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$\mathbb{Z}_k$-code vertex operator algebras

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Abstract. We introduce a simple, self-dual, rational, and C_2-cofinite vertex operator algebra of CFT-type associated with a $\mathbb{Z}_k$-code for $k \geq 2$. Our argument is based on the $\mathbb{Z}_k$-symmetry among the simple current modules for the parafermion vertex operator algebra $K(\mathfrak{sl}_2, k)$. We show that it is naturally realized as the commutant of a certain subalgebra in a lattice vertex operator algebra. Furthermore, we construct all the irreducible modules inside a module for the lattice vertex operator algebra.

1. Introduction.

The parafermion vertex operator algebra $K(\mathfrak{g}, k)$ associated with a finite dimensional simple Lie algebra $\mathfrak{g}$ and a positive integer k is by definition the commutant of the Heisenberg vertex operator algebra generated by the Cartan subalgebra of $\mathfrak{g}$ in $L_{\widehat{\mathfrak{g}}}(k, 0)$, where $L_{\widehat{\mathfrak{g}}}(k, 0)$ is the simple affine vertex operator algebra associated with the affine Kac–Moody Lie algebra $\widehat{\mathfrak{g}}$ at level k. In the case where $\mathfrak{g} = \mathfrak{sl}_2$ and $k \geq 2$, $K(\mathfrak{sl}_2, k)$ is isomorphic to a minimal series principal W-algebra of type A which is a simple, self-dual, rational, and C_2-cofinite vertex operator algebra of CFT-type [**2**], and has exactly k simple currents M^j, $j \in \mathbb{Z}_k$, with $\mathbb{Z}_k$-symmetry. That is, those simple currents form a cyclic group of order k with respect to the fusion product, $M^i \boxtimes_{M^0} M^j = M^{i+j}$ for $i, j \in \mathbb{Z}_k$ with $M^0 = K(\mathfrak{sl}_2, k)$.

In this article we introduce a vertex operator algebra M_D associated with a $\mathbb{Z}_k$-code D of lenght ℓ. Here, a $\mathbb{Z}_k$-code D is an additive subgroup of $(\mathbb{Z}_k)^\ell$. For each codeword $\xi = (\xi_1, \ldots, \xi_\ell) \in D$, we associate the tensor product $M_\xi = M^{\xi_1} \otimes \cdots \otimes M^{\xi_\ell}$ of simple current $K(\mathfrak{sl}_2, k)$-modules M^{ξ_r}, $1 \leq r \leq \ell$. Then the direct sum

$$M_D = \bigoplus_{\xi \in D} M_\xi$$

has a structure of an abelian intertwining algebra [**14**, Theorem 4.1]. Furthermore, M_D becomes a vertex operator algebra if each M_ξ has integral conformal weight [**14**, Theorem 4.2]. Being a D-graded simple current exrension of $M_{\mathbf{0}} = K(\mathfrak{sl}_2, k)^{\otimes \ell}$, the vertex operator algebra M_D is simple, self-dual, rational, C_2-cofinite, and of CFT-type with central charge $2\ell(k-1)/(k+2)$ (Theorem 7.3). Such a construction of M_D was initiated in [**35**] for the case $k = 2$, and the properties of the vertex operator algebra M_D for $k = 2$ have been studied extensively, see [**6**], [**31**], [**36**], [**37**] and the references

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therein. The vertex operator algebra M_D for $k = 3$ was constructed by a slightly different method in [**23**], and its irreducible modules were studied in [**25**].

We realize the vertex operator algebra M_D inside a vertex operator algebra V_{Γ_D} associated with a certain positive definite even lattice Γ_D. Moreover, every irreducible M_D-module is explicitly described inside a module for the lattice vertex operator algebra V_{Γ_D}.

More precisely, consider the lattice vertex operator algebra $V_{\sqrt{2}A_{k-1}}$, which is an extension of the vertex operator algebra $K(\mathfrak{sl}_2, k) \otimes K(\mathfrak{sl}_k, 2)$. There are cosets $N^{(j)}$, $j \in \mathbb{Z}_k$, of $\sqrt{2}A_{k-1}$ in the dual lattice $(\sqrt{2}A_{k-1})^\circ$ such that $N^{(i)} + N^{(j)} = N^{(i+j)}$, and $V_{N^{(j)}}$ contains M^j. For $\xi = (\xi_1, \ldots, \xi_\ell) \in D$, we consider a coset $N(\xi) = N^{(\xi_1)} \times \cdots \times N^{(\xi_\ell)}$ of $(\sqrt{2}A_{k-1})^\ell$ in $((\sqrt{2}A_{k-1})^\circ)^\ell$. The union Γ_D of those cosets is a positive definite even lattice if and only if $(\xi|\xi) = 0$ for all $\xi \in D$ (Lemma 7.1), where $(\cdot|\cdot)$ is the standard inner product on $(\mathbb{Z}_k)^\ell$. Then M_D is realized as the commutant of $K(\mathfrak{sl}_k, 2)^{\otimes \ell}$ in the lattice vertex operator algebra V_{Γ_D} (Equation (7.4)).

We also consider a necessary and sufficient condition on the code D for which Γ_D is a positive definite odd lattice, and M_D is a vertex operator superalgebra.

Using the representation theory of simple current extensions (Section 2.2), we construct all the irreducible M_D-modules inside $V_{(\Gamma_D)^\circ}$, where $(\Gamma_D)^\circ$ is the dual lattice of Γ_D (Theorems 8.7, 8.9, and 8.10). Any linear character χ of the finite abelian group D naturally induces an automorphism of the vertex operator algebra M_D. We discuss irreducible χ-twisted M_D-modules as well. In particular, we obtain the number of inequivalent irreducible χ-twisted M_D-modules (Theorem 8.12). We also study the irreducible M_D-modules in the case where M_D is a vertex operator superalgebra (Theorem 9.1).

The construction of M_D as a commutant of $K(\mathfrak{sl}_k, 2)^{\otimes \ell}$ in the lattice vertex operator algebra V_{Γ_D} was previously discussed in [**3**]. However, the treatment of the simple current $K(\mathfrak{sl}_2, k)$-modules M^j in $V_{N^{(j)}}$, $j \in \mathbb{Z}_k$, was slightly different, and the method there is not suitable for all the irreducible $K(\mathfrak{sl}_2, k)$-modules in $V_{(\sqrt{2}A_{k-1})^\circ}$. In the present paper, we use decompositions of certain irreducible $V_{\sqrt{2}A_{k-1}}$-modules (Proposition 6.3), from which we know how the irreducible $K(\mathfrak{sl}_2, k)$-modules appear in $V_{(\sqrt{2}A_{k-1})^\circ}$ (Proposition 6.4), and it enables us to describe the irreducible M_D-modules inside $V_{(\Gamma_D)^\circ}$.

This paper is organized as follows. Section 2 is devoted to preliminaries, where we recall the representation theory of simple current extensions. In Section 3, we review the properties of the parafermion vertex operator algebra $K(\mathfrak{sl}_2, k)$ for later use. In Sections 4, 5, and 6, we describe the cosets of $N = \sqrt{2}A_{k-1}$ in $N^\circ = (\sqrt{2}A_{k-1})^\circ$, and study how irreducible $K(\mathfrak{sl}_2, k)$-modules appear in the irreducible V_N-modules. The vertex operator algebra M_D is defined in Section 7. In Section 8, we study the irreducible twisted and untwisted modules for M_D, including the classification of irreducible modules, and realizations of the irreducible modules in $V_{(N^\circ)^\ell}$. In Section 9, we discuss the irreducible M_D-modules in the case where M_D is a vertex operator superalgebra. Finally, in Section 10, we mention some known examples of M_D. We calculate the minimal norm of elements in each coset of N in N° in Appendix A.

As to the $P(z)$-tensor product $\boxtimes_{P(z)}$ of [**19**] for a vertex operator algebra V, we only use it with $z = 1$. We write $\boxtimes_V$ for $\boxtimes_{P(1)}$, and call it the fusion product. We also use $\otimes$ to denote the tensor product of vertex operator algebras and their modules as in [**15**].

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2. Preliminaries.

In this section, we recall some basic properties of simple current extensions of vertex operator algebras and their irreducible modules. Our notations for vertex operator algebras and their modules are standard [**15**], [**16**], [**32**].

2.1. Simple current modules.

Let V be a simple, self-dual, rational, and C_2-cofinite vertex operator algebra of CFT-type. Then a fusion product $M \boxtimes_V N$ over V of any V-modules M and N exists [**20**], [**34**]. The fusion product is commutative and associative [**18**, Theorem 3.7].

We denote by $\mathrm{Irr}(V)$ the set of equivalence classes of irreducible V-modules. Then

$$M^1 \boxtimes_V M^2 = \sum_{M^3 \in \mathrm{Irr}(V)} \dim I_V \binom{M^3}{M^1\ M^2} M^3$$

for $M^1, M^2 \in \mathrm{Irr}(V)$, where $I_V\binom{M^3}{M^1\ M^2}$ is the set of all intertwining operators of type $\binom{M^3}{M^1\ M^2}$. An irreducible V-module A is called a simple current if $A \boxtimes_V X$ is an irreducible V-module for any $X \in \mathrm{Irr}(V)$. A set $\{A^\alpha \mid \alpha \in D\}$ of simple current V-modules indexed by a finite abelian group D is said to be D-graded if A^α, $\alpha \in D$, are inequivalent to each other with $A^0 = V$ and $A^\alpha \boxtimes_V A^\beta = A^{\alpha+\beta}$, $\alpha, \beta \in D$. The set $\mathrm{Irr}(V)_{\mathrm{sc}}$ of equivalence classes of simple current V-modules is graded by a finite abelian group [**31**, Corollary 1]. The inverse of $A \in \mathrm{Irr}(V)_{\mathrm{sc}}$ with respect to the fusion product is its contragredient module A'. The fusion product by $A \in \mathrm{Irr}(V)_{\mathrm{sc}}$ induces a permutation

$$X \mapsto A \boxtimes_V X \tag{2.1}$$

on $\mathrm{Irr}(V)$. For a V-module X, we denote its conformal weight by $h(X)$, which is a rational number [**10**, Theorem 11.3]. We define a map $b_V : \mathrm{Irr}(V)_{\mathrm{sc}} \times \mathrm{Irr}(V) \to \mathbb{Q}/\mathbb{Z}$ by

$$b_V(A, X) = h(A \boxtimes_V X) - h(A) - h(X) + \mathbb{Z} \tag{2.2}$$

for $A \in \mathrm{Irr}(V)_{\mathrm{sc}}$ and $X \in \mathrm{Irr}(V)$. The map b_V was introduced in [**14**, Section 3] in the case where $\mathrm{Irr}(V)_{\mathrm{sc}} = \mathrm{Irr}(V)$, see also [**38**, Section 2]. A proof of the following lemma can be found in [**42**, Section 2].

Lemma 2.1. *Let A, $B \in \mathrm{Irr}(V)_{\mathrm{sc}}$, and $X \in \mathrm{Irr}(V)$.*
(1) $b_V(A \boxtimes_V B, X) = b_V(A, X) + b_V(B, X)$.
(2) $b_V(A, B \boxtimes_V X) = b_V(A, B) + b_V(A, X)$.

2.2. Representations of simple current extensions.

Let V be a simple, self-dual, rational, and C_2-cofinite vertex operator algebra of CFT-type. Let $\{V^\alpha \mid \alpha \in D\}$ be a D-graded set of simple current V-modules for a finite abelian group D with $V^0 = V$ and $h(V^\alpha) \in (1/2)\mathbb{Z}$ for all $\alpha \in D$. Then the direct sum $V_D = \bigoplus_{\alpha \in D} V^\alpha$ has a structure of either a simple vertex operator algebra or a simple vertex operator superalgebra which extends the V-module structure on V_D [**5**, Theorem 3.12], see also the references therein. Such a simple vertex operator (super)algebra structure on V_D is unique [**12**, Proposition 5.3]. The vertex operator (super)algebra V_D is called a D-graded simple current extension of V. In this section, we only consider the case in which $h(V^\alpha) \in \mathbb{Z}$ for all $\alpha \in D$, and V_D is a vertex operator algebra. It is known that V_D is simple, self-dual, rational, C_2-cofinite, and of CFT-type [**43**, Theorem 2.14].

We recall the representation theory of V_D from [**24**], [**43**]. As to the notion of a g-twisted module for a vertex operator algebra with respect to its automorphism g, we adopt the definition in [**10**]. Thus a g-twisted module in [**43**] means a g^{-1}-twisted module in this paper.

Let $D^* = \mathrm{Hom}(D, \mathbb{C}^\times)$ be the character group of D. For $\chi \in D^*$, a scalar multiplication by $\chi(\alpha)$ on V^α, $\alpha \in D$, is an automorphism of the vertex operator algebra V_D. That is, D^* naturally acts on V_D, and we can regard D^* as a subgroup of $\mathrm{Aut}\, V_D$. Let M be a χ-twisted V_D-module for $\chi \in D^*$. We say M is D-graded if there is a decomposition $M = \bigoplus_{\alpha \in D} M^\alpha$ as a V-module such that $0 \neq V^\alpha \cdot M^\beta \subset M^{\alpha+\beta}$ for α, $\beta \in D$, where we set $V^\alpha \cdot S = \mathrm{span}\{a_{(n)}v \mid a \in V^\alpha, v \in S, n \in \mathbb{Q}\}$ for a subset S of M.

We consider the action of D on $\mathrm{Irr}(V)$ in (2.1). Let $\mathrm{Irr}(V) = \bigcup_{i \in I} \mathscr{O}_i$ be the D-orbit decomposition. Using the map b_V in (2.2), we define a map $\chi_X : D \to \mathbb{C}^\times$ by

$$\chi_X(\alpha) = \exp(2\pi\sqrt{-1}\, b_V(V^\alpha, X))$$

for $X \in \mathrm{Irr}(V)$. The map χ_X is a linear character of D by (1) of Lemma 2.1. For a D-orbit $\mathscr{O}_i$, (2) of Lemma 2.1 implies that χ_X is independent of the choice of $X \in \mathscr{O}_i$, as $h(V^\alpha) \in \mathbb{Z}$ for all $\alpha \in D$. Thus χ_X is uniquely determined by $\mathscr{O}_i$, so we can write χ_i for χ_X.

We summarize [**24**, Theorem 4.4] and [**43**, Lemma 2.11, Theorems 2.14, 2.19, 3.2, 3.3] as follows.

Theorem 2.2. *Let V_D be a D-graded simple current extension of V, and let $X \in \mathrm{Irr}(V)$.*

(1) *There exists a unique structure of a D-graded χ_X-twisted V_D-module on the space $V_D \boxtimes_V X = \bigoplus_{\alpha \in D} V^\alpha \boxtimes_V X$ which contains $V^0 \boxtimes_V X \cong X$ as a V-submodule.*

(2) *If $M = \bigoplus_{\alpha \in D} M^\alpha$ is a D-graded χ_X-twisted V_D-module such that $X \subset M^\alpha$ as a V-submodule for some $\alpha \in D$, then $V_D \cdot X$ is isomorphic to the D-graded χ_X-twisted V_D-module $V_D \boxtimes_V X$ in the assertion* (1), *where $V_D \cdot X = \mathrm{span}\{a_{(n)}v \mid a \in V_D, v \in X, n \in \mathbb{Q}\} \subset M$.*

(3) *Let $\sigma \in \mathrm{Aut}\, V_D$ such that σ is the identity on V. Assume that there is a σ-twisted V_D-module containing X as a V-submodule. Then $\sigma = \chi_X$, and there exists a surjective V_D-homomorphism from $V_D \boxtimes_V X$ onto $V_D \cdot X$.*

For a D-orbit $\mathscr{O}_i$ in $\mathrm{Irr}(V)$, the structure of a D-graded χ_X-twisted V_D-module on the space $V_D \boxtimes_V X$ in (1) of the above theorem is independent of the choice of $X \in \mathscr{O}_i$, and it is uniquely determined by $\mathscr{O}_i$. The χ_X-twisted V_D-module $V_D \boxtimes_V X$ is not necessarily irreducible. The assertion (3) of the above theorem implies that $V_D \cdot X$ is isomorphic to a direct summand of $V_D \boxtimes_V X$.

Since any irreducible χ-twisted V_D-module for $\chi \in D^*$ is isomorphic to a direct summand of the χ_X-twisted V_D-module $V_D \boxtimes_V X$ with $\chi = \chi_X$ for some $X \in \mathrm{Irr}(V)$ by Theorem 2.2, the study of χ-twisted V_D-modules is reduced to the study of the χ_X-twisted V_D-module $V_D \boxtimes_V X$.

Let $D_X = \{\alpha \in D \mid V^\alpha \boxtimes_V X \cong X\}$ be the stabilizer of $X \in \mathrm{Irr}(V)$ for the action of D on $\mathrm{Irr}(V)$ in (2.1). For a D-orbit $\mathscr{O}_i$, the stabilizer D_X is independent of the choice of $X \in \mathscr{O}_i$, and it is uniquely determined by $\mathscr{O}_i$. Hence we can write D_i for D_X.

In the case where $D_X = 0$, the following assertion holds [**39**, Proposition 3.8].

PROPOSITION 2.3. *If $D_X = 0$, then $V_D \boxtimes_V X$ is an irreducible χ_X-twisted V_D-module.*

If D_X is non-trivial, then the χ_X-twisted V_D-module $V_D \boxtimes_V X$ is reducible, and we need to take some 2-cocycles of D_X into account to obtain its irreducible decomposition as discussed in [**24**], [**43**]. Let $X \in \mathrm{Irr}(V)$, and assume that $D_X \neq 0$. We consider the D_X-graded simple current extension $V_{D_X} = \bigoplus_{\alpha \in D_X} V^\alpha$ of V. Set $V_{\beta+D_X} = \bigoplus_{\alpha\in\beta+D_X} V^\alpha$ for a coset $\beta + D_X \in D/D_X$. Then $V_D = \bigoplus_{\beta+D_X\in D/D_X} V_{\beta+D_X}$ is a D/D_X-graded simple current extension of V_{D_X}. Note that $V_{D_X} \boxtimes_V X \cong X^{\oplus|D_X|}$ as V-modules. Set $Q = \mathrm{Hom}_V(X, V_{D_X} \boxtimes_V X)$. Then $\dim Q = |D_X|$, and we have a canonical isomorphism

$$V_{D_X} \boxtimes_V X \cong X \otimes Q. \tag{2.3}$$

It is shown in [**24**, Theorem 3.10] and [**43**, Theorems 2.14, 2.19] that there exists a 2-cocycle $\epsilon \in Z^2(D_X, \mathbb{C}^\times)$ such that the space Q carries a structure of a module for a twisted group algebra $\mathbb{C}^\epsilon[D_X]$ associated with ϵ [**22**, Chapter 2]. Indeed, Q is isomorphic to the regular representation of $\mathbb{C}^\epsilon[D_X]$. If R is a $\mathbb{C}^\epsilon[D_X]$-submodule of Q, then the subspace $X \otimes R$ of $X \otimes Q$ in (2.3) is a V_{D_X}-submodule of $V_{D_X} \boxtimes_V X$. Thus the irreducible decomposition of $V_{D_X} \boxtimes_V X$ as a V_{D_X}-module is obtained by the irreducible decomposition of Q as a $\mathbb{C}^\epsilon[D_X]$-module.

Let T be an irreducible V_{D_X}-submodule of $V_{D_X} \boxtimes_V X$. Then T is also a direct sum of some copies of X as a V-module, and $V_{\beta+D_X} \boxtimes_V T$, $\beta + D_X \in D/D_X$, are inequivalent irreducible V_{D_X}-modules. Hence the χ_X-twisted V_D-module $V_D \boxtimes_{V_{D_X}} T$ is irreducible by Proposition 2.3. The χ_X-twisted V_D-module structure of $V_D \boxtimes_{V_{D_X}} T$ is uniquely determined by T. Therefore, the irreducible decomposition of $V_D \boxtimes_V X$ as a χ_X-twisted V_D-module is in one-to-one correspondence with the irreducible decomposition of Q in (2.3) as a $\mathbb{C}^\epsilon[D_X]$-module.

The determination of the 2-cocycle ϵ requires more information on the associativity constraints of the fusion products of V-modules [**24**], [**43**]. However, we will only deal with the case where D_X can be regarded as a binary code in this paper. So we make the following assumption on D_X.

HYPOTHESIS 2.4. (1) M^0 is a simple, self-dual, rational, and C_2-cofinite vertex operator algebra of CFT-type.

(2) M^1 is a self-dual simple current M^0-module such that the $\mathbb{Z}_2$-graded simple current extension $M^0 \oplus M^1$ of M^0 is either a simple vertex operator algebra with $h(M^1) \in \mathbb{Z}$ or a simple vertex operator superalgebra with $h(M^1) \in \mathbb{Z} + 1/2$.

(3) For any irreducible M^0-module P, the direct sum $P^0 \oplus P^1$ with $P^0 = P$ and $P^1 = M^1 \boxtimes_{M^0} P$ has a unique structure of a $\mathbb{Z}_2$-graded either untwisted or $\mathbb{Z}_2$-twisted $M^0 \oplus M^1$-module.

(4) $V = (M^0)^{\otimes n}$ for some $n > 0$.

(5) $X \in \mathrm{Irr}(V)$ with $D_X \neq 0$. Moreover, D_X has a structure of a binary code of length n, and $V^\alpha \cong M^{\alpha_1} \otimes \cdots \otimes M^{\alpha_n}$ for $\alpha = (\alpha_1, \ldots, \alpha_n) \in D_X$. In particular,

$$V_{D_X} = \bigoplus_{\alpha=(\alpha_1,\ldots,\alpha_n)\in D_X} M^{\alpha_1} \otimes \cdots \otimes M^{\alpha_n} \subset (M^0 \oplus M^1)^{\otimes n}$$

as an extension of $V = (M^0)^{\otimes n}$.

Suppose V_{D_X} satisfies Hypothesis 2.4. Under this assumption, we can describe the 2-cocycle $\epsilon \in Z^2(D_X, \mathbb{C}^\times)$ explicitly. We divide our argument into two cases.

Case 1. Suppose $M^0 \oplus M^1$ is a simple vertex operator algebra with $h(M^1) \in \mathbb{Z}$. By (3) of Hypothesis 2.4, the 2-cocycle $\epsilon \in Z^2(D_X, \mathbb{C}^\times)$ is cohomologous to a 2-coboundary by [**22**, Chapter 2, Corollary 2.5]. Hence Q is the regular representation of an ordinary group algebra $\mathbb{C}[D_X]$, so that Q is a direct sum of $|D_X|$ inequivalent irreducible $\mathbb{C}[D_X]$-modules. Therefore, $V_{D_X} \boxtimes_V X$ decomposes into a direct sum of $|D_X|$ inequivalent irreducible V_{D_X}-submodules. By considering V_D as a D/D_X-graded simple current extension of V_{D_X}, we see that the irreducible decomposition of $V_D \boxtimes_V X$ as a χ_X-twisted V_D-module is as follows.

Proposition 2.5. *Suppose $D_X \neq 0$ and V_{D_X} satisfies Hypothesis* 2.4. *Suppose further that $M^0 \oplus M^1$ in* (2) *of Hypothesis* 2.4 *is a simple vertex operator algebra with $h(M^1) \in \mathbb{Z}$. Then the irreducible decomposition of the χ_X-twisted V_D-module $V_D \boxtimes_V X$ is given as*

$$V_D \boxtimes_V X = \bigoplus_{j=1}^{|D_X|} U^j,$$

where U^j, $1 \le j \le |D_X|$, are inequivalent irreducible χ_X-twisted V_D-modules. Furthermore, $U^j \cong \bigoplus_{W \in \mathscr{O}_i} W$ as V-modules, where $\mathscr{O}_i$ is the D-orbit in $\mathrm{Irr}(V)$ *containing X.*

Case 2. Suppose $M^0 \oplus M^1$ is a simple vertex operator superalgebra with $h(M^1) \in \mathbb{Z} + 1/2$. In this case, D_X is an even binary code, as the conformal weight of $V^\alpha \cong M^{\alpha_1} \otimes \cdots \otimes M^{\alpha_n}$ is an integer for $\alpha = (\alpha_1, \ldots, \alpha_n) \in D_X$. By (3) of Hypothesis 2.4, we can find the 2-cocycle ϵ inside $Z^2(D_X, \{\pm 1\})$ which satisfies

$$\epsilon(\alpha, \alpha) = (-1)^{\mathrm{wt}(\alpha)/2}, \quad \epsilon(\alpha, \beta)\epsilon(\beta, \alpha) = (-1)^{(\alpha|\beta)} \tag{2.4}$$

for $\alpha, \beta \in D_X$, where $\mathrm{wt}(\alpha)$ is the Hamming weight of α, and $(\,\cdot\,|\,\cdot\,)$ is the standard inner product on $(\mathbb{Z}_2)^n$ [**31**, Section 4.1], see also [**35**], [**36**]. The conditions above uniquely determine the class of ϵ in $H^2(D_X, \{\pm 1\})$ [**16**, Proposition 5.3.3].

It is shown in [**16**, Theorem 5.5.1] that each irreducible representation of $\mathbb{C}^\epsilon[D_X]$ is induced from an irreducible representation of its maximal commutative subalgebra, and the equivalence classes of irreducible $\mathbb{C}^\epsilon[D_X]$-modules are distinguished by their central characters. Let $D_X^\perp = \{\alpha \in (\mathbb{Z}_2)^n \mid (\alpha|D_X) = 0\}$ be the dual code of the binary code D_X, and let E be a maximal self-orthogonal subcode of D_X. It follows from (2.4) that the center of $\mathbb{C}^\epsilon[D_X]$ is $\mathbb{C}^\epsilon[D_X \cap D_X^\perp]$, and $\mathbb{C}^\epsilon[E]$ is a maximal commutative subalgebra of $\mathbb{C}^\epsilon[D_X]$. Since $\mathbb{C}^\epsilon[D_X \cap D_X^\perp] \cong \mathbb{C}[D_X \cap D_X^\perp]$ is an ordinary group algebra, the number of inequivalent irreducible representations of $\mathbb{C}^\epsilon[D_X]$ is equal to that of $\mathbb{C}[D_X \cap D_X^\perp]$, which coincides with the order $|D_X \cap D_X^\perp|$ of $D_X \cap D_X^\perp$. Each irreducible $\mathbb{C}^\epsilon[D_X]$-module has dimension $[D_X : E] = [E : D_X \cap D_X^\perp]$, namely, $[D_X : D_X \cap D_X^\perp]^{1/2}$ [**16**, Theorem 5.5.1]. Since the space Q in (2.3) is isomorphic to the regular representation of $\mathbb{C}^\epsilon[D_X]$, the irreducible decomposition of $V_D \boxtimes_V X$ as a χ_X-twisted V_D-module is as follows.

Proposition 2.6. *Suppose $D_X \neq 0$ and V_{D_X} satisfies Hypothesis 2.4. Suppose further that $M^0 \oplus M^1$ in (2) of Hypothesis 2.4 is a simple vertex operator superalgebra with $h(M^1) \in \mathbb{Z} + 1/2$. Then the irreducible decomposition of the χ_X-twisted V_D-module $V_D \boxtimes_V X$ is given as*

$$V_D \boxtimes_V X = \bigoplus_{j=1}^{|D_X \cap D_X^\perp|} (U^j)^{\oplus m},$$

where $m = [D_X : D_X \cap D_X^\perp]^{1/2}$, and U^j, $1 \le j \le |D_X \cap D_X^\perp|$, are inequivalent irreducible χ_X-twisted V_D-modules. Furthermore, $U^j \cong \bigoplus_{W \in \mathscr{O}_i} W^{\oplus m}$ as V-modules, where $\mathscr{O}_i$ is the D-orbit in $\mathrm{Irr}(V)$ containing X.

3. Parafermion vertex operator algebra $K(\mathfrak{sl}_2, k)$.

In this section, we recall the properties of the parafermion vertex operator algebra $K(\mathfrak{sl}_2, k)$ for $2 \le k \in \mathbb{Z}$. If $k = 2$, then $K(\mathfrak{sl}_2, 2)$ is isomorphic to the Virasoro vertex operator algebra $L(1/2, 0)$ of central charge $1/2$. So we assume that $k \ge 3$ for the rest of this section.

Let $\{h, e, f\}$ be a standard Chevalley basis of the Lie algebra $\mathfrak{sl}_2$. Let $L_{\widehat{\mathfrak{sl}}_2}(k, 0)$ be the simple affine vertex operator algebra associated with $\widehat{\mathfrak{sl}}_2$ and level k. Then $K(\mathfrak{sl}_2, k)$ is defined to be the commutant of the Heisenberg vertex operator algebra generated by $h(-1)\mathbf{1}$ in $L_{\widehat{\mathfrak{sl}}_2}(k, 0)$ [**7**], [**8**], [**9**].

We follow the notaions in [**8**, Section 4]. Let $L = \mathbb{Z}\alpha_1 + \cdots + \mathbb{Z}\alpha_k$ with $\langle \alpha_i, \alpha_j \rangle = 2\delta_{i,j}$ and $\gamma = \alpha_1 + \cdots + \alpha_k$. Let H, E, and $F \in V_L$ be as in [**8**, Section 4]. Then the component operators $H_{(n)}$, $E_{(n)}$, $F_{(n)}$, $n \in \mathbb{Z}$, give a level k representation of $\widehat{\mathfrak{sl}}_2$ under the correspondence $h(n) \leftrightarrow H_{(n)}$, $e(n) \leftrightarrow E_{(n)}$, $f(n) \leftrightarrow F_{(n)}$, and the subalgebra V^{aff} of the vertex operator algebra $V_L \cong L_{\widehat{\mathfrak{sl}}_2}(1, 0)^{\otimes k}$ generated by H, E, and F is isomorphic to $L_{\widehat{\mathfrak{sl}}_2}(k, 0)$. We identify V^{aff} with $L_{\widehat{\mathfrak{sl}}_2}(k, 0)$. We also identify $H_{(n)}$, $E_{(n)}$, and $F_{(n)}$ with $h(n)$, $e(n)$, and $f(n)$, respectively. Let

$$M^j = \{v \in L_{\widehat{\mathfrak{sl}}_2}(k, 0) \mid H_{(n)}v = -2j\delta_{n,0}v \text{ for } n \ge 0\}.$$

Then $M^0 = K(\mathfrak{sl}_2, k)$, and $L_{\widehat{\mathfrak{sl}}_2}(k, 0) = \bigoplus_{j=0}^{k-1} M^j \otimes V_{\mathbb{Z}\gamma - j\gamma/k}$ as $M^0 \otimes V_{\mathbb{Z}\gamma}$-modules [**8**, Lemma 4.2]. The index j of M^j can be considered to be modulo k.

Let $L^\circ = (1/2)L$ be the dual lattice of L, and let v^i, $0 \le i \le k$, and $v^{i,j}$, $0 \le j \le i$, be as in [**8**, Section 4]. Then the V^{aff}-submodule $V^{\mathrm{aff}} \cdot v^i$ of V_{L° generated by v^i is isomorphic to an irreducible $L_{\widehat{\mathfrak{sl}}_2}(k,0)$-module $L_{\widehat{\mathfrak{sl}}_2}(k,i)$ with top level $\mathrm{span}\{v^{i,j} \mid 0 \le j \le i\}$ of conformal weight $i(i+2)/4(k+2)$ [**17**], [**32**, Section 6.2]. Let

$$M^{i,j} = \{v \in V^{\mathrm{aff}} \cdot v^i \mid H_{(n)}v = (i-2j)\delta_{n,0}v \text{ for } n \ge 0\}$$

for $0 \le i \le k$, $0 \le j \le k-1$. Then

$$L_{\widehat{\mathfrak{sl}}_2}(k,i) = \bigoplus_{j=0}^{k-1} M^{i,j} \otimes V_{\mathbb{Z}\gamma+(i-2j)\gamma/2k} \tag{3.1}$$

as $M^0 \otimes V_{\mathbb{Z}\gamma}$-modules [**8**, Lemma 4.3]. The index j of $M^{i,j}$ can be considered to be modulo k. Note that $M^{0,j} = M^j$.

The -1 isometry of the lattice L lifts to an automorphism θ of the vertex operator algebra V_L of order 2. Actually, $\theta(H) = -H$, $\theta(E) = F$, and $\theta(F) = E$.

We summarize the properties of $M^0 = K(\mathfrak{sl}_2, k)$ [**1**], [**2**], [**7**], [**8**], [**13**].

(1) M^0 is a simple, self-dual, rational, and C_2-cofinite vertex operator algebra of CFT-type with central charge $2(k-1)/(k+2)$.

(2) $\mathrm{ch}\, M^0 = 1 + q^2 + 2q^3 + \cdots$.

(3) M^0 is generated by its conformal vector ω and a primary vector W^3 of weight 3.

(4) The automorphism group $\mathrm{Aut}\, M^0$ of M^0 is generated by θ, and $\theta(W^3) = -W^3$.

(5) The irreducible M^0-modules $M^{i,j}$'s are not always inequivalent. In fact,

$$M^{i,j} \cong M^{k-i,j-i}, \quad 0 \le i \le k,\ 0 \le j \le k-1. \tag{3.2}$$

(6) $M^{i,j}$, $0 \le j < i \le k$, form a complete set of representatives of the equivalence classes of irreducible M^0-modules.

(7) The top level of $M^{i,j}$ is a one dimensional space $\mathbb{C}v^{i,j}$, and its weight is

$$h(M^{i,j}) = \frac{1}{2k(k+2)}\big(k(i-2j) - (i-2j)^2 + 2k(i-j+1)j\big) \tag{3.3}$$

for $0 \le j \le i \le k$. Note that (3.3) is valid even when $j = i$. Any irreducible M^0-module except for M^0 itself has positive conformal weight.

(8) The automorphism θ of M^0 induces a permutation $M^{i,j} \mapsto M^{i,j} \circ \theta \cong M^{i,i-j}$ on the irreducible M^0-modules for $0 \le i \le k$, $0 \le j \le k-1$.

(9) M^j, $0 \le j \le k-1$, are the simple currents with $h(M^j) = j(k-j)/k$, and

$$M^{j'} \boxtimes_{M^0} M^{i,j} = M^{i,j+j'}, \quad 0 \le i \le k,\ 0 \le j,\ j' \le k-1. \tag{3.4}$$

The following lemma is a consequence of (3.2) and (3.4).

Lemma 3.1. *$M^{j'} \boxtimes_{M^0} M^{i,j} \cong M^{i,j}$ if and only if $j' = 0$, or k is even and $j' = i = k/2$.*

Let

$$N = \{\alpha \in L \mid \langle \alpha, \gamma \rangle = 0\}.$$

Then $M^0 = \mathrm{Com}_{V^{\mathrm{aff}}}(V_{\mathbb{Z}\gamma}) \subset \mathrm{Com}_{V_L}(V_{\mathbb{Z}\gamma}) = V_N$. The commutant of V^{aff} in V_L is isomorphic to the parafermion vertex operator algebra $K(\mathfrak{sl}_k, 2)$ [**26**]. We denote it by T. Thus $T = \mathrm{Com}_{V_L}(V^{\mathrm{aff}}) = \mathrm{Com}_{V_N}(M^0) \cong K(\mathfrak{sl}_k, 2)$.

4. Cosets $N(j, a)$ of N in N°.

We keep the notations in Section 3. In this section, we describe the cosets of N in its dual lattice N°. For $\boldsymbol{a} = (a_1, \ldots, a_k) \in \{0,1\}^k$, set $\delta_{\boldsymbol{a}} = (1/2)\sum_{p=1}^k a_p\alpha_p$. Then $L^\circ = \bigcup_{\boldsymbol{a}\in\{0,1\}^k}(L + \delta_{\boldsymbol{a}})$ is the coset decomposition of L° by L. Let $\beta_p = \alpha_p - \alpha_{p+1}$, $1 \le p \le k-1$, so $\{\beta_1, \ldots, \beta_{k-1}\}$ is a $\mathbb{Z}$-basis of N. Set $R = N \oplus \mathbb{Z}\gamma$. Then $R \subset L \subset L^\circ \subset R^\circ$ with $R^\circ = N^\circ \oplus (\mathbb{Z}\gamma)^\circ$ and $(\mathbb{Z}\gamma)^\circ = \mathbb{Z}\gamma/2k$. Let

$$\lambda_k = \frac{1}{2k}(\beta_1 + 2\beta_2 + \cdots + (k-1)\beta_{k-1}) = \frac{1}{2k}\gamma - \frac{1}{2}\alpha_k.$$

Then $\langle \beta_p, \lambda_k \rangle = \delta_{p,k-1}$, $1 \le p \le k-1$, and $\langle \lambda_k, \lambda_k \rangle = 1/2 - 1/2k$. The following lemma holds.

LEMMA 4.1. (1) *$\{\beta_2/2, \ldots, \beta_{k-1}/2, \lambda_k\}$ is a $\mathbb{Z}$-basis of N°.*
(2) *The coset decomposition of N° by N is given as*

$$N^\circ = \bigcup_{\substack{0\le i\le 2k-1\\ d_2,\ldots,d_{k-1}\in\{0,1\}}} (N + d_2\beta_2/2 + \cdots + d_{k-1}\beta_{k-1}/2 + i\lambda_k).$$

(3) $N^\circ/N \cong \mathbb{Z}_2^{k-2} \times \mathbb{Z}_{2k}$.

We consider another $\mathbb{Z}$-basis of N°. Let

$$\lambda_p = \lambda_k - \frac{1}{2}\beta_p - \cdots - \frac{1}{2}\beta_{k-1} = \frac{1}{2k}\gamma - \frac{1}{2}\alpha_p, \quad 1 \le p \le k-1.$$

Then $\lambda_p \in N^\circ$ and $2\lambda_p \equiv 2\lambda_k \pmod N$. Note that

$$\lambda_1 + \cdots + \lambda_k = 0. \tag{4.1}$$

Lemma 4.1 implies the next lemma.

LEMMA 4.2. (1) *$\{\lambda_2, \ldots, \lambda_{k-1}, \lambda_k\}$ is a $\mathbb{Z}$-basis of N°.*
(2) *The coset decomposition of N° by N is given as*

$$N^\circ = \bigcup_{\substack{0\le i\le 2k-1\\ d_2,\ldots,d_{k-1}\in\{0,1\}}} \big(N + d_2\lambda_2 + \cdots + d_{k-1}\lambda_{k-1} + i\lambda_k\big).$$

The coset decomposition of L by R is given as

$$L = \bigcup_{j=0}^{k-1} (R - j\alpha_k) = \bigcup_{j=0}^{k-1} \Big(R + 2j\lambda_k - \frac{j}{k}\gamma \Big), \tag{4.2}$$

and $L/R \cong \mathbb{Z}_k$. Moreover, the coset decomposition of R° by L° is given as

$$R^\circ = \bigcup_{j=0}^{k-1} \Big(L^\circ - \frac{j}{2k}\gamma \Big),$$

and $R^\circ/L^\circ \cong \mathbb{Z}_k$.

For $\boldsymbol{a} = (a_1, \dots, a_k) \in \{0,1\}^k$, the support $\mathrm{supp}(\boldsymbol{a})$ is the set of p, $1 \le p \le k$, for which $a_p \neq 0$, and the Hamming weight $\mathrm{wt}(\boldsymbol{a})$ is the number of nonzero entries a_p. Then

$$\delta_{\boldsymbol{a}} = -\sum_{p=1}^{k} a_p \lambda_p + \frac{\mathrm{wt}(\boldsymbol{a})}{2k}\gamma.$$

For $\boldsymbol{a} = (a_1, \dots, a_k) \in \{0,1\}^k$, let

$$N(j, \boldsymbol{a}) = N - \sum_{p=1}^{k} a_p \lambda_p + 2j\lambda_k, \quad 0 \le j \le k-1. \tag{4.3}$$

Since $2k\lambda_k \in N$, we can consider j to be modulo k. We have

$$N(j, \boldsymbol{a}) + N(j', \boldsymbol{a}') = N(j + j' - (\mathrm{wt}(\boldsymbol{a}) + \mathrm{wt}(\boldsymbol{a}') - \mathrm{wt}(\boldsymbol{a} + \boldsymbol{a}'))/2, \boldsymbol{a} + \boldsymbol{a}'),$$

where $\boldsymbol{a}+\boldsymbol{a}'$ is the sum of $\boldsymbol{a}$ and $\boldsymbol{a}'$ as elements of $(\mathbb{Z}_2)^k$, that is, the symmetric difference as subsets of $\{0,1\}^k$. By the definition of λ_p, we also have

$$N(j, \boldsymbol{a}) = N + \frac{1}{2}\sum_{p=1}^{k} a_p \alpha_p - j\alpha_k + \frac{2j - \mathrm{wt}(\boldsymbol{a})}{2k}\gamma. \tag{4.4}$$

Since $2\lambda_k - \gamma/k = -\alpha_k$, this equation implies that

$$R + \delta_{\boldsymbol{a}} + 2j\lambda_k - \frac{j}{k}\gamma = N(j, \boldsymbol{a}) + \Big(\mathbb{Z}\gamma + \frac{\mathrm{wt}(\boldsymbol{a}) - 2j}{2k}\gamma \Big)$$

as subsets of $R^\circ = N^\circ \oplus (\mathbb{Z}\gamma)^\circ$. Hence it follows from (4.2) that

$$L + \delta_{\boldsymbol{a}} = \bigcup_{j=0}^{k-1} \Big(N(j, \boldsymbol{a}) + \Big(\mathbb{Z}\gamma + \frac{\mathrm{wt}(\boldsymbol{a}) - 2j}{2k}\gamma \Big) \Big). \tag{4.5}$$

Lemma 4.3. (1) *For $0 \le j, j' \le k-1$ and $\boldsymbol{a}, \boldsymbol{a}' \in \{0,1\}^k$, we have $N(j, \boldsymbol{a}) = N(j', \boldsymbol{a}')$ if and only if one of the following conditions holds.*

(i) $j \equiv j' \pmod{k}$ *and* $\boldsymbol{a} = \boldsymbol{a}'$.

(ii) $j' \equiv j - \mathrm{wt}(\boldsymbol{a}) \pmod{k}$ *and* $\boldsymbol{a} + \boldsymbol{a}' = (1, \dots, 1)$.

(2) *$N(j, \boldsymbol{a})$, $0 \le j \le k-1$, $\boldsymbol{a} \in \{0,1\}^k$ with $j < \mathrm{wt}(\boldsymbol{a})$, are the distinct cosets of N in N°.*

PROOF. Clearly, $N(j,\boldsymbol{a}) = N(j',\boldsymbol{a}')$ if the condition (i) holds. Suppose the condition (ii) holds. Then $N(j,\boldsymbol{a}) = N(j',\boldsymbol{a}')$ by (4.1) and (4.3). Set $i = \mathrm{wt}(\boldsymbol{a})$ and $i' = \mathrm{wt}(\boldsymbol{a}')$, and assume that $j < i$. Then $0 \le j < i \le k$ and $0 \le i' \le j' < k$. The number of pairs $(j,\boldsymbol{a})$ with $0 \le j \le k-1$ and $\boldsymbol{a} \in \{0,1\}^k$ is $2^k k$. Since $|N^\circ/N| = 2^{k-1}k$, we see that $N(j,\boldsymbol{a}) = N(j',\boldsymbol{a}')$ only if j, j', $\boldsymbol{a}$, and $\boldsymbol{a}'$ satisfy the conditions (i) or (ii). Hence the assertions (1) and (2) hold. □

REMARK 4.4. In Case (ii) of Lemma 4.3 (1), we have $(\mathrm{wt}(\boldsymbol{a}') - 2j') - (\mathrm{wt}(\boldsymbol{a}) - 2j) \equiv k \pmod{2k}$. This agrees with the fact that $N(j,\boldsymbol{a}) + (\mathbb{Z}\gamma + (\mathrm{wt}(\boldsymbol{a}) - 2j)\gamma/2k)$, $0 \le j \le k-1$, $\boldsymbol{a} \in \{0,1\}^k$, in (4.5) are the distinct cosets of R in L°.

The next lemma also holds.

LEMMA 4.5. *The -1 isometry $N^\circ \to N^\circ$; $\alpha \mapsto -\alpha$ transforms $N(j,\boldsymbol{a})$ into $N(\mathrm{wt}(\boldsymbol{a}) - j, \boldsymbol{a})$.*

5. Decomposition of $V_{N(j,\boldsymbol{a})}$.

We keep the notations in Sections 3 and 4. In this section, we study a decomposition of the irreducible V_N-module $V_{N(j,\boldsymbol{a})}$ as a direct sum of irreducible modules for a tensor product of $k-1$ Virasoro vertex operator algebras and M^0. Let

$$c_m = 1 - \frac{6}{(m+2)(m+3)}$$

for $m = 1, 2, \ldots$, and let

$$h^m_{r,s} = \frac{(r(m+3) - s(m+2))^2 - 1}{4(m+2)(m+3)}$$

for $1 \le r \le m+1$, $1 \le s \le m+2$. Then $h^m_{r,s} = h^m_{m+2-r,m+3-s}$, and $L(c_m, h^m_{r,s})$, $1 \le s \le r \le m+1$, form a complete set of representatives of the equivalence classes of irreducible modules for the Virasoro vertex operator algebra $L(c_m, 0)$ [**41**]. We denote the conformal vector of $L(c_m, 0)$ by ω^m.

Recall that ω is the conformal vector of M^0. Let ω_T be the conformal vector of $T = \mathrm{Com}_{V_N}(M^0)$. Then the conformal vector $\omega_N = \omega_T + \omega$ of V_N is a sum of mutually orthogonal Virasoro vectors $\omega^1, \ldots, \omega^{k-1}$, and ω [**11**], [**29**] with $\omega_T = \omega^1 + \cdots + \omega^{k-1}$. The vector ω^m generates $L(c_m, 0)$, so $T \supset L(c_1, 0) \otimes \cdots \otimes L(c_{k-1}, 0)$. The following decomposition is known [**21**], [**27**], [**40**].

LEMMA 5.1. *For $\boldsymbol{a} = (a_1, \ldots, a_k) \in \{0,1\}^k$,*

$$V_{L+\delta_{\boldsymbol{a}}} = \bigoplus_{\substack{0 \le i_s \le s \\ i_s \equiv b_s \pmod 2 \\ 1 \le s \le k}} L(c_1, h^1_{i_1+1, i_2+1}) \otimes \cdots \otimes L(c_{k-1}, h^{k-1}_{i_{k-1}+1, i_k+1}) \otimes L_{\widehat{\mathfrak{sl}}_2}(k, i_k)$$

as $L(c_1, 0) \otimes \cdots \otimes L(c_{k-1}, 0) \otimes L_{\widehat{\mathfrak{sl}}_2}(k, 0)$-modules, where $b_s = \sum_{p=1}^s a_p$.

Combining the decomposition (3.1) with Lemma 5.1, we have

$$V_{L+\delta_{\boldsymbol{a}}} = \bigoplus_{j=0}^{k-1}\Bigl(\bigoplus_{\substack{0\le i_s\le s\\ i_s\equiv b_s \pmod 2\\ 1\le s\le k}} L(c_1, h^1_{i_1+1,i_2+1})\otimes\cdots\otimes L(c_{k-1}, h^{k-1}_{i_{k-1}+1,i_k+1}) \otimes M^{i_k,j}\otimes V_{\mathbb{Z}\gamma+(i_k-2j)\gamma/2k}\Bigr) \quad (5.1)$$

as $L(c_1,0)\otimes\cdots\otimes L(c_{k-1},0)\otimes M^0\otimes V_{\mathbb{Z}\gamma}$-modules.

Since $b_k=\mathrm{wt}(\boldsymbol{a})$, (4.5) implies that

$$V_{L+\delta_{\boldsymbol{a}}} = \bigoplus_{j=0}^{k-1} V_{N(j,\boldsymbol{a})}\otimes V_{\mathbb{Z}\gamma+(b_k-2j)\gamma/2k} \quad (5.2)$$

as $V_N\otimes V_{\mathbb{Z}\gamma}$-modules.

As $V_{\mathbb{Z}\gamma}$-modules, $V_{\mathbb{Z}\gamma+(b_k-2j)\gamma/2k}\cong V_{\mathbb{Z}\gamma+(i_k-2q)\gamma/2k}$ if and only if $q\equiv j+(i_k-b_k)/2 \pmod k$. Here, note that i_k on the right hand side of (5.1) satisfies $i_k\equiv b_k \pmod 2$. Comparing (5.1) and (5.2), we have the following theorem, see [**28**, Proposition 3.4].

Theorem 5.2. *For $0\le j\le k-1$ and $\boldsymbol{a}=(a_1,\dots,a_k)\in\{0,1\}^k$, the irreducible V_N-module $V_{N(j,\boldsymbol{a})}$ decomposes as a direct sum*

$$V_{N(j,\boldsymbol{a})} = \bigoplus_{\substack{0\le i_s\le s\\ i_s\equiv b_s \pmod 2\\ 1\le s\le k}} L(c_1, h^1_{i_1+1,i_2+1})\otimes\cdots\otimes L(c_{k-1}, h^{k-1}_{i_{k-1}+1,i_k+1})\otimes M^{i_k,j+(i_k-b_k)/2} \quad (5.3)$$

of irreducible $L(c_1,0)\otimes\cdots\otimes L(c_{k-1},0)\otimes M^0$-modules, where $b_s=\sum_{p=1}^s a_p$.

The next remark is a restatement of [**28**, Proposition 3.5].

Remark 5.3. $N(j,\boldsymbol{a})=N(j',\boldsymbol{a}')$ for j', $\boldsymbol{a}'$ in Case (ii) of Lemma 4.3 (1) corresponds to the following properties of the highest weights $h^m_{p,q}$ for $L(c_m,0)$ and the irreducible modules $M^{i,j}$ for $K(sl_2,k)$.

(1) $h^m_{p,q}=h^m_{m+2-p,m+3-q}$ for $1\le p\le m+1$, $1\le q\le m+2$.
(2) $M^{i,j}\cong M^{k-i,j-i}$ as $K(sl_2,k)$-modules for $0\le i\le k$, $j\in\mathbb{Z}_k$.

We note that for a given $\boldsymbol{a}\in\{0,1\}^k$, the $L(c_1,0)\otimes\cdots\otimes L(c_{k-1},0)$-modules

$$L(c_1, h^1_{i_1+1,i_2+1})\otimes\cdots\otimes L(c_{k-1}, h^{k-1}_{i_{k-1}+1,i_k+1}),$$

$0\le i_s\le s$, $i_s\equiv b_s \pmod 2$, $1\le s\le k$, in (5.3) are inequivalent to each other.

6. Irreducible $K(\mathfrak{sl}_2,k)$-modules in $V_{N(j,\boldsymbol{a})}$.

In this section, we discuss how irreducible $K(\mathfrak{sl}_2,k)$-modules $M^{i,j}$ appear on the right hand side of (5.3). Since $h^s_{p,q}=0$ if and only if $(p,q)=(1,1)$ or $(s+1,s+2)$, the following lemma holds.

Lemma 6.1. *Let $1 \le m < k$. Then for $a_1, \ldots, a_{m+1} \in \{0,1\}$ and $0 \le i_s \le s$, $1 \le s \le m+1$, the two conditions $i_s \equiv b_s \pmod 2$, $1 \le s \le m+1$, and $h^s_{i_s+1, i_{s+1}+1} = 0$, $1 \le s \le m$, hold only if* (i) *$a_s = 0$ and $i_s = 0$, $1 \le s \le m+1$, or* (ii) *$a_s = 1$ and $i_s = s$, $1 \le s \le m+1$.*

For an arbitrarily given $a_1 \in \{0,1\}$, each coset of N in N° is uniquely expressed as $N(j, \boldsymbol{a})$, $j \in \mathbb{Z}_k$, $\boldsymbol{a} = (a_1, a_2, \ldots, a_k)$, $a_2, \ldots, a_k \in \{0,1\}$ by Lemma 4.3. For the rest of this section, we take $a_1 = 0$. For simplicity of notation, we omit $\mathbf{1} \otimes \cdots \otimes \mathbf{1}$ in an equation as

$$\{v \in V_N \mid \omega^s_{(1)} v = 0, 1 \le s \le k-1\} = \mathbf{1} \otimes \cdots \otimes \mathbf{1} \otimes M^0.$$

The following two propositions are clear from Theorem 5.2 and Lemma 6.1.

Proposition 6.2. *For $j \in \mathbb{Z}_k$, we have*

$$\{v \in V_{N(j,(0,\ldots,0))} \mid \omega^s_{(1)} v = 0, 1 \le s \le k-1\} = M^j.$$

Proposition 6.3. *For $j \in \mathbb{Z}_k$ and $d \in \{0,1\}$, we have*

$$\begin{aligned}\{v \in V_{N(j,(0,\ldots,0,d))} \mid \omega^s_{(1)} v = 0, 1 \le s \le k-2\}& \\ = \bigoplus_{\substack{0 \le i \le k \\ i \equiv d \pmod 2}} L(c_{k-1}, h^{k-1}_{1,i+1}) \otimes M^{i, j+(i-d)/2}.& \qquad (6.1)\end{aligned}$$

The next proposition is a consequence of (3.2).

Proposition 6.4. *Let $d \in \{0,1\}$.*

(1) *If k is odd, then $M^{i,j+(i-d)/2}$, $j \in \mathbb{Z}_k$, $0 \le i \le k$, $i \equiv d \pmod 2$, are inequivalent to each other, and they are the $k(k+1)/2$ inequivalent irreducible modules $M^{i,j}$, $0 \le j < i \le k$.*

(2) *If k is even, then $M^{i,j+(i-d)/2}$, $j \in \mathbb{Z}_k$, $0 \le i \le k$, $i \equiv d \pmod 2$, cover twice the set of inequivalent irreducible modules $M^{i,j}$, $0 \le j < i \le k$ with $i \equiv d \pmod 2$. There are $k(k+2)/4$ (resp. $k^2/4$) inequivalent irreducible modules $M^{i,j}$, $0 \le j < i \le k$ with $i \equiv 0 \pmod 2$ (resp. $i \equiv 1 \pmod 2$). Moreover, for a fixed $j \in \mathbb{Z}_k$, the irreducible modules $M^{i,j+(i-d)/2}$, $0 \le i \le k$, $i \equiv d \pmod 2$, are inequivalent to each other.*

7. Γ_D and M_D for a $\mathbb{Z}_k$-code D.

In this section, we define a vertex operator algebra or a vertex operator superalgebra M_D for a $\mathbb{Z}_k$-code D. The arguments are essentially the same as in Section 3 of [**3**].

Let ℓ be a fixed positive integer. A $\mathbb{Z}_k$-code of length ℓ means an additive subgroup of $(\mathbb{Z}_k)^\ell$. We denote by $(\,\cdot\,|\,\cdot\,)$ the standard inner product $(\xi|\eta) = \xi_1\eta_1 + \cdots + \xi_\ell\eta_\ell \in \mathbb{Z}_k$ for $\xi = (\xi_1, \ldots, \xi_\ell)$, $\eta = (\eta_1, \ldots, \eta_\ell) \in (\mathbb{Z}_k)^\ell$.

For simplicity of notation, set $N^{(j)} = N(j,(0,\ldots,0)) = N + 2j\lambda_k$, $j \in \mathbb{Z}_k$. We consider a coset $N(\xi)$ of N^ℓ in $(N^\circ)^\ell$ defined by

$$N(\xi) = \{(x_1, \ldots, x_\ell) \mid x_r \in N^{(\xi_r)}, 1 \le r \le \ell\} \subset (N^\circ)^\ell \tag{7.1}$$

for $\xi = (\xi_1, \ldots, \xi_\ell) \in (\mathbb{Z}_k)^\ell$. Since $\langle \alpha, \beta \rangle \in -2ij/k + 2\mathbb{Z}$ for $\alpha \in N^{(i)}$, $\beta \in N^{(j)}$, we have

$$\langle \alpha, \beta \rangle \in -\frac{2}{k}(\xi|\eta) + 2\mathbb{Z} \quad \text{for } \alpha \in N(\xi), \beta \in N(\eta). \tag{7.2}$$

Let D be a $\mathbb{Z}_k$-code of length ℓ. We consider two cases.

Case A: $(\xi|\xi) = 0$ for all $\xi \in D$.

Case B: k is even, $(\xi|\eta) \in \{0, k/2\}$ for all $\xi, \eta \in D$, and $(\xi|\xi) = k/2$ for some $\xi \in D$.

Let

$$\Gamma_D = \bigcup_{\xi \in D} N(\xi) \subset (N^\circ)^\ell, \tag{7.3}$$

which is a sublattice of $(N^\circ)^\ell$, as $N(\xi) + N(\eta) = N(\xi + \eta)$ and D is an additive subgroup of $(\mathbb{Z}_k)^\ell$. The following lemma holds by (7.2).

Lemma 7.1. (1) Γ_D *is a positive definite even lattice if and only if* D *is in Case* A. (2) Γ_D *is a positive definite odd lattice if and only if* k *is even and* D *is in Case* B.

If D is in Case A, then V_{Γ_D} is a vertex operator algebra. If k is even and D is in Case B, we set

$$D^0 = \{\xi \in D \mid (\xi|\xi) = 0\}, \quad D^1 = \{\xi \in D \mid (\xi|\xi) = k/2\}.$$

We also set $\Gamma_{D^p} = \bigcup_{\xi \in D^p} N(\xi)$, $p = 0, 1$. Then D^0 is a subgroup of the additive group D of index two, and $D = D^0 \cup D^1$ is the coset decomposition of D by D^0. Moreover, $\Gamma_{D^p} = \{\alpha \in \Gamma_D \mid \langle \alpha, \alpha \rangle \in p + 2\mathbb{Z}\}$, $p = 0, 1$, and $\Gamma_D = \Gamma_{D^0} \cup \Gamma_{D^1}$ with Γ_{D^0} an even sublattice. We have that $V_{\Gamma_D} = V_{\Gamma_{D^0}} \oplus V_{\Gamma_{D^1}}$ is a vertex operator superalgebra.

It follows from (7.1) that $V_{N(\xi)} = V_{N^{(\xi_1)}} \otimes \cdots \otimes V_{N^{(\xi_\ell)}} \subset (V_{N^\circ})^\ell$. We also have $V_{\Gamma_D} = \bigoplus_{\xi \in D} V_{N(\xi)}$ by (7.3). Let

$$M_\xi = \{v \in V_{N(\xi)} \mid (\omega_{T^{\otimes \ell}})_{(1)} v = 0\},$$

where $\omega_{T^{\otimes \ell}}$ is the conformal vector of the vertex operator subalgebra $T^{\otimes \ell}$ of $(V_N)^{\otimes \ell}$. Then $M_\xi = M^{\xi_1} \otimes \cdots \otimes M^{\xi_\ell}$ for $\xi = (\xi_1, \ldots, \xi_\ell) \in (\mathbb{Z}_k)^\ell$ by Proposition 6.2, which is a simple current for $M_{\mathbf{0}} = (M^0)^{\otimes \ell}$ with $\mathbf{0} = (0, \ldots, 0)$ the zero codeword. Since $u_{(n)} v \in V_{N(\xi+\eta)}$ for $u \in V_{N(\xi)}$, $v \in V_{N(\eta)}$, $n \in \mathbb{Z}$, we have $u_{(n)} v \in M_{\xi+\eta}$ for $u \in M_\xi$, $v \in M_\eta$, $n \in \mathbb{Z}$. Thus $M_\xi \boxtimes_{M_{\mathbf{0}}} M_\eta = M_{\xi+\eta}$ for $\xi, \eta \in (\mathbb{Z}_k)^\ell$, and $\mathrm{Irr}(M_{\mathbf{0}})_{\mathrm{sc}} = \{M_\xi \mid \xi \in (\mathbb{Z}_k)^\ell\}$ is $(\mathbb{Z}_k)^\ell$-graded. The top level of M_ξ is one dimensional with $h(M_\xi) = \left(\sum_{r=1}^{\ell} \xi_r\right) - (\xi|\xi)/k$, as $h(M^j) = j - j^2/k$, where ξ_r and $(\xi|\xi)$ are considered to be nonnegative integers.

We have the next proposition by the properties of $M^0 = K(\mathfrak{sl}_2, k)$ in Section 3.

Proposition 7.2. $M_{\mathbf{0}} = (M^0)^{\otimes \ell}$ *is a simple, self-dual, rational, and* C_2*-cofinite vertex operator algebra of CFT-type with central charge* $2\ell(k-1)/(k+2)$*. Any irreducible* $M_{\mathbf{0}}$*-module except for* $M_{\mathbf{0}}$ *itself has positive conformal weight.*

Let M_D be the commutant of $T^{\otimes \ell}$ in V_{Γ_D}. Then

$$M_D = \{v \in V_{\Gamma_D} \mid (\omega_{T^{\otimes \ell}})_{(1)} v = 0\} = \bigoplus_{\xi \in D} M_\xi, \tag{7.4}$$

which is a D-graded simple current extension of $M_{\mathbf{0}}$. The following theorem holds.

THEOREM 7.3. (1) *If D is in Case* A, *then M_D is a simple, self-dual, rational, and C_2-cofinite vertex operator algebra of CFT-type with central charge $2\ell(k-1)/(k+2)$.*

(2) *If k is even and D is in Case* B, *then $M_D = M_{D^0} \oplus M_{D^1}$ is a simple vertex operator superalgebra, whose even part M_{D^0} and odd part M_{D^1} are given by $M_{D^p} = \bigoplus_{\xi \in D^p} M_\xi$, $p = 0, 1$, and $h(M_{D^1}) \in \mathbb{Z} + 1/2$.*

8. Irreducible M_D-modules: Case A.

Let $k \geq 2$, and let D be a $\mathbb{Z}_k$-code of length ℓ satisfying the condition of Case A in Section 7, that is, $(\xi|\xi) = 0$ for all $\xi \in D$. In this section, we classify the irreducible χ-twisted M_D-modules for $\chi \in D^*$. We construct all irreducible untwisted M_D-modules inside $V_{(\Gamma_D)^\circ}$ as well.

8.1. Linear characters of D.

Let

$$P(i,j) = k(i-2j) - (i-2j)^2 + 2k(i-j+1)j.$$

Then $h(M^{i,j}) = P(i,j)/2k(k+2)$ for $0 \leq j \leq i \leq k$ by (3.3). In the case where $0 \leq i \leq j < k$, we have $h(M^{i,j}) = P(k-i, j-i)/2k(k+2)$ by (3.2). We calculate the values of the map $b_{M^0} : \mathrm{Irr}(M^0)_{\mathrm{sc}} \times \mathrm{Irr}(M^0) \to \mathbb{Q}/\mathbb{Z}$ defined by

$$b_{M^0}(M^p, M^{i,j}) = h(M^p \boxtimes_{M^0} M^{i,j}) - h(M^p) - h(M^{i,j}) + \mathbb{Z},$$

where $M^p \boxtimes_{M^0} M^{i,j} = M^{i,j+p}$ by (3.4). If $0 \leq j < i \leq k$, then $0 \leq j < j+1 \leq i \leq k$, and

$$P(i, j+1) - P(i,j) = 2(k+2)(i-2j-1),$$

whereas if $0 \leq i \leq j < k$, then $0 \leq j-i < j+1-i \leq k-i \leq k$, and

$$P(k-i, j+1-i) - P(k-i, j-i) = 2(k+2)(i-2j+k-1).$$

In both cases, we have $b_{M^0}(M^1, M^{i,j}) = (i-2j)/k + \mathbb{Z}$. Thus

$$b_{M^0}(M^p, M^{i,j}) = \frac{p(i-2j)}{k} + \mathbb{Z} \tag{8.1}$$

for $0 \leq i \leq k$, $0 \leq j < k$, and $0 \leq p < k$ by Lemma 2.1.

For $\mu = (\mu_1, \dots, \mu_\ell)$ with $0 \leq \mu_r \leq k$, $1 \leq r \leq \ell$, and $\nu = (\nu_1, \dots, \nu_\ell) \in (\mathbb{Z}_k)^\ell$, let

$$M_{\mu,\nu} = M^{\mu_1,\nu_1} \otimes \cdots \otimes M^{\mu_\ell,\nu_\ell}.$$

Then $M_{\mathbf{0},\xi} = M_\xi$ and

$$\mathrm{Irr}(M_{\mathbf{0}}) = \{M_{\mu,\nu} \mid \mu = (\mu_1, \dots, \mu_\ell),\ \nu = (\nu_1, \dots, \nu_\ell),\ 0 \le \nu_r < \mu_r \le k,\ 1 \le r \le \ell\}. \tag{8.2}$$

It follows from (3.4) that

$$M_\xi \boxtimes_{M_{\mathbf{0}}} M_{\mu,\nu} = M_{\mu,\nu+\xi}. \tag{8.3}$$

Let $b_{M_{\mathbf{0}}} : \mathrm{Irr}(M_{\mathbf{0}})_{\mathrm{sc}} \times \mathrm{Irr}(M_{\mathbf{0}}) \to \mathbb{Q}/\mathbb{Z}$ be a map defined by

$$b_{M_{\mathbf{0}}}(M_\xi, M_{\mu,\nu}) = h(M_\xi \boxtimes_{M_{\mathbf{0}}} M_{\mu,\nu}) - h(M_\xi) - h(M_{\mu,\nu}) + \mathbb{Z}$$

for $\mu = (\mu_1, \dots, \mu_\ell)$ with $0 \le \mu_r \le k$, $1 \le r \le \ell$, $\nu = (\nu_1, \dots, \nu_\ell) \in (\mathbb{Z}_k)^\ell$, and $\xi = (\xi_1, \dots, \xi_\ell) \in (\mathbb{Z}_k)^\ell$. Then (8.1) implies that

$$b_{M_{\mathbf{0}}}(M_\xi, M_{\mu,\nu}) = \sum_{r=1}^{\ell} \frac{\xi_r(\mu_r - 2\nu_r)}{k} + \mathbb{Z}. \tag{8.4}$$

Although μ_r is an integer between 0 and k, we can treat μ_r modulo k on the right hand side of (8.4). Then (8.4) is written as

$$b_{M_{\mathbf{0}}}(M_\xi, M_{\mu,\nu}) = \frac{1}{k}(\xi|\mu - 2\nu) + \mathbb{Z}, \tag{8.5}$$

where $(\cdot|\cdot)$ is the standard inner product on $(\mathbb{Z}_k)^\ell$. In particular,

$$b_{M_{\mathbf{0}}}(M_\xi, M_\eta) = -\frac{2}{k}(\xi|\eta) + \mathbb{Z}. \tag{8.6}$$

Lemma 8.1. *Let $\xi, \eta, \nu \in (\mathbb{Z}_k)^\ell$, and let $\mu = (\mu_1, \dots, \mu_\ell)$ with $0 \le \mu_r \le k$, $1 \le r \le \ell$.*
(1) $b_{M_{\mathbf{0}}}(M_\xi, M_\eta) = 0$ *if $\xi, \eta \in D$.*
(2) $b_{M_{\mathbf{0}}}(M_{\xi+\eta}, M_{\mu,\nu}) = b_{M_{\mathbf{0}}}(M_\xi, M_{\mu,\nu}) + b_{M_{\mathbf{0}}}(M_\eta, M_{\mu,\nu})$.
(3) $b_{M_{\mathbf{0}}}(M_\xi, M_{\mu,\nu+\eta}) = b_{M_{\mathbf{0}}}(M_\xi, M_\eta) + b_{M_{\mathbf{0}}}(M_\xi, M_{\mu,\nu})$.

Proof. Suppose $\xi, \eta \in D$. Then $\xi + \eta \in D$, so $(\xi+\eta|\xi+\eta) = 0$ by our assumption on D. Since $(\xi|\xi) = (\eta|\eta) = 0$, we have $2(\xi|\eta) = 0$. Thus the assertion (1) holds by (8.6). The assertions (2) and (3) are clear from (8.5), see also Lemma 2.1. □

For $\eta \in (\mathbb{Z}_k)^\ell$, let $\chi(\eta)$ be a linear character of the abelian group $(\mathbb{Z}_k)^\ell$ given by

$$\chi(\eta) : (\mathbb{Z}_k)^\ell \to \mathbb{C}^\times; \quad \xi \mapsto \exp(2\pi\sqrt{-1}(\xi|\eta)/k).$$

Then $(\mathbb{Z}_k)^\ell \to \mathrm{Hom}((\mathbb{Z}_k)^\ell, \mathbb{C}^\times)$; $\eta \mapsto \chi(\eta)$ is a group isomorphism. The linear character $\chi_{M_{\mu,\nu}} \in D^*$ is the restriction $\chi(\mu - 2\nu)|_D$ of $\chi(\mu - 2\nu)$ to D by (8.5). That is,

$$\chi_{M_{\mu,\nu}}(\xi) = \exp(2\pi\sqrt{-1}b_{M_{\mathbf{0}}}(M_\xi, M_{\mu,\nu})) = \exp(2\pi\sqrt{-1}(\xi|\mu - 2\nu)/k). \tag{8.7}$$

Let $D^\perp = \{\eta \in (\mathbb{Z}_k)^\ell \mid (D|\eta) = 0\}$. Then $|D||D^\perp| = |(\mathbb{Z}_k)^\ell|$, as $(\cdot|\cdot)$ is a non-degenerate bilinear form.

Lemma 8.2. (1) *The map* $(\mathbb{Z}_k)^\ell \to D^*$; $\eta \mapsto \chi(\eta)|_D$ *is a surjective group homomorphism with kernel* $D^\perp$.

(2) *For any* $\chi \in D^*$, *there exists* $M_{\mu,\mathbf{0}} \in \mathrm{Irr}(M_\mathbf{0})$ *such that* $\chi = \chi_{M_{\mu,\mathbf{0}}}$.

(3) $\chi_{M_{\mu,\nu}} = 1$; *the principal character of* D *if and only if* $\mu - 2\nu \in D^\perp$.

(4) $\chi_{M_{\mu,\nu}} = \chi_{M_{\mu',\nu'}}$ *if and only if* $\mu - 2\nu \equiv \mu' - 2\nu' \pmod{D^\perp}$.

Proof. Non-degeneracy of the bilinear form $(\,\cdot\,|\,\cdot\,)$ implies the assertions (1) and (2). The assertions (3) and (4) are consequences of (8.7) and the definition of $D^\perp$. □

8.2. Irreducible $M_\mathbf{0}$-modules in $V_{(N^\circ)^\ell}$.

Let

$$N(\eta,\delta) = \{(x_1,\dots,x_\ell) \mid x_r \in N(\eta_r,(0,\dots,0,d_r)), 1 \le r \le \ell\} \subset (N^\circ)^\ell$$

for $\eta = (\eta_1,\dots,\eta_\ell) \in (\mathbb{Z}_k)^\ell$ and $\delta = (d_1,\dots,d_\ell) \in \{0,1\}^\ell$.

Proposition 8.3. (1) *Let* $\eta = (\eta_1,\dots,\eta_\ell) \in (\mathbb{Z}_k)^\ell$ *and* $\delta = (d_1,\dots,d_\ell) \in \{0,1\}^\ell$. *Assume that* $\mu = (\mu_1,\dots,\mu_\ell)$ *with* $0 \le \mu_r \le k$, $1 \le r \le \ell$, *and* $\nu = (\nu_1,\dots,\nu_\ell) \in (\mathbb{Z}_k)^\ell$ *satisfy the conditions*

$$\mu_r \equiv d_r \pmod 2, \quad \nu_r = \eta_r + \frac{\mu_r - d_r}{2}, \quad 1 \le r \le \ell. \tag{8.8}$$

Then $V_{N(\eta,\delta)}$ *contains the irreducible* $M_\mathbf{0}$*-module* $M_{\mu,\nu}$.

(2) *Any irreducible* $M_\mathbf{0}$*-module is contained in* $V_{N(\eta,\delta)}$ *for some* η *and* δ. *If* k *is odd, then we can choose* δ *to be* $\delta = (0,\dots,0)$.

Proof. The assertions (1) and (2) hold by Propositions 6.3 and 6.4. □

Lemma 8.4. *Let* $\xi,\eta \in (\mathbb{Z}_k)^\ell$ *and* $\delta \in \{0,1\}^\ell$. *Then* $\langle x,y\rangle \in (\xi|\delta - 2\eta)/k + \mathbb{Z}$ *for* $x \in N(\xi)$ *and* $y \in N(\eta,\delta)$.

Proof. Since $\langle x,y\rangle \in p(d-2j)/k + \mathbb{Z}$ for $x \in N^{(p)}$ and $y \in N(j,(0,\dots,0,d))$, the assertion holds. □

Proposition 8.5. *Let* $\mu = (\mu_1,\dots,\mu_\ell)$ *with* $0 \le \mu_r \le k$, $1 \le r \le \ell$, *and let* $\nu = (\nu_1,\dots,\nu_\ell) \in (\mathbb{Z}_k)^\ell$. *Take* $\eta \in (\mathbb{Z}_k)^\ell$ *and* $\delta \in \{0,1\}^\ell$ *such that the conditions* (8.8) *hold. Then* $b_{M_\mathbf{0}}(M_\xi, M_{\mu,\nu}) = 0$ *for all* $\xi \in D$ *if and only if* $N(\eta,\delta) \subset (\Gamma_D)^\circ$.

Proof. Since $\mu_r - 2\nu_r = d_r - 2\eta_r$ by (8.8), the assertion holds by (8.5) and Lemma 8.4. □

8.3. Irreducible twisted M_D-modules in $V_{(N^\circ)^\ell}$.

Let $X \in \mathrm{Irr}(M_\mathbf{0})$. Then $X = M_{\mu,\nu}$ for some μ and ν by (8.2). Take η and δ such that the conditions (8.8) hold. Then $V_{N(\eta,\delta)}$ contains $M_{\mu,\nu}$ as an $M_\mathbf{0}$-submodule by Proposition 8.3. Since $M_\xi \subset V_{N(\xi)}$, and since $N(\xi) + N(\eta,\delta) = N(\xi+\eta,\delta)$, it follows that $V_{N(\xi+\eta,\delta)}$ contains $M_\xi \boxtimes_{M_\mathbf{0}} M_{\mu,\nu}$. For fixed η and δ, the cosets $N(\xi+\eta,\delta)$, $\xi \in D$, of N^ℓ in $(N^\circ)^\ell$ are all distinct. Hence the $\chi_{M_{\mu,\nu}}$-twisted M_D-module $M_D \cdot M_{\mu,\nu}$ generated by $M_{\mu,\nu}$ in $V_{(N^\circ)^\ell}$ is isomorphic to $M_D \boxtimes_{M_\mathbf{0}} M_{\mu,\nu}$ by (2) of Theorem 2.2.

Furthermore, if $\chi_{M_{\mu,\nu}}(\xi) = 1$ for all $\xi \in D$, then $N(\eta, \delta) \subset (\Gamma_D)^\circ$ by Proposition 8.5, so $M_D \cdot M_{\mu,\nu} \subset V_{(\Gamma_D)^\circ}$. Therefore, the following theorem holds.

Theorem 8.6. *Let $X \in \mathrm{Irr}(M_{\mathbf{0}})$.*

(1) *$V_{(N^\circ)^\ell}$ contains a χ_X-twisted M_D-module isomorphic to $M_D \boxtimes_{M_{\mathbf{0}}} X$.*

(2) *If $\chi_X = 1$, then $V_{(\Gamma_D)^\circ}$ contains an untwisted M_D-module isomorphic to $M_D \boxtimes_{M_{\mathbf{0}}} X$.*

Let W be an irreducible χ-twisted M_D-module for $\chi \in D^*$, and let X be an irreducible $M_{\mathbf{0}}$-submodule of W. Then W is isomorphic to a direct summand of $M_D \boxtimes_{M_{\mathbf{0}}} X$ with $\chi = \chi_X$ by (3) of Theorem 2.2. Thus Theorem 8.6 implies the following theorem.

Theorem 8.7. (1) *$V_{(N^\circ)^\ell}$ contains any irreducible χ-twisted M_D-module for $\chi \in D^*$.*

(2) *$V_{(\Gamma_D)^\circ}$ contains any irreducible untwisted M_D-module.*

Let $\mathrm{Irr}(M_{\mathbf{0}}) = \bigcup_{i \in I} \mathscr{O}_i$ be the D-orbit decomposition of $\mathrm{Irr}(M_{\mathbf{0}})$ for the action of D on $\mathrm{Irr}(M_{\mathbf{0}})$ in (8.3), and let $D_{M_{\mu,\nu}} = \{\xi \in D \mid M_\xi \boxtimes_{M_{\mathbf{0}}} M_{\mu,\nu} \cong M_{\mu,\nu}\}$ be the stabilizer of $M_{\mu,\nu}$. Lemma 3.1 implies the following lemma.

Lemma 8.8. *$M_\xi \boxtimes_{M_{\mathbf{0}}} M_{\mu,\nu} \cong M_{\mu,\nu}$ as $M_{\mathbf{0}}$-modules for some $\xi \neq \mathbf{0}$ if and only if k is even, $\xi = (\xi_1, \ldots, \xi_\ell) \in \{0, k/2\}^\ell$, and $\mu_r = k/2$ for $1 \le r \le \ell$ such that $\xi_r = k/2$.*

The next theorem is a restatement of Proposition 2.3.

Theorem 8.9. *Let $X \in \mathrm{Irr}(M_{\mathbf{0}})$. If $D_X = 0$, then $M_D \boxtimes_{M_{\mathbf{0}}} X$ is an irreducible χ_X-twisted M_D-module.*

Now, suppose $D_X \neq 0$. Then k is even and $D_X \subset \{0, k/2\}^\ell$ by Lemma 8.8. In order to apply the results in Section 2.2, we recall the previous arguments for a special case where the $\mathbb{Z}_k$-code is of length one consisting of two codewords (0) and $(k/2)$. Let $C = \{(0), (k/2)\}$ be such a $\mathbb{Z}_k$-code. Then $\Gamma_C = N \cup N^{(k/2)}$ with $N^{(k/2)} = N + k\lambda_k$, and $M_C = M^0 \oplus M^{k/2}$ is a $\mathbb{Z}_2$-graded simple current extension of M^0 by the self-dual simple current M^0-module $M^{k/2}$ with $h(M^{k/2}) = k/4$.

If $k \equiv 0 \pmod 4$, then $(k/2)^2 \equiv 0 \pmod k$. Hence the $\mathbb{Z}_k$-code C is in Case A in Section 7, and M_C is a simple vertex operator algebra with $h(M^{k/2}) \in \mathbb{Z}$. If $k \equiv 2 \pmod 4$, then $(k/2)^2 \equiv k/2 \pmod k$. Hence C is in Case B in Section 7, and M_C is a simple vertex operator superalgebra with $h(M^{k/2}) \in \mathbb{Z} + 1/2$. In both cases, there exists a unique structure of a $\mathbb{Z}_2$-graded either untwisted or $\mathbb{Z}_2$-twisted M_C-module on the space $P^0 \oplus P^1$ with $P^0 = P$ and $P^1 = M^{k/2} \boxtimes_{M^0} P$ for any irreducible M^0-module P.

Under the correspondence $0 \mapsto 0$ and $k/2 \mapsto 1$, we can regard any additive subgroup of $\{0, k/2\}^\ell \subset (\mathbb{Z}_k)^\ell$ as an additive subgroup of $(\mathbb{Z}_2)^\ell$. In the case where $k \equiv 2 \pmod 4$, this correspondence is the reduction modulo 2, and it in fact gives an isometry from $(\{0, k/2\}^\ell, (\cdot|\cdot))$ to $((\mathbb{Z}_2)^\ell, (\cdot|\cdot))$, where $(\cdot|\cdot)$ is the standard inner product on either $(\mathbb{Z}_k)^\ell$ or $(\mathbb{Z}_2)^\ell$. Hence $D_X \cap D_X^\perp$ in $(\mathbb{Z}_k)^\ell$ corresponds to $D_X \cap D_X^\perp$ in $(\mathbb{Z}_2)^\ell$. Therefore, we obtain the following theorem by Propositions 2.5 and 2.6.

THEOREM 8.10. *Let $X \in \mathrm{Irr}(M_{\mathbf{0}})$. Suppose k is even and $D_X \neq 0$.*

(1) *If $k \equiv 0 \pmod 4$, then the irreducible decomposition of the χ_X-twisted M_D-module $M_D \boxtimes_{M_{\mathbf{0}}} X$ is given as*

$$M_D \boxtimes_{M_{\mathbf{0}}} X = \bigoplus_{j=1}^{|D_X|} U^j,$$

where U^j, $1 \leq j \leq |D_X|$, are inequivalent irreducible χ_X-twisted M_D-modules. Furthermore, $U^j \cong \bigoplus_{W \in \mathscr{O}_i} W$ as $M_{\mathbf{0}}$-modules, where $\mathscr{O}_i$ is the D-orbit in $\mathrm{Irr}(M_{\mathbf{0}})$ containing X.

(2) *If $k \equiv 2 \pmod 4$, then the irreducible decomposition of the χ_X-twisted M_D-module $M_D \boxtimes_{M_{\mathbf{0}}} X$ is given as*

$$M_D \boxtimes_{M_{\mathbf{0}}} X = \bigoplus_{j=1}^{|D_X \cap D_X^{\perp}|} (U^j)^{\oplus m},$$

where $m = [D_X : D_X \cap D_X^{\perp}]^{1/2}$, and U^j, $1 \leq j \leq |D_X \cap D_X^{\perp}|$, are inequivalent irreducible χ_X-twisted M_D-modules. Furthermore, $U^j \cong \bigoplus_{W \in \mathscr{O}_i} W^{\oplus m}$ as $M_{\mathbf{0}}$-modules, where $\mathscr{O}_i$ is the D-orbit in $\mathrm{Irr}(M_{\mathbf{0}})$ containing X.

Since any irreducible χ-twisted M_D-module for $\chi \in D^*$ is isomorphic to a direct summand of the χ_X-twisted M_D-module $M_D \boxtimes_{M_{\mathbf{0}}} X$ with $\chi = \chi_X$ for some $X \in \mathrm{Irr}(M_{\mathbf{0}})$, we obtain a classification of all the irreducible χ-twisted M_D-modules for any $\chi \in D^*$ by Theorems 8.9 and 8.10.

As mentioned in Section 2.2, we can write χ_i for χ_X, and D_i for D_X if X belongs to a D-orbit $\mathscr{O}_i$ in $\mathrm{Irr}(M_{\mathbf{0}})$. Let $I(\chi) = \{i \in I \mid \chi_i = \chi\}$. Then $I = \bigcup_{\chi \in D^*} I(\chi)$. The next lemma follows from (2) of Lemma 8.2.

LEMMA 8.11. *$I(\chi) \neq \emptyset$ for any $\chi \in D^*$.*

Theorems 8.9 and 8.10 imply the next theorem.

THEOREM 8.12. *The number of inequivalent irreducible χ-twisted M_D-modules for $\chi \in D^*$ is given as follows.*

$$\begin{aligned}
&|I(\chi)| && \text{if } k \text{ is odd},\\
&|I(\chi)_0| + \sum_{i \in I(\chi)_1} |D_i| && \text{if } k \equiv 0 \pmod 4,\\
&|I(\chi)_0| + \sum_{i \in I(\chi)_1} |D_i \cap D_i^{\perp}| && \text{if } k \equiv 2 \pmod 4,
\end{aligned}$$

where $I(\chi)_0 = \{i \in I(\chi) \mid D_i = 0\}$ and $I(\chi)_1 = I(\chi) \setminus I(\chi)_0$.

9. Irreducible M_D-modules: Case B.

Let $k \geq 2$, and let D be a $\mathbb{Z}_k$-code of length ℓ satisfying the conditions of Case B in Section 7, that is, k is even, $(\xi|\eta) \in \{0, k/2\}$ for all $\xi, \eta \in D$, and $(\xi|\xi) = k/2$ for some $\xi \in D$. Let D^0 and D^1 be as in Section 7. In this section, we construct all irreducible M_D-modules inside $V_{(\Gamma_{D^0})^\circ}$.

Since D^0 is a $\mathbb{Z}_k$-code of length ℓ in Case A, we can apply the results in Section 8 to the vertex operator algebra M_{D^0}. Let $P \in \mathrm{Irr}(M_{D^0})$. Then P is isomorphic to a direct summand of $M_{D^0} \boxtimes_{M_\mathbf{0}} M_{\mu,\nu}$ for some $M_{\mu,\nu} \in \mathrm{Irr}(M_\mathbf{0})$. Moreover, there are $\eta \in (\mathbb{Z}_k)^\ell$ and $\delta \in \{0,1\}^\ell$ such that $N(\eta,\delta) \subset (\Gamma_{D^0})^\circ$ and $V_{N(\eta,\delta)}$ contains $M_{\mu,\nu}$ as an $M_\mathbf{0}$-submodule.

For simplicity of notation, we identify P with an irreducible direct summand of $M_{D^0} \boxtimes_{M_\mathbf{0}} M_{\mu,\nu}$ isomorphic to P. Then P is a submodule of the M_{D^0}-module $M_{D^0} \boxtimes_{M_\mathbf{0}} M_{\mu,\nu}$, and the M_D-module $M_D \cdot P$ generated by P is isomorphic to $M_D \boxtimes_{M_{D^0}} P$. Thus $M_D \cdot P = P \oplus Q$ as M_{D^0}-modules, where Q is an irreducible M_{D^0}-module isomorphic to $M_{D^1} \boxtimes_{M_{D^0}} P$. Since $\Gamma_D \subset (\Gamma_D)^\circ \subset (\Gamma_{D^0})^\circ$, and since $M_{\mu,\nu} \subset V_{(\Gamma_{D^0})^\circ}$, we have $M_D \cdot P \subset V_{(\Gamma_{D^0})^\circ}$.

If P and Q are inequivalent as M_{D^0}-modules, then there is a unique M_D-module structure on $P \oplus Q$ which extends the M_{D^0}-module structure. If P and Q are equivalent as M_{D^0}-modules, then $P \oplus Q$ is the direct sum of two inequivalent irreducible M_D-modules, both of which are isomorphic to P as M_{D^0}-modules, see [**33**, Proposition 5.2]. Any irreducible M_D-module is obtained in this way. Therefore, the following theorem holds.

Theorem 9.1. *$V_{(\Gamma_{D^0})^\circ}$ contains any irreducible M_D-module.*

10. Examples.

The vertex operator algebra M_D was previously studied for some small k. The first one is the case $k = 2$, where M^0 is the Virasoro vertex operator algebra $L(1/2, 0)$ of central charge $1/2$, and its simple currents are M^0 and $M^1 = L(1/2, 1/2)$. The next one is the case $k = 3$, where M^0 is $L(4/5, 0) \oplus L(4/5, 3)$, and there are three simple currents. These cases were discussed in [**35**] and [**23**], respectively.

In the case $k = 4$, we have $M^0 = V^+_{\sqrt{6}\mathbb{Z}}$ and $M^2 = V^-_{\sqrt{6}\mathbb{Z}}$. So $M_D = V_{\sqrt{6}\mathbb{Z}}$ for $\ell = 1$ and $D = \{(0), (2)\}$. The case $k = 5$ with $\ell = 2$ and $D = \{(00), (12), (24), (31), (43)\}$, and the case $k = 9$ with $\ell = 1$ and $D = \{(0), (3), (6)\}$ were considered in Sections 3.5 and 3.9 of [**30**], respectively.

Let $k = 6$ with $\ell = 1$ and $D = \{(0), (3)\}$. Then

$$M_D = M^0 \oplus M^3 \cong L_{\mathrm{NS}}(5/4, 0) \oplus L_{\mathrm{NS}}(5/4, 3),$$

where $L_{\mathrm{NS}}(c, 0)$ is the simple Neveu–Schwarz algebra of central charge c, and $L_{\mathrm{NS}}(c, h)$ is its irreducible highest weight module with highest weight h, see [**3**, Section 4], [**44**]. In fact, let v be an weight $3/2$ element of M^3 such that $v_{(2)}v = (5/6)\mathbf{1}$. Then $L_n = \omega_{(n+1)}$ and $G_{n-1/2} = v_{(n)}$, $n \in \mathbb{Z}$, satisfy the relations for the Neveu–Schwarz algebra of central charge $5/4$. Thus the subalgebra generated by ω and v in V_{Γ_D} is isomorphic to $L_{\mathrm{NS}}(5/4, 0)$. Moreover, the weight 3 primary vector W^3 of M^0 is a highest weight vector for $L_{\mathrm{NS}}(5/4, 0)$.

Let $k = 8$ with $\ell = 1$ and $D = \{(0), (2), (4), (6)\}$. Then

$$M_D = M^0 \oplus M^2 \oplus M^4 \oplus M^6 \cong L_{\mathrm{NS}}(7/10, 0) \otimes L_{\mathrm{NS}}(7/10, 0)$$

is a simple vertex operator superalgebra, where

$$L_{\mathrm{NS}}(7/10, 0) \cong L(7/10, 0) \oplus L(7/10, 3/2).$$

The even part of M_D is

$$M^0 \oplus M^4 \cong \big(L(7/10, 0) \otimes L(7/10, 0)\big) \oplus \big(L(7/10, 3/2) \otimes L(7/10, 3/2)\big),$$

see [**4**, Theorems 4.14, 4.15], [**30**, Section 3.7].

Appendix. Minimal norm of elements in $N(j, \boldsymbol{a})$.

In this appendix, we calculate the minimal norm of elements in the coset $N(j, \boldsymbol{a})$ of N in N° defined in (4.3). Let $\Omega = \{1, 2, \dots, k\}$, and let $\alpha_S = \sum_{p \in S} \alpha_p$ for a subset S of Ω.

Theorem A.1. *Let $\boldsymbol{a} \in \{0, 1\}^k$ and $0 \le j \le k-1$. Set $I = \mathrm{supp}(\boldsymbol{a})$ and $i = \mathrm{wt}(\boldsymbol{a})$.*
(1) *If $j < i$, then*
 (i) $\min\{\langle \mu, \mu \rangle \mid \mu \in N(j, \boldsymbol{a})\} = (ki - (i - 2j)^2)/2k$,
 (ii) *For $\mu \in N(j, \boldsymbol{a})$, the norm $\langle \mu, \mu \rangle$ is minimal if and only if*

$$\mu = \frac{1}{2}\alpha_I - \alpha_J + \frac{2j - i}{2k}\gamma$$

for some $J \subset I$ with $|J| = j$. There are $\binom{i}{j}$ such μ's.
(2) *If $j \ge i$, then*
 (i) $\min\{\langle \mu, \mu \rangle \mid \mu \in N(j, \boldsymbol{a})\} = (k(k - i) - (k + i - 2j)^2)/2k$,
 (ii) *For $\mu \in N(j, \boldsymbol{a})$, the norm $\langle \mu, \mu \rangle$ is minimal if and only if*

$$\mu = \frac{1}{2}\alpha_I - \alpha_J + \frac{2j - i}{2k}\gamma$$

for some $I \subset J \subset \Omega$ with $|J| = j$. There are $\binom{k-i}{j-i}$ such μ's.

Proof. Any permutation on $\{\alpha_1, \dots, \alpha_k\}$ induces an isometry on $\mathbb{Q} \otimes_{\mathbb{Z}} L$. The isometry fixes γ and leaves L invariant. Since $\lambda_p = \gamma/2k - \alpha_p/2$ and $2\lambda_p \equiv 2\lambda_k \pmod{N}$, $1 \le p \le k$, we may assume that $I = \{1, \dots, i\}$, that is, $a_p = 1$ for $p \le i$, and $a_p = 0$ for $p \ge i + 1$ in (4.3).

Let $d = (2j - i)/2k$. Since $\alpha_p \equiv \alpha_q \pmod{N}$, $1 \le p, q \le k$, and since any element of N is of the form $c_1\alpha_1 + \dots + c_k\alpha_k$ for some $c_1, \dots, c_k \in \mathbb{Z}$ with $c_1 + \dots + c_k = 0$, we see from (4.4) that any element $\mu \in N(j, \boldsymbol{a})$ is of the form

$$\begin{aligned}\mu &= \frac{1}{2}(\alpha_1 + \cdots + \alpha_i) - c_1\alpha_1 - \cdots - c_k\alpha_k + d\gamma \\ &= \sum_{p=1}^{i}(d + 1/2 - c_p)\alpha_p + \sum_{q=i+1}^{k}(d - c_q)\alpha_q\end{aligned}$$

for some $c_1, \ldots, c_k \in \mathbb{Z}$ with $c_1 + \cdots + c_k = j$. Our concern is the minimum of

$$\langle \mu, \mu \rangle/2 = \sum_{p=1}^{i}(d + 1/2 - c_p)^2 + \sum_{q=i+1}^{k}(d - c_q)^2 \tag{A.1}$$

for $c_1, \ldots, c_k \in \mathbb{Z}$ with $c_1 + \cdots + c_k = j$.

We first show the assertion (1). Assume that $0 \le j < i \le k$. Then $-1/2 \le d < 1/2$. If $d = -1/2$, then $i = k$ and $j = 0$. In this case, we have $N(j, \boldsymbol{a}) = N$. Clearly, $\min\{\langle \mu, \mu \rangle \mid \mu \in N\} = 0$, and $\langle \mu, \mu \rangle = 0$ only if $\mu = 0$. Hence the assertion (1) holds in the case $d = -1/2$.

If $d = 0$, then $i = 2j$, and (A.1) reduces to $\langle \mu, \mu \rangle/2 = \sum_{p=1}^{i}(1/2 - c_p)^2 + \sum_{q=i+1}^{k} {c_q}^2$. We see that $(1/2 - c_p)^2$ is $1/4$ if $c_p = 0, 1$, and $9/4$ if $c_p = -1, 2$. Moreover, ${c_q}^2$ is 0 if $c_q = 0$, and 1 if $c_q = \pm 1$. Hence the minimum of $\langle \mu, \mu \rangle/2$ for $c_1, \ldots, c_k \in \mathbb{Z}$ with $c_1 + \cdots + c_k = j$ is attained only when j of $c_1, \ldots, c_i$ are 1, the remaining $i - j$ of $c_1, \ldots, c_i$ are 0, and $c_q = 0$ for $i + 1 \le q \le k$. The minimum of $\langle \mu, \mu \rangle/2$ is $i/4$. Thus the assertion (1) holds in the case $d = 0$.

If $-1/2 < d < 0$, then $0 < d + 1/2 < 1/2$. In this case, $(d + 1/2 - c_p)^2$ belongs to one of the four open intervals $(0, 1/4)$, $(1/4, 1)$, $(1, 9/4)$, or $(9/4, 4)$ according as $c_p = 0$, 1, -1, or 2, respectively. Moreover, $(d - c_q)^2$ belongs to one of the four open intervals $(0, 1/4)$, $(1/4, 1)$, $(1, 9/4)$, or $(9/4, 4)$ according as $c_q = 0$, -1, 1, or -2, respectively. Hence the minimum of (A.1) for $c_1, \ldots, c_k \in \mathbb{Z}$ with $c_1 + \cdots + c_j = j$ is attained only when j of $c_1, \ldots, c_i$ are 1, the remaining $i - j$ of $c_1, \ldots, c_i$ are 0, and $c_q = 0$ for $i + 1 \le q \le k$. The minimum of (A.1) is

$$(d - 1/2)^2 j + (d + 1/2)^2(i - j) + d^2(k - i) = i/4 - (i - 2j)^2/4k.$$

Thus the assertion (1) holds in the case $-1/2 < d < 0$.

If $0 < d < 1/2$, then $1/2 < d + 1/2 < 1$. In this case, $(d + 1/2 - c_p)^2$ belongs to one of the four open intervals $(0, 1/4)$, $(1/4, 1)$, $(1, 9/4)$, or $(9/4, 4)$ according as $c_p = 1$, 0, 2, or -1, respectively. Moreover, $(d - c_q)^2$ belongs to one of the four open intervals $(0, 1/4)$, $(1/4, 1)$, $(1, 9/4)$, or $(9/4, 4)$ according as $c_q = 0$, 1, -1, or 2, respectively. Hence the minimum of (A.1) for $c_1, \ldots, c_k \in \mathbb{Z}$ with $c_1 + \cdots + c_k = j$ is attained only when j of $c_1, \ldots, c_i$ are 1, the remaining $i - j$ of $c_1, \ldots, c_i$ are 0, and $c_q = 0$ for $i + 1 \le q \le k$. Thus the assertion (1) holds in the case $0 < d < 1/2$. We have shown that (1) holds for all $0 \le j < i \le k$.

Next, we show the assertion (2). Assume that $j \ge i$. We use Lemma 4.3. Let $a'_p = 1 - a_p$, $1 \le p \le k$, $\boldsymbol{a}' = (a'_1, \ldots, a'_k)$, and $I' = \operatorname{supp}(\boldsymbol{a}')$. Then $I \cup I' = \Omega$ and $I \cap I' = \emptyset$. Let $i' = \operatorname{wt}(\boldsymbol{a}')$ and $j' = j - i$. Then $i' = k - i$ and $0 \le j' < i' \le k$. The assertion (1) for $N(j', \boldsymbol{a}')$ implies that

(i)$'$ $\min\{\langle \mu, \mu \rangle \mid \mu \in N(j', \boldsymbol{a}')\} = (ki' - (i' - 2j')^2)/2k$, (ii)$'$ For $\mu \in N(j', \boldsymbol{a}')$, the norm $\langle \mu, \mu \rangle$ is minimal if and only if

$$\mu = \frac{1}{2}\alpha_{I'} - \alpha_{J'} + \frac{2j' - i'}{2k}\gamma \tag{A.2}$$

for some $J' \subset I'$ with $|J'| = j'$. There are $\binom{i'}{j'}$ such μ's.

Since $\alpha_{I'} = \gamma - \alpha_I$, and since $2j' - i' = 2j - i - k$, the element μ of (A.2) is equal to

$$\mu = -\frac{1}{2}\alpha_I - \alpha_{J'} + \frac{2j - i}{2k}\gamma.$$

The set $\{J \subset \Omega \mid I \subset J, |J| = j\}$ is in one-to-one correspondence with the set $\{J' \subset \Omega - I \mid |J'| = j - i\}$ by $J \mapsto J - I$ and $J' \mapsto J' \cup I$. Let $J = J' \cup I$. Then $\alpha_J = \alpha_{J'} + \alpha_I$, as $J' \cap I = \emptyset$. Thus the assertion (2) holds. □

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