Random Dynamical Systems of Regular Polynomial Maps on $\mathbb{C}^2$

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Abstract

We introduce the notion of mean stability in i.i.d. random (holomorphic) 2-dimensional dynamical systems. We can see that a **generic** random dynamical system of regular polynomial maps on $\mathbb{P}^2$ (the complex 2-dimensional projective space) having an attractor in the line at infinity, is **mean stable**. If a random holomorphic dynamical system on $\mathbb{P}^2$ is mean stable then for each $z \in \mathbb{P}^2$, for a.e. orbit starting with z, **the Lyapunov exponent is negative**. If a random holomorphic dynamical system on $\mathbb{P}^2$ is mean stable, then for any $z \in \mathbb{P}^2$, the orbit of the Dirac measure at z under the iterations of the dual map of the transition operator **converges to a periodic cycle** of probability measures. Note that the above statements **cannot hold for deterministic dynamics of a single regular polynomial map f with** $\deg(f) \geq 2$.

We see many **randomness-induced phenomena** (phenomena in random dynamical systems which cannot hold for iteration dynamics of single maps). In this talk, we have seen **randomness-induced order**.

Motivation.

- Nature has a lot of random (noise) terms. Thus it is natural and important to consider **random dynamical systems**.

- **Holomorphic dynamical systems** have been deeply investigated. The study of them helps us to investigate real dynamical systems.

- Combining the above two ideas,
  we consider **random holomorphic dynamical systems**.

- We want to find new phenomena (so called **randomness-induced phenomena**) in random dynamical systems which **cannot hold in deterministic iteration dynamical systems of single maps**.

- Other motivations: **Random relaxed Newton's method** (in which we can find roots of polynomials **more easily than the deterministic methods**, see S, [S21]). **The action of holomorphic automorphisms** on complex manifolds. **The action of mapping class groups** of the Riemann surfaces on the character varieties, etc.

Definition 1.

(1) Let $\mathbb{C}^2$ be the 2-dimensional complex Euclidean space.
Let $f : \mathbb{C}^2 \to \mathbb{C}^2$ be a polynomial map, i.e., if we write $f(x, y) = (g(x, y), h(x, y))$, then $g(x, y)$ and $h(x, y)$ are polynomials of (x, y).

We say that f is a regular polynomial map on $\mathbb{C}^2$ if f extends to a holomorphic map on $\mathbb{P}^2$ (the complex 2-dimensional projective space), i.e.,

$$\mathbb{P}^2 = \{[u : v : w] \mid (u, v, w) \in \mathbb{C}^3 \setminus \{(0, 0, 0)\}\}.$$

Note that we regard $\mathbb{C}^2$ as a subset of $\mathbb{P}^2$ via the following canonical identification and inclusion:

$$\mathbb{C}^2 \cong \{[u : v : 1] \in \mathbb{P}^2 \mid (u, v) \in \mathbb{C}^2\} \subset \mathbb{P}^2.$$

Remark: Let $f : \mathbb{C}^2 \to \mathbb{C}^2$ be a polynomial map.
Then f is a regular polynomial map if and only if the following $(*)$ holds.

$(*)$ Let $f(x,y) = (g(x,y), h(x,y))$. Let $g_1(x,y)$ be the highest degree term of $g(x,y)$ and let $h_1(x,y)$ be the highest degree term of $h(x,y)$. Then $\deg(g_1) = \deg(h_1)$ and

$$g_1(x,y) = h_1(x,y) = 0 \Leftrightarrow (x,y) = (0,0).$$

Example: Let $f(x,y) =$
$(a_1x^2 + a_2xy + a_3y^2 + b_1x + b_2y + b_3,\ c_1y^2 + c_2x + c_3y + c_4)$,
where $a_1, c_1 \in \mathbb{C} \setminus \{0\}, a_2, a_3, b_1, b_2, b_3, c_2, c_3, c_4 \in \mathbb{C}$.
Then f is a regular polynomial map on $\mathbb{C}^2$.

(3) If f is a regular polynomial map on $\mathbb{C}^2$,
then we regard f as a holomorphic map on $\mathbb{P}^2$.
We call such a holomorphic map f on $\mathbb{P}^2$
" a regular polynomial map on $\mathbb{P}^2$".

(4) Let X be the space of all regular polynomial maps on $\mathbb{P}^2$ of degree two or more, endowed with the distance η which is defined as $\eta(f,g) = \sup_{z\in\mathbb{P}^2} d(f(z), g(z))$, where d denotes the distance on $\mathbb{P}^2$ induced by the Fubini-Study metric on $\mathbb{P}^2$.

(5) We denote by $\mathfrak{M}_1(X)$ the space of all Borel probabiliy measures on X.
Also, we set $\mathfrak{M}_{1,c}(X) := \{\tau \in \mathfrak{M}_1(X) \mid \operatorname{supp}\tau \text{ is a compact subset of } X\}$.
We endow $\mathfrak{M}_{1,c}(X)$ with a topology $\mathcal{O}$ which satisfies that
$\tau_n \to \tau$ as $n \to \infty$ if and only if

(a) for each bounded continuous function $\varphi : X \to \mathbb{C}$, we have $\int \varphi\, d\tau_n \to \int \varphi\, d\tau$ as $n \to \infty$, and

(b) $\operatorname{supp}\tau_n \to \operatorname{supp}\tau$ as $n \to \infty$ with respect to the Hausdorff metric in the space of all non-empty compact subsets of X.

For each $\tau \in \mathfrak{M}_{1,c}(X)$, we consider i.i.d. random dynamical system on $\mathbb{P}^2$ such that at every step we choose a map $f \in X$ according to τ. This defines a Markov process whose state space is $\mathbb{P}^2$ and whose transition probability $p(z, A)$ from a point $z \in \mathbb{P}^2$ to a Borel subset A of $\mathbb{P}^2$ satisfies $p(z, A) = \tau(\{h \in X \mid h(z) \in A\})$.

(6) For $\forall \tau \in \mathfrak{M}_{1,c}(X)$, let $G_\tau := \{\gamma_n \circ \cdots \circ \gamma_1 \mid n \in \mathbb{N}, \gamma_j \in \operatorname{supp} \tau (\forall j)\}$. This is a **semigroup** with the semigroup operation being the functional composition. (It is important to study the dynamics of G_τ.)

(7) We say that an element $\tau \in \mathfrak{M}_{1,c}(X)$ is mean stable if there exist an $n \in \mathbb{N}$, an $m \in \mathbb{N}$, non-empty open subsets $U_1, \ldots, U_m$ of $\mathbb{P}^2$, a non-empty compact subset K of $\cup_{j=1}^m U_j$, and a constant c with $0 < c < 1$ such that the following (a) and (b) hold.

(a) For each $(\gamma_1, \ldots, \gamma_n) \in (\operatorname{supp} \tau)^n$, we have

$$\gamma_n \circ \cdots \circ \gamma_1(\cup_{j=1}^m U_j) \subset K.$$

Moreover, for each $j = 1, \ldots, m$, for all $x, y \in U_j$ and for each $(\gamma_1, \ldots, \gamma_n) \in (\operatorname{supp} \tau)^n$, we have

$$d(\gamma_n \circ \cdots \circ \gamma_1(x), \gamma_n \circ \cdots \circ \gamma_1(y)) \leq c d(x, y).$$

(b) For each $z \in \mathbb{P}^2$, there exists an element $h_z \in G_\tau$ such that $h_z(z) \in U$.

Remark 2. Let $\mathcal{MS} := \{\tau \in \mathfrak{M}_{1,c}(X) \mid \tau \text{ is mean stable}\}$.
Then $\mathcal{MS}$ is **non-empty and open** in $(\mathfrak{M}_{1,c}(X), \mathcal{O})$.

Example. Let $f_1, f_2 \in X$ be elements defined by

$$f_1(x, y) = (x^2, y^2), \;\; f_2(x, y) = (\frac{1}{4}x^2, \frac{1}{2}y^2), (x, y) \in \mathbb{C}^2.$$

Let $\tau = \frac{1}{2}\delta_{f_1} + \frac{1}{2}\delta_{f_2} \in \mathfrak{M}_{1,c}(X)$, where δ_{f_i} denotes the Dirac measure concentrated at $f_i \in X$ for each $i = 1, 2$. Then $\tau \in \mathcal{MS}$.

Problem 3. (**Open Problem.**) Is $\mathcal{MS}$ dense in $(\mathfrak{M}_{1,c}(X), \mathcal{O})$?
(Remark: This kind of statement is **true for random dynamical systems of** 1**-dimensional complex polynomial maps on** $\mathbb{P}^1 \cong \hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ **(the Riemann sphere) of degree two or more**. S., 2013 ([S13], Adv. Math.), [SW21] (Takayuki Watanabe's presentation).)

Definition 4. Let

$$\mathbb{P}^1_\infty := \{[u : v : 0] \in \mathbb{P}^2 \mid (u, v) \in \mathbb{C}^2 \setminus \{(0,0)\}\}.$$

This is called the line at infinity.

Remark: Let $f \in X$. Then $f(\mathbb{P}^1_\infty) = \mathbb{P}^1_\infty$, $f^{-1}(\mathbb{P}^1_\infty) = \mathbb{P}^1_\infty$, and for each neighborhood B of $\mathbb{P}^1_\infty$, there exists an open neighborhood C of $\mathbb{P}^1_\infty$ with $C \subset B$ such that $\overline{f(C)} \subset C$.

Definition 5. Let Ψ be the set of all $\tau \in \mathfrak{M}_{1,c}(X)$ satisfying the following condition. There exist two non-empty open subsets U, V of $\mathbb{P}^1_\infty$ and an $n \in \mathbb{N}$ such that all of the following (i)(ii)(iii) hold.

(i) $\sharp(\mathbb{P}^1_\infty \setminus U) \geq 3$.

(ii) $\overline{V} \subset U$, where $\overline{V}$ denotes the closure of V in $\mathbb{P}^1_\infty$.

(iii) For each $(\gamma_1, \ldots, \gamma_n) \in (\operatorname{supp} \tau)^n$, we have $\gamma_n \circ \cdots \circ \gamma_1(U) \subset V$.

Remark 6. Ψ is a **non-empty open** subset of $(\mathfrak{M}_{1,c}(X), \mathcal{O})$.

Example: Let Y $(\subset X)$ be the set of all regular polynomial maps $f : \mathbb{P}^2 \to \mathbb{P}^2$ of the form

$$f(x,y) = (a_1 x^2 + a_2 xy + a_3 y^2 + b_1 x + b_2 y + b_3,\ c_1 y^2 + c_2 x + c_3 y + c_4), (x,y) \in \mathbb{C}^2,$$

where $a_1, c_1 \in \mathbb{C} \setminus \{0\}, a_2, a_3, b_1, b_2, b_3, c_2, c_3, c_4 \in \mathbb{C}$.
Note that $Y \cong (\mathbb{C} \setminus \{0\})^2 \times \mathbb{C}^8$.
Let τ be a Borel probability measure on Y with compact support.

Then

$$\tau \in \Psi.$$

In fact, for any $f \in Y$ of the above form, via the identification

$$\mathbb{P}^1_\infty \cong \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}, \quad [z : 1 : 0] \leftrightarrow z \ (z \in \mathbb{C}), \quad [1 : 0 : 0] \leftrightarrow \infty,$$

$f|_{\mathbb{P}^1_\infty} : \mathbb{P}^1_\infty \to \mathbb{P}^1_\infty$ is equal to the map $z \mapsto \frac{1}{c_1}(a_1 z^2 + a_2 z + a_3)(z \in \mathbb{C})$ on $\hat{\mathbb{C}}$, and so $[1 : 0 : 0] \in \mathbb{P}^1_\infty$ is a common attracting fixed point of any $f \in Y$.

Theorem 7. *Let* $\mathcal{A} := \Psi \cap \mathcal{MS} = \{\tau \in \Psi \mid \tau$ *is mean stable*$\}$.
Then $\mathcal{A}$ *is* **open and dense** *in* Ψ.
Moreover, for each mean stable $\tau \in \mathfrak{M}_{1,c}(X)$ *(in particular, for each* $\tau \in \mathcal{A}$*), we have all of the following* (1)–(7).

(1) *There exists a constant* c_τ *with* $c_\tau < 0$ *such that the following holds.*

– *For each* $z \in \mathbb{P}^2$, *there exists a Borel subset* $B_{\tau,z}$ *of* $X^{\mathbb{N}}$ *with* $(\otimes_{n=1}^{\infty}\tau)(B_{\tau,z}) = 1$ *such that for each* $(\gamma_1, \gamma_2 \ldots,) \in B_{\tau,z}$, *we have*

$$\limsup_{n\to\infty} \frac{1}{n} \log \|D(\gamma_n \circ \cdots \circ \gamma_1)_z\| \leq c_\tau < 0.$$

Here, for each $f \in X$ *and each* $z \in \mathbb{P}^2$, *we denote by* $\|Df_z\|$ *the norm of the differential of* f *at* z *with respect to the Fubiny-Study metric in* $\mathbb{P}^2$.

(2) *For each* $z \in \mathbb{P}^2$, *there exists a Borel subset* $C_{\tau,z}$ *of* $X^{\mathbb{N}}$ *with* $(\otimes_{n=1}^{\infty}\tau)(C_{\tau,z}) = 1$ *such that for each* $\gamma = (\gamma_1, \gamma_2, \ldots,) \in C_{\tau,z}$,
there exists a number $r = r(\tau, z, \gamma) > 0$ *satifying that*

$$diam(\gamma_n \circ \cdots \circ \gamma_1(B(z,r))) \to 0 \text{ as } n \to \infty$$

exponentially fast, where $B(z,r)$ *denotes the ball with center* z *and radius* r *with respect to the distance* d *induced by the Fubini-Study metric on* $\mathbb{P}^2$, *and for each subset* A *of* $\mathbb{P}^2$, *we set* $diamA := \sup_{x,y\in A} d(x,y)$.

(3) *Let* $\mathrm{Min}(\tau)$ *be the set of all minimal sets of* τ. *Then,*

$$1 \leq \sharp\mathrm{Min}(\tau) < \infty.$$

Here, we say that a non-empty compact subset L *of* $\mathbb{P}^2$ *is a minimal set of* τ *if for each* $z \in L$, *we have* $L = \overline{\bigcup_{h\in G_\tau}\{h(z)\}}$.

(4) *For each* $z \in \mathbb{P}^2$, *there exists a Borel subset* $D_{\tau,z}$ *of* $X^{\mathbb{N}}$ *with* $(\otimes_{n=1}^{\infty}\tau)(D_{\tau,z}) = 1$ *such that for each* $(\gamma_1, \gamma_2, \ldots) \in D_{\tau,z}$,

$$d(\gamma_n \circ \cdots \circ \gamma_1(z), \cup_{L\in \mathrm{Min}(\tau)} L) \to 0 \quad \text{as } n \to \infty$$

exponentially fast.

(5) *Let $C(\mathbb{P}^2)$ be the Banach space of all continuous complex-valued functions on $\mathbb{P}^2$ endowed with the supremum norm.*
Let $M_\tau : C(\mathbb{P}^2) \to C(\mathbb{P}^2)$ be the linear operator defined by

$$M_\tau(\varphi)(z) = \int_X \varphi(h(z))\ d\tau(h), \text{ for } \varphi \in C(\mathbb{P}^2), z \in \mathbb{P}^2.$$

Then there exists a finite dimensional subspace $W_\tau \neq \{0\}$ of $C(\mathbb{P}^2)$ with $M_\tau(W_\tau) = W_\tau$ such that for each $\varphi \in C(\mathbb{P}^2)$, $\{M_\tau^n(\varphi)\}_{n=1}^\infty$ tends to W_τ as $n \to \infty$.
Also, the map $\nu \mapsto W_\nu$ is **continuous** *on $\mathcal{MS}$ w.r.t. the topology $\mathcal{O}$.*

(6) *There exists a number $0 < \alpha < 1$ such that the following* (a)(b)(c) *hold.*

(a) *The space W_τ in (3) is included in the Banach space $C^\alpha(\mathbb{P}^2)$ of all α-Hölder continuous functions on $\mathbb{P}^2$ endowed with α-Hölder norm.*

(b) *For each $\varphi \in C^\alpha(\mathbb{P}^2)$, $\{M_\tau^n(\varphi)\}_{n=1}^\infty$ tends to W_τ* **exponentially fast**. *(Thus $M_\tau : C^\alpha(\mathbb{P}^2) \to C^\alpha(\mathbb{P}^2)$ has the "***spectral gap property***".)*

(c) *For each $L \in \mathrm{Min}(\tau)$, let $T_{L,\tau} : \mathbb{P}^2 \to [0,1]$ be the function of probability of tending to L. That is,*

$$\begin{aligned} & T_{L,\tau}(z) \\ = \ & (\otimes_{n=1}^\infty \tau)\{(\gamma_1, \gamma_2, \ldots) \in X^{\mathbb{N}} \mid d(\gamma_n \circ \cdots \circ \gamma_1(z), L) \to 0 \text{ as } n \to \infty\}) \end{aligned}$$

for each $z \in \mathbb{P}^2$. Then,

$$T_{L,\tau} \in W_\tau \subset C^\alpha(\mathbb{P}^2).$$

Moreover, for each $z \in \mathbb{P}^2$, we have

$$\Sigma_{L \in \mathrm{Min}(\tau)} T_{L,\tau}(z) = 1.$$

(7) *Let* $F(G_\tau) := \{z \in \mathbb{P}^2 \mid \exists U: \text{ nbd of } z \text{ s.t. } G_\tau \text{ is equicontinuous on } U\}$. *(This is called the Fatou set of semigroup* G_τ*.)*
Then, for each $L \in \mathrm{Min}(\tau)$ *and for each connected component* U *of* $F(G_\tau)$*, there exists a* **constant** $c_U \in [0,1]$ *such that*

$$T_{L,\tau}|_U = c_U \text{ on } U.$$

Thus $T_{L,\tau}$ *is a continuous function on* $\mathbb{P}^2$ *which varies only on* $J(G_\tau) := \mathbb{P}^2 \setminus F(G_\tau)$ *(this* $J(G_\tau)$ *is called the Julia set of* G_τ*).*

Remark 8.

(1) **None of statements** (1)–(6) **in Theorem 7 can hold for deterministic dynamics of a single** $f \in X$. In fact, in the Julia set $J(f|_{\mathbb{P}^1_\infty})$ of $f|_{\mathbb{P}^1_\infty}$, we have a chaotic phenomenon. See Mañé's paper (1988)[Ma88] etc. Therefore, the statements (1)–(6) describe **randomness-induced phenonena** (phenomena in random dynamical systems which cannot hold for iteration dynamics of single maps). In this presentation, we have seen **randomness-induced order**.

(2) Even if a system induced by an element $\tau \in \Psi$ is mean stable and we have randomness-induced order in the system, **the system still may have a kind of complexity**. In fact, if τ has multiple minimal sets, and if it satisfies some conditions, then there exists a minimal set L of τ and a number $\beta \in (0,1)$ such that $T_{L,\tau}$ does not belong to $C^\beta(\mathbb{P}^2)$. It implies that $M_\tau : C^\beta(\mathbb{P}^2) \to C^\beta(\mathbb{P}^2)$ does not have spectral gap property anymore, and there exists an element $\varphi \in C^\beta(\mathbb{P}^2)$ such that $\|M_\tau^n(\varphi)\|_\beta \to \infty$ as $n \to \infty$ (see [S11, S13]).

Thus, in this case, there exists a number $\alpha_\tau \in (0,1)$ such that (i) for each $\alpha \in (0, \alpha_\tau)$, the operator $M_\tau : C^\alpha(\mathbb{P}^2) \to C^\alpha(\mathbb{P}^2)$ has the spectral gap property and for each $\varphi \in C^\alpha(\mathbb{P}^2)$, $\{M_\tau^n(\varphi)\}_{n=1}^\infty$ tends to the finite dimensional subspace W_τ of $C^\alpha(\mathbb{P}^2)$, but, (ii) for each $\alpha \in (\alpha_0, 1)$, the operator $M_\tau : C^\alpha(\mathbb{P}^2) \to C^\alpha(\mathbb{P}^2)$ does not have the spectral gap property anymore (and M_τ might not behave well on $C^\alpha(\mathbb{P}^2)$, e.g., there exists a $\varphi \in C^\alpha(\mathbb{P}^2)$ such that $\|M_\tau^n(\varphi)\|_\alpha \to \infty$ as $n \to \infty$). As we have seen above, even if we have randomness-induced order, we have to check the **gradation between chaos and order**. The above quantitiy α_τ seems to be a kind of quantity which describes the gradation between chaos and order.

Rough idea of the proofs of Theorem 7

To show the density of $\mathcal{A}$ in Ψ, let $\tau \in \Psi$. Let L be a minimal set of τ, i.e., L is a non-empty compact subset of $\mathbb{P}^2$ such that for each $z \in L$, we have $L = \overline{\cup_{h \in G_\tau} \{h(z)\}}$.

- If L is "attracting" for τ, then even if we enlarge the support of τ a little bit and we obtain a new $\nu \in \Psi$, there exists a minimal set L_ν of ν which is close to L and is **still attracting for** ν.
- If L is not attracting for τ, then if we enlarge the support of τ a little bit and we obtain a new $\nu \in \Psi$, then L **is broken** (there is no minimal set of ν around L).

Thus, if we enlarge the support of τ a little bit and obtain a new $\nu \in \Psi$, then **every minimal set** L **of** ν **is attracting for** ν. Then it is easy to see that ν **is mean stable**.

Hence the set of all mean stable $\nu \in \Psi$ is open and **dense** in Ψ.

Summary

(1) We introduce the notion of mean stability in i.i.d. random (holomorphic) 2-dimensional dynamical systems.

(2) We can see that a **generic** random dynamical system of regular polynomial maps on $\mathbb{P}^2$ having an attractor in the line at infinity, is **mean stable**.

(3) If a random holomorphic dynamical system on $\mathbb{P}^2$ is mean stable then for each $z \in \mathbb{P}^2$, for a.e. orbit starting with z, **the Lyapunov exponent is negative**.

(4) If a random holomorphic dynamical system on $\mathbb{P}^2$ is mean stable, then for any $z \in \mathbb{P}^2$, the orbit of the Dirac measure at z under the iterations of the dual map of the transition operator **converges to a periodic cycle** of probability measures.

(5) Note that the statements of (3) and (4) **cannot hold for deterministic dynamics of a single regular polynomial map f with** $\deg(f) \geq 2$.
We see many **randomness-induced phenomena** (phenomena in random dynamical systems which cannot hold for iteration dynamics of single maps). In this presentation, we have seen **randomness-induced order**.

Many kinds of maps in one random dynamical sytstem automatically cooperate together to make the chaoticity weaker. We call such phenomena

Cooperation Principle.

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