# A QUESTION OF JOSEPH RITT FROM THE POINT OF VIEW OF VERTEX ALGEBRAS 

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#### Abstract

Let $k$ be a field of characteristic zero. This paper studies a problem proposed by Joseph F. Ritt in 1950. Precisely, we prove that (1) If $p \geqslant 2$ is an integer, for every integer $i \in \mathbb{N}$, the nilpotency index of the image of $T_{i}$ in the ring $k\{T\} /\left[T^{p}\right]$ equals $(i+1) p-i$. (2) For every pair of integers $(i, j)$, the nilpotency index of the image of $T_{i} U_{j}$ in the ring $k\{T\} /[T U]$ equals $i+j+1$.


## 1. MAIN RESULT

1.1. Let $k$ be a field of characteristic zero. Let us denote by $k\{T\}$ (resp. $k\{T, U\}$ ) the differential $k$-algebra obtained by endowing the $k$-algebra $k\left[T_{i} ; i \in \mathbb{Z} \geqslant 0\right.$ ] (resp. $k\left[T_{i}, U_{j} ; i, j \in \mathbb{Z}_{\geqslant 0}\right]$ ) with the differential $\partial$ defined by $\partial T_{i}=T_{i+1}\left(\right.$ resp. $\partial T_{i}=T_{i+1}$ and $\partial U_{j}=U_{j+1}$ ) for every integer $i \geqslant 0$ (resp. for every pair $(i, j) \in \mathbb{Z}_{\geqslant 0}^{2}$ of integers). In Rit50, Appendix/5], J. F. Ritt asked the following question:

Question. (1) Let $p \geqslant 1$ be an integer. Let $\left[T_{0}^{p}\right]$ be the differential ideal generated by $T_{0}^{p}$. For $p>0, i>0$ what is the least $q(i)$ such that $T_{i}^{q(i)} \equiv 0$ $\bmod \left[T_{0}^{p}\right] ?$
(2) In $k\{T, U\}$, what is the least power $q(i, j)$ of $T_{i} U_{j}$ such that $\left(T_{i} U_{j}\right)^{q(i, j)} \equiv 0$ $\bmod [T U] ?$.

For $i=1$, Ritt states in loc. cit., with no proof, that $q(i)=2 p-1$. In O'K60, K. B. O'Keefe gave a proof of this formula and has shown that, for $i=2$ and $p \geqslant 2$, one has $q(i)=3 p-2$. See also Mea55, Mea73, O'K66] for complements or connected problems. All these works use differential algebra, and are broadly based on the reduction process due to H. Levi (see Lev42]). Despite all these results, to the best of our knowledge, Ritt's question is remained open, in general, up to 2014. In Pog14, G. A. Pogudin indeed provides an answer by showing that

$$
\begin{equation*}
q(i)=(i+1) p-i \tag{1.1}
\end{equation*}
$$

for every integer $i \in \mathbb{Z}_{\geqslant 0}$. The guideline of his proof consists in injecting the differential algebra $k\{T\} /\left[T_{0}^{p}\right]$ into a Grassmann algebra, endowed with a structure of differential algebra, and testing the vanishing of the image of $T_{i}^{q}$ in that algebra (see Pog14, Lemma 2,Theorem 3]). In the direction of the second question, one also deduce from the Levi reduction process the following formula (see [Lev42, III], or, e.g., see $\overline{B S}$ )

$$
\begin{equation*}
q(i, j)=i+j+1 \tag{1.2}
\end{equation*}
$$

1.2. In this article, we provide a proof of formula (1.1) and formula (1.2) using vertex algebra. In this way, our proofs deeply differ from Pog14 and Lev42, BS, since they do not use differential algebra in the sense of Ritt. For the first formula, the key point is, up to passing over $\mathbb{C}$, to identify $\mathbb{C}\{T\} /\left[T_{0}^{p}\right]$ with the FeiginStoyanovsky principal subspace SF94 of the level $(p-1)$ vacuum representation of the affine Kac-Moody algebra $\widehat{\mathfrak{s l}}_{2}$, which is a commutative vertex algebra. This operation allows us to reduce formula (1.1) to a simple fact from representation theory. In MP12], the Feigin-Stoyanovsky principal subspace is generalized to the notion of principal subalgebras of lattice vertex algebras. They are isomorphic to the free vertex algebras in the sense of [B86, R02] (see Kaw15). The answer to the second question can be similarly given using the theory of free vertex algebras and lattice vertex operators.

## 2. Vertex algebras

2.1. Let $V$ be a vector space over $k$. A field on $V$ is a formal power series $f(z) \in$ $\operatorname{End}(V)\left[\left[z, z^{-1}\right]\right]$ such that $f(z) v \in V((z))$ for any $v \in V$. Here, $V((z))$ is the space of formal Laurent series whose coefficients are elements of $V$.
2.2. A vertex algebra is a vector space $V$ equipped with
(1) (Vacuum vector) $|0\rangle \in V$,
(2) (State-field correspondence) $Y(\cdot, z): V \rightarrow \operatorname{End}(V)\left[\left[z, z^{-1}\right]\right]$,
(3) (Translation operator) $T \in \operatorname{End}(V)$,
such that
(1) $T|0\rangle=0$,
(2) $Y(|0\rangle, z)=\mathrm{id}_{V}$,
(3) $Y(v, z)|0\rangle=v+O(z)(v \in V)$,
(4) $[T, Y(v, z)]=\partial_{z} Y(v, z)(v \in V)$,
(5) (locality) for any $u, v \in V$, there exists $N \in \mathbb{Z}$ such that

$$
\begin{equation*}
(z-w)^{-N}(Y(u, z) Y(v, w)-Y(v, w) Y(u, z))=0 \tag{2.1}
\end{equation*}
$$

The biggest number $N \in \mathbb{Z}$ which satisfies (2.1) is called the locality bound for the pair $u, v$. The locality bound $N$ for $u, v$ is the same as the number satisfying

$$
u(N-1) v \neq 0, \quad u(n) v=0 \quad n \geqslant N
$$

Here we have employed the notation $Y(u, z)=\sum_{n \in \mathbb{Z}} u(n) z^{-n-1}$.
We note that the multiplication $v \mapsto u(n) v$ is not associative in general. The monomial $u_{1}\left(n_{1}\right)\left(u_{2}\left(n_{2}\right)\left(\cdots\left(u_{m}\left(n_{m}\right) v\right) \cdots\right)\right) \in V$ with $u_{1}, \ldots, u_{m}, v \in V$ is simply written as $u_{1}\left(n_{1}\right) u_{2}\left(n_{2}\right) \cdots u_{m}\left(n_{m}\right) v$.

## 3. Proof of Formula (1.1)

3.1. Let $k^{\prime}$ be a field extension of $k$. Since, for every $i \geqslant 0$, the polynomials $\partial^{i}\left(T_{0}^{p}\right)$ belong to $k\{T\}$, we observe that, for every integer $q \in \mathbb{Z}_{\geqslant 0}$, the relation $T_{i}^{q} \in\left[T_{0}^{p}\right]$ holds in $k^{\prime}\{T\}$ if and only if it holds in $k\{T\}$. Thus, we may replace $k$ with an algebraic closure of $k$, and, then, by the Lefschetz principle, assume that $k=\mathbb{C}$.
3.2. Let $\mathfrak{g}=\mathfrak{s l}_{2}$ with the standard basis $\{e, h, f\}$, and let $\widehat{\mathfrak{g}}=\mathfrak{g}\left[t, t^{-1}\right] \oplus \mathbb{C} K$ be the affine Kac-Moody algebra associated with $\mathfrak{g}=\mathfrak{s l}_{2}$. The commutation relations of $\widehat{\mathfrak{g}}$ are given by $\left[x_{m}, y_{n}\right]=[x, y]_{m+n}+n \delta_{m+n, 0}(x \mid y) K,[K, \widehat{\mathfrak{g}}]=0$, where $x_{m}=x t^{m}$ and $(x \mid y)=\operatorname{tr}(x y)$ for $x, y \in \mathfrak{g}, m \in \mathbb{Z}$. Let $L_{p-1}(\mathfrak{g})$ be the irreducible vacuum representation of the affine Kac-Moody algebra of level $p-1$, which is the unique simple quotient of the induced module $U(\widehat{\mathfrak{g}}) \otimes_{U(\mathfrak{g}[t] \oplus \mathbb{C} K)} \mathbb{C}_{p-1}$, where $\mathbb{C}_{p-1}$ is the one-dimensional representation of $\mathfrak{g}[t] \oplus \mathbb{C} K$ on which $\mathfrak{g}[t]$ acts trivially and $K$ acts as multiplication by $p-1$. As is well-known ( $[\overline{\mathrm{F} Z 92}$ ), there is a unique vertex algebra structure on $L_{p-1}(\mathfrak{g})$ such that the highest weight vector $|0\rangle$ is the vacuum vector and

$$
Y\left(x_{-1}|0\rangle, z\right)=x(z):=\sum_{n \in \mathbb{Z}} x_{n} z^{-n-1}, \quad x \in \mathfrak{g}
$$

3.3. The Feigin-Stoyanovsky principal subspace $W$ of $L_{p-1}(\mathfrak{g})$ is by definition the commutative vertex subalgebra of $L_{p-1}(\mathfrak{g})$ generated by $e(z)$. Let $\partial$ be the differential of $\mathbb{C}\left[e_{-1}, e_{-2}, e_{-3}, \ldots,\right]$ defined by $\partial e_{-i}=i e_{-i-1}$ for every $i \geqslant 1$. We have a surjective morphism

$$
\begin{equation*}
\mathbb{C}\left[e_{-1}, e_{-2}, e_{-3}, \ldots,\right] \rightarrow W, \quad f \mapsto f|0\rangle \tag{3.1}
\end{equation*}
$$

of differential algebras. According to [SF94 (see also CLM08a, CLM08b, Fei11, $\mathrm{LiH}]$ ), the kernel $J$ of the above map is the ideal generated by the $\partial^{n}\left(e_{-1}^{p}\right)$ for all $n \in \mathbb{Z}_{\geqslant 0}$. Therefore, we have an isomorphism of differential algebras given by:

$$
\begin{equation*}
W=\mathbb{C}\left[e_{-1}, e_{-2}, e_{-3}, \ldots,\right] / J \cong \mathbb{C}\{T\} /\left[T_{0}^{p}\right], \quad e_{-i} \mapsto T_{i-1} /(i-1)! \tag{3.2}
\end{equation*}
$$

Remark 3.4. Let us stress that the character of $W$ coincides with that of the Virasoro $(2,2 p+1)$-minimal model vertex algebra $\operatorname{Vir}_{2,2 p+1}$ (SF94, Fei11). Let us stress that, if $X=\operatorname{Spec}\left(\mathbb{C}[T] /\left\langle T^{p}\right\rangle\right)$, the $k$-algebra $\mathbb{C}\{T\} /\left[T_{0}^{p}\right]$ is isomorphic to the algebra $\mathcal{O}\left(J_{\infty} X\right)$ of the arc scheme $J_{\infty} X$ associated with $X$. Let us then mention that the identification of $\mathcal{O}\left(J_{\infty} X\right)$ with $\operatorname{gr} \operatorname{Vir}_{2,2 p+1}$ has been previously established in vEH .
3.5. Formula (1.1) immediately follows from (3.2) and Proposition 3.6,

Proposition 3.6. We have $e_{-i}^{i(p-1)}|0\rangle \neq 0$ and $e_{-i}^{i(p-1)+1}=0$ on $W$ for all $i \geqslant 1$.
Proof. We have $\left[f_{i}, e_{-i}\right]=-h_{0}+i K$, and so, $\left\{f_{i},-h_{0}+i K, e_{-i}\right\}$ forms an $\mathfrak{s l}_{2}$-triple inside $\widehat{\mathfrak{g}}$. Since $L_{1}(\mathfrak{g})$ is integrable and $\left(-h_{0}+i K\right)|0\rangle=i(p-1)|0\rangle,|0\rangle$ generates an $(i(p-1)+1)$-dimensional representation over the $\mathfrak{s l}_{2}$-triple. Therefore, $e_{-i}^{i(p-1)}|0\rangle \neq 0$ and $e_{-i}^{i(p-1)+1}|0\rangle=0$.

## 4. Proof of Formula (1.2)

In order to give a proof of second question, we need to introduce lattice vertex algebras and free vertex algebras.
4.1. As in subsection 3.1 we may assume that $k=\mathbb{C}$.
4.2. Let $L$ be an integral lattice with the $\mathbb{Z}$-bilinear form $(\cdot, \cdot): L \times L \rightarrow \mathbb{Z}$ on $L$. We set $\mathfrak{h}=k \otimes_{\mathbb{Z}} L$ with the extended $k$-bilinear form $(\cdot, \cdot): \mathfrak{h} \times \mathfrak{h} \rightarrow k$ and let $k[L]=\bigoplus_{\alpha \in L} k e^{\alpha}$ be the group algebra of $L$. Then the vector space

$$
V_{L}=M(1) \otimes k[L]
$$

admits a natural vertex superalgebra structure, called the lattice vertex superalgebra associated with $L$. Here, $M(1)$ is the Heisenberg vertex algebra (Fock space) attached to $\mathfrak{h}$. We have $|0\rangle=1 \otimes 1$, where we write 1 for the vacuum vector of $M(1)$. The state-field correspondence for $1 \otimes e^{\alpha}$ is

$$
Y\left(1 \otimes e^{\alpha}, z\right)=E^{-}(-\alpha, z) E^{+}(-\alpha, z) \otimes e_{\alpha} z^{\alpha}
$$

where

$$
E^{ \pm}(-\alpha, z)=\exp \left(\sum_{n \in \pm \mathbb{Z}} \frac{-\alpha(n)}{n} z^{-n}\right)
$$

$z^{\alpha}$ is defined to be $z^{(\alpha, \beta)}$ on $M(1) \otimes e^{\beta}$, and $e_{\alpha}: k[L] \rightarrow k[L]$ is defined by $e_{\alpha}\left(e^{\beta}\right)=$ $\eta(\alpha, \beta) e^{\alpha+\beta}$ with a certain cocycle $\eta$ on $L$. The superalgebra $V_{L}$ is a vertex algebra if and only if $L$ is an even lattice. Here $L$ is called even if $(\alpha, \alpha)$ is even for any $\alpha \in L$.

Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra of type ADE with the root lattice $Q$. It is known that the affine vertex algebra $L_{1}(\mathfrak{g})$ of level 1 is isomorphic to the lattice vertex algebra $V_{Q}$.
4.3. Let $C$ be a $\mathbb{Z}$-basis of $L$. The vertex subsuperalgebra $W=W(C, L)$ of $V_{L}$ generated by $\left\{e^{\alpha} ; \alpha \in C\right\}$ is called a principal subalgebra of $V_{L}$ MP12.

For instance, the Feigin-Stoyanovsky principal subspaces of $L_{1}(\mathfrak{g})$ are isomorphic to the principal subalgebras $W(\Phi, Q)$ of lattice vertex algebras $V_{Q}$, where $\Phi$ is a base of the root system of $\mathfrak{g}$.
4.4. Let $B$ be a set and $N: B \times B \rightarrow \mathbb{Z}$ a symmetric function. The free vertex superalgebra $F=F(B, N)$ is freely generated by $B$ such that for any $a, b \in B$, the number $N(a, b)$ is the locality bound for the pair $a, b$. It has the following universal property: any vertex superalgebra generated by $B$ satisfying

$$
\begin{equation*}
(z-w)^{-N(a, b)}(Y(a, z) Y(b, w)-Y(b, w) Y(a, z))=0 \quad \text { with } a, b \in B \tag{4.1}
\end{equation*}
$$

is a sujective image of $F$ (see R02, Kaw15). The free vertex algebras were first mentioned in [866 and constructed in R02. The construction in R02 basically proceeds as follows:

- Consider the free associative algebra $A$ generated by the symbols $a(n)$ with $a \in B$ and $n \in \mathbb{Z}$.
- Take an appropriate completion $\widehat{A}$ of $A$ so that we can take the quotient of $\widehat{A}$ by the two-sided ideal generated by the left-hand sides of (4.1), which are infinite sums in general.
- Again quotient it by the left ideal generated by $a(n)$ for $a \in B$ and $n \geqslant 0$, which corresponds to axiom (3) in the definition of vertex algebras. This is the free vertex algebra $F=F(B, N)$.
- The set $B$ is embedded in $F$ by $a \mapsto a(-1)$.

See also Kaw15 for an account of Roitman's construction. There, the completion is taken as a generalization of the degreewise completion in the sense of MNT10, which is used to define the universal enveloping algebras of vertex algebras.

Let $L=L(B, N)$ be the free abelian group generated by $B$ with the $\mathbb{Z}$-bilinear form $(\cdot, \cdot): L \times L \rightarrow \mathbb{Z}$ defined by bilinearly extending the assignment $(a, b)=$ $-N(a, b)$ for $a, b \in B$. Then the free vertex (super)algebra $F(B, N)$ is isomorphic to the principal subalgebra $W(B, L)$ (see R02, Kaw15]).
4.5. We recall combinatorial bases of free vertex (super)algebras from R02, MP12, Kaw15. Let $B$ be a set and $N: B \times B \rightarrow \mathbb{Z}$ a symmetric function. Suppose that $B$ is totally ordered with the order $<$. The free vertex superalgebra $F=F(B, N)$ has the $\mathbb{C}$-basis which consists of the monomials of the form

$$
a_{m}\left(n_{m}+\sum_{i=1}^{m-1} N\left(a_{m}, a_{i}\right)\right) \cdots a_{2}\left(n_{2}+N\left(a_{2}, a_{1}\right)\right) a_{1}\left(n_{1}\right)|0\rangle
$$

with $m \geqslant 0, a_{1} \leqslant a_{2} \leqslant \ldots \leqslant a_{m} \in B$ and $n_{1}, n_{2}, \ldots, n_{m} \in \mathbb{Z}_{<0}$ such that $n_{i} \leqslant n_{i-1}$ if $a_{i}=a_{i-1}$ for $1<i \leqslant m$.
4.6. Let us take over the notation of the previous subsection and suppose from now on that $N(a, a) \in 2 \mathbb{Z}$ for every $a \in B$. In this case, the superalgebra $F=F(B, N)$ is a vertex algebra.

Recall that a vertex algebra $V$ is called commutative if $a(n) b=0$ for any $a, b \in V$ and $n \geqslant 0$. In this case, $V$ is a differential algebra with the multiplication $a \cdot b=$ $a(-1) b$ and differential $\partial=T$, the translation operator of $V$.

By the construction of free vertex algebras, we see that $F$ is commutative if and only if $N(a, b) \leqslant 0$.

From now on let us assume that $F$ is commutative.
Let $R_{F}$ be Zhu's Poisson algebra associated with $F$ :

$$
R_{F}:=F / C_{2}(F), \quad C_{2}(F):=\operatorname{span}_{\mathbb{C}}\{u(-2) v ; u, v \in F\}
$$

Let $u$ and $v$ be elements of $F$. Note that $u(n) v \in C_{2}(F)$ for any $n \leqslant-2$. We write by $\bar{u} \in R_{F}$ the image of $u$ under the canonical surjection $F \rightarrow R_{F}$. The product on $R_{F}$ is defined by $\bar{u} \bar{v}=\overline{u(-1) v}$ and the Poisson bracket is $\{\bar{u}, \bar{v}\}=\overline{u(0) v}$. As we assume that $F$ is commutative, the Poisson bracket is trivial and $R_{F}$ is a commutative associative algebra.

By using the combinatorial basis given in Subsection 4.5, we observe that

$$
\begin{equation*}
R_{F} \cong \mathbb{C}[B] /(a b ; a, b \in B, N(a, b) \leqslant-1) \tag{4.2}
\end{equation*}
$$

as algebras, where $\mathbb{C}[B]$ is the polynomial algebra with the set $B$ of independent variables.
4.7. In this section, we describe free vertex algebras as lifts of quotients of differential algebras. After finishing this work, we found that recent preprint LiH proves Theorem 4.8 and Corollary 4.10 including super-cases using theory of principal subspaces of lattice vertex superalgebras. We however would like to keep the present proofs which use the theory of free vertex algebras as it seems to be remarkably short.

Let $B$ be a set and $N: B \times B \rightarrow \mathbb{Z}$ a symmetric function. Suppose that $(a, a) \in 2 \mathbb{Z}$ for every $a \in B$. We assume that $N(a, b) \leqslant 0$ for all $a, b \in B$ so that $F=F(B, N)$ is commutative. Recall that $k\{B\}=k\left[a_{i} ; a \in B, i \in \mathbb{Z}_{\geqslant 0}\right]$ denotes the differential $k$-algebra with the differential $\partial$.

Theorem 4.8. Let $I$ be the differential ideal generated by $\left\{a_{-m-1} b_{0} ; a, b \in B, N(a, b) \leqslant\right.$ $m \leqslant-1\}$. Then, we have

$$
\begin{equation*}
F(B, N) \cong k\{B\} / I \tag{4.3}
\end{equation*}
$$

Proof. By the universality of $F$, we have the surjection $F \rightarrow k\{B\} / I$ since $k\{B\} / I$ is commutative. On the other hand, we have the surjection $\pi: k\{B\} / I \rightarrow W(B, L)$ from $k\{B\} / I$ to the principal subalgebra of the lattice vertex algebra $V_{L}$ with the lattice $L=L(B, N)$. Since $F \cong W(B, L)$, we have the assertions.

Remark 4.9. When the lattice $L(B, N)$ is positive definite, Theorem 4.8 is proved in [P14, Theorem 2].

Corollary 4.10. The free vertex algebra $F=F(B, N)$ is isomorphic to the jet lift of Zhu's Poisson algebra $R_{F}$

$$
\begin{equation*}
F(B, N) \cong k\{B\} /\left(a_{0} b_{0} ; a, b \in B, N(a, b) \leqslant-1\right) \tag{4.4}
\end{equation*}
$$

if and only if $N(a, a) \in\{0,-2\}$ for every $a \in B$ and $N(a, b) \in\{0,-1\}$ for every pair $(a, b)$ with $a \neq b \in B$.

Proof. Write $J$ the divisor of the right-hand side of (4.4). As $\partial\left(a_{0} a_{0}\right)=2 a_{1} a_{0}$, we see that $I=J$ if $N(a, a) \in\{0,-2\}$ for any $a \in B$ and $N(a, b) \in\{0,-1\}$ for all $a \neq b \in B$. Since $\partial^{2}\left(a_{0} a_{0}\right)=2 a_{2} a_{0}+a_{1} a_{1}$ and $\partial\left(a_{0} b_{0}\right)=a_{1} b_{0}+a_{0} b_{1}$, we have the second assertion.

Example 4.11. Let $B=\{a, b\}$ with $N(a, a)=N(b, b)=0$ and $N(a, b)=N(b, a)=$ -1. Then $R_{F} \cong k[a, b] /(a b)$ as algebras and $F \cong k\{a, b\} /\left(a_{0} b_{0}\right)$ as differential algebras, where $F=F(B, N)$.
4.12. We now apply Corollary 4.10 to give an answer to the second problem of Ritt.

Let $L$ be a lattice with the $\mathbb{Z}$-basis $C=\{\alpha, \beta\}$ and the symmetric bilinear form defined by

$$
(\alpha, \alpha)=0, \quad(\beta, \beta)=0, \quad(\alpha, \beta)=1
$$

Let us consider the lattice vertex algebra $V_{L}$ with $\mathfrak{h}=\mathbb{C} \otimes_{\mathbb{Z}} L$. Note that we have

$$
e_{\gamma} \cdot e^{\delta}(z)=z^{-(\gamma, \delta)} e^{\delta}(z) \cdot e_{\gamma} \quad \gamma, \delta \in L
$$

Note also that

$$
\begin{equation*}
E^{+}(h, z) 1 \otimes e^{\gamma}=1 \otimes e^{\gamma} \tag{4.5}
\end{equation*}
$$

for any $h \in \mathfrak{h}$ and $\gamma \in L$.
Lemma 4.13. (cf. [LL12, Proposition 6.3.14]) For any $\gamma, \delta \in L$,

$$
E^{+}(\gamma, z) E^{-}(\delta, w)=(1-w / z)^{(\gamma, \delta)} E^{-}(\delta, w) E^{+}(\gamma, z)
$$

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The vertex algebra $V_{L}$ is graded by conformal weights: $V_{L}=\bigoplus_{\Delta \in \mathbb{Z}}\left(V_{L}\right)_{\Delta}$, where $\left(V_{L}\right)_{\Delta}$ is the subspace spanned by the vectors of conformal weight $\Delta$ and the conformal weight of $h_{1}\left(-n_{1}\right) \cdots h_{m}\left(-n_{m}\right) \otimes e^{\gamma}$ is given by

$$
n_{1}+\cdots n_{m}+\frac{(\gamma, \gamma)}{2}
$$

Note that

$$
\begin{equation*}
e^{\alpha}(n) V_{\Delta} \subset V_{\Delta-n-1} . \tag{4.6}
\end{equation*}
$$

Now, the key point is that the free vertex algebra $F(B, N)$ defined in Example 1 is isomorphic to the following principal subspace $W$ of $V_{L}$ :

$$
W=W_{L}(C)=\left\langle 1 \otimes e^{\alpha}, 1 \otimes e^{\beta}\right\rangle \subset V_{L}
$$

It is defined by the assignment $a \mapsto 1 \otimes e^{\alpha}$ and $b \mapsto 1 \otimes e^{\beta}$.
Theorem 4.14. Let $i, j$ be non-negative integers. Then in $W$ we have

$$
\left(e^{\alpha}(-i-1) e^{\beta}(-j-1)\right)^{n}|0\rangle \neq 0 \quad \text { if and only if } n \leqslant i+j
$$

Proof. The element $\left(e^{\alpha}(-i-1) e^{\beta}(-j-1)\right)^{n}|0\rangle$ belongs to $M(1) \otimes \mathbb{C} e^{n(\alpha+\beta)} \subset V_{L}$. Since the conformal weight of $e^{n(\alpha+\beta)}$ is $n^{2}$, the conformal weight of any homogenous vector $v$ of $M(1) \otimes \mathbb{C} e^{n(\alpha+\beta)}$ is equal to or greater than $n^{2}$, and it equals to $n^{2}$ if and only if $v$ coincides with $e^{n(\alpha+\beta)}$ up to constant multiplication. On the other hand, $\left(e^{\alpha}(-i-1) e^{\beta}(-j-1)\right)^{n}|0\rangle$ is homogenous of conformal weight $(i+j) n$, see (4.6). Hence $\left(e^{\alpha}(-i-1) e^{\beta}(-j-1)\right)^{n}|0\rangle=0$ for $n>i+j$, and

$$
\begin{equation*}
\left(e^{\alpha}(-i-1) e^{\beta}(-j-1)\right)^{i+j}|0\rangle=c \otimes e^{(i+j)(\alpha+\beta)}, \quad c \in \mathbb{C} . \tag{4.7}
\end{equation*}
$$

for some $c \in \mathbb{C}$. It remains to show that $c \neq 0$. Since $W$ is commutative, the vector $\left(e^{\alpha}(-i-1) e^{\beta}(-j-1)\right)^{n}|0\rangle$ coincides with

$$
\begin{equation*}
\operatorname{Res}\left(z_{1}^{-i-1} \cdots z_{i+j}^{-i-1} w_{1}^{-j-1} \cdots w_{i+j}^{-j-1} e^{\alpha}\left(z_{1}\right) \cdots e^{\alpha}\left(z_{i+j}\right) e^{\beta}\left(w_{1}\right) \cdots e^{\beta}\left(w_{i+j}\right)|0\rangle\right) \tag{4.8}
\end{equation*}
$$

Here Res $f\left(z_{1}, \ldots, z_{i+j}, w_{1}, \ldots, w_{i+j}\right)$ denotes the coefficient of $z_{1}^{-1} \ldots z_{i+1}^{-1} w_{1}^{-1} \ldots w_{i+j}^{-1}$ in $f$. Using Lemma 4.13 repeatedly, we have up to some non-zero multiple from 2-cocycle $\varepsilon$ that

$$
\begin{aligned}
& e^{\alpha}\left(z_{1}\right) \cdots e^{\alpha}\left(z_{i+j}\right) e^{\beta}\left(w_{1}\right) \cdots e^{\beta}\left(w_{i+j}\right) \\
& =z_{1}^{i+j} \cdots z_{i+j}^{i+j} \prod_{k, \ell=1}^{i+j}\left(1-w_{k} / z_{\ell}\right) E^{-}\left(-\alpha, z_{1}\right) \cdots E^{-}\left(-\alpha, z_{i+j}\right) E^{-}\left(-\beta, w_{1}\right) \cdots E^{-}\left(-\beta, w_{i+j}\right) \\
& \cdot E^{+}\left(-\alpha, z_{1}\right) \cdots E^{+}\left(-\alpha, z_{i+j}\right) E^{+}\left(-\beta, w_{1}\right) \cdots E^{+}\left(-\beta, w_{i+j}\right) \otimes e_{(i+j) \alpha}\left(z_{1} \ldots z_{i+r}\right)^{\alpha} e_{(i+j) \beta}\left(w_{1} \ldots w_{i+j}\right)^{\beta} .
\end{aligned}
$$

Hence by (4.5), we find that up to some nonzero multiple from 2-cocycle (4.8) is equal to

$$
\begin{align*}
& \operatorname{Res}\left(z_{1}^{j-1} \cdots z_{i+j}^{j-1} w_{1}^{-j-1} \cdots w_{i+j}^{-j-1} \prod_{k, \ell=1}^{i+j}\left(1-w_{k} / z_{\ell}\right)\right.  \tag{4.9}\\
& \left.\quad E^{-}\left(-\alpha, z_{1}\right) \cdots E^{-}\left(-\alpha, z_{i+j}\right) E^{-}\left(-\beta, w_{1}\right) \cdots E^{-}\left(-\beta, w_{i+j}\right) \otimes e^{(i+j)(\alpha+\beta)}\right)
\end{align*}
$$

Since it must be equal to $1 \otimes e^{(i+j)(\alpha+\beta)}$ up to constant multiplication, from weight consideration we conclude that (4.9) coincides with

$$
\operatorname{Res}\left(z_{1}^{j-1} \cdots z_{i+j}^{j-1} w_{1}^{-j-1} \cdots w_{i+j}^{-j-1} \prod_{k, \ell=1}^{i+j}\left(1-w_{k} / z_{\ell}\right) 1 \otimes e^{(i+j)(\alpha+\beta)}\right)
$$

If $j=1$, then this equals $-(i+j)!\otimes e^{(i+j)(\alpha+\beta)}$. In a similar way, we see that it is equal to $(-1)^{j} P_{i+j, j} \otimes e^{(i+j)(\alpha+\beta)}$, where $P_{i+j, j}$ is the number of arrangements of 0 and 1 on the $(i+j) \times(i+j)$-square such that any row and any column have exactly $j$ tuples of 1. Since $P_{i+j, j} \neq 0$, we have Theorem4.14.

It follows immediately from Theorem 4.14 that the answer to the second question of Ritt is given by $i+j+1$.

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