NON-AUTONOMOUS CONFORMAL ITERATED FUNCTION SYSTEMS WITH OVERLAPS

YUTO NAKAJIMA

1. Introduction

A Non-autonomous Iterated Function System (NIFS) $\Phi = (\{\phi_i^{(j)}\}_{i\in I^{(j)}})_{j=1}^{\infty}$ on a compact subset $X \subset \mathbb{R}^m$ consists of a sequence of finite collections of uniformly contracting maps $\phi_i^{(j)} : X \to X$, where $I^{(j)}$ is a finite set. The system Φ is an Iterated Function System (for short, IFS) if the collections $\{\phi_i^{(j)}\}_{i\in I^{(j)}}$ are independent of j. In comparison to usual IFSs, we allow the contractions $\phi_i^{(j)}$ applied at each step j to vary as j changes.

Rempe-Gillen and Urbański [9] introduced Non-autonomous Conformal Iterated Function Systems (NCIFSs). An NCIFS $\Phi = (\{\phi_i^{(j)}\}_{i\in I^{(j)}})_{j=1}^{\infty}$ on a compact subset $X \subset \mathbb{R}^m$ consists of a sequence of collections of uniformly contracting conformal maps $\phi_i^{(j)} : X \to X$ satisfying some mild conditions containing the Open Set Condition (OSC) which is defined as follows. We say that a sequence $(\{\phi_i^{(j)}\}_{i\in I^{(j)}})_{j=1}^{\infty}$ of finite collections of maps on a compact subset X with $\mathrm{int}(X) \neq \emptyset$ satisfies the OSC if for all $j \in \mathbb{N}$ and all distinct indices $a, b \in I^{(j)}$,

$$\phi_a^{(j)}(\mathrm{int}(X)) \cap \phi_b^{(j)}(\mathrm{int}(X)) = \emptyset. \tag{1}$$

Then the limit set of the NCIFS $\Phi = (\{\phi_i^{(j)}\}_{i\in I^{(j)}})_{j=1}^{\infty}$ is defined as the set of possible limit points of sequences $\phi_{\omega_1}^{(1)}(\phi_{\omega_2}^{(2)}...(\phi_{\omega_i}^{(i)}(x))...))$, $\omega_j \in I^{(j)}$ for all $j \in \{1, 2, ..., i\}$, $x \in X$. Rempe-Gillen and Urbański introduced *the lower pressure function* $\underline{P}_\Phi : [0,\infty) \to [-\infty,\infty]$ of the NCIFS Φ. Then *the Bowen dimension* s_Φ of the NCIFS Φ is defined by $s_\Phi = \sup\{s \geq 0 \ : \ \underline{P}_\Phi(s) > 0\} = \inf\{s \geq 0 \ : \ \underline{P}_\Phi(s) < 0\}$. Rempe-Gillen and Urbański proved that the Hausdorff dimension of the limit set is the Bowen dimension of the NCIFS ([9, 1.1 Theorem]). For related results for non-autonomous systems, see [2].

In this paper, we study NIFSs with overlaps on $\mathbb{R}^m$. Here, we do not assume the OSC. We introduce transversal families of non-autonomous conformal iterated function systems on $\mathbb{R}^m$. We show that if a $d-$parameter family of such systems satisfies the transversality condition, then for almost every parameter value the Hausdorff dimension of the limit set is the minimum of m and the Bowen dimension. Moreover, we give an example of a family $\{\Phi_t\}_{t\in U}$ of parameterized NIFSs such that $\{\Phi_t\}_{t\in U}$ satisfies the transversality condition but Φ_t does not satisfy the OSC for any $t \in U$. The method of transversality is utilized for parametrized IFSs involving some complicated overaps (e.g., [8], [11], [4], [5], [10]). For some general family of functions with the transversality condition, see [10], [6], [13].

2. Main result

In this section we present the framework of transversal families of non-autonomous conformal iterated function systems and we present the main results on them. For each $j \in \mathbb{N}$, let $I^{(j)}$ be a finite set. For any $n, k \in \mathbb{N}$ with $n \leq k$, we set

$$I_n^k := \prod_{j=n}^{k} I^{(j)}, I_n^\infty := \prod_{j=n}^{\infty} I^{(j)}, I^n := \prod_{j=1}^{n} I^{(j)}, \text{and } I^\infty := \prod_{j=1}^{\infty} I^{(j)}.$$

Let $U \subset \mathbb{R}^d$. For any $t \in U$, let $\Phi_t = (\Phi_t^{(j)})_{j=1}^\infty$ be a sequence of collections of maps on a set $X \subset \mathbb{R}^m$, where

$$\Phi_t^{(j)} = \{\phi_{i,t}^{(j)} : X \to X\}_{i \in I^{(j)}}.$$

Let $n, k \in \mathbb{N}$ with $n \leq k$. For any $\omega = \omega_n \omega_{n+1} \cdots \omega_k \in I_n^k$, we set

$$\phi_{\omega,t} := \phi_{\omega_n,t}^{(n)} \circ \cdots \circ \phi_{\omega_k,t}^{(k)}.$$

Let $n \in \mathbb{N}$. For any $\omega = \omega_n \omega_{n+1} \cdots \in I_n^\infty$ and any $j \in \mathbb{N}$, we set

$$\omega|_j := \omega_n \omega_{n+1} \cdots \omega_{n+j-1} \in I_n^{n+j-1}.$$

Let $V \subset \mathbb{R}^m$ be an open set and let $\phi : V \to \phi(V)$ be a diffeomorphism. We denote by $D\phi(x)$ the derivative of ϕ evaluated at x. We say that ϕ is *conformal* if for any $x \in V$ $D\phi(x) : \mathbb{R}^m \to \mathbb{R}^m$ is a similarity linear map, that is, $D\phi(x) = c_x \cdot A_x$, where $c_x > 0$ and A_x is an orthogonal matrix. For any conformal map $\phi : V \to \phi(V)$, we denote by $|D\phi(x)|$ its scaling factor at x, that is, if we set $D\phi(x) = c_x \cdot A_x$ we have $|D\phi(x)| = c_x$. For any set $A \subset V$, we set

$$||D\phi||_A := \sup\{|D\phi(x)| \ : \ x \in A\}.$$

We denote by $\mathcal{L}_d$ the d-dimensional Lebesgue measure on $\mathbb{R}^d$. We introduce *the transversal family of non-autonomous conformal iterated function systems* by employing the settings in [9] and [10].

Definition 2.1 (Transversal family of non-autonomous conformal iterated function systems)**.** Let $m \in \mathbb{N}$ and let $X \subset \mathbb{R}^m$ be a non-empty compact convex set. Let $d \in \mathbb{N}$ and let $U \subset \mathbb{R}^d$ be an open set. For each $j \in \mathbb{N}$, let $I^{(j)}$ be a finite set. Let $t \in U$. For any $j \in \mathbb{N}$, let $\Phi_t^{(j)}$ be a collection $\{\phi_{i,t}^{(j)} : X \to X\}_{i \in I^{(j)}}$ of maps $\phi_{i,t}^{(j)}$ on X. Let $\Phi_t = (\Phi_t^{(j)})_{j=1}^\infty$. We say that $\{\Phi_t\}_{t \in U}$ is a Transversal family of Non-autonomous Conformal Iterated Function Systems (TNCIFS) if $\{\Phi_t\}_{t \in U}$ satisfies the following six conditions.

1. *Conformality* : There exists an open connected set $V \supset X$(independent of i, j and t) such that for any i, j and $t \in U$, $\phi_{i,t}^{(j)}$ extends to a C^1 conformal map on V such that $\phi_{i,t}^{(j)}(V) \subset V$.
2. *Uniform contraction* : There is a constant $0 < \gamma < 1$ such that for any $t \in U$, any $n \in \mathbb{N}$, any $\omega \in I_n^\infty$ and any $j \in \mathbb{N}$,
$$|D\phi_{\omega|_j,t}(x)| \leq \gamma^j$$
for any $x \in X$.
3. *Bounded distortion* : There exists a Borel measurable locally bounded function $K : U \to [1, \infty)$ such that for any $t \in U$, any $n \in \mathbb{N}$, any $\omega \in I_n^\infty$ and any $j \in \mathbb{N}$,
$$|D\phi_{\omega|_j,t}(x_1)| \leq K(t)|D\phi_{\omega|_j,t}(x_2)| \tag{2}$$
for any $x_1, x_2 \in V$.

4. *Distortion continuity* : For any $\eta > 0$ and $t_0 \in U$, there exists $\delta = \delta(\eta, t_0) > 0$ such that for any $t \in U$ with $|t - t_0| \leq \delta$, for any $n, j \in \mathbb{N}$ and for any $\omega \in I_n^\infty$,

$$\exp(-j\eta) \leq \frac{||D\phi_{\omega|_j,t_0}||_X}{||D\phi_{\omega|_j,t}||_X} \leq \exp(j\eta). \tag{3}$$

We define the *address map* as follows. Let $t \in U$. For all $n \in \mathbb{N}$ and all $\omega \in I_n^\infty$,

$$\bigcap_{j=1}^{\infty} \phi_{\omega|_j,t}(X)$$

is a singleton by the uniform contraction property. It is denoted by $\{y_{\omega,n,t}\}$. The map

$$\pi_{n,t} \colon I_n^\infty \to X$$

is defined by $\omega \mapsto y_{\omega,n,t}$. Then $\pi_{n,t}$ is called *the n-th address map corresponding to t*. Note that for any $t \in U$ and $n \in \mathbb{N}$ the map $\pi_{n,t}$ is continuous with respect to the product topology on I_n^∞.

5. *Continuity* : Let $n \in \mathbb{N}$. The function $I_n^\infty \times U \ni (\omega, t) \mapsto \pi_{n,t}(\omega)$ is continuous.
6. *Transversality condition* : For any compact subset $G \subset U$ there exists a sequence of positive constants $\{C_n\}_{n=1}^\infty$ with

$$\lim_{n\to\infty} \frac{\log C_n}{n} = 0$$

such that for all $\omega, \tau \in I_n^\infty$ with $\omega_n \neq \tau_n$ and for all $r > 0$,

$$\mathcal{L}_d\left(\{t \in G \ : \ |\pi_{n,t}(\omega) - \pi_{n,t}(\tau)| \leq r\}\right) \leq C_n r^m.$$

Remark 2.2. If $m \geq 2$, the Conformality condition implies the Bounded distortion condition. For the details, see [9, page. 1984 Remark].

Remark 2.3. Let $n \in \mathbb{N}$ and let $t \in U$. Then for any $\omega \in I_n^\infty$,

$$\pi_{n,t}(\omega) = \lim_{j\to\infty} \phi_{\omega|_j,t}(x),$$

where $x \in X$.

Remark 2.4. In the case of usual IFSs, the constants C_n in the transversality condition are independent of n since the n-th address maps $\pi_{n,t}$ are independent of n.

Let $\{\Phi_t\}_{t\in U}$ be a TNCIFS. For any $n \in \mathbb{N}$ and $t \in U$, *the n-th limit set $J_{n,t}$* of Φ_t is defined by

$$J_{n,t} := \pi_{n,t}(I_n^\infty).$$

For any $t \in U$, we define the lower pressure function $\underline{P}_t : [0, \infty) \to [-\infty, \infty]$ of Φ_t as the following. For any $s \geq 0$ and $n \in \mathbb{N}$, we set

$$Z_{n,t}(s) := \sum_{\omega\in I^n} (||D\phi_{\omega,t}||_X)^s,$$

and

$$\underline{P}_t(s) := \liminf_{n\to\infty} \frac{1}{n} \log Z_{n,t}(s) \in [-\infty, \infty].$$

By [9, Lemma 2.6], the lower pressure function has the following monotonicity. If $s_1 < s_2$, then either both $\underline{P}_t(s_1)$ and $\underline{P}_t(s_2)$ are equal to ∞, both are equal to $-\infty$, or $\underline{P}_t(s_1) > \underline{P}_t(s_2)$. Then for any $t \in U$, we set

$$s(t) := \sup\{s \geq 0 \ : \ \underline{P}_t(s) > 0\} = \inf\{s \geq 0 \ : \ \underline{P}_t(s) < 0\},$$

where we set $\sup \emptyset = 0$ and $\inf \emptyset = \infty$. The value $s(t)$ is called the Bowen dimension of Φ_t. We set $J_t := J_{1,t}$ for any $t \in U$. We now give the main result of this paper.

Main Theorem. Let $\{\Phi_t\}_{t \in U}$ be a TNCIFS. Suppose that the function $t \mapsto s(t)$ is a real-valued and continuous function on U. Then

$$\dim_H(J_t) = \min\{m, s(t)\}$$

for $\mathcal{L}_d$-a.e. $t \in U$.

Main Theorem is a generalization of [10, Theorem 3.1 (i)].

3. Example

In this section, we give an example of a family $\{\Phi_t\}_{t \in U}$ of parameterized NCIFSs such that $\{\Phi_t\}_{t \in U}$ satisfies the transversality condition but Φ_t does not satisfy the open set condition for any $t \in U$. We set $\mathbb{D} := \{z \in \mathbb{C} \ : \ |z| < 1\}$. For any holomorphic function f on $\mathbb{D}$, we denote by $f'(z)$ the complex derivative of f evaluated at $z \in \mathbb{D}$. For the transversality condition, we now give a slight variation of [11, Lemma 5.2]. For the reader's convenience we include the proof in Appendix.

Lemma 3.1. *Let $\mathcal{H}$ be a compact subset of the space of holomorphic functions on $\mathbb{D}$ endowed with the compact open topology. We set*

$$\tilde{\mathcal{M}}_H := \{\lambda \in \mathbb{D} \ : \textit{there exists } f \in \mathcal{H} \text{ such that } f(\lambda) = f'(\lambda) = 0\}.$$

Let G be a compact subset of $\mathbb{D} \backslash \tilde{\mathcal{M}}_H$. Then there exists $K = K(\mathcal{H}, G) > 0$ such that for any $f \in \mathcal{H}$ and any $r > 0$,

$$\mathcal{L}_2\left(\{\lambda \in G : |f(\lambda)| \leq r\}\right) \leq Kr^2. \tag{4}$$

We now give a family $\{\Phi_t\}_{t \in U}$ of parametrized systems such that $\{\Phi_t\}_{t \in U}$ is a TNCIFS but Φ_t does not satisfy the open set condition (1) for any $t \in U$. In order to do that, we set

$$U := \{t \in \mathbb{C} \ : \ |t| < 2 \times 5^{-5/8}, \ t \notin \mathbb{R}\}.$$

Note that $2 \times 5^{-5/8} \approx 0.73143 > 1/\sqrt{2}$. Let $t \in U$. For each $j \in \mathbb{N}$, we define

$$\Phi_t^{(j)} = \{z \mapsto \phi_{1,t}^{(j)}(z), z \mapsto \phi_{2,t}^{(j)}(z)\} := \left\{z \mapsto tz, z \mapsto tz + \frac{1}{j}\right\}.$$

Proposition 3.2. *For any $t \in U$, the system $(\Phi_t^{(j)})_{j=1}^{\infty}$ does not satisfy the open set condition.*

Proof. Suppose that the system $(\Phi_t^{(j)})_{j=1}^{\infty}$ satisfies the open set condition (1). Then there exists a compact subset $X \subset \mathbb{C}$ with $\mathrm{int}(X) \neq \emptyset$ such that $\phi_{1,t}^{(j)}(\mathrm{int}(X)) \cap \phi_{2,t}^{(j)}(\mathrm{int}(X)) = \emptyset$. Hence there exist $x \in X$ and $r > 0$ such that

$$\phi_{1,t}^{(j)}(B(x,r)) \cap \phi_{2,t}^{(j)}(B(x,r)) = B(tx, |t|r) \cap B(tx + 1/j, |t|r) = \emptyset.$$

In particular, we have for all $j \in \mathbb{N}$,

$$2|t|r < \frac{1}{j}.$$

This is a contradiction. □

We set

$$X := \left\{ z \in \mathbb{C} \ : \ |z| \le \frac{1}{1 - 2 \times 5^{-5/8}} \right\}.$$

Then we have that for any $t \in U$, for any $j \in \mathbb{N}$ and for any $i \in I^{(j)} := \{1, 2\}$, $\phi_{i,t}^{(j)}(X) \subset X$. We set $b_1^{(j)} = 0$ and $b_2^{(j)} = 1/j$ for each j. Let $n, j \in \mathbb{N}$. We give the following lemma.

Lemma 3.3. *Let $t \in U$. For any $\omega = \omega_n \cdots \omega_{n+j-1} \in I_n^{n+j-1}$ and any $z \in X$ we have*

$$\phi_{\omega,t}(z) = \phi_{\omega_n,t}^{(n)} \circ \cdots \circ \phi_{\omega_{n+j-1},t}^{(n+j-1)}(z) = t^j z + \sum_{i=1}^{j} b_{\omega_{n+i-1}}^{(n+i-1)} t^{i-1},$$

where $b_{\omega_{n+i-1}}^{(n+i-1)} \in \{0, \frac{1}{n+i-1}\}$. In particular, for any $\omega = \omega_n \cdots \omega_{n+j-1} \cdots \in I_n^{\infty}$,

$$\pi_{n,t}(\omega) = \sum_{i=1}^{\infty} b_{\omega_{n+i-1}}^{(n+i-1)} t^{i-1}.$$

Proof. This can be shown by induction on j. □

We can show that the family $\{\Phi_t\}_{t \in U}$ of systems is a TNCIFS as follows.

1. *Conformality* : Let $t \in U$. For any $j \in \mathbb{N}$ and any $i \in I^{(j)}$, $\phi_{i,t}^{(j)}(z) = tz + b_i^{(j)}$ is a similarity map on $\mathbb{C}$.
2. *Uniform Contraction* : We set $\gamma = 2 \times 5^{-5/8}$. Then for any $\omega \in I_n^{n+j-1}$ and $z \in X$,
$$|D\phi_{\omega,t}(z)| = |t|^j \le \gamma^j$$
by Lemma 3.3.
3. *Bounded distortion* : By Lemma 3.3, for any $\omega = \omega_n \cdots \omega_{n+j-1} \in I_n^{n+j-1}$ and $z \in \mathbb{C}$, $|D\phi_{\omega,t}(z)| = |t|^j$. We define the Borel measurable locally bounded function $K : U \to [1, \infty)$ by $K(t) = 1$. Then for any $\omega \in I_n^{n+j-1}$,
$$|D\phi_{\omega,t}(z_1)| \le K(t) |D\phi_{\omega,t}(z_2)|$$
for all $z_1, z_2 \in \mathbb{C}$.
4. *Distortion continuity* : Fix $t_0 \in U$. Since the map $t \mapsto \log |t|$ is continuous at $t_0 \in U$, for any $\eta > 0$ there exists $\delta = \delta(\eta, t_0) > 0$ such that for any $t \in U$ with $|t_0 - t| < \delta$,
$$|\log |t_0| - \log |t|| < \eta.$$
Hence we have
$$|\log |t_0|^j / |t|^j| < j\eta,$$
which implies that for any $\omega \in I_n^{n+j-1}$,
$$\exp(-j\epsilon) < \frac{||D\phi_{\omega,t_0}||}{||D\phi_{\omega,t}||} = \exp(\log |t_0|^j / |t|^j) < \exp(j\epsilon).$$
5. *Continuity* : By Lemma 3.3, we have for any $t \in U$ and any $\omega \in I_n^{\infty}$,
$$\pi_{n,t}(\omega) = \sum_{i=1}^{\infty} b_{\omega_{n+i-1}}^{(n+i-1)} t^{i-1}.$$
Hence the map $(\omega, t) \mapsto \pi_{n,t}(\omega)$ is continuous on $I_n^{\infty} \times U$.

6. *Transversality condition* : We introduce a set $\mathcal{G}$ of holomorphic functions on $\mathbb{D}$ and the set $\tilde{\mathcal{O}}_2$ of double zeros in $\mathbb{D}$ for functions which belong to $\mathcal{G}$.

$$\mathcal{G} := \left\{ f(t) = \pm 1 + \sum_{j=1}^{\infty} a_j t^j : a_j \in [-1,1] \right\},$$

$$\tilde{\mathcal{O}}_2 := \{t \in \mathbb{D} \ : \text{there exists } f \in \mathcal{G} \text{ such that } f(t) = f'(t) = 0\}.$$

Note that $\mathcal{G}$ is a compact subset of the space of holomorphic functions on $\mathbb{D}$ endowed with the compact open topology. Let $n \in \mathbb{N}$. Then we have for any $t \in U$ and any $\omega, \tau \in I_n^\infty$ with $\omega_n \neq \tau_n$,

$$\begin{aligned}
\pi_{n,t}(\omega) - \pi_{n,t}(\tau) &= \sum_{i=1}^{\infty} b_{\omega_{n+i-1}}^{(n+i-1)} t^{i-1} - \sum_{i=1}^{\infty} b_{\tau_{n+i-1}}^{(n+i-1)} t^{i-1} \\
&= b_{\omega_n}^{(n)} - b_{\tau_n}^{(n)} + \sum_{i=2}^{\infty} \left(b_{\omega_{n+i-1}}^{(n+i-1)} - b_{\tau_{n+i-1}}^{(n+i-1)} \right) t^{i-1} \\
&= \frac{1}{n} \left(\pm 1 + \sum_{i=2}^{\infty} n \left(b_{\omega_{n+i-1}}^{(n+i-1)} - b_{\tau_{n+i-1}}^{(n+i-1)} \right) t^{i-1} \right).
\end{aligned}$$

Then the function $t \mapsto \pm 1 + \sum_{i=2}^{\infty} n(b_{\omega_{n+i-1}}^{(n+i-1)} - b_{\tau_{n+i-1}}^{(n+i-1)}) t^{i-1}$ is a holomorphic function which belongs to $\mathcal{G}$. Let $G \subset \mathbb{D} \backslash \tilde{\mathcal{O}}_2$ be a compact subset. By Lemma 3.1, there exists $K = K(\mathcal{G}, G) > 0$ such that for any $\omega, \tau \in I_n^\infty$ with $\omega_n \neq \tau_n$ and any $r > 0$,

$$\begin{aligned}
&\mathcal{L}_2(\{t \in G \ : \ |\pi_{n,t}(\omega) - \pi_{n,t}(\tau)| \leq r\}) \\
&= \mathcal{L}_2(\{t \in G \ : \ |\pm 1 + \sum_{i=2}^{\infty} n(b_{\omega_{n+i-1}}^{(n+i-1)} - b_{\tau_{n+i-1}}^{(n+i-1)}) t^{i-1}| \leq nr\}) \\
&\leq K(nr)^2.
\end{aligned}$$

If we set $C_n := Kn^2$ for any $n \in \mathbb{N}$, we have

$$\mathcal{L}_2(\{t \in G \ : \ |\pi_{n,t}(\omega) - \pi_{n,t}(\tau)| \leq r\}) \leq C_n r^2$$

and

$$\frac{1}{n} \log C_n = \frac{1}{n} \log K + \frac{2}{n} \log n \to 0$$

as $n \to \infty$.

Finally, we use the following theorem.

Theorem 3.4. [12, Proposition 2.7] *A power series of the form* $1 + \sum_{j=1}^{\infty} a_j z^j$, *with* $a_j \in [-1,1]$, *cannot have a non-real double zero of modulus less than* $2 \times 5^{-5/8}$.

By using the above theorem, we have that $U = \{t \in \mathbb{C} \ : \ |t| < 2 \times 5^{-5/8}, \ t \notin \mathbb{R}\} \subset \mathbb{D} \backslash \tilde{\mathcal{O}}$. Hence the family $\{\Phi_t\}_{t \in U}$ satisfies the transversality condition.

By the above arguments, we get the following.

Proposition 3.5. *The family* $\{\Phi_t\}_{t \in U}$ *of parametrized systems is a TNCIFS.*

We calculate the lower pressure function $\underline{P}_t$ for Φ_t, $t \in U$ as the following. For any $s \in [0, \infty)$,

$$\begin{aligned}\underline{P}_t(s) &= \liminf_{n\to\infty} \frac{1}{n} \log \sum_{\omega\in I^n} ||D\phi_{\omega,t}||^s \\ &= \liminf_{n\to\infty} \frac{1}{n} \log \sum_{\omega\in I^n} |t|^{ns} \\ &= \liminf_{n\to\infty} \frac{1}{n} \log(2^n|t|^{ns}) \\ &= \log 2 + s\log|t|.\end{aligned}$$

Hence for each $t \in U$, $\underline{P}_t(s)$ has the zero

$$s(t) = \frac{\log 2}{-\log|t|}$$

and the function $t \mapsto s(t)$ is continuous on U. Let J_t be the (1st) limit set corresponding to t. Then by Main Theorem, we have

$$\dim_H(J_t) = \min\{2, s(t)\} = s(t)$$

for a.e. $t \in \{t \in \mathbb{C} \;:\; |t| \leq 1/\sqrt{2}, t \notin \mathbb{R}\}$ and

$$\dim_H(J_t) = \min\{2, s(t)\} = 2$$

for a.e. $t \in \{t \in \mathbb{C} \;:\; 1/\sqrt{2} \leq |t| < 2 \times 5^{-5/8}, t \notin \mathbb{R}\}$.

Appendix

In order to prove Lemma 3.1, we give some definition and remark.

Definition 3.6. Let G be a compact subset of $\mathbb{R}^d$. We say that a family of balls $\{B(x_i, r_i)\}_{i=1}^k$ in $\mathbb{R}^d$ is *packing for* G if for each $i \in \{1, ..., k\}$, $x_i \in G$ and for each $i, j \in \{1, ..., k\}$ with $i \neq j$, $B(x_i, r_i) \cap B(x_j, r_j) = \emptyset$.

Remark 3.7. Let G be a compact subset of $\mathbb{R}^d$, let $r > 0$ and let $\{B(x_i, r)\}_{i=1}^k$ be a family of balls in $\mathbb{R}^d$. If $\{B(x_i, r)\}_{i=1}^k$ is packing for G, then there exists $N \in \mathbb{N}$ which depends only on G and r such that $k \leq N$.

Proof. There exists a finite covering $\{B(y_j, r/2)\}_{j=1}^N$ for G since G is compact. Here, N depends only on G and r. Since $x_i \in G$ for each i, there exists j_i such that $x_i \in B(y_{j_i}, r/2)$. Since $\{B(x_i, r)\}_{i=1}^k$ is a disjoint family, if $i \neq l \in \{1, ..., k\}$, then $j_i \neq j_l$. Thus $k \leq N$. □

We give a proof of Lemma 3.1.

(*proof of Lemma* 3.1)**.** Since $\mathcal{H}$ is compact and the set $\tilde{\mathcal{M}}_H$ is the set of possible double zeros, we have that there exists $\delta = \delta_{\mathcal{H},G} > 0$ such that for any $f \in \mathcal{H}$,

$$|f(\lambda)| < \delta \Rightarrow |f'(\lambda)| > \delta \;\text{ for } \lambda \in G. \tag{5}$$

We assume that $r < \delta$, otherwise (4) holds with $K = \mathcal{L}_2(G)/\delta^2$. Let

$$\Delta_r := \{\lambda \in G : |f(\lambda)| \leq r\}.$$

Let Co(G) be the convex hull of G. We set $M = M_G := \sup\{|g''(\lambda)| \in [0,\infty) : \lambda \in \mathrm{Co}(G), g \in \mathcal{H}\}$. Since Co$(G)$ is compact and $\mathcal{H}$ is compact, $M < \infty$. Fix $z_0 \in \Delta_r$. By Taylor's formula, for $z \in G$,

$$|f(z) - f(z_0)| = |f'(z_0)(z - z_0) + \int_{z_0}^{z} (z - \xi) f''(\xi) d\xi|,$$

where the integration is performed along the straight line path from z_0 to z. Then $|f'(z_0)| > \delta$ by (5). Hence

$$|f(z) - f(z_0)| \geq |f'(z_0)||z - z_0| - M|z - z_0|^2 > \delta|z - z_0| - M|z - z_0|^2.$$

Now if we set

$$A_{z_0,r} := \left\{ z \in \mathbb{D}^* \ : \ \frac{4r}{\delta} < |z - z_0| < \frac{\delta}{2M} \right\},$$

then for any $z \in A_{z_0,r}$,

$$\delta|z - z_0| - M|z - z_0|^2 = |z - z_0|(\delta - M|z - z_0|) > \frac{4r}{\delta}\frac{\delta}{2} = 2r,$$

and $|f(z)| \geq |f(z) - f(z_0)| - |f(z_0)| > r$. It follows that the annulus $A_{z_0,r}$ does not intersect Δ_r.

Assume that $4r/\delta \leq \delta/4M$, otherwise (4) holds with $K = \mathcal{L}_2(G)(16M/\delta^2)^2$. Then the disc $B(z_0, \delta/4M)$ centered at z_0 with the radius $\delta/4M$ covers $\Delta_r \cap \{z : |z - z_0| < \delta/2M\}$. Then fix $z_1 \in \Delta_r \backslash \{z : |z - z_0| < \delta/2M\}$. Since the annulus $A_{z_1,r}$ does not intersect Δ_r, $B(z_1, \delta/4M)$ covers $(\Delta_r \backslash \{z : |z - z_0| < \delta/2M\}) \cap \{z : |z - z_1| < \delta/2M\}$ and $B(z_0, \delta/4M) \cap B(z_1, \delta/4M) = \emptyset$. If we repeat the procedure, we get a finite covering $\{B(z_i, \delta/4M)\}_{i=0}^{k}$ for Δ_r since Δ_r is compact. Then $\{B(z_i, \delta/4M)\}_{i=0}^{k}$ is packing for G. By Remark 3.7, there exists $N \in \mathbb{N}$ which depends only on $\mathcal{H}$ and G such that $k \leq N$. Since the annulus $A_{z_i,r}$ does not intersect Δ_r for each $i \in \{0, ..., k\}$, $\{B(z_i, 4r/\delta)\}_{i=0}^{k}$ is also a covering for Δ_r. Hence we have

$$\begin{aligned} \mathcal{L}_2(\Delta_r) &\leq \mathcal{L}_2(\bigcup_{i=0}^{k} \{B(z_i, 4r/\delta)\}) \\ &= \sum_{i=0}^{k} \mathcal{L}_2(\{B(z_i, 4r/\delta)\}) \\ &\leq NC(\frac{4r}{\delta})^2 = NC(\frac{4}{\delta})^2 r^2, \end{aligned}$$

where the constant C does not depend on r. If we set $K := NC(4/\delta)^2$, we get the desired inequality.

Acknowledgement. This study is supported by JSPS KAKENHI Grant Number JP 19J21038. This work was supported by the Research Institute for Mathematical Sciences,an International Joint Usage/Research Center located in Kyoto University.

References

[1] K. Falconer, *Fractal geometry-Mathematical foundations and applications(Third edition)*, WILEY, 2014.

[2] M. Holland and Y. Zhang, *Dimension results for inhomogeneous Moran set constructions*, Dyn. Syst. 28(2) (2013), 222-250.

[3] P. Mattila, *Geometry of sets and measures in Euclidean spaces. Fractals and rectifiability,* Cambridge Studies in Advanced Mathematics, 44. Cambridge University Press, Cambridge, 1995.

[4] T. Jordan, *Dimension of fat Sierpiński gaskets*, Real Anal. Exchange, 31(1) (2005/06), 97-110.

[5] T. Jordan and M. Pollicott, *Properties of measures supported on fat Sierpiński carpets*, Ergodic Theory Dynam. Systems, 26(3) (2006), 739-754.

[6] E. Mihailescu and M. Urbański, *Transversal families of hyperbolic skew- products*, Discrete and Continuous Dynamical Systems A, 2008, 21 (3): 907-928.

[7] R. D. Mauldin and M. Urbański, *Graph directed Markov systems*, Geometry and dynamics of limit sets, Cambridge Tracts in Mathematics, vol. 148, Cambridge University Press, Cambridge, 2003.

[8] M. Pollicott and K. Simon, *The Hausdorff dimension of λ-expansions with deleted digits*, Trans. Amer. Math. Soc., 347, no. 3 (1995), 967-983.

[9] L. Rempe-Gillen and M. Urbański, *Non-autonomous conformal iterated function systems and Moran-set constructions*, Trans. Amer. Math. Soc., 368(3) (2016), 1979-2017.

[10] K. Simon, B. Solomyak and M. Urbański, *Hausdorff dimension of limit sets for parabolic IFS with overlaps*, Pacific J. Math. 201 (2001), 441-478.

[11] B. Solomyak, *Measure and dimension for some fractal families,* Math. Proc. Camb. Phil. Soc. 124, 531-46 (1998).

[12] B. Solomyak and H. Xu, *On the 'Mandelbrot set' for a pair of linear maps and complex Bernoulli convolutions*, Nonlinearity 16 , no. 5, 1733-1749 (2003).

[13] H. Sumi and M. Urbański, *Transversality family of expanding rational semigroups*, Adv. Math. 234 (2013) 697-734.

Yuto Nakajima
Course of Mathematical Science, Department of Human Coexistence,
Graduate School of Human and Environmental Studies, Kyoto University
Yoshida Nihonmatsu-cho, Sakyo-ku, Kyoto, 606-8501, Japan
E-mail: nakajima.yuuto.32n@st.kyoto-u.ac.jp