PILLAY'S ALTERNATIVE DEFINITION FOR WEAK ELIMINATION OF IMAGINARIES COINCIDES WITH POIZAT'S ORIGINAL ONE

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Abstract. The smallest algebraically closed set which appears in Poizat's original definition for WEI coincides with algebraic closure of finite real tuple which appears in Pillay's alternative definition for WEI.

1. Two definitions for WEI

Let $\mathcal{M}$ be a sufficiently saturated model of T. $\bar{a}, \bar{b}, \bar{c}, \dots$ denote finite tuples in $\mathcal{M}$ and $a, b, c, \dots$ denote elements of $\mathcal{M}$. $L(\bar{a})$ denotes the set of L-formulas with parameter $\bar{a}$. For $\varphi(\bar{x}, \bar{a}) \in L(\bar{a})$, $\varphi(\bar{x}, \bar{a})^{\mathcal{M}} := \{\bar{m} \subset \mathcal{M} : \mathcal{M} \models \varphi(\bar{m}, \bar{a})\}$. We work in $\mathcal{M}^{\mathrm{eq}} := \{\bar{a}/E : \bar{a}/E$ is the E-class of $\bar{a}$, where $\bar{a}$ is a finite tuple of $\mathcal{M}$ and $E(\bar{x}, \bar{y})$ is a ϕ-definable equivalence relation with $\mathrm{lh}(\bar{x}) = \mathrm{lh}(\bar{y}) = \mathrm{lh}(\bar{a})$ $\}$. Let $A \subset \mathcal{M}^{\mathrm{eq}}$. For $\bar{a}/E \in \mathcal{M}^{\mathrm{eq}}$, we write $\bar{a}/E \in \mathrm{acl}^{\mathrm{eq}}(A)$ if the orbit of $\bar{a}/E$ by automorphisms fixing A pointwise is finite, and $\bar{a}/E \in \mathrm{dcl}^{\mathrm{eq}}(A)$ if $\bar{a}/E$ is fixed by automorphisms fixing A pointwise. For $\bar{a} \subset \mathcal{M}$, we write $\bar{a} \in \mathrm{acl}(A)$ if the orbit of $\bar{a}$ by automorphisms fixing A pointwise is finite, and $\bar{a} \in \mathrm{dcl}(A)$ if $\bar{a}$ is fixed by automorphisms fixing A pointwise.

Definition 1.1. We say that T admits weak elimination of imaginaries (WEI) in the sense of B.Poizat (See pp.321-322 in [Po2]), for any $\varphi(\bar{x}, \bar{a}) \in L(\bar{a})$ we have the smallest algebraically closed set B such that $\varphi(\bar{x}, \bar{a})$ is definable over B.

Fact 1.2. *Theorem 16.15 in* [Po2]*: T admits WEI in the sense of B.Poizat if and only if for any $\varphi(\bar{x}, \bar{a}) \in L(\bar{a})$ there exists an $\emptyset$-definable formula $\psi_{\bar{a}}(\bar{x}, \bar{z})$ such that $1 \leq |\{\bar{b} \subset \mathcal{M} : \varphi(\bar{x}, \bar{a})^{\mathcal{M}} = \psi_{\bar{a}}(\bar{x}, \bar{b})^{\mathcal{M}}\}| < \omega$. Note that $1 = |\{\bar{b} \subset \mathcal{M} : \varphi(\bar{x}, \bar{a})^{\mathcal{M}} = \psi_{\bar{a}}(\bar{x}, \bar{b})^{\mathcal{M}}\}|$ is equivalent to elimination of imaginaries.*

Proof. Although the proof is given in [Po2], by using our notations, we give the proof for the sake of completeness.
($\Rightarrow$): Let $B = \mathrm{acl}(B)$ be the smallest algebraically closed set defining $\varphi(\bar{x}, \bar{a})$. By compactness there exist $\bar{b} \subset B$ and $\emptyset$-definable formula $\psi'_{\bar{a}}(\bar{x}, \bar{z})$ such that $\varphi(\bar{x}, \bar{a})^{\mathcal{M}} = \psi'_{\bar{a}}(\bar{x}, \bar{b})^{\mathcal{M}}$. By way of contradiction suppose that for each $n < \omega$ there exist distinct $\bar{b}_i (1 \leq i \leq n)$ such that $\bar{b} = \bar{b}_1, \mathrm{tp}(\bar{b}) = \mathrm{tp}(\bar{b}_i)$ and $\psi_{\bar{a}}(\bar{x}, \bar{b})^{\mathcal{M}} = \psi_{\bar{a}}(\bar{x}, \bar{b}_i)^{\mathcal{M}}$. By compactness there exists $\bar{b}'$ such that $\mathrm{tp}(\bar{b}) = \mathrm{tp}(\bar{b}')$, $b' \notin \mathrm{acl}(\bar{b})$ and $\psi_{\bar{a}}(\bar{x}, \bar{b})^{\mathcal{M}} = \psi_{\bar{a}}(\bar{x}, \bar{b}')^{\mathcal{M}}$. As $\varphi(\bar{x}, \bar{a})^{M} = \psi_{\bar{a}}(\bar{x}, \bar{b}')^{\mathcal{M}}$, by the smallestness of B, we

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have $\bar b \subset B \subseteq \mathrm{acl}(\bar b')$ and $\bar b' \notin \mathrm{acl}(\bar b)$. Note that $\bar b'$ witnesses that $\mathrm{acl}(\bar b)$ is strictly contained in $\mathrm{acl}(\bar b')$. As $\mathrm{tp}(\bar b) = \mathrm{tp}(\bar b')$, there exist $\sigma \in \mathrm{Aut}(\mathcal{M})$ such that $\sigma(\bar b') = \bar b$. As $\psi_{\bar a}(\bar x, \bar b)^{\mathcal{M}} = \psi_{\bar a}(\bar x, \bar b')^{\mathcal{M}}$, we have $\psi'_{\bar a}(\bar x, \sigma(\bar b))^{\mathcal{M}} = \psi'_{\bar a}(\bar x, \bar b)^{\mathcal{M}} = \varphi(\bar x, \bar a)^{\mathcal{M}}$ and $\mathrm{acl}(\sigma(\bar b))$ is strictly contained in $\mathrm{acl}(\bar b) \subseteq B$. This contradicts the smallestness of B. So we have that $1 \le |\{\bar b' \subset \mathcal{M} : \mathrm{tp}(\bar b) = \mathrm{tp}(\bar b'), \psi_{\bar a}(\bar x, \bar b)^{\mathcal{M}} = \psi_{\bar a}(\bar x, \bar b')^{\mathcal{M}}\}| < \omega$. By compactness there exists $\rho(\bar z) \in \mathrm{tp}(\bar b)$ such that $1 \le |\{\bar b' \subset \mathcal{M} : \rho(\bar b') \wedge \forall \bar x(\psi_{\bar a}(\bar x, \bar b) \leftrightarrow \psi_{\bar a}(\bar x, \bar b'))\}| < \omega$. Put $\psi_{\bar a}(\bar x, \bar z) :\equiv \rho(\bar z) \wedge \psi'_{\bar a}(\bar x, \bar z)$ as desired.
($\Leftarrow$): Suppose that $\varphi(\bar x, \bar a)$ is definable over C. For any $\sigma \in \mathrm{Aut}(\mathcal{M}/C)$ we have that for any $\bar b \in \{\bar b \subset \mathcal{M} : \varphi(\bar x, \bar a)^{\mathcal{M}} = \psi_{\bar a}(\bar x, \bar b)^{\mathcal{M}}\}$ which is a non-empty finite set, we have $\psi_{\bar a}(\bar x, \bar b)^{\mathcal{M}} = \varphi(\bar x, \bar a)^{\mathcal{M}} = \varphi(\bar x, \sigma(\bar a))^{\mathcal{M}} = \psi_{\bar a}(\bar x, \sigma(\bar b))^{\mathcal{M}}$. So we have that $\sigma(\{\bar b \subset \mathcal{M} : \varphi(\bar x, \bar a)^{\mathcal{M}} = \psi_{\bar a}(\bar x, \bar b)^{\mathcal{M}}\}) = \{\bar b \subset \mathcal{M} : \varphi(\bar x, \bar a)^{\mathcal{M}} = \psi_{\bar a}(\bar x, \bar b)^{\mathcal{M}}\} =: \{\bar b_i : 1 \le i \le n\}$ for some $1 \le n < \omega$. We see that $\mathrm{acl}(\bar b_1, \ldots, \bar b_n) \subseteq \mathrm{acl}(C)$. As $\varphi(\bar x, \bar a)$ is definable over $\bar b_i$ for each $1 \le i \le n$, we see that $\mathrm{acl}(\bar b_1, \ldots, \bar b_n) = \mathrm{acl}(\bar b_i)$ is the smallest algebraically closed subset defining $\varphi(\bar x, \bar a)$. □

Proposition 1.3. *T admits WEI in the sense of A.Pillay (See pp.63 in* [Pi2]*); for any $\bar a/E \in \mathcal{M}^{\mathrm{eq}}$ there exist a finite tuple $\bar b \subset \mathcal{M}$ such that $\bar a/E \in \mathrm{dcl}^{\mathrm{eq}}(\bar b)$ and $\bar b \in \mathrm{acl}^{\mathrm{eq}}(\bar a/E)$ if and only if T admits WEI in the sense of B.Poizat.*

Proof. ($\Leftarrow$): Let $E(\bar x, \bar y)$ be an $\emptyset$-definable equivalence relation. By Fact 1.2 there exists an $\emptyset$-definable formula $\psi_{\bar a}(\bar x, \bar z)$ such that $1 \le |\{\bar b \subset \mathcal{M} : E(\bar x, \bar a)^{\mathcal{M}} = \psi_{\bar a}(\bar x, \bar b)^{\mathcal{M}}\}| < \omega$. Take $\bar b \in \{\bar b \subset \mathcal{M} : E(\bar x, \bar a)^{\mathcal{M}} = \psi_{\bar a}(\bar x, \bar b)^{\mathcal{M}}\}$. For any $\sigma \in \mathrm{Aut}(\mathcal{M}/\bar b)$ we have $E(\bar x, \bar a)^{\mathcal{M}} = \psi_{\bar a}(\bar x, \bar b)^{\mathcal{M}} = E(\bar x, \sigma(\bar a))^{\mathcal{M}}$. Put $e := \bar a/E$. So we see that $e \in \mathrm{dcl}^{\mathrm{eq}}(\bar b)$. On the other hand for any $\tau \in \mathrm{Aut}(\mathcal{M}^{\mathrm{eq}}/e)$, we have $\psi_{\bar a}(\bar x, \bar b)^{\mathcal{M}} = E(\bar x, \bar a)^{\mathcal{M}} = E(\bar x, \tau(\bar a))^{\mathcal{M}} = \psi_{\bar a}(\bar x, \tau(\bar b))^{\mathcal{M}}$. As $1 \le |\{\bar b \subset \mathcal{M} : E(\bar x, \bar a)^{\mathcal{M}} = \psi_{\bar a}(\bar x, \bar b)^{\mathcal{M}}\}| < \omega$, we see that $\bar b \in \mathrm{acl}^{\mathrm{eq}}(e)$.
($\Rightarrow$): Let $\varphi(\bar x, \bar a) \in L(\bar a)$ and put $E(\bar y, \bar z) :\equiv \forall \bar x(\varphi(\bar x, \bar y) \leftrightarrow \varphi(\bar x, \bar z))$. By assumption there exists a finite tuple $\bar b \subset \mathcal{M}$ such that $\bar a/E \in \mathrm{dcl}^{\mathrm{eq}}(\bar b)$ and $\bar b \in \mathrm{acl}^{\mathrm{eq}}(\bar a/E)$. Put $e := \bar a/E$. We show that $\mathrm{acl}(\bar b)$ is the smallest algebraically closed set over which $\varphi(\bar x, \bar a)$ is definable.

Claim 1. *Minimality: Suppose that $\varphi(\bar x, \bar a)$ is definable over C. Then $\mathrm{acl}(\bar b) \subseteq \mathrm{acl}(C)$.*

Let $\{\bar b_1, \ldots, \bar b_n\}$ be the set of e-conjugates of $\bar b$, where $\bar b_1 = \bar b$. As $e \in \mathrm{dcl}^{\mathrm{eq}}(\bar b_i)$ for each $1 \le i \le n$, we see that $\varphi(\bar x, \bar a)$ is definable over $\bar b_i$ for each $1 \le i \le n$. Suppose that $\varphi(\bar x, \bar a)$ is definable over C. Let $\sigma \in \mathrm{Aut}(\mathcal{M}/C)$. As $\varphi(\bar x, \bar a)$ is definable over C, we have $\varphi(\bar x, \bar a)^{\mathcal{M}} = \varphi(\bar x, \sigma(\bar a))^{\mathcal{M}}$, so we see that $\mathcal{M} \models E(\bar a, \sigma(\bar a))$, so we have $e \in \mathrm{dcl}^{\mathrm{eq}}(C)$. Therefore for any $\sigma \in \mathrm{Aut}(\mathcal{M}/C)$ and each $1 \le i \le n$ we have $\mathrm{tp}(\bar b/e) = \mathrm{tp}(\bar b_i/e) = \mathrm{tp}(\sigma(\bar b_i)/e)$. So we have $\{\sigma(\bar b_i) : \sigma \in \mathrm{Aut}(\mathcal{M}/C)\} \subseteq \{\bar b_1, \ldots, \bar b_n\}$. So we see that $\bar b_i \in \mathrm{acl}(C)$ for each $1 \le i \le n$. In particular we see that $\mathrm{acl}(\bar b) \subseteq \mathrm{acl}(C)$.

Claim 2. *Uniqueness: Let $\bar c \subset \mathcal{M}$ be such that $e \in \mathrm{dcl}^{\mathrm{eq}}(\bar c)$ and $\bar c \in \mathrm{acl}^{\mathrm{eq}}(e)$. Then $\mathrm{acl}(\bar b) = \mathrm{acl}(\bar c)$.*

Suppose that $\varphi(\bar x, \bar a)$ is definable over B. Then $\mathrm{acl}(\bar c) \subseteq \mathrm{acl}(B)$ by claim 1. As $\varphi(\bar x, \bar a)$ is definable over each $\bar b$ and $\bar c$, by using Claim 1 twice, we see $\mathrm{acl}(\bar b) \subseteq \mathrm{acl}(\bar c) \subseteq \mathrm{acl}(\bar b)$ as desired. □

2. THE ELIMINATION OF $\exists^\infty$ IS ORTHOGONAL TO WEI

Remark 2.1. (1) Elimination of $\exists^\infty$ does not imply WEI: Any o-minimal theory (which is a dense linear ordered set without endpoints) admits elimination of $\exists^\infty$; any definable set $X \subset \mathcal{M}$ in an o-minimal structure $\mathcal{M}$ is of form that $X = \bigcup_{1\le i<j\le n}(a_i, a_j) \cup \bigcup_{1\le k\le m}\{b_j\}$, where $a_i < a_j$ for each $1 \le i < j \le n$. If $|X|$ is bigger than the number of parameters, X is an infinite set. If the elimination of $\exists^\infty$ implies WEI, any o-minimal theory admits elimination of imaginaries by lexicographic ordering of $\bar{a}/E$-conjugates of $\bar{b}$ in Proposition 1.3, a contradiction, since A.Pillay gives an o-minimal structure without elimination of imaginaries on pp.714 in [Pi1].

(2) When we want to deduce WEI from the elimination of $\exists^\infty$, we try to show this by induction on the length of parameters as follows: By induction hypothesis we have $\{\bar{b}_i : 1 \le i \le n\} := \{\bar{b} \subset \mathcal{M} : \forall\bar{x}\forall y(\varphi(\bar{x}, y, \bar{a}) \leftrightarrow \psi_{\bar{a}}(\bar{x}, y, \bar{b})\}| < \omega$. For $a \in \mathcal{M}$, suppose that $\{c \subset \mathcal{M} : \forall\bar{x}(\varphi(\bar{x}, a, \bar{a}) \leftrightarrow \psi_{\bar{a}}(\bar{x}, c, \bar{b}_i))\}| < \omega$ for each $i = 1, \dots, n$, this case is as desired. Otherwise, suppose there exist infinitely many c such that $\forall\bar{x}(\varphi(\bar{x}, a, \bar{a}) \leftrightarrow \psi_{\bar{a}}(\bar{x}, c, \bar{b}_i))$ for some $1 \le i \le n$. However, we only have $\mathcal{M} \models \exists^\infty y\forall\bar{x}(\varphi(\bar{x}, a, \bar{a}) \leftrightarrow \psi_{\bar{a}}(\bar{x}, y, \bar{b}_i))$, we do not know that

$$\mathcal{M} \models \forall\bar{x}(\varphi(\bar{x}, a, \bar{a}) \leftrightarrow \exists^\infty y\psi_{\bar{a}}(\bar{x}, y, \bar{b}_i)).$$

(3) A.Pillay points out that WEI does not imply elimination of $\exists^\infty$: Let T be a stable theory with finite cover property. Then T^{eq} has EI and does not eliminate $\exists^\infty$.

3. FACTS ON WEI AND GEI

Definition 3.1. (1) T admits geometric elimination of imaginaries if for any $\bar{a}/E \in \mathcal{M}^{\mathrm{eq}}$ there exists $\bar{b} \subset \mathcal{M}$ such that $\bar{a}/E \in \mathrm{acl}^{\mathrm{eq}}(\bar{b})$ and $\bar{b} \in \mathrm{acl}^{\mathrm{eq}}(\bar{a}/E)$. (See [Hr].)

(2) T admits elimination of imaginaries if for any $\bar{a}/E \in \mathcal{M}^{\mathrm{eq}}$ there exists $\bar{b} \subset \mathcal{M}$ such that $\bar{a}/E \in \mathrm{dcl}^{\mathrm{eq}}(\bar{b})$ and $\bar{b} \in \mathrm{dcl}^{\mathrm{eq}}(\bar{a}/E)$. (See [Po2].)

(3) T has finite set property if $F = \{\bar{a}_1, \dots, \bar{a}_n\}$ is a finite set of finite sequences of $\mathcal{M}$, then there exists $\bar{b} \subset \mathcal{M}$ such that $\sigma(F) = F \Leftrightarrow \sigma|\bar{b} = id_{\bar{b}}$ for any $\sigma \in \mathrm{Aut}(\mathcal{M})$. (See Definition 2.3. in [T].)

Remark 3.2. (1) T admits elimination of imaginaries if and only if T admits weak elimination of imaginaries and has finite set property. Proposition 1.6 in [CF]

(2) T admits WEI $\Rightarrow$ if $X \subset \mathcal{M}^n$ is definable over each $A = \mathrm{acl}(A)$ and $B = \mathrm{acl}(B)$, then X is definable over $A \cap B$. Does the converse implication hold?

(3) $< \mathrm{Aut}(\mathcal{M}/A), \mathrm{Aut}(\mathcal{M}/B) >= \mathrm{Aut}(\mathcal{M}/A \cap B)$ for $A, B \subset \mathcal{M} \Rightarrow$ if $X \subset \mathcal{M}^n$ is definable over each A and B, then X is definable over $A \cap B$.

Fact 3.3. (1) If T is ω-categorical and $M \models T$ is countable, then T admits WEI $\Leftrightarrow < \mathrm{Aut}(M/A), \mathrm{Aut}(M/B) >= \mathrm{Aut}(M/A \cap B)$ for finite $A = \mathrm{acl}(A), B = \mathrm{acl}(B) \subset M$. (See Lemma 1.3 in [EH].)

(2) D.M.Hoffmann shows that the simple theory CCMA(=Compact Complex Manifolds with an Automorphism) having GEI and finite set property in Theorem 4.3.6 and Lemma 4.3.7 in [Ho]. But CCMA does not having EI

by Corollary 3.6 in [BHM]. As WEI+finite set property=EI by Proposition 1.6 in [CF], CCMA does not have WEI.

(3) Let T be a rosy theory having weak canonical bases with respect to a strict independence relation $\downarrow$. Suppose that any type over algebraically closed sets in the real sort is $\downarrow$-stationary. THEN T is stable, non-forking relation coincides with $\downarrow$, and geometric elimination of imaginaries implies weak elimination of imaginaries. (See [Y].)

Question 3.4. (1) *If T is stable, do we have that T admits WEI $\Leftrightarrow < \mathrm{Aut}(\mathcal{M}/A), \mathrm{Aut}(\mathcal{M}/B) >= \mathrm{Aut}(\mathcal{M}/A \cap B)$ for $A = \mathrm{acl}(A), B = \mathrm{acl}(B) \subset \mathcal{M}$?*

(2) *Find a stable theory which admits GEI but does not admit WEI.*

(3) *Hrushovski's new strongly minimal set admits WEI (See* [Hr]*) but does not have finite set property (See* [BV]*), so does not have EI. Find a new strongly minimal set D which geometrically eliminates imaginaries but does not weakly eliminate imaginaries, and determine the natural number n that D is n-ample but not $(n+1)$-ample.*

(4) $\mathrm{SCF_e}$ *for each $e \in \omega \cup \{\infty\}$ in the language of fields does not eliminate imaginaries (See Remark 5.3 in* [M]*) and has finite set property, so it does not have WEI. Is* $\mathrm{SCF_e}$ *for each $e \in \omega \cup \{\infty\}$ in the language of fields stable? Does* $\mathrm{SCF_e}$ *for each $e \in \omega \cup \{\infty\}$ in the language of fields geometrically eliminate imaginaries?*

4. Question 3.4 (2) and beautiful pair

First of all, we recall the definition of beautiful pair for stable L-theories as in [Po1].

Definition 4.1. Let T be an L-theory and P be a new unary predicate symbol. L_P denotes $L \cup \{P\}$. Let $M \models T$. We say that $(M, P(M))$ is a beautiful pair of T if

(1) $P(M) \prec M$ is $|T|^+$-saturated elementary substructure of M.

(2) If A is a finite subset of M and any L-type $p \in \mathrm{S}_L(P(M)A)$, then $p^M \neq \emptyset$.

If T is stable, any two beautiful pairs of T are L_P-elementary equivalent.
If T is stable without finite cover property, the complete theory T_P of beautiful pairs is again stable. The following fact is in [PiV].

Fact 4.2. *Let $T = T^{\mathrm{eq}}$ be stable without finite cover property. Then the following are equivalent.*

(1) *Any imaginary $e \in (M, P(M))^{\mathrm{eq}}$ is L_P-interdefinable with some $\bar{f} \in M$.*

(2) *Any imaginary $e \in (M, P(M))^{\mathrm{eq}}$ is L_P-interalgebraic with some $\bar{f} \in M$.*

(3) *No infinte group is definable in any model of T.*

If $T := ACF_p$, we have $T = T^{\mathrm{eq}}$ does not have finite cover property. The following fact in [Pi3].

Fact 4.3. *Let (K, F) be a saturated model of beautiful pair of ACF_p. Let $e \in (K, F)^{\mathrm{eq}}$. Then there are a connected algebraic group G, an irreducible variety V and a rational (in sense of algebraic geometry) action of G on V, all defined (in sense of algebraic geometry) over $F = P(K)$, such that*

(1) *for a generic point $v \in V(K)$ over $F = P(K)$ and $1_G \neq g \in G(F)$ we have $g \cdot v \neq v$. (i.e. generically free action.)*
(2) *for some $v \in V(K)$ generic over $F = P(K)$, e is L_P-interalgebraic with a canonical parameter of $G(F) \cdot v$.*

Remark 4.4. (1) By Fact 4.2, $(ACF_p)_P$ does not admit GEI.
(2) By Fact 4.3, $(ACF_p)_P$ admits GEI in some additional sorts: algebraic principal homogeneneous spaces, say $(G(F), G(F) \cdot v)$, where $v \in V(K)$ is some generic over $F = P(K)$. In such additional sorts, does $(ACF_p)_P$ admit WEI? Does it have finite set property with respect to L_P-automorphisms?

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