Some results related to Keisler-Shelah isomorphism theorem

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1 Introduction and Preliminaries

Under GCH, Keisler [3] showed that if $M \equiv N$ are L-structures of size $\leq 2^{|L|+\aleph_0}$, then there is an ultrafilter U for which their ultrapowers are isomorphic. Later, Shelah showed that the GCH condition can be eliminated. In particular, for a countable language, if $M \equiv N$ and $|M|, |N| \leq 2^{\aleph_0}$, then there is an ultrafilter U for which their ultrapowers are isomorphic.

In the original paper by Keisler, it was shown that his ultrafilter U can chosen as an ultrafilter on ω (under CH). If T is stable, without assuming CH, U can be chosen as an ultrafilter on ω.

Golshani and Shelah [5] showed that, under $\neg CH$, there are models $M, N \models Th(\mathbb{Q}, <)$ of size $\leq 2^{\aleph_0}$ such that for no U on ω, $M^\omega/U \cong N^\omega/U$.

Given the history of studies related to isomorphism theorem described above, we give the following conjecture:

Conjecture ($\neg CH$) *If T is countable unstable, then there are models $M, N \models T$ of size $\leq 2^{\aleph_0}$ such that, for no U on ω, $M^\omega/U \cong N^\omega/U$.*

Although the conjecture has not been completely solved so far, the purpose of this paper is to report the partial results. Unstable theories are known to have either the strict order property or the independence property. The most typical theories with strict order property are those with a linear order. We show that our conjecture is true in this case. One of the most important

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theory with the independence property is the theory of random graphs. It is also shown that the conjecture is true for this theory. Some parts of this article are joint work with Kota Takeuchi.

2 Definable Linear Orders

Remark 1. Let $\varphi(\bar{x}, \bar{y}; \bar{a})$ be a formula defining an infinite linear preorder on some definable set. We can choose an increasing $\bar{a}$-indiscernible sequence $\{\bar{b}_i\}_{i<\omega}$ in the domain. Let $p(\bar{x}, \bar{z}) = \mathrm{tp}(\bar{b}_0, \bar{a}) \in S(T)$. Then $\varphi(\bar{x}, \bar{y}, \bar{a})$ linearly preorders $p(\mathcal{M}, \bar{a})$, and no maximal element exists in $p(\mathcal{M}, \bar{a})$. Define $\psi(\bar{x}\bar{z}, \bar{y}\bar{w})$ as the formula

$$\varphi(\bar{x}, \bar{y}, \bar{z}) \wedge \bar{z} = \bar{w}.$$

Then ψ, having no parameters, defines a preorder on the type-definable set $D = \{\bar{b}\bar{c} : \bar{b}\bar{c} \models p(\bar{x}, \bar{y})\}$. Moreover, we have:

1. $E(\bar{x}\bar{z}, \bar{y}\bar{w}) :\equiv (\bar{z} = \bar{w})$ defines an equivalence relation on D;
2. On each E-class, $\psi(\bar{x}\bar{z}, \bar{y}\bar{w})$ defines a linear preorder;
3. Elements from different E-classes are not comparable (using ψ).

Let T^* be the Skolemization of T, and let dcl^* denote the definable closure in T^*. We can naturally expand $\mathcal{M}$ to a model of $M^* \models T^*$.

Theorem 2. *Suppose $\neg CH$. Let T be unstable and suppose that, in T^{eq}, there is an infinite definable linear order. Then, there are two models M, and N of size $\leq 2^\omega$ such that their ultrapowers M^ω/U and N^ω/U are not isomorphic, for any ultrafilter U on ω.*

Proof. According to Remark 1, we assume there is a type $p(x) \in S(T)$ and formulas $E(x, y)$ and $<$ such that

1. E gives an equivalence relation on the set defined by $p(x)$;
2. Each E-class is linearly preodered by $<$.
3. Different E-classes are $<$-incomparable.

x may not be a single variable. We construct a sequence of sets $A_i \subset \mathcal{M}$ such that

- $|A_i| \leq |i| + \omega$,
- For each $d \in \mathrm{dcl}^*(A_{<i})$, there is $e \in A_i$ such that $E(\mathrm{dcl}^*(A_{<i}), d) < e$. (If $x \in \mathrm{dcl}^*(A_{<i})$ is E-equivalent to d, then $x < e$.)

We put $M = \mathrm{dcl}^*(A_{<\omega_1})$ and $N = \mathrm{dcl}^*(A_{<\omega_2})$.

Claim A. *In M^ω/U, there is an unbounded increasing sequence of length ω_1.*

Proof. Choose a $<$-increasing sequence $\{a_i\}_{i<\omega_1}$ of M such that

- a_i belongs to $A_i \setminus \mathrm{dcl}^*(A_{<i})$, for all $i < \omega_1$;
- $E(\mathrm{dcl}^*(A_{<i}), a_0) < a_i$.

Let σ be the natural embedding of M into M^ω/U. Then $\{\sigma(a_i)\}_{i<\omega_1}$ is clearly an increasing sequence in M^ω/U. We show that it is unbounded. Suppose, for a contradiction, that $b = [(b_0, b_1, \dots)]_U$ bounded $\{\sigma(a_i)\}_{i<\omega_1}$. Choose $\alpha < \omega_1$ such that all b_n's are contained in $\mathrm{dcl}^*(A_{<\alpha})$. Since $\sigma(a_0) < b$ holds, there must be a set $X \in U$ such that $a_0 < b_n$ holds for all $n \in X$. So, $E(a_0, b_n)$ holds for all $n \in X$. By the choice of A_α and a_α, we must have $b_n < a_\alpha$. A contradiction. End of Proof of Claim

Claim B. *Every increasing sequence in N^ω/U of length ω_1 is bounded.*

Proof. Let $\{d_i\}_{i<\omega_1}$ be an increasing sequence in N^ω/U, where each d_i has the form $[(d_i(n))_{n<\omega}]_U$. There is $\alpha < \omega_2$ such that all the $d_i(n)$'s ($i < \omega_1$, $n < \omega$) are in $\mathrm{dcl}^*(A_{<\alpha})$. For each $n < \omega$, choose $e(n) \in \mathrm{dcl}^*(A_\alpha)$ with

$$E(\mathrm{dcl}^*(A_{<\alpha}), d_0(n)) < e(n).$$

We put $e = [(e(0), e(1), \dots)]_U$. Now we show that e bounds all d_i ($i < \omega_1$). Let $i < \omega_1$ be arbitrary. Since $d_0 < d_i$ holds in N^ω/U, there is $X \in U$ such that $E(d_i(n), d_0(n))$ holds for all $n \in X$. By the choice of $e(n)$, we must have $d_i(n) < e(n)$ for $n \in X$. Thus we have $d_i < e$ in N^ω/U. End of Proof of Claim

$\square$

3 Random Graph

We show that a similar situation holds for the theory of random graphs. We continue to assume $\neg CH$.

Let $\mathcal{G} = (\mathcal{G}, R)$ be a saturated random graph. We write $xy \lhd uv$ if $R(yu) \wedge \neg R(xv)$. $\lhd$ is antisymmetric, i.e., $xy \lhd uv$ implies $uv \not\lhd xy$. For a set $X \subset \mathcal{G}$, we also write $X \lhd uv$, if $xy \lhd uv$ holds for all $x, y \in X$.

Choose $\bar{a}_i = a_i^0 a_i^1$ $(i < \omega_2)$ such that

$$\mathrm{dcl}^*(\bar{a}_{<i}) \lhd \bar{a}_i \quad (i < \omega_2).$$

Let $M = \mathrm{dcl}^*(\bar{a}_{<\omega_1})$ and $N = \mathrm{dcl}^*(\bar{a}_{<\omega_2})$.

Claim C. *In M^ω/U, there is a sequence $\{\bar{b}_i : i < \omega_1\}$ such that*

(*) for any tuple $\bar{c} = de$ there is $i < \omega_1$ with $de \lhd \bar{b}_i$.

Proof. Let σ be the natural embedding of M into M^ω/U and let $\bar{b}_i = \sigma(\bar{a}_i)$ $(i < \omega_1)$. We show (*) for this sequence. Since $\bar{c}$ belongs to M^ω/U, we can write $\bar{c}$ as $\bar{c} = [(d_i)_{i<\omega}]_U\, [(e_i)_{i<\omega}]_U$. We can choose $\alpha < \omega_1$ such that all d_i, e_i $(i < \omega)$ are in $\mathrm{dcl}^*(\bar{a}_\alpha)$. Then we see $d_i e_i \lhd a_\alpha$ $(i \in \omega)$. This implies that, in M^ω/U, $\bar{c} \lhd \sigma(a_\alpha)$ holds. □

Claim D. *In N^ω/U, there is no sequence $\{\bar{b}_i : i < \omega_1\}$ such that*

(**) for any tuple $\bar{c} = de$ there is $i < \omega_1$ with $de \lhd \bar{b}_i$.

Proof. Suppose there were such a sequence $\{\bar{b}_i : i < \omega_1\}$. Each $\bar{b}_i$ has the form $\bar{b}_i = [(\bar{b}_{i0}, \bar{b}_{i1}, \bar{b}_{i2}, \dots)]_U$. Let τ be the natural embedding of N into N^ω/U. So, $\tau(\bar{a}_i) = [(\bar{a}_i, \bar{a}_i, \dots)]_U$. Since there are ω_2-many $\tau(\bar{a}_j)$'s, we can find $i^* < \omega_1$ such that $\tau(\bar{a}_j) \lhd \bar{b}_{i^*}$ holds in N^ω/U for ω_2-many $j < \omega_1$. Now choose $\alpha < \omega_2$ such that

1. $\tau(\bar{a}_\alpha) \lhd \bar{b}_{i^*}$;
2. $\bar{b}_{i^*0}, \bar{b}_{i^*1}, \bar{b}_{i^*2}, \dots \in \mathrm{dcl}^*(\bar{a}_{<\alpha})$ (in N).

By 2, and by our choice of $\bar{a}_\alpha$, we have $\bar{b}_{i^*j} \lhd \bar{a}_\alpha$ for all $j < \omega$. So, we must have $\bar{b}_{i^*} \lhd \tau(\bar{a}_\alpha)$ in N^ω/U. However, this contradicts 1, since $\lhd$ is antisymmetric. □

References

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