Asymptotic behaviours of pressure functionals and statistical representations of the coefficients

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Abstract

We study an asymptotic behaviour of the pressure functional

$$P(\varphi + t\psi) = P(\varphi) + p_1 t + p_2 t^2 + \cdots + p_n t^n + o(t^n) \quad (t \to 0)$$

with potentials φ and ψ on a countable Markov shift X. We show that if the transition matrix of X is finitely primitive, the potentials φ and ψ are real-valued locally Hölder continuous functions on X, and a sufficient condition for an asymptotic expansion of $t \mapsto P(\varphi + t\psi)$ is satisfied, then the 3-th coefficient p_3 of this expansion has a limit representation which looks like the asymptotic variance p_2 well. The form of the coefficient p_q ($q \geq 4$) is also investigated under a special condition for ψ.

1 Introduction

Let X be a countable Markov shift with finitely primitive transition matrix (see Section 3 for definition). Take real-valued locally Hölder continuous functions φ and ψ on X. We study an asymptotic behaviour of the pressure functional, namely

$$P(\varphi + t\psi) = P(\varphi) + p_1 t + p_2 t^2 + \cdots + p_n t^n + o(t^n) \quad (t \to 0).$$

Here $P(\varphi + t\psi)$ is called the topological pressure of $\varphi + t\psi$ [4] which is defined by (3.1) in Section 3. It is known that if $P(\varphi + t\psi)$ has the expansion $P(\varphi) + p_1 t + p_2 t^2 + o(t^2)$ and suitable conditions hold then p_1 and p_2 have the forms [8]

$$p_1 = \int_X \psi \, d\mu \tag{1.1}$$

$$p_2 = \frac{1}{2!} \lim_{n \to \infty} \frac{1}{n} \int_X \Big(\sum_{k=0}^{n-1} (\psi - \int_X \psi \, d\mu) \circ \sigma^k \Big)^2 d\mu, \tag{1.2}$$

where μ is the σ-invariant Borel probability Gibbs measure of the potential φ and σ is the shift transformation. We are interested in whether p_k ($k \geq 3$) has the expression similar to the asymptotic variance p_2.

Our main results is as follows: if we fix an integer $n \geq 3$ and assume that a sufficient condition ((4.1) in Section 4) for asymptotic expansion of order n of pressure functional is satisfied, then we give a explicit formula for the limit of $\mu((\sum_{i=0}^{m-1}(\psi-\int_X \psi\, d\mu)\circ\sigma^i)^k)/m^{[k/2]}$ as $m \to \infty$ for each $2 \le k \le n$ (Theorem 4.3), where $[k/2]$ denotes the largest integer i with $i \le k/2$. As a corollary, we obtain the formula of the 3-th coefficient:

$$p_3 = \frac{1}{3!} \lim_{m \to \infty} \frac{1}{m} \int_X \Big(\sum_{k=0}^{m-1} (\psi - \int_X \psi\, d\mu) \circ \sigma^k \Big)^3 d\mu \tag{1.3}$$

(Corollary 4.4). On the other hand, each p_q ($q \ge 4$) does not have a similar form in general (Remark 4.6). Nevertheless, under the cases $p_2 = \cdots = p_{q-1} = 0$, we have a similar form for the coefficient p_q as well as (1.3) (Proposition 4.7). By using this proposition, we see that if ψ is cohomologous to a constant, then all coefficients p_k ($2 \le k \le n$) are zero.

Note that in the infinite state case, the function $t \mapsto P(\varphi + t\psi)$ may be not analytic at $t = 0$ in general though the finite state case implies that the function $t \mapsto P(\varphi + t\psi)$ is analytic on $\mathbb{R}$ by using analytic perturbation theory for suitable transfer operators [5]. Then we need a new method for obtaining the asymptotic expansion in the infinite state case. By developing the asymptotic analysis of the pressure of perturbed potential for the finite state case to the countable state case, we obtain the higher asymptotic expansion of $P(\varphi + t\psi)$ (Proposition 4.1). Furthermore, we also extend a transfer operator technique of Kotani and Sunada [3] in subshift of finite type to a technique in countable Markov shift. In this study, we need the exponential decay of correlation for φ in countable state space and it is guaranteed by a spectral gap property of suitable transfer operators [1] (see also Theorem 3.6). By using these arguments, our main results are showed.

In a future work, we shall study the asymptotic expansion of the Hausdorff dimension of the limit sets generated by a perturbed infinite graph Markov systems using our results and techniques in this paper. Remark that in a suitable limit set, its Hausdorff dimension is given by a solution s of $P(s\varphi) = 0$ of a physical potential φ which is sometime called a *Bowen's formula*. We shall investigate representations of the higher-order coefficients. In another application, information of the higher-order coefficient of pressure functionals will be useful for improving of convergence of a central limit theorem.

The next section 2 is mentioned about previous related results which is given by [3]. In Section 3, we recall the notion of symbolic dynamics with countable state space and the notion of thermodynamic formalism which need to state our results. The main results are precisely given in Section 4. In particular, the outlines of proofs are stated in the same section. In the finial section 5, we give a simple concrete example.

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2 Related results

In this section, we recall the asymptotic expansion of pressure functional on compact smooth manifold with topological mixing Anosov diffeomorphism [3]. Let X be a compact smooth manifold and $T : X \to X$ a topologically mixing Anosov diffeomorphism (see [2] for definition). Denoted by μ the T-invariant probability Gibbs measure on X. For $\varphi_0, \varphi_1, \ldots, \varphi_k \in C^\infty(X \to \mathbb{R})$, let

$$\mathrm{Cor}(\varphi_0, \varphi_1, \ldots, \varphi_k) = \frac{\partial^{k+1}}{\partial t_0 \cdots \partial t_k} \log \left(\int_X \exp\left(\sum_{i=0}^{k} t_i \varphi_i \right) d\mu \right) \bigg|_{t_0 = \cdots = t_k = 0}.$$

Theorem 2.1 ([3, Theorem 1 and Theorem 2]) *Let X be a compact smooth manifold, $T : X \to X$ topologically mixing Anosov diffeomorphism and take φ of $C^\infty(X \to \mathbb{R})$. Denoted by μ the T-invariant Gibbs measure for φ on X. Then for $\varphi_0, \ldots, \varphi_k \in C^\infty(X \to \mathbb{R})$,*

$$\frac{\partial^{k+1}}{\partial t_0 \cdots \partial t_k} P(\varphi + t_0\varphi_0 + \cdots + t_k\varphi_k) \bigg|_{t_0 = \cdots = t_k = 0} = \sum_{n_1, \ldots, n_k = -\infty}^{\infty} \mathrm{Cor}(\varphi_0, \varphi_1 \circ T^{n_1}, \ldots, \varphi_k \circ T^{n_k}),$$

where the right hand side converges absolutely.

Corollary 2.2 ([3, Corollary 1]) *Under the same condition of Theorem 2.1, for $\psi \in C^\infty(X)$, the pressure functional has an infinite series at $t = 0$:*

$$P(\varphi + t\psi) = P(\varphi) + p_1 t + \cdots + p_k t^k + \cdots$$

with

$$\begin{aligned} p_1 &= \mu(\psi) \\ p_2 &= \frac{1}{2} \sum_{n=-\infty}^{\infty} \mathrm{Cov}(\psi, \psi \circ T^n) = \frac{1}{2} \sum_{n=-\infty}^{\infty} \left(\int_X \psi\psi \circ T^n \, d\mu - \int_X \psi \, d\mu \int_X \psi \, d\mu \right) \\ p_k &= \frac{1}{k!} \sum_{n_1, \ldots, n_{k-1} = -\infty}^{\infty} \mathrm{Cov}(\psi, \psi \circ T^{n_1}, \cdots, \psi \circ T^{n_{k-1}}) \quad \textit{for } k \geq 2. \end{aligned}$$

Remark 2.3 It is not hard to show that the coefficient p_2 in the above corollary has the form (1.2) replacing σ by T. On the other hand, it is not easy to check that the coefficient p_3 in the above has the form (1.3).

3 Preliminarily

We start with the notion of symbolic dynamics with countable state space [4, 6, 7]. Let S be a countable set with distinct topology and $A = (A(ij))_{S \times S}$ a zero-one matrix, i.e.

$A(ij) = 0$ or $A(ij) = 1$ for all $i, j \in S$. The set

$$X = \{\omega = \omega_0\omega_1 \cdots \in \prod_{n=0}^{\infty} S \,:\, A(\omega_k\omega_{k+1}) = 1 \text{ for any } k \geq 0\}$$

endowed with product topology induced by the discrete topology on S, and endowed with the shift transformation $\sigma : X \to X$ is defined by $(\sigma\omega)_n = \omega_{n+1}$ for any $n \geq 0$. This is called a *countable Markov shift* (or *topological Markov shift*) with state space S and with transition zero-one matrix A. In what follows, we assume $X \neq \emptyset$. An element ω of X denotes $\omega = \omega_0\omega_1\omega_2\cdots$ with $\omega_0, \omega_1, \omega_2, \cdots \in S$.

Word $w = w_1w_2 \ldots w_n \in S^n$ is *admissible* if $A(w_1w_2) = A(w_2w_3) = \cdots = A(w_{n-1}w_n) = 1$. For admissible word $w \in S^n$, *cylinder set* is defined by $[w] := \{\omega \in X : \omega_0 \cdots \omega_{n-1} = w\}$. The transition matrix A is *finitely irreducible* if there exists a finite subset $F \subset \bigcup_{n=1}^{\infty} S^n$ such that for any $a, b \in S$, there is $w \in F$ so that $a \cdot w \cdot b$ is admissible, where $a \cdot w$ is the concatenation of a and w. The matrix A is called *finitely primitive* if there exists a finite subset $F \subset S^N$ with an integer $N \geq 1$ so that for any $a, b \in S$, $a \cdot w \cdot b$ is admissible for some $w \in F$ [4]. Note that finitely primitively implies finitely irreducibility. The matrix A has the *big images and pre-images (BIP) property* if there is a finite set $S_0 = \{a_1, \cdots, a_N\}$ of S such that for any $b \in S$, there exist $1 \leq i, j \leq N$ such that $A(a_ib) = A(ba_j) = 1$.

Remark 3.1 The matrix A is finitely irreducible if and only if A is irreducible and has the BIP property. Similarity, the matrix A is finitely primitive if and only if X is topologically mixing and A has the BIP property [7].

For $\theta \in (0, 1)$, a metric $d_\theta : X \times X \to \mathbb{R}$ is defined by

$$d_\theta(\omega, \upsilon) = \begin{cases} \theta^{\min\{n \geq 0 \,:\, \omega_n \neq \upsilon_n\}} & (\omega \neq \upsilon) \\ 0 & (\omega = \upsilon). \end{cases}$$

Remark 3.2 The metric topology induced by d_θ coincides with the product topology induced by the discrete topology on S.

Remark 3.3 (X, d_θ) is a complete and separable metric space. Moreover, if $\{a \in S : [a] \neq \emptyset\}$ is an infinite set, then X is not compact.

Next we introduce some function spaces and the notion of thermodynamic formalism [4, 8]. Put $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. A function $f : X \to \mathbb{K}$ is *θ-locally Lipschitz continuous* if $c(f) := \sup_{e \in S} \sup\{|f(\omega) - f(\upsilon)|/d_\theta(\omega, \upsilon) \,:\, \omega, \upsilon \in [e],\ \omega \neq \upsilon\} < \infty$. A function $f : X \to \mathbb{K}$ is called a *locally Hölder continuous* function if f is a θ-locally Lipschitz continuous for some $\theta \in (0,1)$. We set

$$\begin{aligned} C(X,\mathbb{K}) =& \{f : X \to \mathbb{K} : \text{continuous functions}\} \\ C_b(X,\mathbb{K}) =& \{f \in C(X,\mathbb{K}) : \|f\|_\infty < \infty\} \quad \text{with} \quad \|f\|_\infty := \sup_{\omega\in X}|f(\omega)| \\ F_\theta(X,\mathbb{K}) =& \{f : X \to \mathbb{K} : \theta\text{-locally Lipschitz continuous }\} \\ F_{\theta,b}(X,\mathbb{K}) =& \{f \in F_\theta(X,\mathbb{K}) : \|f\|_\infty < \infty\} \quad \text{with} \quad \|f\|_\theta := \|f\|_\infty + c(f). \end{aligned}$$

The spaces $(C_b(X,\mathbb{K}), \|\cdot\|_\infty)$ and $(F_{\theta,b}(X,\mathbb{K}), \|\cdot\|_\theta)$ are Banach spaces. Note the inclusion $F_\theta(X,\mathbb{K}) \subset F_{\theta'}(X,\mathbb{K})$ for $\theta < \theta'$. The symbol $\mathbb{K}$ may be omitted from these definitions when $\mathbb{K} = \mathbb{C}$.

For function $\varphi : X \to \mathbb{R}$, the *topological pressure* $P(\varphi)$ of φ is given by

$$P(\varphi) = \lim_{n\to\infty}\frac{1}{n}\log\sum_{w\in S^n : [w]\neq\emptyset}\exp(\sup_{\omega\in[w]}\sum_{k=0}^{n-1}\varphi(\sigma^k\omega)) \tag{3.1}$$

if it exists [4].

Remark 3.4 If the transition matrix of X is finitely irreducible and φ is in $F_\theta(X,\mathbb{R})$ with $P(\varphi) < \infty$, then $P(\varphi)$ coincides with the Gurevich pressure $P_G(\varphi)$ of φ which is introduced in [6].

A σ-invariant Borel probability measure μ on X is said to be a *Gibbs measure* of the potential $\varphi : X \to \mathbb{R}$ if there exist $c \geq 1$ and $P \in \mathbb{R}$ such that for any $\omega \in X$ and $n \geq 1$

$$c^{-1} \leq \frac{\mu([\omega_0\omega_1\ldots\omega_{n-1}])}{\exp(-nP + \sum_{k=0}^{n-1}\varphi(\sigma^k\omega))} \leq c.$$

Remark 3.5 If A is finitely irreducible and φ is in $F_\theta(X,\mathbb{R})$ with $P(\varphi) < \infty$, then the Gibbs measure of φ uniquely exists and P equals $P(\varphi)$ [4].

For a real-valued function φ on X, the Ruelle operator $\mathcal{L}_\varphi$ associated to φ is defined by

$$\mathcal{L}_\varphi f(\omega) = \sum_{e\in S : t(e)=i(\omega_0)} e^{\varphi(e\cdot\omega)}f(e\cdot\omega)$$

if this series converges in $\mathbb{C}$ for a complex-valued function f on X and for $\omega \in X$. It is known that if the incidence matrix is finitely irreducible and φ is in $F_\theta(X,\mathbb{R})$ with finite topological pressure, then $\mathcal{L}_\varphi$ becomes a bounded linear operator both on the Banach spaces $F_{\theta,b}(X)$ and $C_b(X)$. We state a version of Ruelle-Perron-Frobenius Theorem as follows.

Theorem 3.6 ([1]) *Let X be a countable Markov shift with finitely primitive transition matrix. Let $\varphi \in F_\theta(X,\mathbb{R})$ with finite pressure. Then the Ruelle operator $\mathcal{L}_\varphi : F_{\theta,b}(X) \to$*

$F_{\theta,b}(X)$ of φ has the spectral decomposition

$$\mathcal{L}_\varphi = \lambda\mathcal{P} + \mathcal{R}$$

such that

(1) λ is the spectral radius of $\mathcal{L} : F_{\theta,b}(X) \to F_{\theta,b}(X)$ and is a simple eigenvalue of $\mathcal{L}_\varphi$;

(2) $\mathcal{P}$ is the eigenprojection of the eigenvalue λ of $\mathcal{L}$ onto the one-dimensional eigenspace. In particular,

$$\mathcal{P}f = \int_X fh\,d\nu,$$

where h is the corresponding positive eigenfunction of the eigenvalue λ and ν is the corresponding positive eigenvector of λ of the dual operator $\mathcal{L}_\varphi^* : F_{\theta,b}(X)^* \to F_{\theta,b}(X)^*$ with $\nu(h) = 1$;

(3) $\mathcal{PR} = \mathcal{RP} = \emptyset$ and the spectral radius of $\mathcal{R} : F_{\theta,b}(X) \to F_{\theta,b}(X)$ is less than λ;

(4) $P(\varphi)$ equals $\log\lambda$ and $h\nu$ becomes the Gibbs measure of the potential φ.

For the sake of convenience, we call the number λ in above the *Perron eigenvalue* of $\mathcal{L}_\varphi$.

Remark 3.7 When the transition matrix A is finitely irreducible, the assertions of Theorem 3.6 are correct excluding (3). In fact, finitely irreducibility implies that $\mathcal{R}$ has the eigenvalue $\lambda e^{2\pi i\sqrt{-1}/p}$ for each $i = 1, 2, \ldots, p-1$ with the period p of A as nonnegative irreducible matrices.

4 Main results

Recall that X is a countable Markov shift with countable state space S and with finitely primitive transition zero-one matrix A. Fix an integer $n \geq 1$. Let $\varphi, \psi \in F_\theta(X, \mathbb{R})$. We introduce a sufficient condition for the asymptotic expansion of order n of $[0, \infty) \ni t \mapsto P(\varphi + t\psi)$:

$(\Phi)_n$ There exists $t_0 > 0$ such that for any $k = 0, 1, \ldots, n$,

$$\sum_{i\in S} \sup_{\omega\in[i]} \sup_{0\leq t\leq t_0} e^{\varphi(\omega)+t\psi(\omega)}|\psi(\omega)|^k < \infty. \tag{4.1}$$

Proposition 4.1 ([9]) *Let X be a countable Markov shift with finitely irreducible incidence matrix. Assume that the condition $(\Phi)_n$ is satisfied for a fixed integer $n \geq 1$. Then $P(\varphi + t\psi)$ has the form*

$$P(\varphi + t\psi) = P(\varphi) + p_1 t + p_2 t^2 + \cdots + p_n t^n + o(t^n) \text{ in } \mathbb{R} \tag{4.2}$$

as $t \to 0$ for some numbers $p_k \in \mathbb{R}$.

Outline of proof. Note that the number $P(\varphi + t\psi)$ is finite for all $0 \le t \le t_0$ from the condition $(\Phi)_n$ putting $k = 0$. The Taylor expansion $e^{\varphi + t\psi} = \sum_{k=0}^{\infty} e^{\varphi}\psi^k t^k/k!$ implies the asymptotic expansion of the operator $\mathcal{L}_{\varphi+t\psi}$

$$\mathcal{L}_{\varphi+t\psi} = \mathcal{L}_{\varphi} + \mathcal{L}_{\varphi,1}t + \cdots + \mathcal{L}_{\varphi,n}t^n + \tilde{\mathcal{L}}_n(t,\cdot)t^n$$

with the coefficient $\mathcal{L}_{\varphi,k} = \mathcal{L}_{\varphi} \circ (\psi^k/k!)$ and the remainder $\tilde{\mathcal{L}}_n(t,\cdot)$. The condition $(\Phi)_n$ implies the following two properties: (i) each $\mathcal{L}_{\varphi,k}$ is a bounded linear operator acting on $F_{\theta,b}(X)$; (ii) the remainder $\|\tilde{\mathcal{L}}_n(t,\cdot)\|_\infty$ vanishes as $t \to 0$. Consequently, it follows from (i)(ii) in above that the Perron eigenvalue $\eta(t)$ of $\mathcal{L}_{\varphi+t\psi}$ has the expansion

$$\eta(t) = \lambda + \eta_1 t + \cdots + \eta_n t^n + o(t^n) \tag{4.3}$$

as $t \to 0$ with coefficients $\eta_k \in \mathbb{R}$ $(k = 1, 2, \ldots, n)$, where λ is the Perron eigenvalue of $\mathcal{L}_{\varphi}$. Hence the function $t \mapsto P(\varphi + t\psi) = \log \eta(t)$ satisfies the expansion (4.2) with the coefficient

$$p_k = \sum_{l=1}^{k} \frac{(-1)^{l-1}}{l \cdot \lambda^l} \sum_{\substack{i_1,\ldots,i_l \ge 1: \\ i_1+\cdots+i_l = k}} \eta_{i_1} \cdots \eta_{i_l} \qquad (k = 1, \ldots, n).$$

□

Remark 4.2 Assume that the condition $(\Phi)_n$ is satisfied for an integer $n \ge 2$. Then the coefficients p_1 and p_2 have the forms (1.1) and (1.2), respectively.

Denoted by $\lambda(t)$ the Perron eigenvalue of the operator $\mathcal{L}_{\varphi+t(\psi-\mu(\psi))}$. Note that if $(\Phi)_n$ is satisfied, then so is for $\psi := \psi - \mu(\psi)$. Therefore we have the asymptotic expansion of the Perron eigenvalue $\lambda(t)$ of $\mathcal{L}_{\varphi+t(\psi-\mu(\psi))}$

$$\lambda(t) = \lambda + \lambda_1 t + \lambda_2 t^2 + \cdots + \lambda_n t^n + o(t^n) \text{ in } \mathbb{R}$$

as $t \to 0$ with coefficients $\lambda_k \in \mathbb{R}$ $(k = 1, 2, \ldots, n)$. In particular, $\lambda_1 = \mu(\psi - \mu(\psi)) = 0$ holds.

Now we are in a position to state one of our main results:

Theorem 4.3 ([9]) *Let X be a countable Markov shift with finitely primitive transition matrix. Assume that the condition $(\Phi)_n$ is satisfied for a fixed integer $n \ge 2$. Denoted by μ the Gibbs measure of φ. Then for any $k = 2, 3, \ldots, n$*

$$\lim_{m \to 0} \frac{1}{m^{[k/2]}} \frac{1}{k!} \int_X \Big(\sum_{i=0}^{m-1} (\psi - \int_X \psi \, d\mu) \circ \sigma^i \Big)^k d\mu = \begin{cases} \frac{1}{[k/2]!} (p_2)^{[k/2]} & (k \textit{ is even}) \\ \frac{1}{([k/2]-1)!} (p_2)^{[k/2]-1} p_3 & (k \textit{ is odd}), \end{cases} \tag{4.4}$$

where $[k/2]$ means the largest integer i with $i \le k/2$.

Outline of proof. Let $\varphi(t,\cdot)=\varphi+t(\psi-\mu(\psi))$. We consider the following three steps:
Step I: The m-th iteration of $\mathcal{L}_{\varphi(t,\cdot)}$ has the asymptotic expansion

$$\mathcal{L}^m_{\varphi(t,\cdot)}=\mathcal{L}^m_\varphi+\mathcal{M}_{1,m}t+\cdots+\mathcal{M}_{n,m}t^n+\tilde{\mathcal{M}}_{n,m}(t,\cdot)t^n$$

for each $m\geq 1$ with $\mathcal{M}_{k,m}=\mathcal{L}^m_\varphi\circ(\sum_{i=0}^{m-1}(\psi-\mu(\psi))\circ\sigma^i)^k/k!$ and with $\|\tilde{\mathcal{M}}_{n,m}(t,\cdot)\|_\infty\to 0$.

Indeed, this assertion is obtained by calculating m-iteration of the both side of the expansion $\mathcal{L}_{\varphi(t,\cdot)}=\mathcal{L}_\varphi+\mathcal{M}_1t+\cdots+\mathcal{M}_nt^n+\tilde{\mathcal{M}}_n(t,\cdot)t^n$ with $\tilde{\mathcal{M}}_n(t,\cdot)$ satisfying $\|\tilde{\mathcal{M}}_n(t,\cdot)\|_\infty\to 0$.

Step II: The asymptotic expansion of m-th power of eigenvalue $\lambda(t)$ of $\mathcal{L}_{\varphi(t,\cdot)}$ has the expansion

$$\lambda(t)^m=\lambda^m+\lambda_{1,m}t+\cdots+\lambda_{n,m}t^n+o(t^n)$$

and each $\lambda_{k,m}$ satisfies

$$\frac{\lambda_{k,m}}{\lambda^m}=\frac{\nu(\mathcal{M}_{k,m}h)}{\lambda^m}+o(m^{[k/2]}). \tag{4.5}$$

This assertion follows from Step I, an argument of [3] and the property of spectral gap in Theorem 3.6(3).

Step III:

$$\frac{\lambda_{k,m}}{\lambda^m}=\begin{cases}\frac{1}{[k/2]!}(p_2)^{[k/2]} & (k \text{ is even})\\ \frac{1}{([k/2]-1)!}(p_2)^{[k/2]-1}p_3 & (k \text{ is odd})\end{cases} + o(m^{[k/2]}). \tag{4.6}$$

Indeed, since $\lambda_{k,m}$ is the k-th coefficient of $(\lambda+\lambda_2t^2+\cdots+\lambda_nt^n+o(t^n))^m$, we obtain

$$\begin{aligned}\frac{\lambda_{k,m}}{\lambda^m}=&\sum_{i=1}^{[k/2]}\frac{m!}{(m-i)!}\sum_{\substack{j_2,\ldots,j_k\geq 0:\\ j_2+\cdots+j_k=i\\ 2j_2+\cdots+kj_k=k}}\prod_{l=2}^{k}\frac{1}{j_l!}\left(\frac{\lambda_l}{\lambda}\right)^{j_l}\\ =&a_1^{k,m}m+a_2^{k,m}m^2+\cdots+a_{[k/2]}^{k,m}m^{[k/2]}\end{aligned}$$

with some numbers $a_i^{k,m}\in\mathbb{R}$ $(i=1,2,\ldots,[k/2])$. In particular, $a_{[k/2]}^{k,m}$ is equal to the right hand side of (4.4). Hence the assertion of the theorem holds from (4.5) and (4.6) together with the equations $p_2=\lambda_2/\lambda$ and $p_3=\lambda_3/\lambda$. □

Corollary 4.4 ([9]) *Let X be a countable Markov shift with finitely primitive transition matrix. Assume that the condition $(\Phi)_n$ is satisfied for a fixed integer $n\geq 3$. Then the 3-th coefficient of the expansion of $t\mapsto P(\varphi+t\psi)$ has the form (1.3).*

Proof. The assertion immediately follows from Theorem 4.3 by putting $k = 3$. □

Remark 4.5 Theorem 4.3 says that if $p_2 \neq 0$ and $p_3 \neq 0$, then the speed of convergence $\mu((\sum_{i=0}^{m-1}(\psi - \mu(\psi)) \circ \sigma^i)^k) \to 0$ has exactly the order $1/m^{[k/2]}$ for any $k \geq 2$.

Remark 4.6 The coefficient p_k for $k \geq 4$ is not displayed as well as (1.3) in general. In fact, if either k is even and $p_2 \neq 0$ or k is odd and $p_2 p_3 \neq 0$, then the formula (4.4) implies

$$\lim_{m \to \infty} \frac{1}{m} \left| \int_X \Big(\sum_{i=0}^{m-1} (\psi - \int_X \psi \, d\mu) \circ \sigma^i \Big)^k d\mu \right| = +\infty.$$

We next consider the case $p_2 = \cdots = p_{q-1} = 0$ and $p_q \neq 0$ for some $q \geq 3$:

Proposition 4.7 ([9]) *Let X be a countable Markov shift with finitely primitive transition zero-one matrix. Assume that the condition $(\Phi)_n$ is satisfied for a fixed integer $n \geq 3$, and $p_2 = \cdots = p_{q-1} = 0$ for some $3 \leq q \leq n$. Then for each $k = q, q+1, \ldots, n$*

$$\frac{1}{m^{[k/q]}} \int_X \Big(\sum_{i=0}^{m-1} (\psi - \int_X \psi \, d\mu) \circ \sigma^i \Big)^k d\mu \to \begin{cases} \dfrac{k!}{[k/q]!} p_q^{[k/q]} & (k \equiv 0 \ (mod\ q)) \\ \dfrac{k!}{([k/q]-1)!} p_q^{[k/q]-1} p_{q+1} & (k \equiv 1 \ (mod\ q)) \\ O(1) & (otherwise). \end{cases} \tag{4.7}$$

as $m \to \infty$. Consequently,

$$\begin{aligned} p_q &= \frac{1}{q!} \lim_{m \to \infty} \frac{1}{m} \int_X \Big(\sum_{i=0}^{m-1} (\psi - \int_X \psi \, d\mu) \circ \sigma^i \Big)^q d\mu \\ p_{q+1} &= \frac{1}{(q+1)!} \lim_{m \to \infty} \frac{1}{m} \int_X \Big(\sum_{i=0}^{m-1} (\psi - \int_X \psi \, d\mu) \circ \sigma^i \Big)^{q+1} d\mu. \end{aligned}$$

Outline of proof. Note that the assumptions $p_2 = \cdots = p_{q-1} = 0$ imply $\lambda_2 = \cdots = \lambda_{q-1} = 0$. Thus we rewrite

$$\frac{\lambda_{k,m}}{\lambda^m} = a_1^{k,m} m + a_2^{k,m} m^2 + \cdots + a_{[k/p]}^{k,m} m^{[k/p]}$$

with

$$a_{[k/p]}^{k,m} = \sum_{\substack{j_q, \ldots, j_k \geq 0 : \\ j_q + \cdots + j_k = [k/q] \\ q j_q + \cdots + k j_k = k}} \prod_{l=q}^{k} \frac{(p_l)^{j_l}}{j_l!}.$$

It is directly checked that the number $a^{k,m}_{[k/p]}$ equals the right hand side of (4.7) in the case $k \equiv i \pmod{q}$ ($i = 0$ or 1). A proof of the remainder part is similar to the proof of Theorem 4.3. $\square$

Remark 4.8 By virtue of Proposition 4.7, we see that if ψ is cohomologous to a constant, then $p_k = 0$ for each $k = 2, 3, \dots, n$, where ψ is cohomologous to a constant if there exist a continuous function $u : X \to \mathbb{R}$ and a constant $c \in \mathbb{R}$ such that $\psi = u - u \circ u + c$.

5 An example

We remark that in the finite state case (i.e. $\sharp S < \infty$), the function $t \mapsto P(\varphi + t\psi)$ is analytic on $\mathbb{R}$ for any $\varphi, \psi \in F_0(X, \mathbb{R})$. In particular, the condition $(\Phi)_n$ holds for any $n \geq 1$ in this case. On the other hand, in the infinite state case, the function $t \mapsto P(\varphi + t\psi)$ may be not analytic at a point.

For example, put $n \geq 3$, $S = \{1, 2, \dots\}$, $X = \prod_{n=0}^{\infty} S$, $p(i) = 1/(i(\log i)^{n+2})$ and $\varphi(\omega) = \psi(\omega) = \log(p(\omega_0 + 2))$ for $\omega \in X$.

Proposition 5.1 ([9]) *Under the notation $n, S, X, p(i), \varphi, \psi$ in above, the condition $(\Phi)_n$ is fulfilled and $(\Phi)_{n+1}$ does not hold. In particular, $P(\varphi + t\psi)$ is not differentiable at $t = 0$. Moreover, the coefficient p_k $(k = 1, 2, 3)$ has the form*

$$p_1 = \sum_{l=3}^{\infty} \frac{p(l) \log p(l)}{\lambda}$$

$$p_2 = \frac{1}{2} \lim_{m \to \infty} \sum_{i_1, \cdots, i_m = 3}^{\infty} \frac{\prod_{j=1}^{m} p(i_j)}{\lambda^m} \left(\sum_{k=1}^{m} \left(\log p(i_k) - \sum_{l=3}^{\infty} \frac{p(l) \log p(l)}{\lambda} \right) \right)^2$$

$$p_3 = \frac{1}{6} \lim_{m \to \infty} \sum_{i_1, \cdots, i_m = 3}^{\infty} \frac{\prod_{j=1}^{m} p(i_j)}{\lambda^m} \left(\sum_{k=1}^{m} \left(\log p(i_k) - \sum_{l=3}^{\infty} \frac{p(l) \log p(l)}{\lambda} \right) \right)^3,$$

where $\lambda = \sum_{i=3}^{\infty} p(i)$.

Proof. We will check the validity of the condition $(\Phi)_n$. Note that $p(i + 2) < 1$ for any $i \in S$ and $n \geq 1$. For $k \geq 1$, the series in the condition $(\Phi)_k$ satisfies

$$\begin{aligned} \sum_{i \in S} \sup_{\omega \in [i]} \sup_{0 \leq t \leq t_0} e^{\varphi(\omega) + t\psi(\omega)} |\psi(\omega)|^k &= \sum_{i=3}^{\infty} p(i)(-\log p(i))^k \\ &= \sum_{l=0}^{k} \binom{k}{l} (n+2)^{k-l} \sum_{i=3}^{\infty} \frac{(\log \log i)^{k-l}}{i(\log i)^{n+2-l}}. \end{aligned}$$

The series $\sum_{i=3}^{\infty}(\log\log i)^{k-l}/(i(\log i)^{n+2-l})$ converges for any $n+2-l \geq 2$, i.e. $n \geq l$. Therefore $(\Phi)_n$ is fulfilled. On the other hand, If $k = n+1$ and $l = n+1$, then this series does not converge. This means that the condition $(\Phi)_{n+1}$ fails. Note that $P(\varphi + t\psi)$ is not differentiable at $t = 0$ by $P(\varphi + t\psi) = +\infty$ for any $t < 0$. □

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