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Vector bundles on the stack of G-zips and partial Hasse invariants

By

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Introduction

The stack of G-zips is an object in the realm of group-theory, which was introduced by Moonen–Wedhorn ([15]) and more thoroughly studied by Pink–Wedhorn–Ziegler in [16, 17]. One of the main applications of this stack is to study stratifications in moduli spaces in positive characteristic. Let k be an algebraic closure of $\mathbf{F}_p$. Let G be a connected reductive group over $\mathbf{F}_p$ and $\mu : \mathbf{G}_k \to G_k$ a cocharacter. Pink–Wedhorn–Ziegler attach to (G, μ) an algebraic stack $G\text{-}\mathtt{Zip}^{\mu}$ over k. Its underlying topological space is finite and admits an explicit parametrization in terms of the Weyl group of G (see Theorem 2.1). This stack appears in the theory of Shimura varieties. If S_K is the special fiber of a Hodge-type Shimura variety with good reduction, then Zhang showed ([19]) that there is a smooth (surjective) map $\zeta : S_K \to G\text{-}\mathtt{Zip}^{\mu}$, where (G, μ) denotes the reductive group over $\mathbf{F}_p$ and the cocharacter $\mu : \mathbf{G}_{m,k} \to G_k$ deduced from the Shimura datum. The fibers of the map ζ are the Ekedahl–Oort strata of S_K.

The stack $G\text{-}\mathtt{Zip}^{\mu}$ itself is an interesting algebraic object, endowed with a natural stratification, as well as a family of vector bundles. Denote by P the parabolic subgroup deduced from the cocharacter μ (see §2 for the precise definition) and let $L \subset P$ be the Levi subgroup given by the centralizer of μ. Any algebraic P-representation (V, ρ) gives rise to a vector bundle $\mathcal{V}(\rho)$ on $G\text{-}\mathtt{Zip}^{\mu}$. In the paper [13], we studied line bundles on the stack $G\text{-}\mathtt{Zip}^{\mu}$ and showed the existence of generalized μ-ordinary Hasse invariants. This result was generalized to all strata in [7]. In the paper [12], we studied vector bundles of the form $\mathcal{V}_I(\lambda)$ for $\lambda \in X^*(T)$. The vector bundle $\mathcal{V}_I(\lambda)$ is the vector bundle attached to the P-representations $V_I(\lambda) := \mathrm{Ind}_B^P(\lambda)$ where B is a Borel subgroup contained in P. These vector bundles arise naturally in the context of automorphic forms. Indeed, the

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global sections of $\mathcal{V}_I(\lambda)$ over S_K are automorphic forms modulo p of level K and weight λ. By pullback via the map $\zeta : S_K \to G\text{-}\mathtt{Zip}^\mu$, global sections of $\mathcal{V}_I(\lambda)$ over $G\text{-}\mathtt{Zip}^\mu$ can also be viewed as such automorphic forms. Therefore, it is relevant to study the space $H^0(G\text{-}\mathtt{Zip}^\mu, \mathcal{V}_I(\lambda))$. When P is defined over $\mathbf{F}_p$, we determined this space in terms of the representation $V_I(\lambda)$ in [12, Theorem 3.7.2]. In the general case, we give an explicit formula for the space $H^0(G\text{-}\mathtt{Zip}^\mu, \mathcal{V}(\rho))$ for an arbitrary P-representation (V, ρ) in [9, Theorem 3.4.1]. Returning to vector bundles of the form $\mathcal{V}_I(\lambda)$, we are interested in the set

$$C_{\mathrm{zip}} := \{\lambda \in X^*(T) \mid H^0(G\text{-}\mathtt{Zip}^\mu, \mathcal{V}_I(\lambda)) \neq 0\}.$$

This set is a cone in $X^*(T)$ (i.e. an additive submonoid). For a cone $C \subset X^*(T)$, write $\langle C\rangle$ for the saturated cone of C, i.e. the set of $\lambda \in X^*(T)$ such that some positive multiple of λ lies in C. It is conjectured that the cone $\langle C_{\mathrm{zip}}\rangle$ controls the possible weights of modulo p automorphic forms (see Conjecture 6.1).

The goal of this proceedings paper is to present some new results regarding the set C_{zip} that constitute part of the work in progress [6] in collaboration with Imai and Goldring. It is inspired by results of Diamond–Kassaei in [3, 4] for Hilbert–Blumenthal Shimura varieties, which show (among other results) that the weight of any nonzero Hilbert modular form in characteristic p is spanned over $\mathbf{Q}_{>0}$ by the weights of certain partial Hasse invariants constructed by Andreatta–Goren in [1]. We introduce a general notion of partial Hasse invariants, for arbitrary reductive groups G. To explain it, recall the stack of G-zip flags $G\text{-}\mathtt{ZipFlag}^\mu$ defined in [7]. It admits a natural projection map

$$\pi : G\text{-}\mathtt{ZipFlag}^\mu \to G\text{-}\mathtt{Zip}^\mu.$$

For any character $\lambda \in X^*(T)$, there is a line bundle $\mathcal{V}_{\mathrm{flag}}(\lambda)$ such that $\pi_*(\mathcal{V}_{\mathrm{flag}}(\lambda)) = \mathcal{V}_I(\lambda)$. Furthermore, the stack $G\text{-}\mathtt{ZipFlag}^\mu$ admits a natural stratification $(\mathcal{C}_w)_{w\in W}$. Write Δ for the set of simple roots of G. The codimension one strata are of the form $(\mathcal{C}_{w_0 s_\alpha})_{\alpha\in\Delta}$, where w_0 is the longest element of W and s_α is the reflection along α. For each $\alpha \in \Delta$, there exists a section $H_\alpha \in H^0(G\text{-}\mathtt{ZipFlag}^\mu, \mathcal{V}_{\mathrm{flag}}(\lambda_\alpha))$ for a certain character $\lambda_\alpha \in X^*(T)$, whose vanishing locus is precisely the Zariski closure of the codimension one stratum $\mathcal{C}_{w_0 s_\alpha}$. Note that H_α, λ_α are not completely uniquely determined by α, but the small ambiguity in the choice is irrelevant. Since $\pi_*(\mathcal{V}_{\mathrm{flag}}(\lambda_\alpha)) = \mathcal{V}_I(\lambda_\alpha)$, the partial Hasse invariant H_α can also be interpreted as a global section of $\mathcal{V}_I(\lambda_\alpha)$ over $G\text{-}\mathtt{Zip}^\mu$.

Inspired by the result of Diamond–Kassaei mentioned above, we introduce the cone $C_{\mathrm{Hasse}} \subset X^*(T)$ generated by the weights $(\lambda_\alpha)_{\alpha\in\Delta}$ of the partial Hasse invariants. From the definition of C_{zip}, one has $C_{\mathrm{Hasse}} \subset C_{\mathrm{zip}}$. The natural group-theoretical generalization of Diamond–Kassaei's result would be the equality $\langle C_{\mathrm{zip}}\rangle = \langle C_{\mathrm{Hasse}}\rangle$.

However, this equality is false in general (see §7 for a counter-example). In the work in progress [6], we determine exactly for which pairs (G, μ) this equality holds by giving an explicit characterization (Theorem 8.1). If this condition holds, we say that (G, μ) is of Hasse-type. Therefore, one can hope to generalize the results of [3, 4] to Shimura varieties such that (G, μ) is of Hasse-type.

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§ 1. The F-zip attached to an abelian variety

Let p be a prime number and denote by k an algebraic closure of $\mathbf{F}_p$. Let $\sigma : k \to k$, $x \mapsto x^p$ be the p-power Frobenius homomorphism. If A is an abelian variey over k, then the p-torsion $H = A[p]$ is a finite, commutative k-group scheme killed by p. By Dieudonne theory, there is an equivalence of categories $H \mapsto \mathbf{D}(H)$ between such objects and triples (M, F, V), where

(i) M is a finite-dimensional k-vector space,

(ii) $F : M \to M$ is a σ-linear endomorphism,

(iii) $V : M \to M$ is a σ^{-1}-linear endomorphism,

subject to the conditions $FV = 0$ and $VF = 0$. If the triple (M, F, V) satisfies furthermore $\mathrm{Ker}(F) = \mathrm{Im}(V)$ and $\mathrm{Ker}(V) = \mathrm{Im}(F)$, then we call it a Dieudonne space. For group schemes of the form $A[p]$, the associated triple (M, F, V) is a Dieudonne space. If $g = \dim(A)$, then $\dim_k(M) = 2g$ and F, V have rank g. It is easy to see that there are only finitely many isomorphism classes of Dieudonne spaces of dimension $2g$, let $\{H_1, \dots, H_N\}$ be a set of representatives.

Similarly, let S be a scheme of characteristic p and $\mathcal{A} \to S$ an abelian scheme over S of relative dimension g. For each point $s \in S$, we can consider the abelian variety $\mathcal{A}_s := \mathcal{A} \otimes_S \overline{\kappa(s)}$ where $\kappa(s)$ is the field of definition of s and $\overline{\kappa(s)}$ is an algebraic closure. We can then study how the isomorphism class of $\mathcal{A}_s[p]$ varies for $s \in S$. We obtain a finite decomposition

$$S = \bigsqcup_{i=1}^{N} S_i$$

where S_i is the set of $s \in S$ such that $\mathcal{A}_s[p] \simeq H_i$. For example, the ordinary locus of S is the set of $s \in S$ for which

$$\mathcal{A}_s[p] \simeq \mu_p^g \times (\mathbf{Z}/p\mathbf{Z})^g. \tag{1.1}$$

We now explain a useful way to think about this decomposition. Consider the relative algebraic de Rham cohomology $\mathcal{M} := H^1_{\mathrm{dR}}(\mathcal{A}/S)$. It is a locally free $\mathcal{O}_S$-module of rank $2g$, equipped with the following structure:

(i) A Hodge filtration $0 \subset \Omega \subset \mathcal{M}$, where Ω is a locally free $\mathcal{O}_S$-submodule of rank g,

(ii) an $\mathcal{O}_S$-linear map $F : \mathcal{M}^{(p)} \to \mathcal{M}$,

(iii) an $\mathcal{O}_S$-linear map $V : \mathcal{M} \to \mathcal{M}^{(p)}$.

Furthermore, $(\mathcal{M}, F, V)$ satisfies $\mathrm{Ker}(F) = \mathrm{Im}(V) = \Omega^{(p)}$ and $\mathrm{Ker}(V) = \mathrm{Im}(F)$. When $S = \mathrm{Spec}(k)$, this is simply the Dieudonne space attached to an abelian variety, as we explained above.

We note that there is a natural equivalence between such triples and quadruples $(\mathcal{M}, \mathcal{C}, \mathcal{D}, \iota_\bullet)$, where

(i) $\mathcal{M}$ is a locally free $\mathcal{O}_S$-module of rank $2g$,

(ii) $\mathcal{C} \subset \mathcal{M}$ and $\mathcal{D} \subset \mathcal{M}$ are locally free $\mathcal{O}_S$-submodules of rank g,

(iii) $\iota_0 : \mathcal{C}^{(p)} \to \mathcal{M}/\mathcal{D}$ and $\iota_1 : (\mathcal{M}/\mathcal{C})^{(p)} \to \mathcal{D}$ are isomorphisms of $\mathcal{O}_S$-modules.

This equivalence is given by sending $(\mathcal{M}, F, V)$ to $(\mathcal{M}, \Omega, \mathrm{Im}(F), \iota_\bullet)$ where ι_0, ι_1 are the isomorphisms naturally deduced from F and V. We call such a quadruple $(\mathcal{M}, \mathcal{C}, \mathcal{D}, \iota_\bullet)$ an F-zip of rank g over S. Consider the stack $\mathtt{F\text{-}Zip}_g$ over $\mathbf{F}_p$ which classifies F-zips of rank g. In other words, for any $\mathbf{F}_p$-scheme T, morphisms $T \to \mathtt{F\text{-}Zip}_g$ correspond bijectively to F-zips over T.

Recall that we started with an abelian scheme $\mathscr{A} \to S$ and attached an F-zip of rank g on S. In particular, we obtain a natural morphism of stacks $\zeta : S \to \mathtt{F\text{-}Zip}_g$. By definition, the underlying topological space of $\mathtt{F\text{-}Zip}_g$ is the set of equivalence classes of maps $\mathrm{Spec}(K) \to \mathtt{F\text{-}Zip}_g$ where K is an algebraically closed field. Hence, they correspond to isomorphism classes of F-zips over algebraically closed fields of characteristic p. Over such a field, an F-zip is simply a Dieudonne space, so we deduce that the underlying topological space of $\mathtt{F\text{-}Zip}_g$ is in bijection with the set $\{H_1, \dots, H_N\}$. Furthermore, the locus $S_i \subset S$ defined earlier coincides with the fiber of the map $\zeta : S \to \mathtt{F\text{-}Zip}_g$ above the point of $\mathtt{F\text{-}Zip}_g$ corresponding to H_i.

§ 2. More general reductive groups

One often considers abelian varieties endowed with some extra structure. For example, let S be an $\mathbf{F}_p$-scheme and $(\mathcal{A}, \xi)$ a principally polarized abelian variety over S. Let $(\mathcal{M}, \Omega, F, V)$ be the F-zip attached to $\mathcal{A}$. The principal polarization ξ induces a perfect pairing $\langle -, - \rangle : \mathcal{M} \times \mathcal{M} \to \mathcal{O}_S$. Furthermore, it is compatible with F, V in the sense that $\langle Fx, y \rangle = \langle x, Vy \rangle^{(p)}$, where $\langle -, - \rangle^{(p)}$ denotes the induced pairing on $\mathcal{M}^{(p)}$. The stack that classifies tuples $(\mathcal{M}, \Omega, F, V, \langle -, - \rangle)$ is called the stack of symplectic F-zips of rank g.

More generally, in order to study F-zips with additional structure, it is convenient to consider the stack of G-zips, for any connected reductive $\mathbf{F}_p$-group G. Fix a cocharacter $\mu : \mathbf{G}_{m,k} \to G_k$. This cocharacter gives rise to a pair of opposite parabolics $P_{\pm}$, where P_+ (resp. P_-) is the parabolic subgroup of G_k whose Lie algebra is $\bigoplus_{n \geq 0} \mathfrak{g}_n$ (resp. $\bigoplus_{n \leq 0} \mathfrak{g}_n$), where $\mathfrak{g}_n \subset \mathrm{Lie}(G_k)$ is the subspace where $x \in \mathbf{G}_{m,k}$ acts by multiplication with x^n via μ. The intersection $L = P_+ \cap P_-$ is a common Levi subgroup, equal to the centralizer of μ. Set $P := P_-$, $Q = (P_+)^{(p)}$, and $M = L^{(p)}$. The stack of G-zips of type μ is the stack $G\text{-}\mathtt{Zip}^{\mu}$ such that for any k-scheme S, $G\text{-}\mathtt{Zip}^{\mu}(S)$ parametrizes tuples (I, I_P, I_Q, ι), where

(i) I is a G-torsor over S,

(ii) $I_P \subset I$ is a P-torsor over S,

(iii) $I_Q \subset I$ is a Q-torsor over S,

(iv) $\iota : (I_P/U)^{(p)} \to I_Q/V$ is an isomorphism of M-torsors.

We recall an important result of Pink–Wedhorn–Ziegler. If H is an algebraic group, denote by $R_{\mathrm{u}}(H)$ the unipotent radical of H. For $x \in P$, we can write uniquely $x = \overline{x}u$ with $\overline{x} \in L$ and $u \in R_{\mathrm{u}}(P)$. This defines a projection map $\theta^P_L \colon P \to L;\ x \mapsto \overline{x}$. Similarly, we have a projection $\theta^Q_M \colon Q \to M$. Denote by $\varphi \colon G \to G$ the Frobenius homomorphism. Since $M = L^{(p)}$, it induces a map $\varphi \colon L \to M$. The zip group is the subgroup of $P \times Q$ defined by

$$E := \{(x, y) \in P \times Q \mid \varphi(\theta^P_L(x)) = \theta^Q_M(y)\}.$$

Let E act on the left on G_k by the rule $(x, y) \cdot g := xgy^{-1}$ for all $(x, y) \in E$ and all $g \in G_k$. Then, by [17, Th. 1.5], there is an isomorphism of k-stacks

$$G\text{-}\mathtt{Zip}^{\mu} \simeq [E \backslash G_k]\,. \tag{2.1}$$

In particular, the underlying topological space of $G\text{-}\mathtt{Zip}^{\mu}$ coincides with the set of E-orbits in G_k. We explain a parametrization of these orbits from [16]. Fix a Borel pair

(B,T) satisfying $B \subset P$ and $T \subset L$, and suppose for simplicity that (B,T) is defined over $\mathbf{F}_p$. After possibly changing μ to a conjugate cocharacter, such a Borel pair always exists. Write Φ for the set of T-roots. Let $\Phi_+ \subset \Phi$ denote the positive roots (where positivity is defined with respect to the Borel subgroup opposite to B). Finally, let Δ be the set of simple roots. Recall that there is a bijection between subsets of Δ and conjugacy classes of parabolic subgroups of G_k (Borel subgroups corresponding to the empty set). Let $I, J \subset \Delta$ denote the types of P, Q respectively. We put $\Delta^P := \Delta \setminus I$. Note that since $B \subset P$, the set I coincides with the set Δ_L of simple roots of L. Let W be the Weyl group of T and $\ell : W \to \mathbf{Z}_{\geq 0}$ the length function. Write w_0 for the longest element in W. For a subset $K \subset \Delta$, let $W_K \subset W$ be the subgroup generated by $\{s_\alpha \mid \alpha \in K\}$, and let $w_{0,K}$ be the longest element of W_K. Define W^K as the set of elements $w \in W$ which are of minimal length in the coset wW_K. For $w \in W$, choose a representative $\dot{w} \in N_G(T)$, such that $(w_1 w_2)^{\cdot} = \dot{w}_1 \dot{w}_2$ whenever $\ell(w_1 w_2) = \ell(w_1) + \ell(w_2)$ (this is possible by choosing a Chevalley system, see [2], Exp. XXIII, §6). Define $z := w_0 w_{0,J}$. For $w \in W$, define G_w as the E-orbit of $\dot{w}\dot{z}^{-1}$. The E-orbits in G form a stratification of G by locally closed subsets.

Theorem 2.1 ([16, Th. 11.3]). *The map $w \mapsto G_w$ induces a bijection from W^J onto the set of E-orbits in G. Furthermore, for $w \in W^J$, one has*

$$\dim(G_w) = \ell(w) + \dim(P).$$

We explain the connection with F-zips, symplectic F-zips and G-zips. For this, let $\mathrm{Sp}(2g)$ be the symplectic group over $\mathbf{F}_p$ attached to the matrix

$$\Psi := \begin{pmatrix} & -J \\ J & \end{pmatrix} \qquad \text{where} \qquad J := \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix}.$$

Let $B \subset \mathrm{Sp}(2g)$ be the Borel subgroup of lower-triangular matrices in $\mathrm{Sp}(2g)$ and $T \subset B$ the maximal torus given by diagonal matrices in $\mathrm{Sp}(2g)$. Consider the cocharacter $\mu_g : \mathbf{G}_m \to \mathrm{Sp}(2g)$, $z \mapsto \begin{pmatrix} zI_g & 0 \\ 0 & z^{-1}I_g \end{pmatrix}$. We may also view μ_g as a cocharacter of $\mathrm{GL}_{2g,\mathbf{F}_p}$. Then, through the correspondence between vector bundles and torsors for the general linear group, F-zips of rank g identify naturally with GL_{2g}-zips of type μ_g. Similarly, symplectic F-zips of rank g identifiy with $\mathrm{Sp}(2g)$-zips of type μ_g.

§ 3. Vector bundles on $G\text{-}\mathtt{Zip}^{\mu}$

For an algebraic group H over k, write $\mathrm{Rep}(H)$ for the category of algebraic representations of H, i.e. morphisms $\rho : H \to \mathrm{GL}(V)$ where V is a finite-dimensional k-vector space.

Let G be a reductive group over $\mathbf{F}_p$ and $\mu : \mathbf{G}_{m,k} \to G_k$ a cocharacter. Write again P, Q, L, M for the algebraic groups defined in §2. Let $\rho : P \to \mathrm{GL}(V)$ be an algebraic representation. By definition, the stack $G\text{-}\mathtt{Zip}^{\mu}$ carries a universal P-torsor I_P, thus by applying ρ to this P-torsor, we obtain a vector bundle $\mathcal{V}(\rho)$ on $G\text{-}\mathtt{Zip}^{\mu}$. This construction gives rise to a functor

$$\mathrm{Rep}(P) \to \mathfrak{VB}(G\text{-}\mathtt{Zip}^{\mu})$$

where the notation $\mathfrak{VB}(\mathcal{X})$ (for a stack $\mathcal{X}$) denotes the category of vector bundles on $\mathcal{X}$. The natural projection $\theta_L^P : P \to L$ induces a fully faithful functor $(\theta_L^P)^* : \mathrm{Rep}(L) \to \mathrm{Rep}(P)$. Hence, we view $\mathrm{Rep}(L)$ as the full subcategory of $\mathrm{Rep}(P)$ of P-representations which are trivial on the unipotent radical $R_{\mathrm{u}}(P)$. In particular, we are interested in the following kind of representations.

Since we assumed $T \subset L$, the group $B_L := B \cap L$ is a Borel subgroup of L. For a character $\lambda \in X^*(T)$, define an L-representation $V_I(\lambda)$ by

$$V_I(\lambda) = \mathrm{Ind}_{B_L}^{L}(\lambda).$$

Denote by $\mathcal{V}_I(\lambda)$ the vector bundle on $G\text{-}\mathtt{Zip}^{\mu}$ attached to $V_I(\lambda)$. We call $\mathcal{V}_I(\lambda)$ the *automorphic vector bundle associated to the weight* λ on $G\text{-}\mathtt{Zip}^{\mu}$. This terminology comes from the theory of Shimura varieties. Indeed, let S_K be the special fiber of the Kisin–Vasiu integral model of a Hodge-type Shimura variety with good reduction at p, and let G be the reductive group over $\mathbf{F}_p$ deduced from the Shimura datum. Then Zhang showed in [19] that there is a smooth map $\zeta : S_K \to G\text{-}\mathtt{Zip}^{\mu}$. Then, the pullback $\zeta^*\mathcal{V}_I(\lambda)$ is an automorphic bundle, and its global sections over S_K are automorphic forms modulo p of level K and weight λ. Note that if $\lambda \in X^*(T)$ is not L-dominant, then $V_I(\lambda) = 0$ and hence $\mathcal{V}_I(\lambda) = 0$.

In the example of $G = \mathrm{Sp}(2g)$, $\mu = \mu_g$, we can make this question much more explicit. Recall that in this case, the stack $G\text{-}\mathtt{Zip}^{\mu}$ parametrizes tuples $(\mathcal{M}, \Omega, F, V, \langle -, - \rangle)$ (see §1). Identify $X^*(T) = \mathbf{Z}^g$ and for $\lambda = (k_1, \ldots, k_g)$, write $\mathcal{V}_I(k_1, \ldots, k_g)$ for $\mathcal{V}_I(\lambda)$. The family of vector bundles $\mathcal{V}_I(k_1, \ldots, k_g)$ is obtained by applying Schur functors to Ω. Another way to think about it is via the stack of zip flags. For a general $\mathbf{F}_p$-reductive group G and cocharacter $\mu : \mathbf{G}_{m,k} \to G_k$, it is defined as follows. It is the stack that parametrizes pairs $(\underline{I}, J)$ where $\underline{I} = (I, I_P, I_Q, \iota)$ is a G-zip and $J \subset I_P$ is a B-torsor.

We denote this stack by $G\text{-}\mathtt{ZipFlag}^{\mu}$. There is a natural projection map

$$\pi : G\text{-}\mathtt{ZipFlag}^{\mu} \to G\text{-}\mathtt{Zip}^{\mu}$$

given by $(\underline{I}, J) \mapsto \underline{I}$. For any representation $(V,\rho) \in \mathrm{Rep}(B)$, by applying the universal B-torsor on $G\text{-}\mathtt{ZipFlag}^{\mu}$, we obtain a vector bundle $\mathcal{V}_{\mathrm{flag}}(\rho)$. We have the identification

$$\pi_*(\mathcal{V}_{\mathrm{flag}}(\rho)) = \mathcal{V}(\mathrm{Ind}_B^P(\rho)).$$

In particular, we can think of the vector bundle $\mathcal{V}_I(\lambda)$ on $G\text{-}\mathtt{Zip}^{\mu}$ as the push-forward of the line bundle $\mathcal{V}_{\mathrm{flag}}(\lambda)$. Let us return to the example of the symplectic group. In this case, the stack of zip flags parametrizes tuples $(\mathcal{M}, \Omega, \mathcal{F}_\bullet, F, V, \langle -,- \rangle)$, where $(\mathcal{M}, \Omega, F, V, \langle -,- \rangle)$ is a symplectic F-zip, and $\mathcal{F}_\bullet$ is a full flag of Ω. Specifically, it is a filtration

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_{g-1} \subset \mathcal{F}_g = \Omega$$

where $\mathcal{F}_i$ is a locally free $\mathcal{O}_S$-module of rank i, locally direct factor of Ω. In this description, we used the fact that for the group $\mathrm{Sp}(2g)$, a B-torsor contained in I_P corresponds to a symplectic flag refining the Hodge filtration, and by using the pairing $\langle -,- \rangle$, it is equivalent to give a full flag of Ω (with no condition). Define the line bundle $\mathcal{L}_i := \mathcal{F}_i/\mathcal{F}_{i-1}$ on $G\text{-}\mathtt{ZipFlag}^{\mu}$ for $1 \leq i \leq g$. For $\lambda = (k_1, \dots, k_g) \in \mathbf{Z}^g$, the line bundle $\mathcal{V}_{\mathrm{flag}}(\lambda)$ on $G\text{-}\mathtt{ZipFlag}^{\mu}$ is then concretely given by

$$\mathcal{L}(k_1, \dots, k_g) := \bigotimes_{i=1}^{g} \mathcal{L}_i^{-k_i}.$$

Similarly, the vector bundle $\mathcal{V}_I(k_1, \dots, k_g)$ is the push-forward of $\mathcal{L}(k_1, \dots, k_g)$ via π.

§ 4. Global sections of vector bundles

In the paper [9], we determine the space of global sections $H^0(G\text{-}\mathtt{Zip}^{\mu}, \mathcal{V}(\rho))$ for an arbitrary representation $(V,\rho) \in \mathrm{Rep}(P)$. This space can be expressed in terms of the part of the Brylinski–Kostant filtration of V which is invariant under a certain finite group scheme (see [9, Theorem 3.4.1]). To simplify, we will assume here that P is defined over $\mathbf{F}_p$ and we will only consider representations in $\mathrm{Rep}(L)$. For $(V,\rho) \in \mathrm{Rep}(L)$, write $V = \bigoplus_{\chi \in X^*(T)} V_\chi$ for the T-weight decomposition of V. Recall that $\Delta^P := \Delta \setminus I$. Define a subspace $V_{\geq 0}^{\Delta^P} \subset V$ as the sum of weight spaces V_χ such that $\langle \chi, \alpha^\vee \rangle \leq 0$ for all $\alpha \in \Delta^P$.

Theorem 4.1. *Let $(V, \rho) \in \mathrm{Rep}(L)$. There is an identification*

$$H^0(G\text{-}\mathtt{Zip}^\mu, \mathcal{V}(\rho)) = V^{L(\mathbf{F}_p)} \cap V_{\geq 0}^{\Delta^P}.$$

In particular, this formula applies to the L-representations $V_I(\lambda)$, which are of particular interest for us. In the papers [8, 12], we studied global sections of the vector bundle $\mathcal{V}_I(\lambda)$. In particular, we investigated for which $\lambda \in X^*(T)$, this vector bundle admits nonzero global sections on $G\text{-}\mathtt{Zip}^\mu$. From the point of view of representation theory, it seems very difficult to determine when the intersection $V_I(\lambda)^{L(\mathbf{F}_p)} \cap V_I(\lambda)_{\geq 0}^{\Delta^P}$ is nonzero. We will study this question in the next section.

Again, let us consider the case $G = \mathrm{Sp}(2g)$, $\mu = \mu_g$. As we explained, we have $\pi_*\mathcal{L}(k_1, \dots, k_g) = \mathcal{V}_I(k_1, \dots, k_g)$, hence the space $H^0(G\text{-}\mathtt{Zip}^\mu, \mathcal{V}_I(k_1, \dots, k_g))$ identifies with global sections of $\mathcal{L}(k_1, \dots, k_g)$ on $G\text{-}\mathtt{ZipFlag}^\mu$. Recall also that $G\text{-}\mathtt{ZipFlag}^\mu$ parametrizes tuples $(\mathcal{M}, \Omega, \mathcal{F}_\bullet, F, V, \langle -, - \rangle)$. Let us give examples of sections of the line bundles $\mathcal{L}(k_1, \dots, k_g)$. Fix an integer $1 \leq i \leq g$. By restricting the Verschiebung map $V : \Omega \to \Omega^{(p)}$ to $\mathcal{F}_i$ and composing with the projection $\Omega^{(p)} \to (\Omega/\mathcal{F}_{g-i})^{(p)}$, we obtain a map $V_i : \mathcal{F}_i \to (\Omega/\mathcal{F}_{g-i})^{(p)}$ of vector bundles of rank i. Taking the determinant, we obtain a map

$$H_i := \det(V_i) : \mathcal{L}_1 \otimes \cdots \otimes \mathcal{L}_i \to (\mathcal{L}_{g-i+1} \otimes \cdots \otimes \mathcal{L}_g)^p \tag{4.1}$$

In other words, H_i is a section of the line bundle $\mathcal{L}(\lambda_i)$ where

$$\lambda_i = (1, \dots, 1, 0 \dots, 0) - (0, \dots, 0, p \dots, p)$$

(both 1 and p appear i times). In particular, for $i = g$, the section H_g is the classical Hasse invariant. Write $\omega = \bigwedge^g \Omega$, hence we have $\mathcal{V}_I(\lambda_g) = \omega^{p-1}$. Let $\mathscr{A}_g$ be the moduli stack of principally polarized abelian varieties over $\mathbf{F}_p$. As we explained, there is a natural map $\zeta : \mathscr{A}_g \to G\text{-}\mathtt{Zip}^\mu$. Then, the pullback of H_g by ζ is the classical Hasse invariant of $\mathscr{A}_g$, whose non-vanishing locus is the ordinary locus of $\mathscr{A}_g$. More generally, the sections H_i $(1 \leq i \leq g)$ are called partial Hasse invariants. We explain this terminology in the next section. We give the vanishing loci of the other sections H_i in §7.

§ 5. Flag strata and partial Hasse invariants

Let G be a reductive group over $\mathbf{F}_p$ and $\mu : \mathbf{G}_{m,k} \to G_k$ a cocharacter. There is a natural stratification $(\mathcal{C}_w)_{w \in W}$ of $G\text{-}\mathtt{ZipFlag}^\mu$ which corresponds to the Bruhat stratification of G. Specifically, if we write $G\text{-}\mathtt{Zip}^\mu = [E \backslash G_k]$ as in (2.1), then the stack $G\text{-}\mathtt{ZipFlag}^\mu$ is isomorphic to $[E' \backslash G_k]$, where $E' := E \cap (B \times Q)$ acts on G_k by restricting the action of E. Furthermore, it is easy to see that $E' \subset B \times {}^zB$ (recall that $z = w_0 w_{0,J}$). Composing with the map $g \mapsto gz$, we finally obtain a morphism

$$\psi : G\text{-}\mathtt{ZipFlag}^\mu \to [B\backslash G/B].$$

The Bruhat stratification $(BwB)_{w\in W}$ gives a natural stratification of the stack $[B\backslash G/B]$. By pulling back via ψ, we obtain a locally closed stratification $(\mathcal{C}_w)_{w\in W}$ of $G\text{-}\mathtt{ZipFlag}^\mu$. The codimension of $\mathcal{C}_w$ coincides with the colength of the element $w \in W$ (defined as $\ell(w_0) - \ell(w)$). In particular, there are exacty $|\Delta|$ strata of codimension one, corresponding to the elements $w_0 s_\alpha$ for $\alpha \in \Delta$.

Let us come back to the case $(G, \mu) = (\mathrm{Sp}(2g), \mu_g)$. Recall the definition of the flag space $\mathscr{F}_g$ of $\mathscr{A}_g$. Similarly to $G\text{-}\mathtt{ZipFlag}^\mu$, it parametrizes tuples $(A, \xi, \mathcal{F}_\bullet)$ where $(A, \xi) \in \mathscr{A}_g$ and $\mathcal{F}_\bullet \subset \Omega_A$ is a full flag. This space was first introduced by Ekedahl–Van der Geer in [5]. The space $\mathscr{F}_g$ can also be viewed as the fiber product

$$\mathscr{F}_g = \mathscr{A}_g \times_{G\text{-}\mathtt{Zip}^\mu} G\text{-}\mathtt{ZipFlag}^\mu\,.$$

By pullback from $G\text{-}\mathtt{ZipFlag}^\mu$, we obtain a stratification $(S_w)_{w\in W}$ of $\mathscr{F}_g$. For a more concrete description of the stratum S_w in this case, see [5, §4]. The sections H_i $(1 \le i \le g)$ constructed in (4.1) have the following property. Identifying $X^*(T) = \mathbf{Z}^g$ as usual, write $\alpha_i = e_i - e_{i+1}$ for $i = 1, \dots, g-1$ and $\alpha_g = 2e_g$. Then, the vanishing locus of the section $H_i \in H^0(G\text{-}\mathtt{ZipFlag}^\mu, \mathcal{L}(\lambda_i))$ coincides with the Zariski closure of $\mathcal{C}_{w_0 s_{\alpha_i}}$. For this reason, we call these sections partial Hasse invariants on $G\text{-}\mathtt{ZipFlag}^\mu$. The cone in $\mathbf{Z}^g$ generated by the weights λ_i $(1 \le i \le g)$ is called the Hasse cone, and is denoted by $C_{\mathrm{Hasse}} \subset \mathbf{Z}^g$.

Similarly, for an arbitrary pair (G, μ), there exist characters $\lambda_\alpha \in X^*(T)$ and sections $h_\alpha \in H^0(G\text{-}\mathtt{ZipFlag}^\mu, \mathcal{V}_{\mathrm{flag}}(\lambda_\alpha))$ such that the vanishing locus of h_α is $\mathcal{C}_{w_0 s_\alpha}$. See [10] for a general study of partial Hasse invariants and their properties. Again, we denote by $C_{\mathrm{Hasse}} \subset X^*(T)$ the cone generated by the characters λ_α. Concretely, the cone C_{Hasse} can also be defined as the image of the set of dominant characters $X^*(T)_+$ by the linear map

$$h\colon X^*(T) \to X^*(T), \quad \lambda \mapsto \lambda - p\sigma(zw_0\lambda)$$

where σ indicates the action of Frobenius on $X^*(T)$.

§ 6. The zip cone

Again, let (G, μ) be an arbitrary cocharacter datum, with attached groups P, L, Q, M. Fix also a Borel pair (B, T) defined over $\mathbf{F}_p$ as in §2. The zip cone is defined as the set of $\lambda \in X^*(T)$ such that $\mathcal{V}_I(\lambda)$ admits nonzero sections over $G\text{-}\mathtt{Zip}^\mu$, in other words:

$$C_{\mathrm{zip}} := \{\lambda \in X^*(T) \mid H^0(G\text{-}\mathtt{Zip}^\mu, \mathcal{V}_I(\lambda)) \neq 0\}.$$

By Theorem 4.1, the set C_{zip} is also the locus where the L-representaton $V_I(\lambda)$ satisfies that $V_I(\lambda)^{L(\mathbf{F}_p)} \cap V_I(\lambda)^{\Delta^P}_{\geq 0} \neq 0$. Using the identification of $H^0(G\text{-}\mathtt{Zip}^\mu, \mathcal{V}_I(\lambda))$ with $H^0(G\text{-}\mathtt{ZipFlag}^\mu, \mathcal{V}_{\mathrm{flag}}(\lambda))$, it follows from the formula $\mathcal{V}_{\mathrm{flag}}(\lambda+\lambda') = \mathcal{V}_{\mathrm{flag}}(\lambda) \otimes \mathcal{V}_{\mathrm{flag}}(\lambda')$ for all $\lambda, \lambda' \in X^*(T)$ that C_{zip} is stable under addition. One has also obviously $0 \in C_{\mathrm{zip}}$. For a cone $C \subset X^*(T)$, denote by $\langle C \rangle$ the saturated cone of C, i.e. the set of $\lambda \in X^*(T)$ such that some positive multiple of λ lies in C. We have the inclusions

$$C_{\mathrm{Hasse}} \subset C_{\mathrm{zip}} \subset X^*_{+,I}(T)$$

where $X^*_{+,I}(T)$ denotes the set of L-dominant characters, i.e. characters λ satisfying $\langle \lambda, \alpha^\vee \rangle \geq 0$ for all $\alpha \in I$. The first inclusion follows from the definition, and the second one from the fact that $\mathcal{V}_I(\lambda) = 0$ if $\lambda \notin X^*_{+,I}(T)$.

Even though C_{zip} is completely defined in group-theoretical terms, it is useful to return to the theory of Shimura varieties to understand C_{zip} intuitively. Recall that a Shimura variety comes as a tower of algebraic varieties $Sh = (Sh_K)_K$ defined over some number field F, where K varies in the set of compact open subgroups of $\mathbf{G}(\mathbf{A}_f)$ (here $\mathbf{G}$ is the corresponding connected reductive group over $\mathbf{Q}$). Assume that Sh is of Hodge-type, and that $\mathbf{G}_{\mathbf{Q}_p}$ is unramified. Furthermore, fix a hyperspecial subgroup $K_p \subset \mathbf{G}(\mathbf{Q}_p)$. Then, Kisin ([11]) and Vasiu ([18]) constructed a canonical model $\mathcal{S} = (\mathcal{S}_{K^p})_{K^p}$ of the tower $Sh_{K_p} = (Sh_{K_pK^p})_{K^p}$ over $\mathcal{O}_{F_\mathfrak{p}}$, for any place $\mathfrak{p}|p$ in F. For K of the form K_pK^p (where $K^p \subset \mathbf{G}(\mathbf{A}^p_f)$), let S_K be the special fiber of $\mathcal{S}_K$. It is defined over the residual field κ of $\mathfrak{p}$. As we explained, there is a smooth surjective map $\zeta_K : S_K \to G\text{-}\mathtt{Zip}^\mu$ (where G denotes the special fiber of a $\mathbf{Z}_p$-reductive model of $\mathbf{G}_{\mathbf{Q}_p}$). Furthermore, the maps ζ_K commute with change of level. It is natural to define a set $C_K(k)$ as follows

$$C_K(k) := \{\lambda \in X^*(T) \mid H^0(S_K \otimes_\kappa k, \mathcal{V}_I(\lambda)) \neq 0\}.$$

Here, we denoted again by $\mathcal{V}_I(\lambda)$ its pullback via ζ_K. The set $C_K(k)$ indicates the possible weights of nonzero automorphic forms over k, which is an important question. The set $C_K(k)$ highly depends on the level K. However, since the change of level maps are finite etale, one can show that the saturated cone $\langle C_K(k) \rangle$ is independent of K. For this reason, we conjectured the following:

Conjecture 6.1 ([8, Conjecture 2.1.6]). *One has*

$$\langle C_K(k) \rangle = \langle C_{\mathrm{zip}} \rangle.$$

Note that the inclusion $C_{\mathrm{zip}} \subset C_K(k)$ is obvious. We proved this conjecture in several cases in *loc. cit.*. Since the vector bundles $\mathcal{V}_I(\lambda)$ admit natural models over $\mathcal{O}_{F_\mathfrak{p}}$, one can also define a set $C_K(\mathbf{C})$ in a similar way. By the same argument, $\langle C_K(\mathbf{C})\rangle$ is independent of K. Let C_{GS} denote the set of characters $\lambda \in X^*(T)$ satisfying the conditions

$$\langle \lambda, \alpha^\vee\rangle \geq 0 \quad \text{for } \alpha \in I,$$
$$\langle \lambda, \alpha^\vee\rangle \leq 0 \quad \text{for } \alpha \in \Phi_+ \setminus \Phi_{L,+}.$$

For example, in the case of $\mathrm{Sp}(2g)$, the set C_{GS} is given by the tuples $(k_1, \dots, k_g)$ such that $0 \geq k_1 \geq \cdots \geq k_g$. By work of Griffiths–Schmid, one has

$$\langle C_K(\mathbf{C})\rangle = C_{\mathrm{GS}}.$$

Furthermore, by reducing sections modulo p, one can see that one has always an inclusion $\langle C_K(\mathbf{C})\rangle \subset \langle C_K(k)\rangle$ (see [12, Proposition 1.8.3]). Hence, if Conjecture 6.1 is correct, we should have an inclusion $C_{\mathrm{GS}} \subset \langle C_{\mathrm{zip}}\rangle$, which is now a purely group-theoretical statement. We indeed verify this prediction for an arbitrary pair (G, μ) in the work in progress [6] (generalizing [12, Corollary 3.5.6]):

Theorem 6.2. *For arbitrary (G, μ), one has $C_{\mathrm{GS}} \subset \langle C_{\mathrm{zip}}\rangle$.*

Hence, Theorem 6.2 substantiates Conjecture 6.1, since the inclusion $C_{\mathrm{GS}} \subset \langle C_{\mathrm{zip}}\rangle$ is predicted by Conjecture 6.1 (at least for groups attached to Shimura varieties of Hodge-type). We now explain in more detail the proof of Theorem 6.2. For $\lambda \in X^*_{+,I}(T)$, let $f_\lambda \in V_I(\lambda)$ be a nonzero element of the highest weight line in the L-representation $V_I(\lambda)$. We define the norm $\mathrm{Norm}(f_\lambda)$ of f_λ. For simplicity, we explain its construction in the case when P is defined over $\mathbf{F}_p$. It is defined by taking the product of the $s \cdot f_\lambda$ over $s \in L(\mathbf{F}_p)$, and corresponds to an element

$$\mathrm{Norm}(f_\lambda) \in V(d\lambda)^{L(\mathbf{F}_p)}$$

where $d = |L(\mathbf{F}_p)|$. Hence, by Theorem 4.1, if $\mathrm{Norm}(f_\lambda)$ lies in the subspace $V_I(\lambda)^{\Delta^P}_{\geq 0}$, then this element defines a global section over $G\text{-}\mathtt{Zip}^\mu$ of weight $d\lambda$. We explain the result in the general case (here we do not assume that P is defined over $\mathbf{F}_p$). Let $L_0 \subset L$ be the largest Levi subgroup containing T and defined over $\mathbf{F}_p$.

Theorem 6.3 ([6]). *The element* $\mathrm{Norm}(f_\lambda)$ *defines a (nonzero) global section over $G\text{-}\mathtt{Zip}^\mu$ if and only if for all $\alpha \in \Delta^P$, the following holds:*

$$\sum_{w \in W_{L_0}(\mathbf{F}_p)} \sum_{i=0}^{r_\alpha - 1} p^{i+\ell(w)} \ \langle w\lambda, \sigma^i(\alpha^\vee)\rangle \leq 0.$$

When P is defined over $\mathbf{F}_p$, Theorem 6.3 is enough to show the inclusion $C_{\mathrm{GS}} \subset \langle C_{\mathrm{zip}} \rangle$. Indeed, in this case and for $\lambda \in C_{\mathrm{GS}}$, all summands of the above sum are ≤ 0, hence the sum is ≤ 0. Therefore, $\mathrm{Norm}(f_\lambda)$ defines a nonzero section of weight $d\lambda$, which shows that $\lambda \in \langle C_{\mathrm{zip}} \rangle$. In other words, denote by C_{hw} the set of $\lambda \in X^*_{+,I}(T)$ such that the inequalities of Theorem 6.3 are satisfied (here, "hw" stands for "highest weight"). Then we have $C_{\mathrm{GS}} \subset C_{\mathrm{hw}} \subset \langle C_{\mathrm{zip}} \rangle$. However, when P is not defined over $\mathbf{F}_p$, the inclusion $C_{\mathrm{GS}} \subset C_{\mathrm{hw}}$ may not hold (on the other hand, the inclusion $C_{\mathrm{hw}} \subset \langle C_{\mathrm{zip}} \rangle$ always holds). To show $C_{\mathrm{GS}} \subset \langle C_{\mathrm{zip}} \rangle$ in the general case, we study in detail the case when G is a Weil restriction. Then, we embed diagonally G in $\mathrm{Res}_{\mathbf{F}_{p^m}/\mathbf{F}_p}(G_{\mathbf{F}_{p^m}})$ for an appropriate $m \geq 1$ and deduce the result for G. To sum up, we have the following inclusions

$$
\begin{array}{ccccccc}
 & & & \langle C_{\mathrm{Hasse}} \rangle & & & \\
 & & & \searrow & & & \\
X^*_-(L) & \hookrightarrow & C_{\mathrm{hw}} & \hookrightarrow & \langle C_{\mathrm{zip}} \rangle & \hookrightarrow & X^*_{+,I}(T) \\
 & \searrow & & \nearrow & & & \\
 & & C_{\mathrm{GS}} & & & &
\end{array}
$$

Here $X^*(L)_-$ denotes the set $X^*(L) \cap X^*(T)_-$, where $X^*(T)_-$ is the set of anti-dominant characters. We recall results of [13] about μ-ordinary Hasse invariants. In *loc. cit.*, we considered the set

$$X^*(L)_{-,\mathrm{reg}} = \{\lambda \in X^*(L) \mid \langle \lambda, \alpha^\vee \rangle < 0, \ \forall \alpha \in \Delta^P\}. \tag{6.1}$$

We showed ([13, Theorem 1]) that if $\lambda \in X^*(L)_{-,\mathrm{reg}}$, then there exists a section $H_\mu \in H^0(G\text{-}\mathtt{Zip}^\mu, \mathcal{V}(N\lambda))$ (some integer $N \geq 1$), such that the non-vanishing locus of H_μ is the unique open stratum of $G\text{-}\mathtt{Zip}^\mu$. In particular, it implies $X^*(L)_{-,\mathrm{reg}} \subset \langle C_{\mathrm{zip}} \rangle$. Hence, the present discussion is a vast generalization of the results of [13].

§ 7. Example: The case Sp(6)

Let us focus on the case $(\mathrm{Sp}(2g), \mu_g)$ for $g = 3$. We retain the notations introduced in §3. We constructed partial Hasse invariants, which are sections over $G\text{-}\mathtt{ZipFlag}^\mu$ of weights $\lambda_1 = (1, 0, -p)$, $\lambda_2 = (1, 1-p, -p)$ and $\lambda_3 = (1-p, 1-p, 1-p)$ respectively. It is possible to construct more complicated sections. Consider the map $V : \Omega \to \Omega^{(p)}$. By twisting, we also have a map $V^{(p)} : \Omega^{(p)} \to \Omega^{(p^2)}$. By composition, we have $V^{(p)} \circ V : \Omega \to \Omega^{(p^2)}$. Now, take the tensor product of the maps $V^{(p)} \circ V|_{\mathcal{L}_1} : \mathcal{L}_1 \to \Omega^{(p^2)}$ and

$V^{(p)}|_{\mathcal{L}_1^p} : \mathcal{L}_1^p \to \Omega^{(p^2)}$. We obtain a map

$$f : \mathcal{L}_1 \otimes \mathcal{L}_1^p \to \Omega^{(p^2)} \otimes \Omega^{(p^2)}.$$

Compose this map with the natural map $\wedge : \Omega^{(p^2)} \otimes \Omega^{(p^2)} \to \bigwedge^2 \Omega^{(p^2)}$ and the projection $\bigwedge^2 \Omega^{(p^2)} \to \bigwedge^2 (\Omega/\mathcal{F}_1)^{(p^2)}$. Since $\bigwedge^2(\Omega/\mathcal{F}_1) = \mathcal{L}_2 \otimes \mathcal{L}_3$, we obtain finally a map

$$f_1 : \mathcal{L}_1 \otimes \mathcal{L}_1^p \to (\mathcal{L}_2 \otimes \mathcal{L}_3)^{p^2},$$

hence a section of $\mathcal{L}(p+1, -p^2, -p^2)$. This section f_1 is an example of section of the form $\mathrm{Norm}(f_\lambda)$ (see Theorem 6.3). It seems very difficult to grasp the definition of f_1, however its vanishing locus has a simple interpretation. View this section on the flag space $\mathscr{F}_g$ by pullback, and let $x = (A, \xi, \mathcal{F}_\bullet)$ be a point of $\mathscr{F}_g(k)$. Write $M = \mathbf{D}(A[p])$ for the Dieudonne space of A. The Hodge filtration corresponds to $0 \subset VM \subset M$. Furthermore, VM is endowed with a filtration

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 \subset \mathcal{F}_3 = VM$$

given by $\mathcal{F}_\bullet$. Then we have an equivalence

$$f_1(x) \neq 0 \Longleftrightarrow \mathcal{F}_1 \oplus V(\mathcal{F}_1) \oplus V^2(\mathcal{F}_1) = VM.$$

In other words, the non-vanishing locus corresponds to the points where the three k-lines $\mathcal{F}_1$, $V(\mathcal{F}_1)$ and $V^2(\mathcal{F}_1)$ are linearly independent. There is also a section f_2 of weight $(1, 1, -(p^2+p))$ whose non-vanishing locus is given by a similar condition for the dual $M^\vee$. The construction of f_2 is similar to f_1, we refer the interested reader to [12, §6.4]. For arbitrary $g \geq 1$, we can also give the vanishing locus for the partial Hasse invariants H_i $(1 \leq i \leq g)$. One has:

$$H_i(x) \neq 0 \Longleftrightarrow \mathcal{F}_{g-i} \oplus V(\mathcal{F}_i) = VM.$$

In particular for $i = g$, the section H_g is the classical Hasse invariant. Its non-vanishing locus coincides with the ordinary locus by the following easy lemma.

Lemma 7.1. *The following conditions are equivalent.*

(i) A is ordinary.

(ii) One has $VM \oplus FM = M$.

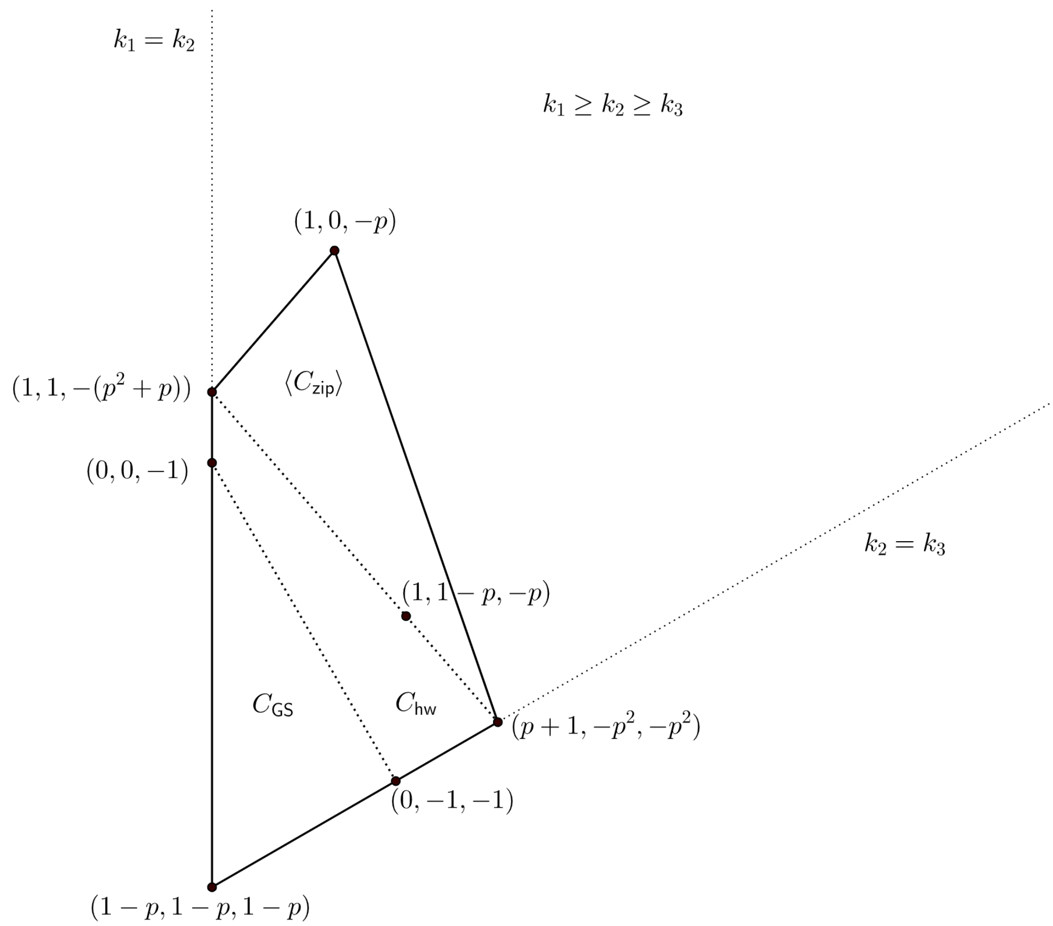
(iii) One has $V(VM) = VM$.

Proof. By (1.1), A is ordinary if and only if $M \simeq \mu_p^g \times (\mathbf{Z}/p\mathbf{Z})^g$. Via the Dieudonne equivalence explained in §1, this amounts to $M = VM \oplus FM$, which shows the equivalence between (i) and (ii). Moreover, this implies immediately $V(VM) = VM$. Conversely, if $V(VM) = VM$ then V is injective on VM by dimension reasons, hence $VM \cap FM = 0$ and thus $M = VM \oplus FM$. This terminates the proof. $\square$

For Sp(6), the cones are given by the following equations

$$C_{\text{Hasse}} = \mathbf{N}(1, 0-p) + \mathbf{N}(1, 1-p, -p) + \mathbf{N}(1-p, 1-p, 1-p).$$
$$C_{\text{GS}} = \{(k_1, k_2, k_3),\ 0 \geq k_1 \geq k_2 \geq k_3\}$$
$$C_{\text{hw}} = \{(k_1, k_2, k_3),\ p^2 k_1 + p k_2 + k_3 \leq 0\}.$$

Let us represent graphically these cones. In $\mathbf{R}^3$, we choose a generic affine hyperplane that cuts all the cones, and represent the intersections with this hyperplane. Hence, a point represents a half-line from the origin. As explained, all cones are contained in the set of L-dominant characters, i.e. the set of $(k_1, k_2, k_3) \in \mathbf{Z}^3$ with $k_1 \geq k_2 \geq k_3$. We represent the weights of the two sections f_1, f_2 defined above, as well as the weights of the three partial Hasse invariants.



To avoid cluttering the picture, we did not represent the Hasse cone, which is generated by $(1,0-p)$, $(1,1-p,-p)$ and $(1-p,1-p,1-p)$. Note that it intersects both C_{GS} and C_{hw} and there is no inclusion relation between these three cones.

§ 8. G-zips of Hasse-type

In the case Sp(6), the above diagram shows explicitly the cone $\langle C_{\mathrm{zip}}\rangle$. However, for $g \geq 4$ and for most reductive groups G, this cone is still undetermined. We give in this section a family of cases where we can determine $\langle C_{\mathrm{zip}}\rangle$. Via Conjecture 6.1, this potentially will apply to the study of automorphic forms in characteristic p.

This work is inspired from the papers [3, 4] of Diamond–Kassaei. They show as a corollary of [4, Theorem 8.1], that for Hilbert–Blumenthal Shimura varieties (also in ramified cases), one has an equality

$$\langle C_K(k)\rangle = \langle C_{\mathrm{Hasse}}\rangle. \tag{8.1}$$

We also proved this result using different techniques in [8]. We showed moreover that a similar equality holds for Siegel threefolds ($G = \mathrm{Sp}(4)_{\mathbf{F}_p}$), and Picard surfaces at split primes ($G = GL_{3,\mathbf{F}_p}$). Since we have in general $C_{\mathrm{Hasse}} \subset C_{\mathrm{zip}} \subset C_K(k)$, the cones of (8.1) also coincide with $\langle C_{\mathrm{zip}}\rangle$. However, we saw that for Sp(6), the inclusion $\langle C_{\mathrm{Hasse}}\rangle \subset \langle C_{\mathrm{zip}}\rangle$ was strict, so we cannot expect such a result to hold for general groups G.

To explain the second result of [6], we must first recall the topological properties of the various cones. For a cone $C \subset X^*(T)$, write $C_{\mathbf{R}_{\geq 0}}$ for the cone generated over $\mathbf{R}_{\geq 0}$ by C inside $X^*(T) \otimes_{\mathbf{Z}} \mathbf{R}$. In what follows, endow the subset $X^*_{+,I}(T)_{\mathbf{R}_{\geq 0}}$ with the subspace topology inherited from $X^*(T) \otimes_{\mathbf{Z}} \mathbf{R}$. Also, recall the definition of $X^*(L)_{-,\mathrm{reg}}$ given in (6.1). We explained the inclusion $X^*(L)_{-,\mathrm{reg}} \subset \langle C_{\mathrm{zip}}\rangle$. We note that:

Fact. *The set $C_{\mathrm{zip},\mathbf{R}_{\geq 0}}$ is a neighborhood of $X^*(L)_{-,\mathrm{reg}}$ inside $X^*_{+,I}(T)_{\mathbf{R}_{\geq 0}}$.*

For example, in the case Sp(6), the set $X^*(L)_{-,\mathrm{reg}}$ is the half-line $\mathbf{R}_{\geq 0}(-1,-1,-1)$, which contains the weight of the classical Hasse invariant $\lambda_3 = (1-p,1-p,1-p)$. The above fact can be proven separately, but can also be deduced immediately from the (much more difficult) inclusion $C_{\mathrm{GS}} \subset C_{\mathrm{zip}}$. Indeed, it is clear that $C_{\mathrm{GS},\mathbf{R}_{\geq 0}}$ is a neighborhood of $X^*(L)_{-,\mathrm{reg}}$ inside $X^*_{+,I}(T)_{\mathbf{R}_{\geq 0}}$, thus so is $C_{\mathrm{zip},\mathbf{R}_{\geq 0}}$. One can ask whether the Hasse cone $C_{\mathrm{Hasse},\mathbf{R}_{\geq 0}}$ is also a neighborhood of $X^*(L)_{-,\mathrm{reg}}$. First of all, it can happen that $X^*(L)_{-,\mathrm{reg}}$ is not contained in $C_{\mathrm{Hasse},\mathbf{R}_{\geq 0}}$. Secondly, even when the inclusion $X^*(L)_{-,\mathrm{reg}} \subset C_{\mathrm{Hasse},\mathbf{R}_{\geq 0}}$ holds, it can happen that this cone is not a neighborhood of $X^*(L)_{-,\mathrm{reg}}$. This can be observed in the case Sp(6) explained in §7.

Theorem 8.1 ([6]). *Let (G,μ) be an arbitrary cocharacter datum, with attached groups P, L, Q, M. The following properties are equivalent:*

(i) $C_{\mathrm{Hasse},\mathbf{R}_{\geq 0}}$ *is a neighborhood of* $X^*(L)_{-,\mathrm{reg}}$ *inside* $X^*_{+,I}(T)_{\mathbf{R}_{\geq 0}}$.

(ii) The inclusion $C_{\mathrm{GS}} \subset C_{\mathrm{Hasse}}$ *holds.*

(iii) One has the equality $\langle C_{\mathrm{zip}} \rangle = \langle C_{\mathrm{Hasse}} \rangle$.

(iv) The parabolic P *is defined over* $\mathbf{F}_p$*, and the Frobenius* σ *acts on* I *by* $-w_{0,I}$.

In Property (iv), note that since P is defined over $\mathbf{F}_p$, the subset $I \subset \Delta$ is stable by the action of σ. Note also that the element $-w_{0,I}$ preserves I as well. We say that (G, μ) is of Hasse-type if any of the above conditions is satisfied. For example, in the case of Hilbert–Blumenthal Shimura varieties considered by Diamond–Kassaei, we have $I = \emptyset$, so it is obviously of Hasse-type. The case $(\mathrm{Sp}(2g), \mu_g)$ is of Hasse-type if and only if $g \leq 2$. The case (GL_3, μ) where $\mu : z \mapsto \mathrm{diag}(z, z, 1)$ is also of Hasse-type.

Returning to Shimura varieties, we may ask when the equality (8.1) of Diamond–Kassaei generalizes. If this equality holds, then a fortiori $\langle C_{\mathrm{zip}} \rangle = \langle C_{\mathrm{Hasse}} \rangle$, hence (G, μ) must be of Hasse-type. Conversely, we conjecture that for Hodge-type Shimura varieties such that (G, μ) is of Hasse-type, the equality (8.1) holds. Beside the cases already mentioned treated in [8], the Hodge-type Shimura varieties attached to spinor groups $\mathrm{GSpin}(2n + 1, 2)$ are also of Hasse-type. Therefore, Diamond–Kassaei's results potentially generalize to these Shimura varieties.

References

[1] Andreatta, F. and Goren, E. Hilbert modular forms: mod p and p-adic aspects, *Mem. Amer. Math Soc.*, **173** (2005).

[2] Artin, M. and Bertin, J. E. and Demazure, M. and Gabriel, P. and Grothendieck, A. and Raynaud, M. and Serre, J.-P., SGA3: Schémas en groupes, *Institut des Hautes Études Scientifiques*, (1965/66).

[3] Diamond, F. and Kassaei, Payman L., Minimal weights of Hilbert modular forms in characteristic p, *Compos. Math.*, **153**, 9 (2017), 1769–1778.

[4] Diamond, F. and Kassaei, Payman L., The cone of minimal weights for mod p Hilbert modular forms, *preprint, arXiv:2004.13227*, (2020).

[5] Ekedahl, T. and van der Geer, G. Cycle classes of the E-O stratification on the moduli of abelian varieties, In: Tschinkel, Y., Zarhin, Y. (eds.) Algebra, Arithmetic and Geometry. *Progress in Mathematics*, **269** (2009), 567-–636. Birkhäuser, Boston, MA.

[6] Goldring, W., Imai, N. and Koskivirta, J.-S., Weights of mod p automorphic forms and partial Hasse invariants, *available soon* (2021).

[7] Goldring, W. and Koskivirta, J.-S., Strata Hasse invariants, Hecke algebras and Galois representations, *Invent. Math. 217*, No. **3** (2019), 887–984.

[8] Goldring, W. and Koskivirta, J.-S., Automorphic vector bundles with global sections on G-Zip$^{\mathcal{Z}}$-schemes, *Compositio Math.*, **154** (2018), 2586–2605.

[9] Imai, N. and Koskivirta, J.-S., Automorphic vector bundles on the stack of G-zips, *Forum Math. Sigma 9* (2021), Paper No. e37, 31 pp.

[10] Imai, N. and Koskivirta, J.-S., Partial Hasse invariants for Shimura varieties of Hodge-type, *preprint, arXiv:2109.11117*, (2021).

[11] Kisin, M., Integral models for Shimura varieties of abelian type, *J. Amer. Math. Soc.*, **23**, 4 (2010), 967–1012.

[12] Koskivirta, J.-S., Automorphic forms on the stack of G-Zips, *Results in Mathematics*, no. **3**, Art. 91 (2019), 52 pp.

[13] Koskivirta, J.-S. and Wedhorn, T., Generalized μ-ordinary Hasse invariants, *J. Algebra*, **502** (2018), 98–119.

[14] Moonen, B., Group schemes with additional structures and Weyl group cosets, *Moduli of abelian varieties (Texel Island, 1999), Progr. Math.*, **195**, (2001), 255–298.

[15] Moonen, B. and Wedhorn, T., Discrete invariants of varieties in positive characteristic, *IMRN*, **72** (2004), 3855–3903.

[16] Pink, R. and Wedhorn, T. and Ziegler, P., Algebraic zip data, *Doc. Math.*, **16** (2011), 253–300.

[17] Pink, R. and Wedhorn, T. and Ziegler, P., F-zips with additional structure, *Pacific J. Math.*, **274** (2015), 183–236.

[18] Vasiu, A., Integral canonical models of Shimura varieties of preabelian type, *Asian J. Math.*, **3** (1999), 401–518.

[19] Zhang, C., Ekedahl-Oort strata for good reductions of Shimura varieties of Hodge type, *Canad. J. Math.*, **70**, 2 (2018), 451–480.