 - 0.6514i$



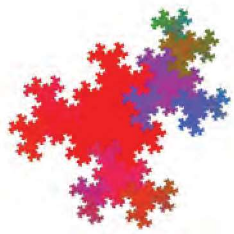
$\mathcal{F}(\alpha, \alpha^4, \alpha^6|_c)$
$c \sim 0.0283 + 0.5102i$



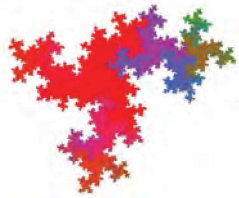
$\mathcal{F}(\alpha, -\alpha^4, \alpha^6|_c)$
$c \sim 1.8067 + 0.4236i$



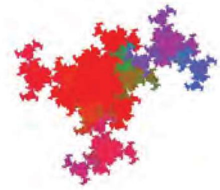
$\mathcal{F}(\alpha, -\alpha^4, \alpha^6|_c)$
$c \sim 1.1084 - 1.9541i$



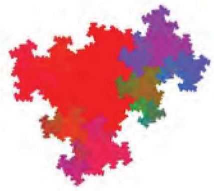
$\mathcal{F}(-\alpha, \alpha^4, \alpha^6|_c)$
$c \sim 0.6588 + 1.1615i$



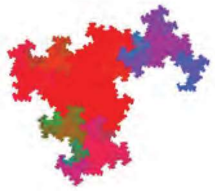
$\mathcal{F}(-\alpha, \alpha^4, \alpha^6|_c)$
$c \sim 1.2328 + 0.7926i$



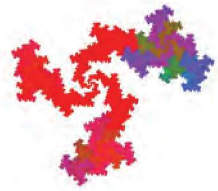
$\mathcal{F}(-\alpha, \alpha^4, \alpha^6|_c)$
$c \sim 0.3428 + 0.1701i$



$\mathcal{F}(-\alpha, \alpha^4, -\alpha^6|_c)$
$c \sim 0.5289 - 0.2113i$



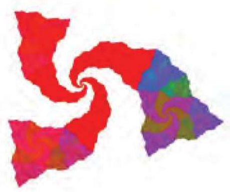
$\mathcal{F}(-\alpha, \alpha^4, -\alpha^6|_c)$
$c \sim -0.2906 - 0.3700i$



$\mathcal{F}(-\alpha, \alpha^4, -\alpha^6|_c)$
$c \sim 0.8195 + 0.1587i$

$\mathcal{F}(-\alpha, -\alpha^4, \alpha^6|_c)$
$c \sim 0.7123 - 0.4816i$

$\mathcal{F}(-\alpha, -\alpha^4, \alpha^6|_c)$
$c \sim 0.3161 + 0.9915i$



$\mathcal{F}(-\alpha, -\alpha^4, \alpha^6|_c)$
$c \sim 1.2595 - 0.0289i$

FIGURE 18. 3-similarity tilings of type $(1, 4, 6)$ associated with $1 + z^2 + z^3$. $(\alpha, \alpha^4, \alpha^6) \sim (0.232786 + 0.792552i, 0.193265 - 0.423563i, 0.045366 + 0.314416i)$.

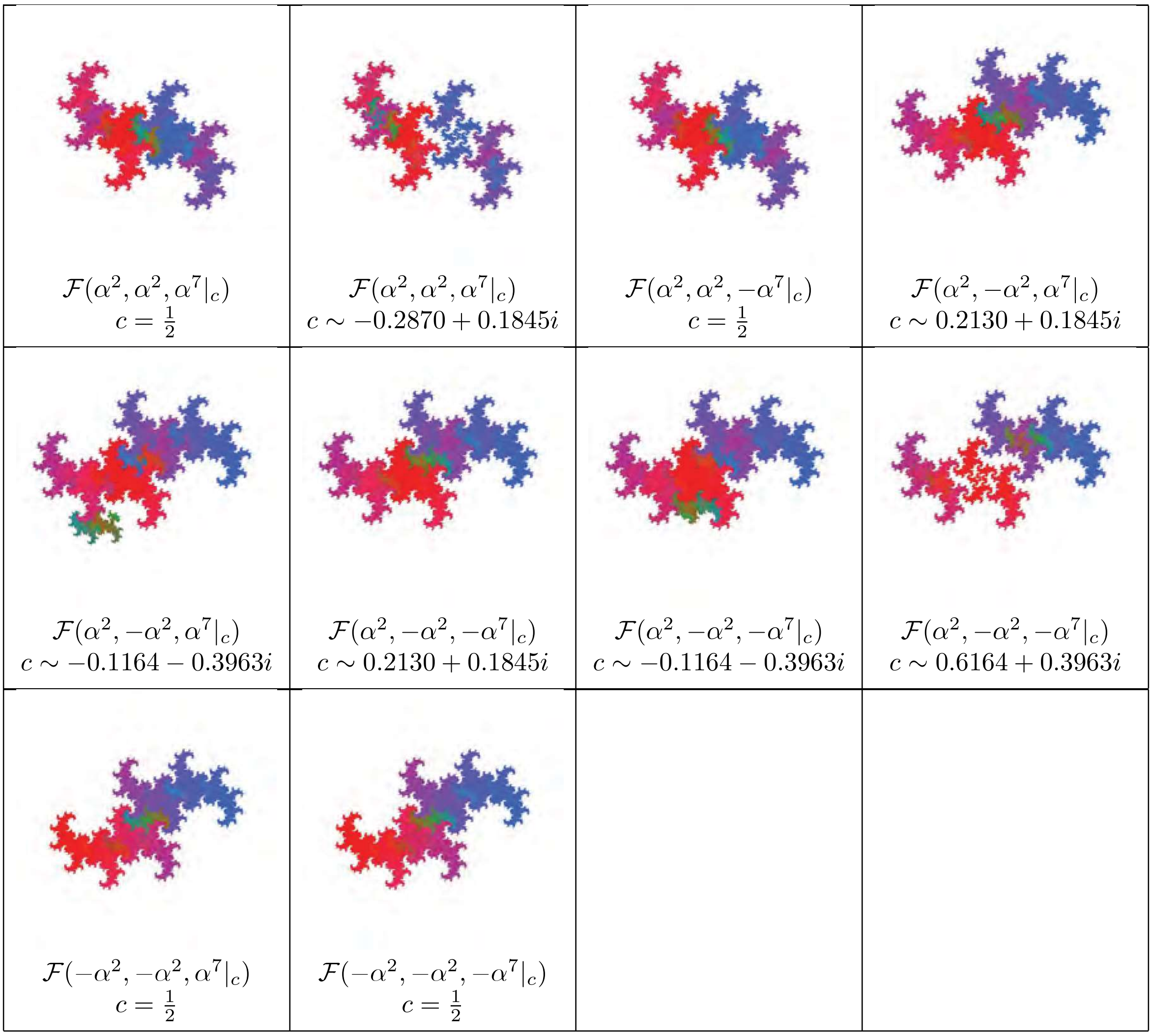


FIGURE 19. 3-similarity tilings of type $(2, 2, 7)$ associated with $1 + z^2 + z^3$. $(\alpha^2, \alpha^2, \alpha^7) \sim (-0.573950 + 0.368989i, -0.573950 + 0.368989i, -0.238631 + 0.109146i)$.

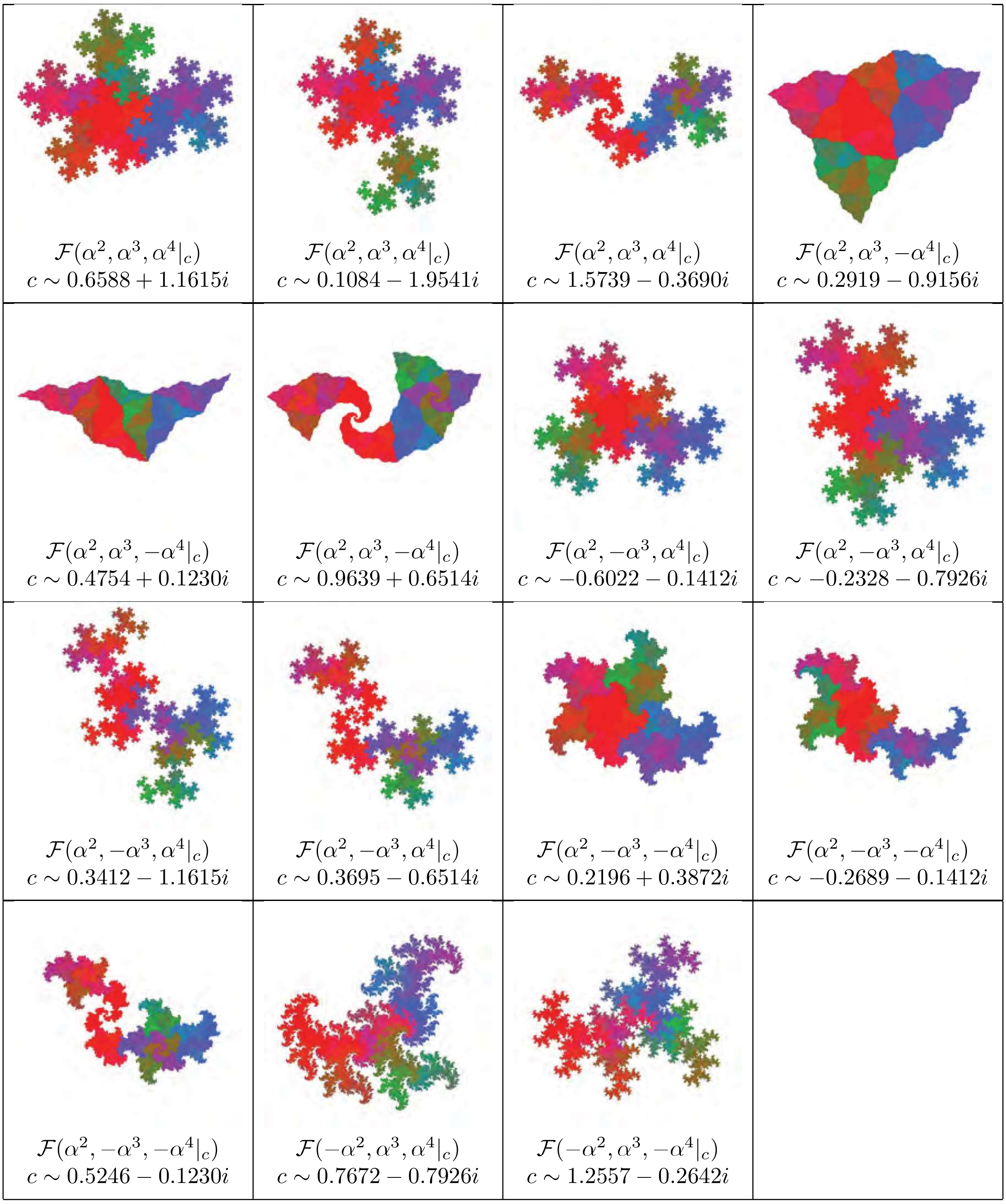


FIGURE 20. 3-similarity tilings of type $(2, 3, 4)$ associated with $1 + z^2 + z^3$. $(\alpha^2, \alpha^3, \alpha^4) \sim (-0.573950 + 0.368989i, -0.426050 - 0.368989i, 0.193265 - 0.423563i)$.

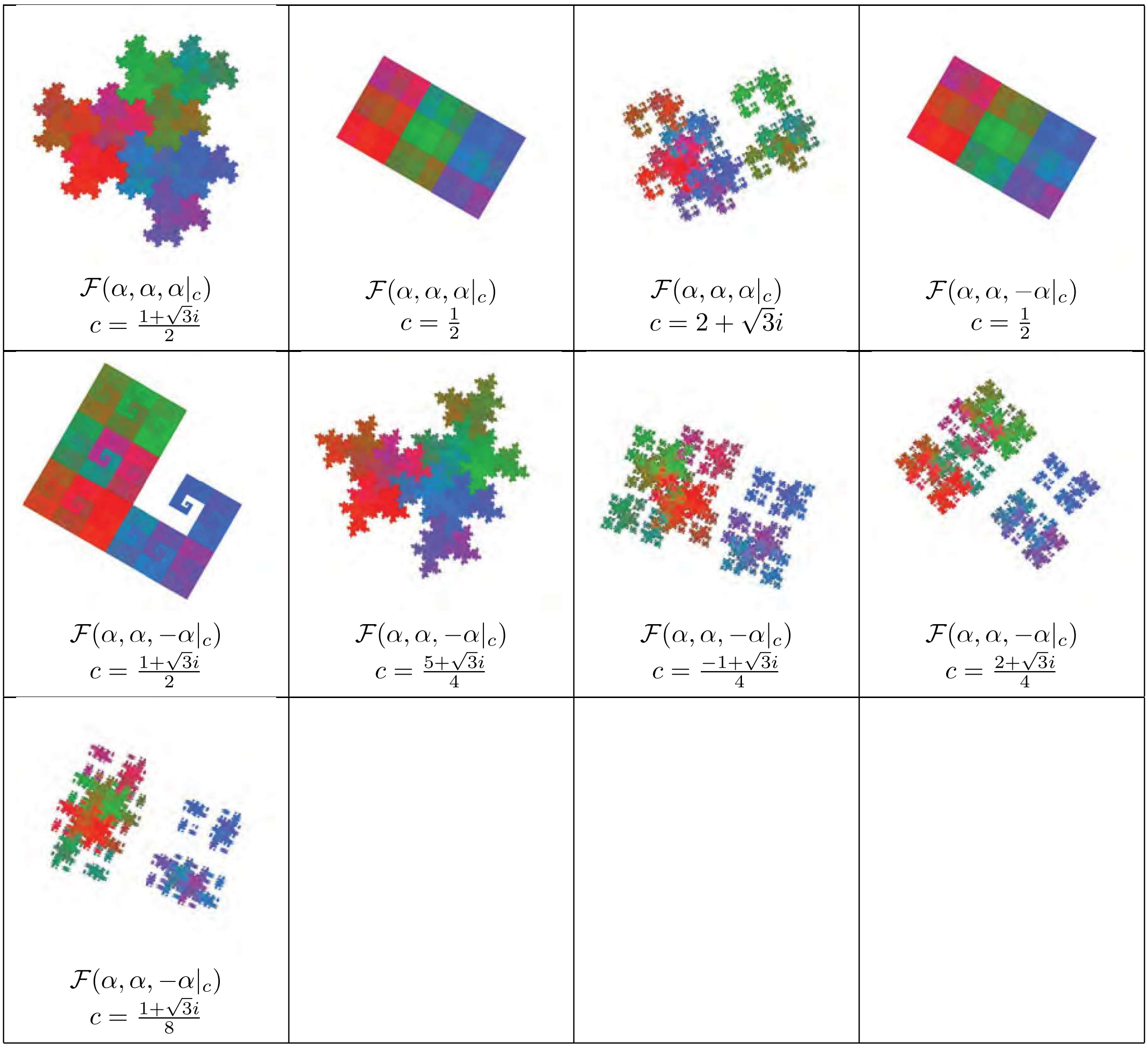


FIGURE 21. 3-similarity tilings associated with $1 + 3z^2$, $\alpha = \frac{i}{\sqrt{3}}$.

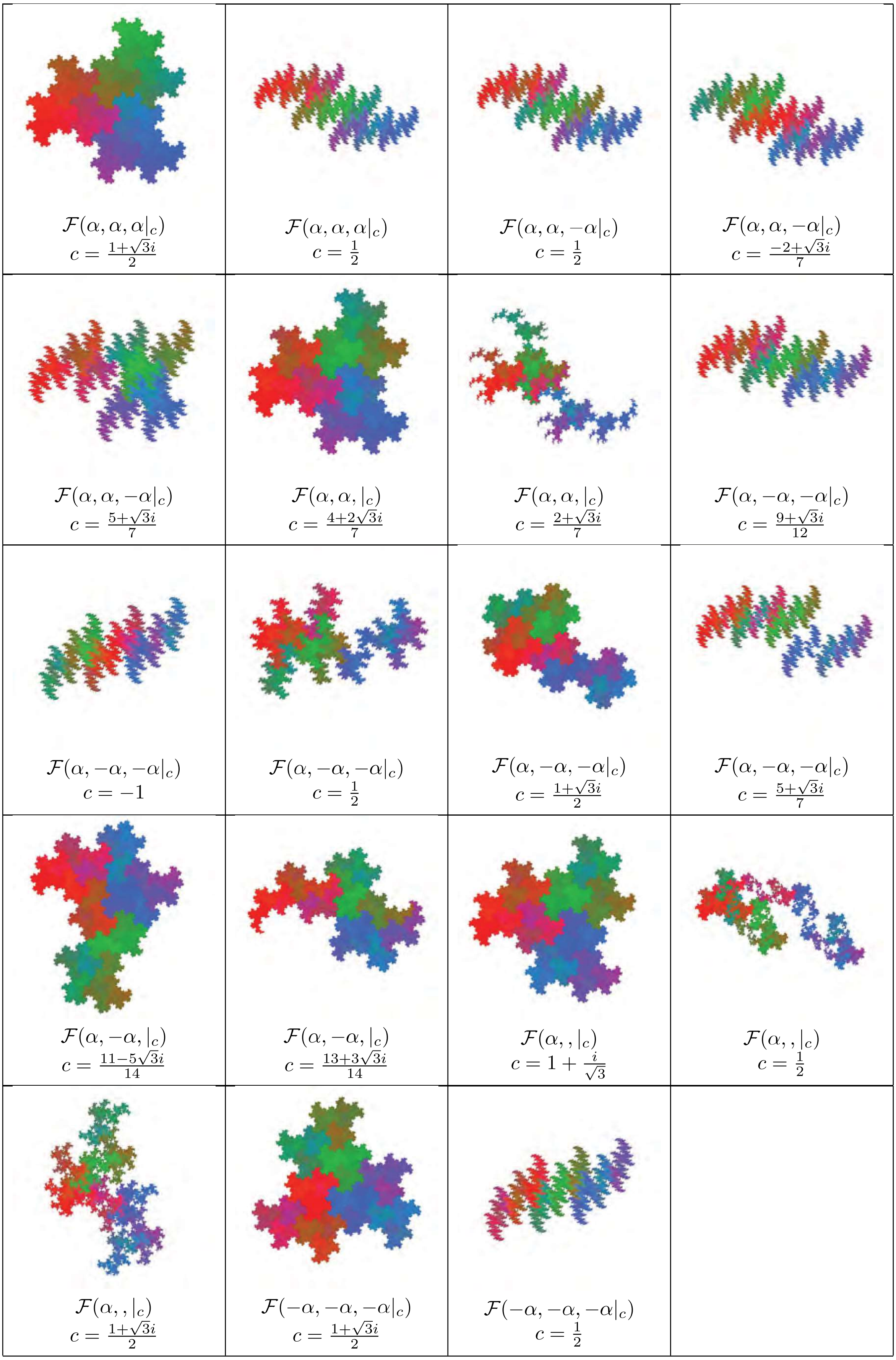


FIGURE 22. 3-similarity tilings associated with $1 - 3z + 3z^2$, $\alpha = \frac{3+\sqrt{3}i}{6}$.

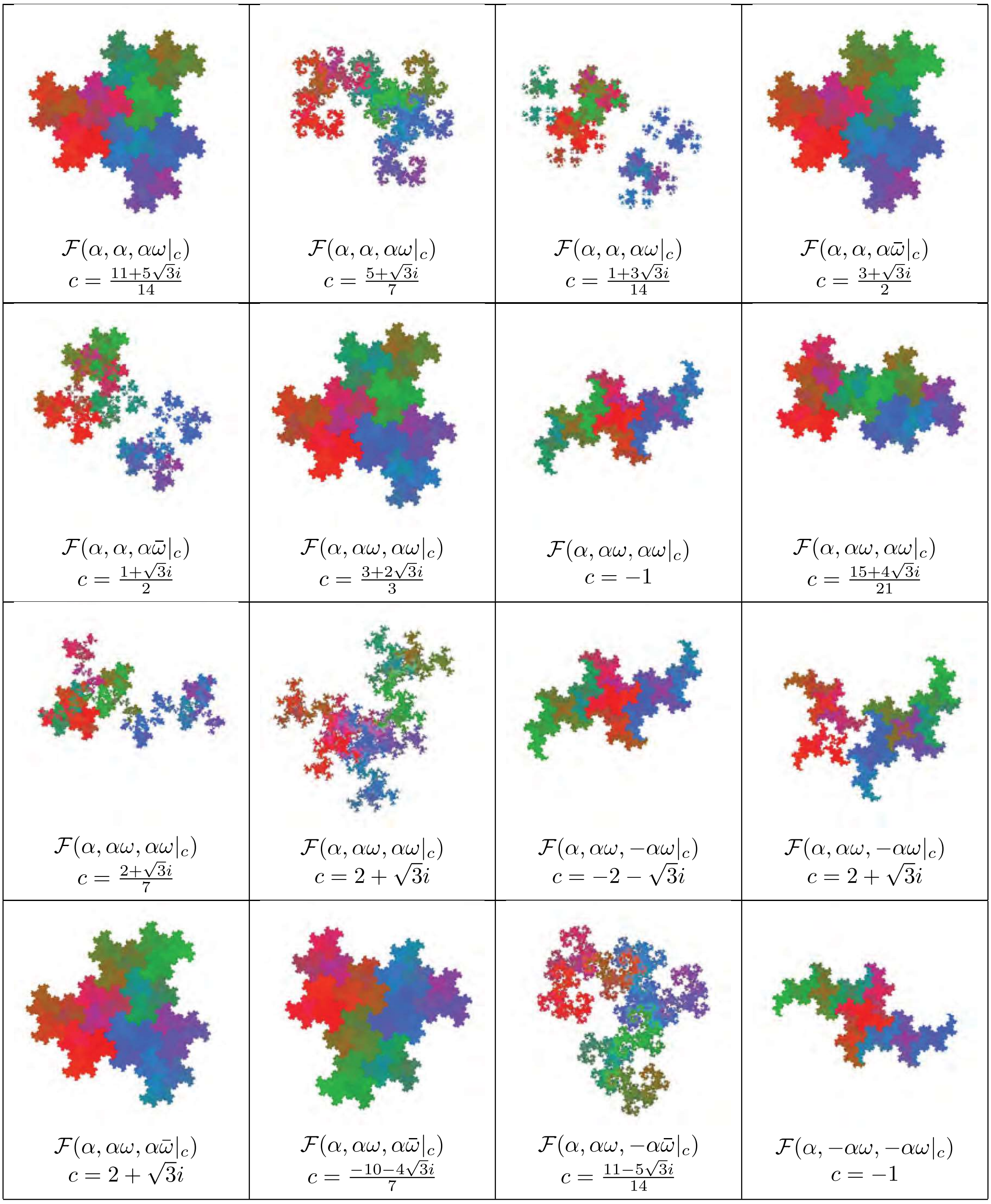


FIGURE 23. 3-similarity tilings related to both $1+3z^2$ and $1-3z+3z^2$, $\alpha=\frac{i}{\sqrt{3}}$, $\omega=\frac{1+\sqrt{3}i}{2}$. (1)

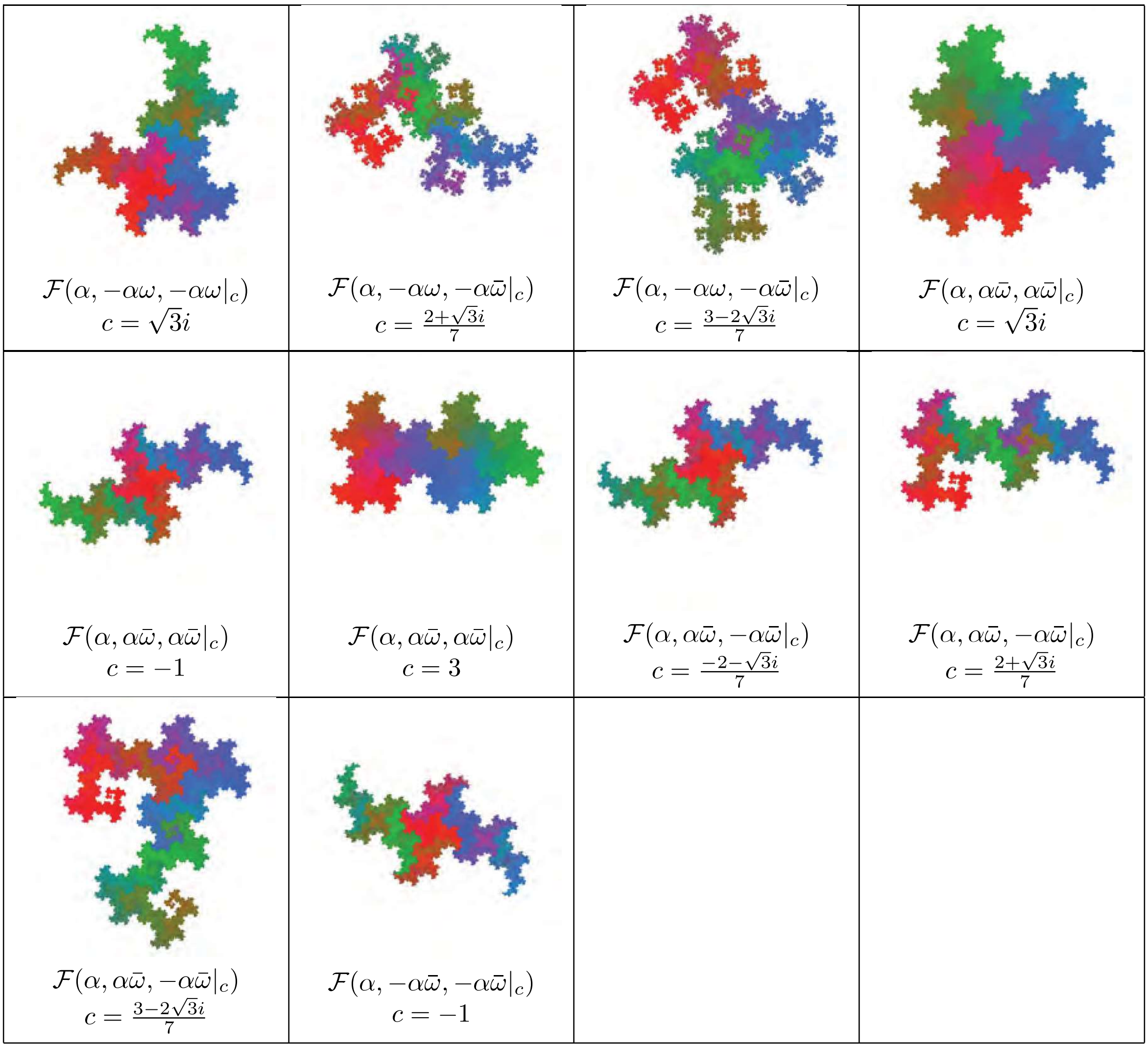


FIGURE 24. 3-similarity tilings related to both $1+3z^2$ and $1-3z+3z^2$, $\alpha = \frac{i}{\sqrt{3}}$, $\omega = \frac{1+\sqrt{3}i}{2}$. (2)

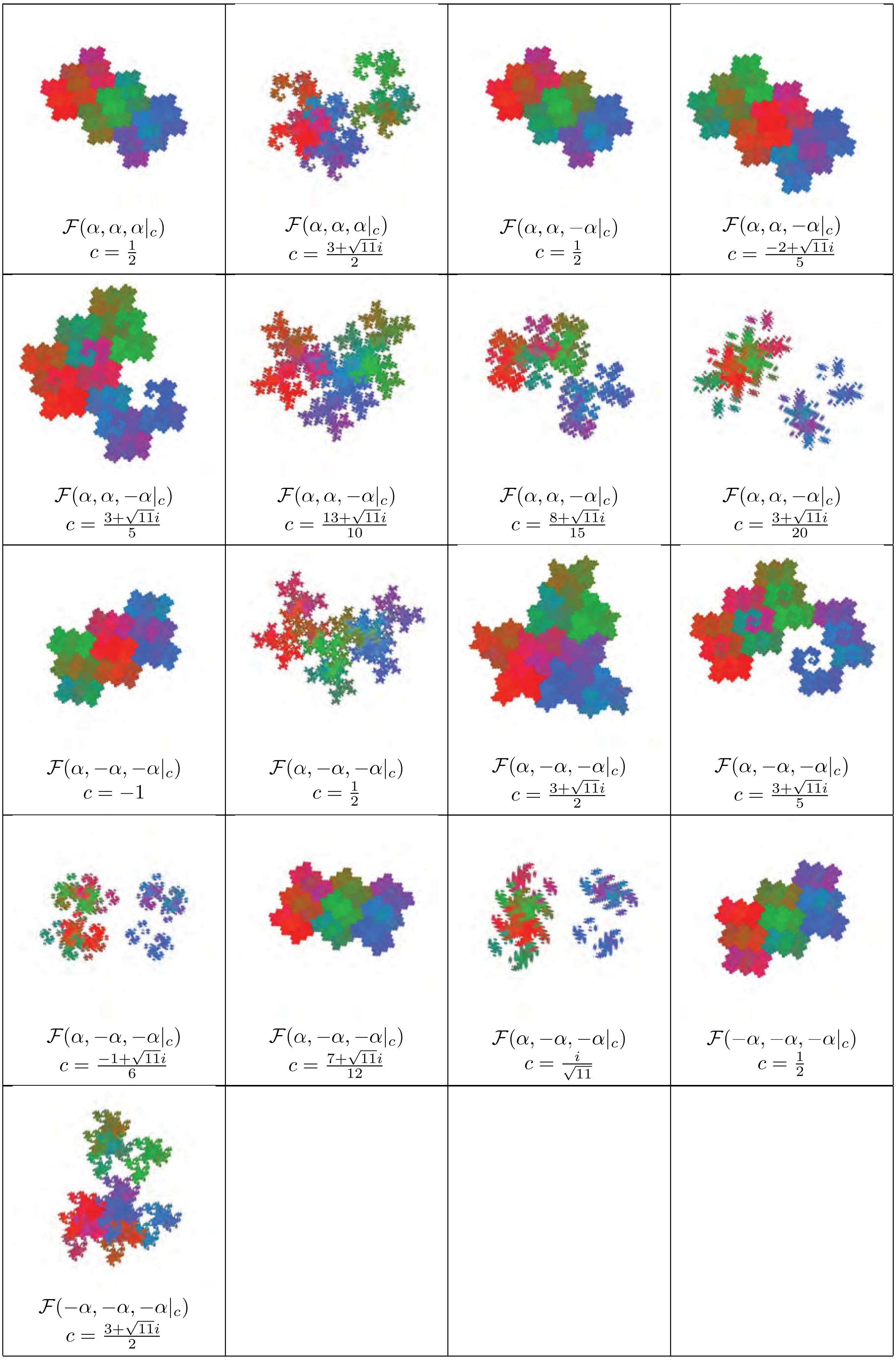


FIGURE 25. 3-similarity tilings associated with $1-z+3z^2$, $\alpha=\frac{1+\sqrt{11}i}{6}$.

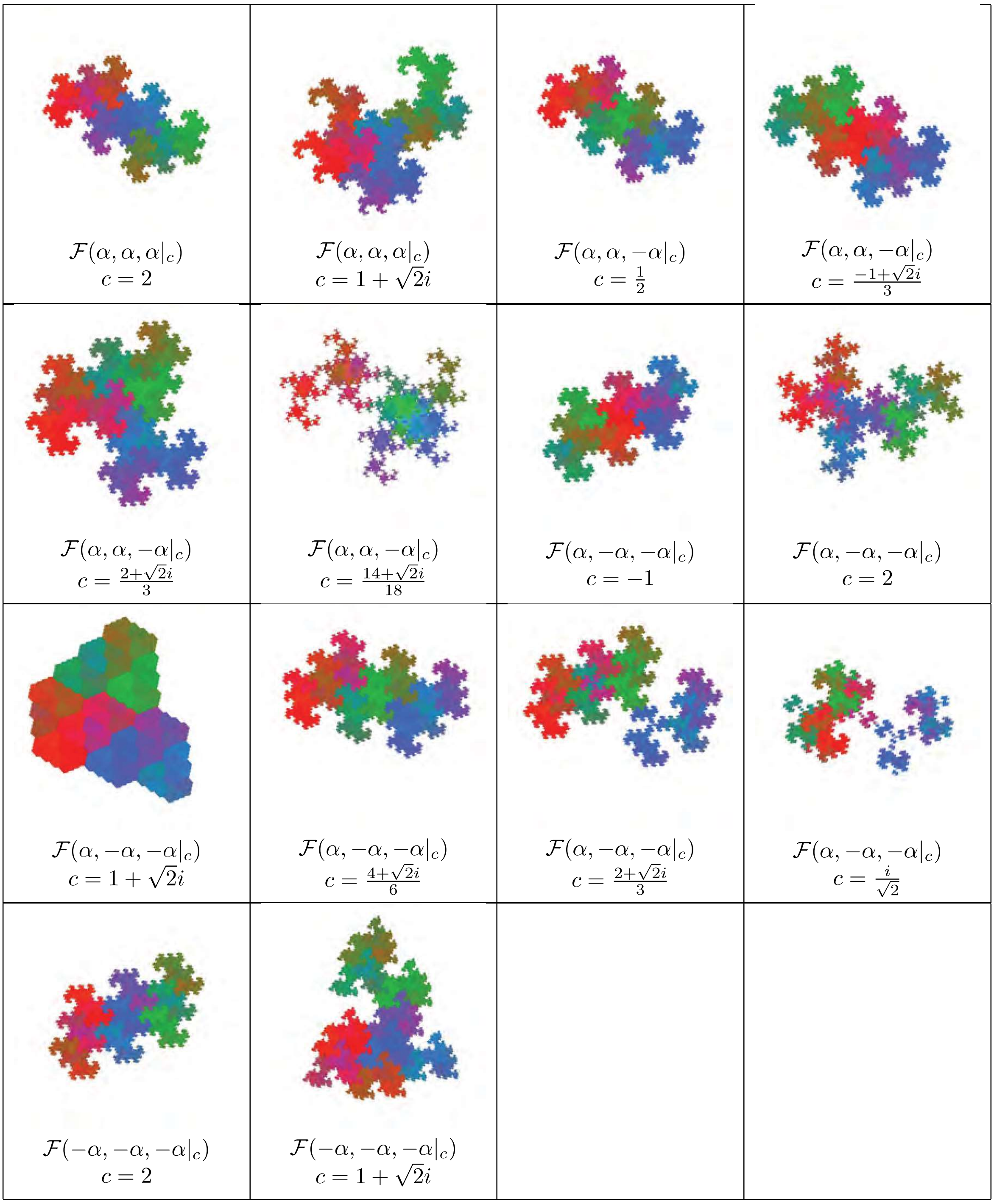


FIGURE 26. 3-similarity tilings associated with $1 - 2z + 3z^2$, $\alpha = \frac{1+\sqrt{2}i}{3}$.

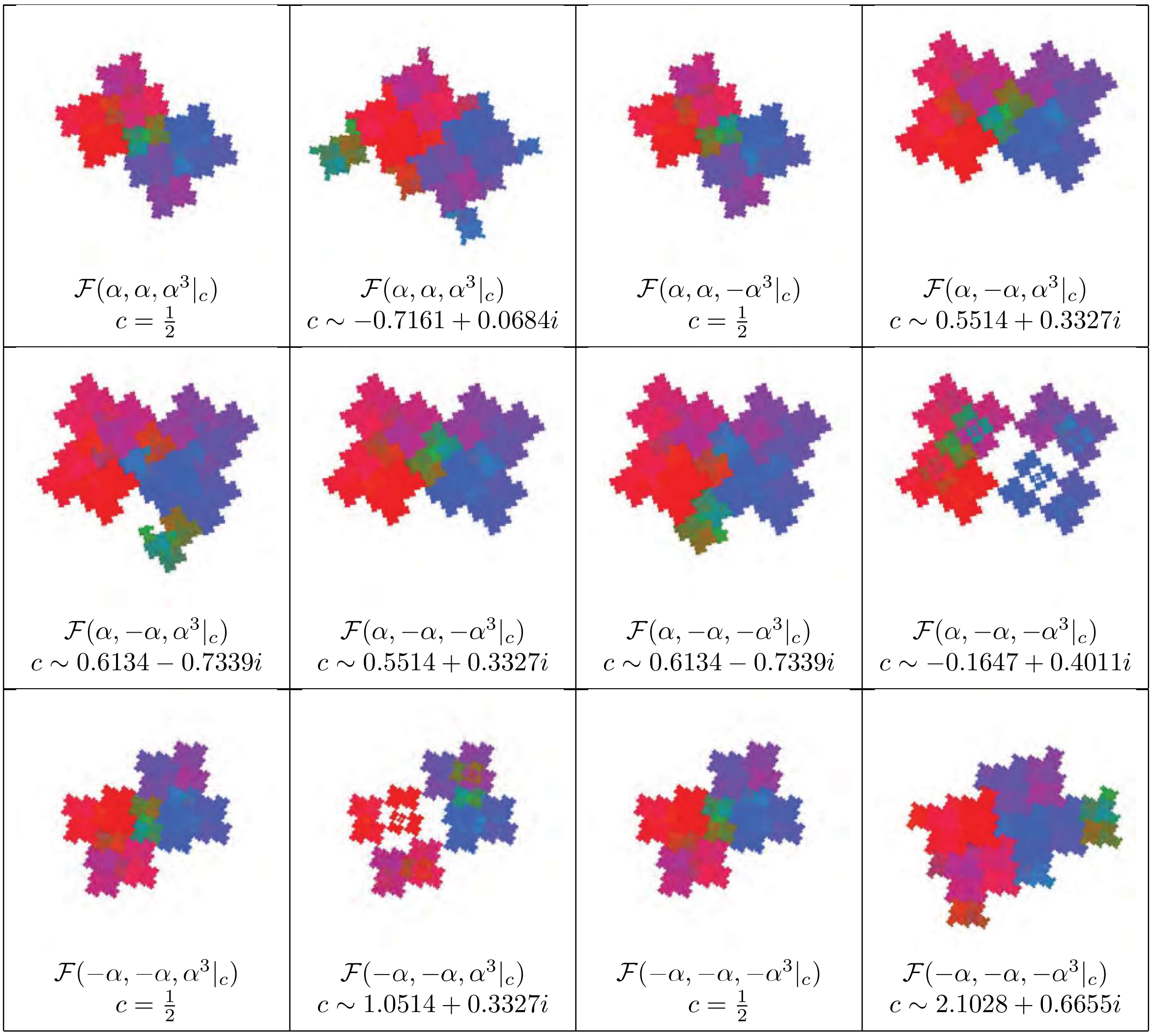


FIGURE 27. 3-similarity tilings associated with $1 + 2z^2 + z^3$, $\alpha \sim 0.102785 + 0.665457i$.

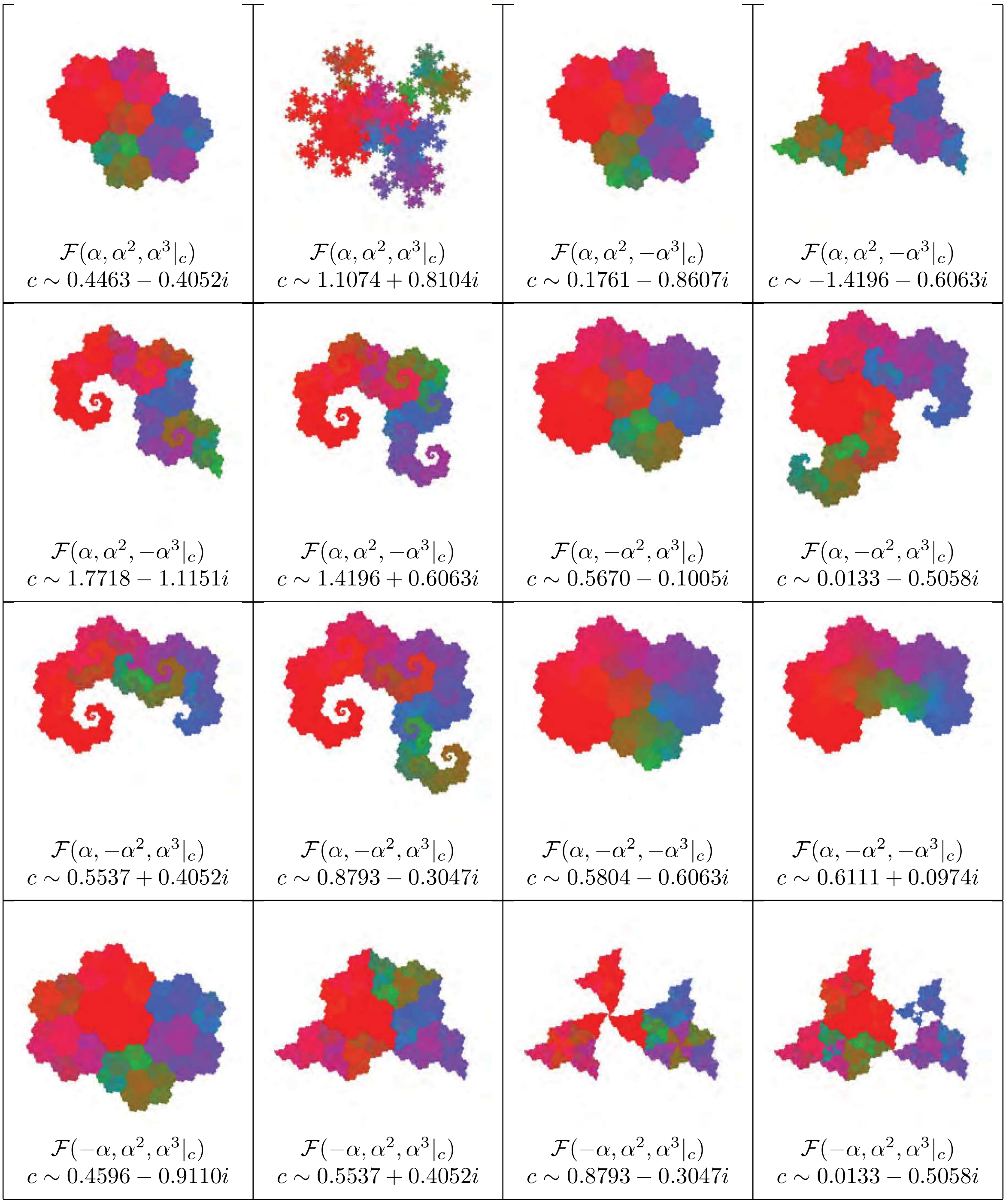


FIGURE 28. 3-similarity tilings associated with $1 - z + z^2 + z^3$, $\alpha \sim 0.419643 + 0.606291i$. (1)

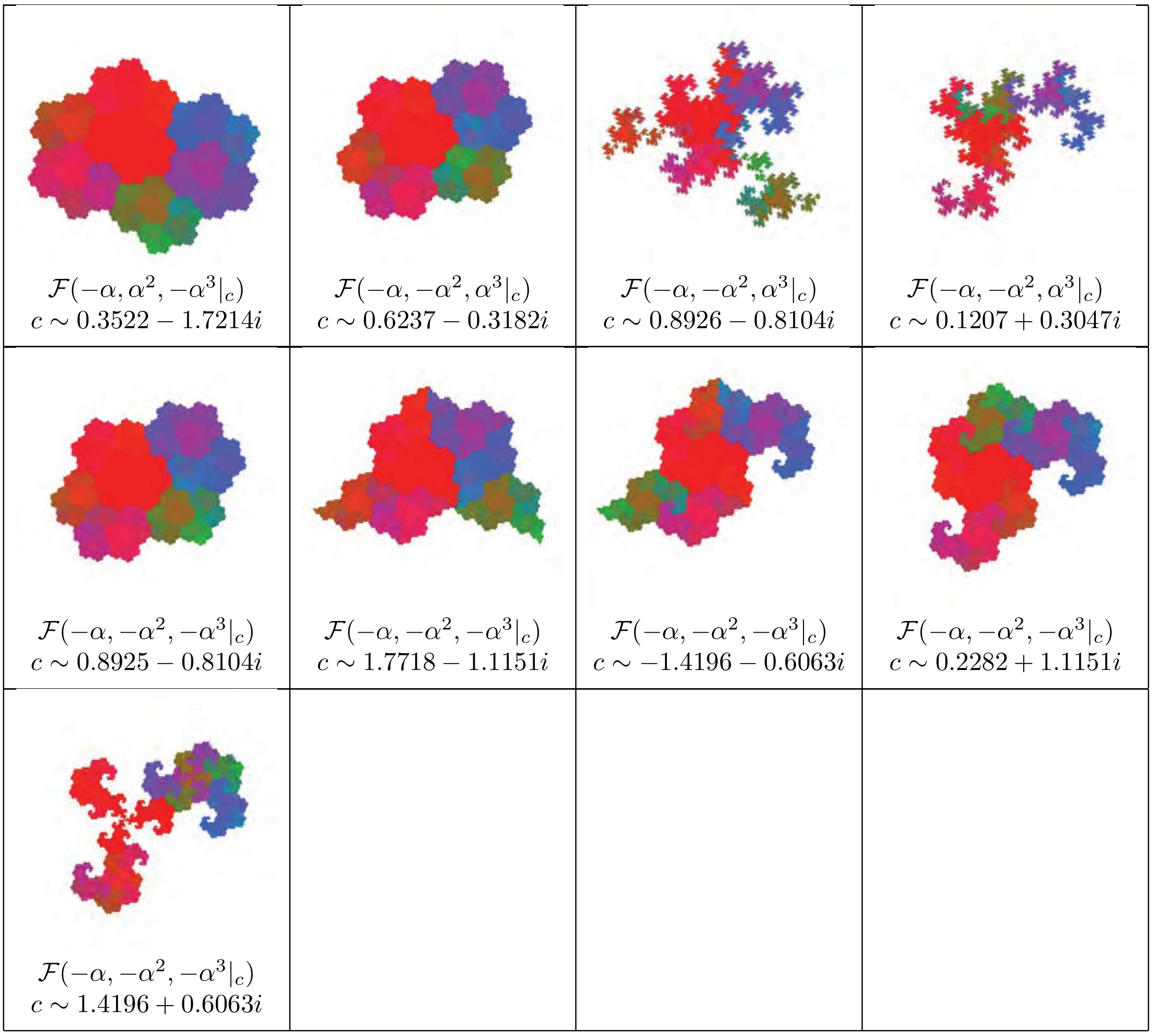


FIGURE 29. 3-similarity tilings associated with $1 - z + z^2 + z^3$, $\alpha \sim 0.419643 + 0.606291i$. (2)

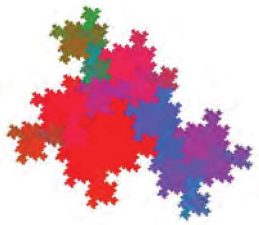
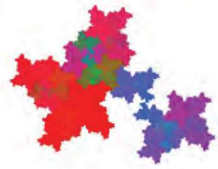
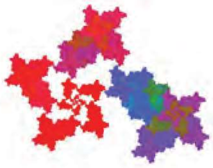
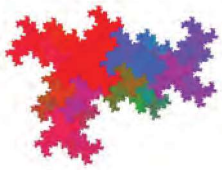
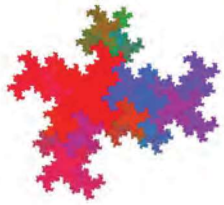
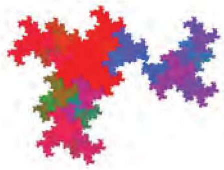
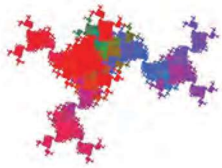
 $\mathcal{F}(\alpha, \alpha^2, \alpha^4\|_c)$ $c \sim 1.6624 + 0.5623i$	 $\mathcal{F}(\alpha, \alpha^2, \alpha^4\|_c)$ $c \sim -0.0925 + 2.0520$	 $\mathcal{F}(\alpha, \alpha^2, -\alpha^4\|_c)$ $c \sim 0.6652 - 0.1976i$	 $\mathcal{F}(\alpha, \alpha^2, -\alpha^4\|_c)$ $c \sim 0.2287 + 0.4712i$
 $\mathcal{F}(\alpha, \alpha^2, -\alpha^4\|_c)$ $c \sim 1.2168 + 0.0455i$	 $\mathcal{F}(-\alpha, \alpha^2, \alpha^4\|_c)$ $c \sim 0.8774 - 0.7449i$	 $\mathcal{F}(-\alpha, \alpha^2, \alpha^4\|_c)$ $c \sim 0.7849 + 1.3071i$	 $\mathcal{F}(-\alpha, \alpha^2, \alpha^4\|_c)$ $c \sim -0.6624 - 0.5623i$
 $\mathcal{F}(-\alpha, \alpha^2, -\alpha^4\|_c)$ $c \sim 0.3348 + 0.1976i$	 $\mathcal{F}(-\alpha, -\alpha^2, \alpha^4\|_c)$ $c \sim 0.6624 + 0.5623i$		

FIGURE 30. 3-similarity tilings associated with $1 - z + 2z^2 - z^3$, $\alpha \sim 0.122561 + 0.744862i$.

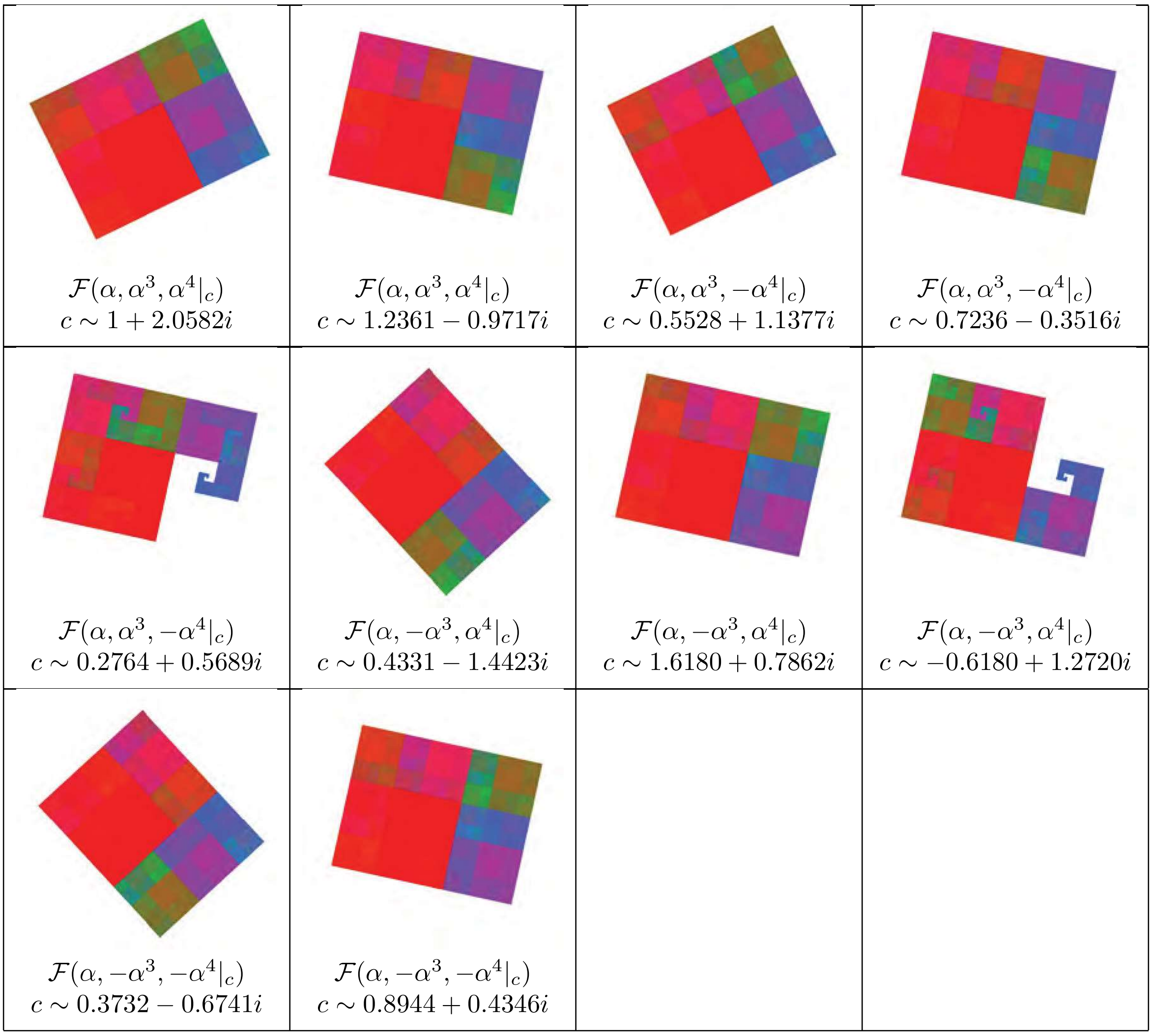


FIGURE 31. 3-similarity tilings associated with $1 + z^2 - z^4$, $\alpha \sim 0.786151i$.