Some remarks on groups definable in certain generic structures

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1 Introduction

We use notation and terminology from Kikyo [8], Baldwin-Shi [2] and Wagner [11]. We also use some terminology from graph theory [4].

Suppose A is a graph. $V(A)$ denotes the set of vertices of A, and $E(A)$ the set of edges of A. If $X \subseteq V(A)$, $A|X$ denotes the substructure B of A such that $V(B) = X$. If there is no ambiguity, X denotes $A|X$. We usually follow this convention. $B \subseteq A$ means that B is a substructure of A. A substructure of a graph is an induced subgraph in graph theory. $A|X$ is the same as $A[X]$ in Diestel's book [4].

We say that X is *connected* in A if X is a connected graph in the graph theoretical sense [4]. A maximal connected substructure of A is a *connected component* of A.

Let A, B, C be graphs such that $A \subseteq C$ and $B \subseteq C$. AB denotes $C|(V(A) \cup V(B))$, $A \cap B$ denotes $C|(V(A) \cap V(B))$, and $A - B$ denotes $C|(V(A) - V(B))$. We also write $X - Y$ in general for the relative compliment of Y in X also known as the set difference of X and Y. If $A \cap B = \emptyset$, $E(A,B)$ denotes the set of edges xy such that $x \in A$ and $y \in B$. We put $e(A,B) = |E(A,B)|$. $E(A,B)$ and $e(A,B)$ depend on the graph in which we are working.

Let D be a graph and A, B, and C substructures of D. We write $D = B \oplus_A C$ if $D = BC$, $B \cap C = A$, and $E(D) = E(B) \cup E(C)$. $E(D) = E(B) \cup E(C)$ means that there are no edges between $B - A$ and $C - A$. D is called a *free amalgam of B and C over A*. If A is empty, we write $D = B \oplus C$, and D is also called a *free amalgam of B and C*.

Definition 1.1 Let α be a real number such that $0 < \alpha < 1$.

(1) For a finite graph A, we define a predimension function δ_α by

$$\delta_\alpha(A) = |A| - e(A)\alpha.$$

(2) Let A and B be substructures of a common graph. Put

$$\delta_\alpha(A/B) = \delta_\alpha(AB) - \delta_\alpha(B).$$

Definition 1.2 Let A and B be graphs with $A \subseteq B$, and suppose A is finite.

$A <_\alpha B$ if whenever $A \subsetneq X \subseteq B$ with X finite then $\delta_\alpha(A) < \delta_\alpha(X)$.

We say that A is *closed* in B if $A <_\alpha B$. We also say that B is a *strong extension* of A.

Let $\mathbf{K}_\alpha$ be the class of all finite graphs A such that $\emptyset <_\alpha A$.

Some facts about $<_\alpha$ appear in [2, 11, 12]. Some proofs are given in [8].

Fact 1.3 *Let A and B be disjoint substructures of a common graph. Then*

$$\delta_\alpha(A/B) = \delta_\alpha(A) - e(A,B)\alpha.$$

Fact 1.4 *If $A <_\alpha B \subseteq D$ and $C \subseteq D$ then $A \cap C <_\alpha B \cap C$.*

Fact 1.5 *Let $D = B \oplus_A C$.*

(1) $\delta_\alpha(D/A) = \delta_\alpha(B/A) + \delta_\alpha(C/A)$.

(2) *If $A <_\alpha C$ then $B <_\alpha D$.*

(3) *If $A <_\alpha B$ and $A <_\alpha C$ then $A <_\alpha D$.*

Let B, C be graphs and $g : B \to C$ a graph embedding. g is a *closed embedding* of B into C if $g(B) <_\alpha C$. Let A be a graph with $A \subseteq B$ and $A \subseteq C$. g is a *closed embedding over* A if g is a closed embedding and $g(x) = x$ for any $x \in A$.

In the rest of the paper, $\mathbf{K}$ denotes a class of finite graphs closed under isomorphisms.

Definition 1.6 Let $\mathbf{K}$ be a subclass of $\mathbf{K}_\alpha$. $(\mathbf{K}, <_\alpha)$ has the *amalgamation property* if for any finite graphs $A, B, C \in \mathbf{K}$, whenever $g_1 : A \to B$ and $g_2 : A \to C$ are closed embeddings then there is a graph $D \in \mathbf{K}$ and closed embeddings $h_1 : B \to D$ and $g_2 : C \to D$ such that $h_1 \circ g_1 = h_2 \circ g_2$.

$\mathbf{K}$ has the *hereditary property* if for any finite graphs A, B, whenever $A \subseteq B \in \mathbf{K}$ then $A \in \mathbf{K}$.

$\mathbf{K}$ is an *amalgamation class* if $\emptyset \in \mathbf{K}$ and $\mathbf{K}$ has the hereditary property and the amalgamation property.

A countable graph M is a *generic structure* of $(\mathbf{K}, <_\alpha)$ if the following conditions are satisfied:

(1) If $A \subseteq M$ and A is finite then there exists a finite graph $B \subseteq M$ such that $A \subseteq B <_\alpha M$.

(2) If $A \subseteq M$ then $A \in \mathbf{K}$.

(3) For any A, $B \in \mathbf{K}$, if $A <_\alpha M$ and $A <_\alpha B$ then there is a closed embedding of B into M over A.

Let A be a finite structure of M. There is a smallest B satisfying $A \subseteq B <_\alpha M$, written $\mathrm{cl}(A)$. The set $\mathrm{cl}(A)$ is called the *closure* of A in M.

Fact 1.7 ([2, 11, 12]) *Let $(\mathbf{K}, <_\alpha)$ be an amalgamation class. Then there is a generic structure of $(\mathbf{K}, <_\alpha)$. Let M be a generic structure of $(\mathbf{K}, <_\alpha)$. Then any isomorphism between finite closed substructures of M can be extended to an automorphism of M.*

Definition 1.8 Let $\mathbf{K}$ be a subclass of $\mathbf{K}_\alpha$. $(\mathbf{K}, <_\alpha)$ has the *free amalgamation property* if whenever $D = B \oplus_A C$ with $B, C \in \mathbf{K}$, $A <_\alpha B$ and $A <_\alpha C$ then $D \in \mathbf{K}$.

By Fact 1.5 (2), we have the following.

Fact 1.9 *Let $\mathbf{K}$ be a subclass of $\mathbf{K}_\alpha$. If $(\mathbf{K}, <_\alpha)$ has the free amalgamation property then it has the amalgamation property.*

Definition 1.10 Let $\mathbb{R}^+$ be the set of non-negative real numbers. Suppose $f : \mathbb{R}^+ \to \mathbb{R}^+$ is a strictly increasing concave (convex upward) unbounded function. Assume that $f(0) = 0$, and $f(1) \leq 1$. We assume that f is piecewise smooth. $f'_+(x)$ denotes the right-hand derivative at x. We have $f(x+h) \leq f(x) + f'_+(x)h$ for $h > 0$. Define $\mathbf{K}_f$ as follows:

$$\mathbf{K}_f = \{A \in \mathbf{K}_\alpha \mid B \subseteq A \Rightarrow \delta_\alpha(B) \geq f(|B|)\}.$$

Note that if $\mathbf{K}_f$ is an amalgamation class then the generic structure of $(\mathbf{K}_f, <_\alpha)$ has a countably categorical theory [12].

A graph X is *normal to* f if $\delta(X) \geq f(|X|)$. A graph A belongs to $\mathbf{K}_f$ if and only if U is normal to f for any substructure U of A.

Fact 1.11 ([8]) *Suppose $1 > p/q > 0$ where p and q are coprime positive integers. Then there is a tree W with the following properties: Let L be the set of all nodes of W and F the set of all leaves of W.*

(1) *L is a path in W with p vertices and $p-1$ edges.*

(2) *$|F| = q - p + 1$. Every leaf is adjacent to some vertex in L.*

(3) *$\delta_{p/q}(W/F) = 0$.*

(4) *$B \cap F <_{p/q} B$ for any proper substructure B of W.*

We call W a twig *for p/q.*

2 On classes defined by bounded control functions

If the control function f is a constant function $f(x) = 0$, then $\mathbf{K}_f = \mathbf{K}_\alpha$. The generic structures of $(\mathbf{K}_\alpha, <_\alpha)$ have a very rich (wild) structure (Brody-Laskowski, Evans-Wong).

Fact 2.1 (Evans, Wong[5]) *Let α be a rational number with $1 > \alpha > 0$. Let M be the generic structure of $(\mathbf{K}_\alpha, <_\alpha)$. Then any finite graphs are definable in M (the domain and the edge relation are definable with parameters). More strongly, there are two formulas $\varphi_v(x;z)$ and $\varphi_e(x,y;z)$ such that for any finite graph G there is a tuple m_G in M such that $(\varphi_v(M;m_G), \varphi_e(x,y;m_G))$ is isomorphic to G. Here, x, y, and z are tuples of variables.*

Similarly, the finite bipartite graphs are uniformly definable in M.

Proof. Evans and Wong gave a proof in the case of $\mathbf{K}_1$ where the members of $\mathbf{K}_1$ are structures with one ternary relation which represents 3-hyperedges for the sake of simplicity. We give a proof for our case. We show that all finite graphs are uniformly definable in M with parameters.

Let n be a natural number. Let $G_n = W_1 \oplus_F W_2 \oplus_F \cdots \oplus_F W_n$ where all W_i are twigs for α and F is the set of leaves of W_i. Note that W_i are isomorphic over F.

Let c_i be the tuple of nodes of W_i. Let $V = \{c_1, \ldots, c_n\}$. We code edges on V as follows. To put an edge between c_i and c_j with $i \neq j$, attach a twig W_{ij} for α so that some leaf of W_{ij} is identified with a vertex in c_i and another leaf of W_{ij} is identified with a vertex in c_j, and the rest of leaves of W_{ij} are identified with vertices in F. Let G'_n be an extension of G_n obtained by this way. Then G'_n belongs to $\mathbf{K}_\alpha$. Embed G'_n into M so that the isomorphic image of G'_n in M is closed in M. Then there are no extension of the isomorphic image of G'_n in M by attaching some twig for α to it. Note that the set of "vertices" $V = \{c_1, \ldots, c_n\}$ is definable over F and also the set of edges are definable over F in a uniform way. Hence, all finite graphs are uniformly definable in M with parameters. □

It is likely that the following conjecture holds:

Conjecture 2.2 *Let α be a rational number with $1 > \alpha > 0$. Assume $(\mathbf{K}_f, <_\alpha)$ has FAP and $f : \mathbb{R}^+ \to \mathbb{R}^+$ is bounded. let M be the generic structure of $(\mathbf{K}_f, <_\alpha)$. Then any finite graphs are uniformly definable in M.*

At the RIMS meeting 2021, the author announced that this conjecture is true, but it is not clear that the all substructures of the proposed structure belong to the class $\mathbf{K}_f$.

Theorem 2.3 *let M be the generic structure of $(\mathbf{K}_\alpha, <_\alpha)$. Then an infinite group is definable in some elementary extension of M.*

Proof. A Desarguesian projective plane is a two sorted structure with a sort for points, a sort for lines, and an incidence relation between points and lines. It can be represented as a bipartite graph. So, any finite Desarguesian projective plane are definable in M in a uniform way. In a Desarguesian projective plane, a group structure is definable on the set of points on a line by a formula independent of a particular projective plane.

Since there are arbitrarily large Desarguesian projective planes, an infinite group is definable in some elementary extension of M. □

3 On classes defined by unbounded control functions

We begin this section by some facts.

Fact 3.1 *Assume that $(\mathbf{K}_f, <_\alpha)$ has the free amalgamation property. Let M be the generic structure of $(\mathbf{K}_f, <_\alpha)$.*

If f is unbounded, then $Th(M)$ is $\aleph_0$-categorical.

Let A, B be finite substructures of M. If $A <_\alpha M$, $B <_\alpha M$ and $\sigma : A \to B$ is a graph isomorphism then σ can be extended to an automorphism of M.

Hence, $\mathrm{qftp}(A) = \mathrm{qftp}(B)$ *with $A <_\alpha M$, $B <_\alpha M$ implies* $\mathrm{tp}(A) = \mathrm{tp}(B)$.

$\mathrm{tp}(A)$ *is determined by* $\mathrm{qftp}(\mathrm{cl}(A))$.

The following is the main theorem.

Theorem 3.2 *Let α be a rational number with $1 > \alpha > 0$. Assume $(\mathbf{K}_f, <_\alpha)$ has FAP and $f : \mathbb{R}^+ \to \mathbb{R}^+$ is unbounded. let M be the generic structure of $(\mathbf{K}_f, <_\alpha)$. Then no infinite groups are definable in any elementary extensions of M.*

Proof. Note that $Th(M)$ is $\aleph_0$-categorical. Suppose a formula $G(x,a)$ defines an infinite group in an elementary extension of M, where a is a parameter. Since $Th(M)$ is $\aleph_0$-categorical, we can assume that $a \in M$. Let $\overline{a}$ be the closure of a in M. Let g be a non-algebraic element of M over a satisfying $G(x,a)$. Consider $\overline{g,a}$. We have $\overline{a} <_\alpha \overline{g,a}$ with $g \notin \overline{a}$. Let $D = D_1 \oplus_{\overline{a}} D_2$ where $D_i \cong_{\overline{a}} \overline{g,a}$. D belongs to $\mathbf{K}_f$ by FAP. Embed D over $\overline{a}$ so that the isomorphic image of D is closed in M. Let g_1, g_2 be isomorphic images in D_1 and D_2 of g respectively.

Let $g_3 = g_1 \cdot g_2$ be the product in the group.

$\overline{g_1,a} \cup \overline{g_2,a}$ is closed. g_3 is definable over g_1 and g_2. g_3 belongs to $\overline{g_1,a} \cup \overline{g_2,a}$ because the algebraic closure and the closure in M are the same. Hence g_3 belongs to $\overline{g_1,a}$ or $\overline{g_2,a}$, say $g_3 \in \overline{g_1,a}$. Since g_2 is definable over g_1 and g_3, this implies that $g_2 \in \overline{g_1,a}$. But this is a contradiction by the construction of D and the choice of g_1 and g_2. □

The following question is natural but the author has no idea at the moment.

Question 3.3 *Is the following statement true? Let α be a rational number with $1 > \alpha > 0$. Assume $(\mathbf{K}_{\alpha,f}, <)$ has FAP and $f : \mathbb{R}^+ \to \mathbb{R}^+$ is unbounded. let M be the generic structure of $(\mathbf{K}_{\alpha,f}, <)$. Then no infinite groups are interpretable in any elementary extensions of M.*

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