Mean ergodic theorem for linear operator cocycles and random invariant densities

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1 Introduction

This paper concerns a cocycle generated by linear operators, called a *linear operator cocycle*. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\sigma : \Omega \to \Omega$ be an invertible $\mathbb{P}$-preserving ergodic transformation. For a measurable space Σ, we say that a measurable map $\Phi : \mathbb{N}_0 \times \Omega \times \Sigma \to \Sigma$ is a *random dynamical system* on Σ over the driving system σ if

$$\varphi_\omega^{(0)} = \mathrm{id}_\Sigma \quad \text{and} \quad \varphi_\omega^{(n+m)} = \varphi_{\sigma^m\omega}^{(n)} \circ \varphi_\omega^{(m)}$$

for each $n, m \in \mathbb{N}_0$ and $\omega \in \Omega$, with the notation $\varphi_\omega^{(n)} = \Phi(n, \omega, \cdot)$ and $\sigma\omega = \sigma(\omega)$, where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. A standard reference for random dynamical systems is the monograph by Arnold [1]. It is easy to check that

$$\varphi_\omega^{(n)} = \varphi_{\sigma^{n-1}\omega} \circ \varphi_{\sigma^{n-2}\omega} \circ \cdots \circ \varphi_\omega \tag{1.1}$$

with the notation $\varphi_\omega = \Phi(1, \omega, \cdot)$. Conversely, for each measurable map $\varphi : \Omega \times \Sigma \to \Sigma : (\omega, x) \mapsto \varphi_\omega(x)$, the measurable map $(n, \omega, x) \mapsto \varphi_\omega^{(n)}(x)$ given by (1.1) is a random dynamical system. We call it a random dynamical system induced by φ over σ, and simply denote it by (φ, σ). When Σ is a Banach space (with its Borel measurable sets from its strong norm) and $\varphi_\omega : \Sigma \to \Sigma$ is $\mathbb{P}$-almost surely linear, (φ, σ) is called a *linear operator cocycle*.

As a one of interesting class of the linear operator cocycle, we introduce a *Markov operator cocycle* defined as follows. Let $(X, \mathcal{A}, m)$ be a probability space and $L^1(X, m)$ the space of all m-integrable functions on X endowed with the usual L^1-norm $\| \cdot \|_{L^1(X)}$. Let $D(X, m)$ be the set of all density functions, i.e., a subset of $L^1(X, m)$ defined by

$$D(X, m) = \left\{ f \in L^1(X, m) : f \geq 0 \ m\text{-almost everywhere}, \|f\|_{L^1(X)} = 1 \right\}.$$

We say that $P : L^1(X, m) \to L^1(X, m)$ is a Markov operator if $P(D(X, m)) \subset D(X, m)$ holds. One of the most important examples of Markov operators is the *Perron-Frobenius operator* induced by a measurable and non-singular transformation $T : X \to X$ (that is, the probability measure $m \circ T^{-1}$ is absolutely continuous with respect to m). The Perron-Frobenius operator $\mathcal{L}_T : L^1(X, m) \to L^1(X, m)$ of T is defined by

$$\int_X \mathcal{L}_T f g dm = \int_X f g \circ T dm \qquad \text{for } f \in L^1(X, m) \text{ and } g \in L^\infty(X, m). \tag{1.2}$$

We say that a linear operator cocycle (P, σ) induced by a measurable map $P : \Omega \times L^1(X, m) \to L^1(X, m)$ over σ is called a *Markov operator cocycle* if $P_\omega = P(\omega, \cdot) : L^1(X, m) \to L^1(X, m)$ is a Markov operator for $\mathbb{P}$-almost every $\omega \in \Omega$.

Markov operators naturally appear in the study of dynamical systems as Perron-Frobenius operators, Markov processes as integral operators with stochastic kernels of the processes, and annealed type random dynamical systems as integrations of Perron-Frobenius operators over environmental parameters (see [5,7] for details). A Markov operator cocycle is given by compositions of potentially different Markov operators which are provided with the environment $\{\sigma^n(\omega)\}_{n\geq 0}$ driven by a measure-preserving transformation σ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$,

$$\mathbb{N} \times \Omega \times L^1(X, m) \to L^1(X, m) : (n, \omega, f) \mapsto P_{\sigma^{n-1}(\omega)} \circ P_{\sigma^{n-2}(\omega)} \circ \cdots \circ P_\omega f.$$

Then, it essentially possesses two kinds of randomness:

(i) The evolution of densities at each time are dominated by Markov operators P_ω,

(ii) The selection of each Markov operators is driven by the base dynamics σ.

Thus, by considering Markov operator cocycles, we expect to understand more complicated phenomena in multi-stochastic systems. The study of Markov operator cocycles follows measurable random dynamical systems in the sense of [1]. We also refer to [8].

Now we recall the definition of invariant densities for linear operator cocycles (P, σ), called random invariant densities.

Definition 1.1. A measurable map $h : \Omega \to L^1(X, m)$ with $h(\omega) = h_\omega$ is called a *random invariant density* if $h_\omega \in D(X, m)$ and $P_\omega h_\omega = h_{\sigma\omega}$ hold for $\mathbb{P}$-almost every $\omega \in \Omega$.

In this note, we summarize the mean ergodic theorem for a linear operator cocycle on a general Banach space (Theorem 1), which guarantees the existence of random invariant density under certain conditions. The conventional mean ergodic theorem provides that the average of the sequence $\{P^n f\}_n$ converges in strong, and the limit point becomes an invariant density. The classical mean ergodic theorem for a single linear operator by von Neumann deals only with a reflexive Banach space, and after that, Yosida and Kakutani [10] generalized the theorem to the case of a general Banach space under the assumption of weak precompactness of Cesàro average of time evolution. As known in [2], the theorem for a linear operator cocycle is fulfilled if the Banach space is reflexive. Then, giving an appropriate definition of *weak precompactness* for the cocycle, we succeeded to obtain a general result for mean ergodic theorem of linear operator cocycles, that guarantees the existence of invariant measures for linear operator cocycles.

See [9] for more precise descriptions including the proofs.

2 The lift operator and weak precompactness

In this section, we introduce our key tools: the lift operator $\mathscr{P}$ of a linear operator cocycle (P, σ) and weak precompactness of functions in fiberwise and global sense in order to construct a random invariant density for the linear operator cocycle. We first prepare the Banach space of Bochner integrable functions over a Banach space $\mathfrak{X}$ (with norm $\|\cdot\|_{\mathfrak{X}}$) denoted by $L^1(\Omega, \mathfrak{X})$, based on [3,6]. Then, we define the lift operator $\mathscr{P}$ over $L^1(\Omega, \mathfrak{X})$ associated with the linear operator cocycle and relate it with a random invariant density.

Let us define

$$\mathscr{L}^1(\Omega, \mathfrak{X}) = \{f : \Omega \to \mathfrak{X}, \text{strongly measurable and integrable}\},$$

$$\mathscr{N} = \left\{f : \Omega \to \mathfrak{X},\ \text{strongly measurable and}\ \ \|\varphi(\omega)\|_{\mathfrak{X}} = 0,\ \mathbb{P}\text{-a.e.}\omega \in \Omega\right\},$$

where $f:\Omega\to\mathfrak{X}$ is called *strongly measurable* provided that there exists a sequence of simple functions $f_n=\sum_{i=1}^{N}1_{F_i}v_i$ for some $N=N(n)\in\mathbb{N}$, $\{F_i=F_i(n):i=1,\dots,N\}\in\mathscr{F}$ and $\{v_i=v_i(n):i=1,\dots,N\}\subset\mathfrak{X}$ such that $\lim_{n\to\infty}\|f(\omega)-f_n(\omega)\|_{\mathfrak{X}}=0$ for $\mathbb{P}$-almost every $\omega\in\Omega$. Then we define

$$L^1(\Omega,\mathfrak{X}):=\mathscr{L}^1(\Omega,\mathfrak{X})/\mathscr{N}.$$

Note that if $\mathfrak{X}=L^1(X,m)$ then $L^1\left(\Omega,L^1(X,m)\right)$ is isometric to $L^1(\Omega\times X,\mathbb{P}\times m)$ (see Lemma 4.1). The space $L^1(\Omega,\mathfrak{X})$ is equipped with the usual norm $|||\cdot|||_1$ given by

$$|||f|||_1:=\int_\Omega\|f_\omega\|_{\mathfrak{X}}\,d\mathbb{P}(\omega)\quad\text{for } f\in L^1(\Omega,\mathfrak{X}).$$

The lift operator of a give linear operator cocycle is defined as follows.

Definition 2.1. For a linear operator cocycle (P,σ) over a Banach space $\mathfrak{X}$ where $P_\omega:\mathfrak{X}\to\mathfrak{X}$ is bounded uniformly in ω, the *lift operator* $\mathscr{P}:L^1(\Omega,\mathfrak{X})\to L^1(\Omega,\mathfrak{X})$ is defined by

$$(\mathscr{P}f)(\omega):=P_{\sigma^{-1}\omega}f_{\sigma^{-1}\omega}$$

for $f\in L^1(\Omega,\mathfrak{X})$ and $\mathbb{P}$-almost every $\omega\in\Omega$ so that for each $n\in\mathbb{N}$ we have

$$(\mathscr{P}^nf)(\omega)=P^{(n)}_{\sigma^{-n}\omega}f_{\sigma^{-n}\omega}$$

for $\mathbb{P}$-almost every $\omega\in\Omega$.

Remark 2.1. (I) The above lift operator is a well-defined bounded linear operator over $L^1(\Omega,\mathfrak{X})$. Indeed, if $f:\Omega\to\mathfrak{X}$ is strongly measurable then f is approximated by $f_n=\sum_{i=1}^{n}1_{F_i}v_i$ and

$$\mathscr{P}f_n=\sum_{i=1}^{n}1_{\sigma F_i}P_{\sigma^{-1}\omega}v_i$$

leads to strong measurability of $\mathscr{P}f$. Moreover if $f,\tilde{f}\in L^1(\Omega,\mathfrak{X})$ and $f-\tilde{f}\in\mathscr{N}$, then we have

$$\begin{aligned}\left\|\mathscr{P}\left(f-\tilde{f}\right)(\omega)\right\|_{\mathfrak{X}}&=\left\|P_{\sigma^{-1}\omega}\left(f_{\sigma^{-1}\omega}-\tilde{f}_{\sigma^{-1}\omega}\right)\right\|_{\mathfrak{X}}\\&\le M\left\|f_{\sigma^{-1}\omega}-\tilde{f}_{\sigma^{-1}\omega}\right\|_{\mathfrak{X}}=0\end{aligned}$$

for $\mathbb{P}$-almost every $\omega\in\Omega$ where M is the supremum of the operator norm of P_ω and $\mathscr{P}f=\mathscr{P}\tilde{f}$ $\mathbb{P}$-almost everwhere. We also have

$$|||\mathscr{P}f|||_1=\int_\Omega\|P_{\sigma^{-1}\omega}f_{\sigma^{-1}\omega}\|_{\mathfrak{X}}\,d\mathbb{P}(\omega)\le\int_\Omega M\,\|f_{\sigma^{-1}\omega}\|_{\mathfrak{X}}\,d\mathbb{P}(\omega)=M|||f|||_1,$$

which implies that $\mathscr{P}$ is a bounded operator. In particular, if $\|P_\omega\|\le 1$ for $\mathbb{P}$-almost every $\omega\in\Omega$ then $\mathscr{P}$ is a contraction operator over $L^1(\Omega,\mathfrak{X})$.

(II) We note that $h\in L^1\left(\Omega,L^1(X,m)\right)$ is a random invariant density if and only if $\mathscr{P}h=h$ (see Proposition 4.2 (2) more precisely). □

Recall that a subset $\mathscr{F}\subset L^1(X,m)$ is called weak precompact if for any sequence $\{f_n\}_n\subset\mathscr{F}$ there is a further subsequence $\{f_{n_k}\}_k$ which converges weakly in $L^1(X,m)$. Now we define weak precompactness in $L^1(\Omega,\mathfrak{X})$ in two senses.

Definition 2.2. A set $\mathscr{F}\subset L^1(\Omega,\mathfrak{X})$ is called *fiberwise weakly precompact* if for every sequence $\{f_n\}_n\subset\mathscr{F}$, there exists $h\in L^1(\Omega,\mathfrak{X})$ such that for $\mathbb{P}$-almost every $\omega\in\Omega$, there exists a subsequence $\{n_k\}_k=\{n_k(\omega)\}_k\subset\mathbb{N}$ such that $\{(f_{n_k})(\omega)\}_k$ converges weakly to $h(\omega)$.

A set $\mathscr{F}\subset L^1(\Omega,\mathfrak{X})$ is called *globally weakly precompact* if for every sequence $\{f_n\}_n\subset\mathscr{F}$, there is a further subsequence $\{f_{n_k}\}_k$ which converges weakly in $L^1(\Omega,\mathfrak{X})$.

Remark 2.2. Several sufficient conditions for weak precompactness are known as follows (IV.8, [4]). It reads that $\left\{P^{(n)}_{\sigma^{-n}\omega}1_X\right\}_n$ is weakly precompact if one of the following three conditions holds:

(i) There exists $g_\omega \in L^1_+(X,m) := \left\{f \in L^1(X,m) : f \geq 0\right\}$ such that for any $n \geq 1$

$$\left|P^{(n)}_{\sigma^{-n}\omega}1_X(x)\right| \leq g_\omega(x) \quad m\text{-almost every} x \in X;$$

(ii) There exists $M_\omega > 0$ and $p_\omega > 1$ such that

$$\left\|P^{(n)}_{\sigma^{-n}\omega}1_X\right\|_{L^{p_\omega}(X,m)} \leq M_\omega;$$

(iii) $\left\{P^{(n)}_{\sigma^{-n}\omega}1_X\right\}_n$ is uniformly integrable, namely, for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$m(A) < \delta \quad \text{implies} \quad \int_A P^{(n)}_{\sigma^{-n}\omega}1_X dm < \varepsilon \quad \text{for all } n \geq 1.$$

3 Mean ergodic theorem for linear operator cocycles

Let $\mathfrak{X}$ be a weakly sequential complete Banach space and $P : \Omega\times\mathfrak{X} \to \mathfrak{X}$ a linear operator cocycle which is almost surely contraction. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and σ be an invertible $\mathbb{P}$-preserving ergodic (i.e., $\sigma^{-1}E = E$ (mod $\mathbb{P}$) implies $E = \emptyset$ or Ω (mod $\mathbb{P}$)) transformation on Ω. We define the operator $\mathscr{A}^n$ meaning the average of $\mathscr{P}^n$ by

$$(\mathscr{A}^n f)(\omega) := \frac{1}{n}\sum_{k=0}^{n-1}(\mathscr{P}^k f)(\omega) = \frac{1}{n}\sum_{k=0}^{n-1}P^k_{\sigma^{-k}\omega}f_{\sigma^{-k}\omega}$$

for $f \in L^1(\Omega,\mathfrak{X})$ and $\mathbb{P}$-almost every $\omega \in \Omega$. Recall that a sequence $\{(\mathscr{A}^n f)\}_n$ is fiberwise weakly precompact for $f \in L^1(\Omega,\mathfrak{X})$ if there exists $h \in L^1(\Omega,\mathfrak{X})$ such that for $\mathbb{P}$-almost every $\omega \in \Omega$, there exists a subsequence $\{n_k\}_k \subset \mathbb{N}$, $n_k = n_k(\omega, f)$, such that $(\mathscr{A}^{n_k} f)(\omega)$ converges weakly to $h(\omega)$ for $\mathbb{P}$-almost every $\omega \in \Omega$.

Theorem 1. *Let $\mathfrak{X}$ be a weakly sequential complete Banach space, σ an invertible $\mathbb{P}$-preserving ergodic transformation over the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and P_ω a linear operator which maps $\mathfrak{X}$ into itself. Assume that $\|P_\omega\| \leq 1$ for $\mathbb{P}$-almost every $\omega \in \Omega$ and $\{\mathscr{A}^n f\}_n$ is fiberwise weakly precompact for any $f \in L^1(\Omega,\mathfrak{X})$. Then there exists $h \in L^1(\Omega,\mathfrak{X})$ such that*

$$\lim_{n\to\infty} \|(\mathscr{A}^n f)(\omega) - h(\omega)\|_{\mathfrak{X}} = 0,$$

and $P_\omega h_\omega = h_{\sigma\omega}$ for $\mathbb{P}$-almost every $\omega \in \Omega$.

4 Skew product for the case $\mathfrak{X} = L^1(X)$

In this section, we introduce some useful facts for the case $\mathfrak{X} = L^1(X)$. We first show the following isometric isomorphism between $L^1(\Omega, L^1\ (X,m))$ and $L^1(\Omega\times X, \mathbb{P}\times m)$, that identifies a random invariant density $h \in L^1\left(\Omega, L^1(X,m)\right)$ as a function in $L^1(\Omega \times X, \mathbb{P} \times m)$.

Proposition 4.1. $L^1\left(\Omega, L^1(X,m)\right) \cong L^1(\Omega \times X, \mathbb{P} \times m)$ *holds.*

From the proposition, we have

$$L^1(\Omega, D(X,m)) \subset L^1\left(\Omega, L^1(X,m)\right) \cong L^1(\Omega \times X, \mathbb{P} \times m)$$

and we frequently identify $h \in L^1(\Omega, D(X,m))$ as a function in $L^1(\Omega \times X, \mathbb{P} \times m)$. We can characterize a random invariant density $h \in L^1(\Omega, L^1(X,m))$ as a fixed point of $\mathscr{P}$ as a function of $L^1(\Omega \times X, \mathbb{P} \times m)$.

Proposition 4.2. *The following statements are true:*

1. *The lift operator $\mathscr{P}$ can be naturally identified with a Markov operator over $L^1(\Omega \times X, \mathbb{P} \times m)$ (this operator is also denoted by the same symbol);*
2. *$h \in L^1(\Omega, D(X,m))$ is a random invariant density if and only if $\mathscr{P}h = h$ as a function of $D(\Omega \times X, \mathbb{P} \times m)$;*
3. *the following diagram commutes:*

$$\begin{array}{ccc} L^1(\Omega, L^1(X,m)) & \xrightarrow{\mathscr{P}} & L^1(\Omega, L^1(X,m)) \\ \downarrow \iota & \circlearrowleft & \downarrow \iota \\ L^1(\Omega \times X, \mathbb{P} \times m) & \xrightarrow[\mathscr{P}]{} & L^1(\Omega \times X, \mathbb{P} \times m) \end{array}$$

where ι is the isometry arises in Proposition 4.1.

An important example of the lift operator $\mathscr{P}$ of a Markov operator cocycle is the Perron-Frobenius operator of a skew product transformation of a random transformations.

Proposition 4.3. *Let Θ be a $\mathbb{P} \times m$ non-singular skew product transformation over $\Omega \times X$ given by*

$$\Theta(\omega, x) = (\sigma\omega, T_\omega x)$$

where $T_\omega : X \to X$ is a non-singular transformation for $\omega \in \Omega$ and $\sigma : \Omega \to \Omega$ is an invertible ergodic measure-preserving transformation. Then the lift operator associated with the cocyle of $\mathcal{L}_\omega$ the Perron-Frobenius operator of T_ω is the Perron-Frobenius operator of Θ.

Example 4.1. Let X and Ω be a unit interval $[0,1]$. Set $\beta = \frac{\sqrt{5}+1}{2}$, that is, $\beta^2 - \beta - 1 = 0$ holds. Consider the transformations T_1 and T_2 on X defined by

$$T_1(x) = \beta x \pmod 1, \quad T_2(x) = \begin{cases} \beta x & (x \in [0, 1/\beta)) \\ \frac{\beta}{\beta - 1}(x-1)+1 & (x \in [1/\beta, 1]) \end{cases}. \tag{4.1}$$

Next, let $\sigma : \Omega \to \Omega$ be an irrational rotation with angle $1/\beta$, namely, $\sigma(\omega) = \omega + 1/\beta \pmod 1$.

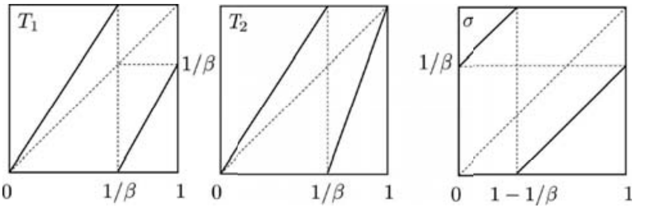

Figure 1: Illustrations of the map T_1, T_2 and σ.

Let P_i be a Perron-Frobenius operator corresponding to T_i, $i = 1, 2$. We define P_ω by

$$P_\omega = \begin{cases} P_1 & \text{if } \omega \in [0, 1 - 1/\beta) \\ P_2 & \text{if } \omega \in [1 - 1/\beta, 1] \end{cases}. \tag{4.2}$$

Then, the Markov operator cocycle given by above setting admits a random invariant density $h \in D(\Omega, L^1(X))$, that is, $P_\omega h_\omega = h_{\sigma\omega}$ holds for $\mathbb{P}$-almost every $\omega \in \Omega$. Moreover, h is given by

$$h_\omega(x) = \begin{cases} h_1(x) & \text{if } \omega \in [0, 1/\beta) \\ h_2(x) & \text{if } \omega \in [1/\beta, 1) \end{cases} \tag{4.3}$$

with

$$h_1(x) = 1_{[0,1]}(x), \quad h_2(x) = \frac{2}{\beta} 1_{[0,1/\beta)}(x) + \frac{1}{\beta} 1_{[1/\beta,1]}.$$

Indeed, putting $I_1 = [0, 1 - 1/\beta)$, $I_2 = [1 - 1/\beta, 1/\beta)$ and $I_3 = [1/\beta, 1]$, We know that

$$\sigma(I_1) = I_3, \quad \sigma(I_2) \subset I_1, \quad \sigma(I_3) \subset I_1 \cup I_2.$$

Moreover, we immediately find that $P_1 h_1 = h_2$, $\quad P_2 h_1 = P_2 h_2 = h_1$. Then, we can check the fact through the following three cases.

Case 1: if $\omega \in I_1$, then we have $P_\omega h_\omega = P_1 h_1 = h_2$. Moreover, $h_{\sigma\omega} = h_2$ since $\sigma(\omega) \in I_3$.

Case 2: if $\omega \in I_2$, then we have $P_\omega h_\omega = P_2 h_1 = h_1$. Moreover, $h_{\sigma\omega} = h_1$ since $\sigma(\omega) \in I_1$.

Case 3: if $\omega \in I_3$, then we have $P_\omega h_\omega = P_2 h_2 = h_1$. Moreover, $h_{\sigma\omega} = h_1$ since $\sigma(\omega) \in I_1 \cup I_2$.

Therefore, the invariance $P_\omega h_\omega = h_{\sigma\omega}$ is proven for $\mathbb{P}$-almost every $\omega \in \Omega$.

We finally consider the skew product transformation $F : X \times \Omega \to X \times \Omega$ defined by

$$F(x, \omega) = (T_\omega(x), \sigma(\omega)), \tag{4.4}$$

and let P_F be a Perron-Frobenius operator for F. Then the function $h \in D(X \times \Omega)$ given by

$$h(x, \omega) = 1_{[0,1]\times[0,1/\beta)}(x, \omega) + \frac{2}{\beta} \cdot 1_{[0,1/\beta]\times[1/\beta,1)}(x, \omega) + \frac{1}{\beta} \cdot 1_{[1/\beta,1]\times[1/\beta,1)}(x, \omega)$$

satisfies $P_F h = h$ because of Figure 2.

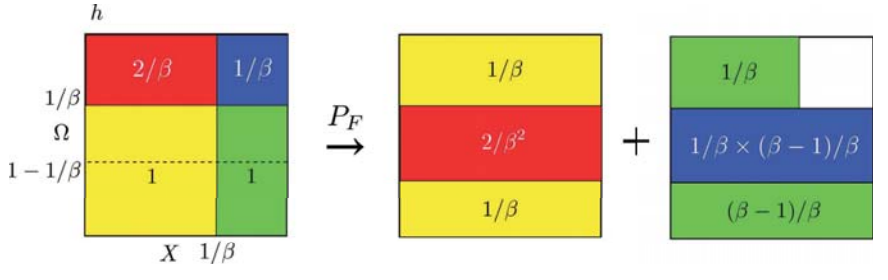

Figure 2: Illustration of the evolution of density h by P_F on $X \times \Omega$. The numbers in each square denote the hight of density.

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