

KIM-FORKING AND KIM-DIVIDING IN NATP THEORIES

JOONHEE KIM
YONSEI UNIVERSITY

1. INTRODUCTION

This note is based on the author's talk in RIMS Symposia, "Model theoretic aspects of the notion of independence and dimension", held on December 13–15, 2021. Kim's lemma has been a key observation in the study of forking and dividing, especially in the proof of the equivalence of forking (Kim-forking) and dividing (Kim-dividing) in simple theories and NTP_2 theories (NSOP_1 theories, resp.). In a collaboration with Bonghun Lee and Hyoyoon Lee, now we are trying to show that Kim-forking and Kim-dividing are equivalent in NATP theories. Following the strategy of [CK12], we obtained a partial result which is expected to play a role in proving the equivalence, namely

Theorem 1.1. *In NATP theories, if a formula Kim-divides over a model M , then it coheir-divides over M .*

2. PRELIMINARY

We summarize notations and basic facts of the subjects to be covered in this note. First we recall notations and definitions of tree properties.

Notation 2.1. Let κ and λ be cardinals.

- (i) By κ^λ , we mean the set of all functions from λ to κ .
- (ii) By $\kappa^{<\lambda}$, we mean $\bigcup_{\alpha < \lambda} \kappa^\alpha$ and call it a tree. If $\kappa = 2$, we call it a binary tree. If $\kappa \geq \omega$, then we call it an infinitary tree.
- (iii) By \emptyset or $\langle \rangle$, we mean the empty string in $\kappa^{<\lambda}$, which means the empty set (recall that every function can be regarded as a set of ordered pairs).

Let $\eta, \nu \in \kappa^{<\lambda}$.

- (iv) By $\eta \leq \nu$, we mean $\eta \subseteq \nu$. If $\eta \trianglelefteq \nu$ or $\nu \trianglelefteq \eta$, then we say η and ν are comparable.
- (v) By $\eta \perp \nu$, we mean that $\eta \not\trianglelefteq \nu$ and $\nu \not\trianglelefteq \eta$. We say η and ν are incomparable if $\eta \perp \nu$.
- (vi) By $\eta \wedge \nu$, we mean the maximal $\xi \in \kappa^{<\lambda}$ such that $\xi \trianglelefteq \eta$ and $\xi \trianglelefteq \nu$.
- (vii) By $l(\eta)$, we mean the domain of η .
- (viii) By $\eta <_{lex} \nu$, we mean that either $\eta \leq \nu$, or $\eta \perp \nu$ and $\eta(l(\eta \wedge \nu)) < \nu(l(\eta \wedge \nu))$.
- (ix) By $\eta \frown \nu$, we mean $\eta \cup \{(i + l(\eta), \nu(i)) : i < l(\nu)\}$.

Let $X \subseteq \kappa^{<\lambda}$.

- (x) By $\eta \frown X$ and $X \frown \eta$, we mean $\{\eta \frown x : x \in X\}$ and $\{x \frown \eta : x \in X\}$ respectively.

Let $\eta_0, \dots, \eta_n \in \kappa^{<\lambda}$.

- (xi) By $\text{cl}(\eta_0, \dots, \eta_n)$, we mean a tuple $(\eta_0 \wedge \eta_0, \dots, \eta_0 \wedge \eta_n, \dots, \eta_n \wedge \eta_0, \dots, \eta_n \wedge \eta_n)$.
- (xii) We say a subset X of $\kappa^{<\lambda}$ is an *antichain* if the elements of X are pairwise incomparable, *i.e.*, $\eta \perp \nu$ for all $\eta, \nu \in X$.

Definition 2.2. Let $\varphi(x, y)$ be an \mathcal{L} -formula.

- (i) We say $\varphi(x, y)$ has the *tree property* (TP) if there exists a tree-indexed set $(a_\eta)_{\eta \in \omega^{<\omega}}$ of parameters and $k \in \omega$ such that
 - $\{\varphi(x, a_{\eta \upharpoonright n})\}_{n \in \omega}$ is consistent for all $\eta \in \omega^\omega$ (path consistency), and
 - $\{\varphi(x, a_{\eta \upharpoonright i})\}_{i \in \omega}$ is k -inconsistent for all $\eta \in \omega^{<\omega}$, *i.e.*, any subset of $\{\varphi(x, a_{\eta \upharpoonright i})\}_{i \in \omega}$ of size k is inconsistent.
- (ii) We say $\varphi(x, y)$ has the *tree property of the first kind* (TP₁) if there is a tree-indexed set $(a_\eta)_{\eta \in \omega^{<\omega}}$ of parameters such that
 - $\{\varphi(x, a_{\eta \upharpoonright n})\}_{n \in \omega}$ is consistent for all $\eta \in \omega^\omega$, and
 - $\{\varphi(x, a_\eta), \varphi(x, a_\nu)\}$ is inconsistent for all $\eta \perp \nu \in \omega^{<\omega}$.
- (iii) We say $\varphi(x, y)$ has the *tree property of the second kind* (TP₂) if there is an array-indexed set $(a_{i,j})_{i,j \in \omega}$ of parameters such that
 - $\{\varphi(x, a_{n,\eta(n)})\}_{n \in \omega}$ is consistent for all $\eta \in \omega^\omega$, and
 - $\{\varphi(x, a_{i,j}), \varphi(x, a_{i,k})\}$ is inconsistent for all $i, j, k \in \omega$ with $j \neq k$.
- (iv) We say $\varphi(x, y)$ has the *1-strong order property* (SOP₁) if there is a binary-tree-indexed set $(a_\eta)_{\eta \in 2^{<\omega}}$ of parameters such that
 - $\{\varphi(x, a_{\eta \upharpoonright n})\}_{n \in \omega}$ is consistent for all $\eta \in 2^\omega$,
 - $\{\varphi(x, a_{\eta \upharpoonright 1}), \varphi(x, a_{\eta \upharpoonright 0 \upharpoonright \nu})\}$ is inconsistent for all $\eta, \nu \in 2^{<\omega}$.
- (v) We say $\varphi(x, y)$ has the *2-strong order property* (SOP₂) if there is a binary-tree-indexed set $(a_\eta)_{\eta \in 2^{<\omega}}$ of parameters such that
 - $\{\varphi(x, a_{\eta \upharpoonright n})\}_{n \in \omega}$ is consistent for all $\eta \in 2^\omega$,
 - $\{\varphi(x, a_\eta), \varphi(x, a_\nu)\}$ is inconsistent for all $\eta \perp \nu \in 2^{<\omega}$.
- (vi) We say a theory has TP if there is a formula having TP with respect to its monster model of the theory. Sometimes we say that the theory is TP, and we call the theory an TP theory. We define TP₁ theory, TP₂ theory, SOP₁ theory, and SOP₂ theory in the same manner.
- (vii) We say a theory is NTP if the theory is not TP, and we call the theory NTP theory. We define NTP₁ theory, NTP₂ theory, NSOP₁ theory, and NSOP₂ theory in the same manner.

The following facts are well known (*cf.* [Con12], [DS04], [KK11], and [She90]).

- Fact 2.3.**
- (i) A theory has TP₁ if and only if it has SOP₂.
 - (ii) A theory has TP if and only if it has TP₁ or TP₂.
 - (iii) A theory has TP if and only if it has SOP₁ or TP₂.
 - (iv) If a theory has SOP₂, then it has SOP₁.

Definition 2.4. [AK20, Definition 4.1] We say a formula $\varphi(x, y)$ has the *antichain tree property* (ATP) if there exists a tree indexed set of parameters $(a_\eta)_{\eta \in 2^{<\omega}}$ such that

- (i) for any antichain X in $2^{<\omega}$, the set $\{\varphi(x, a_\eta) : \eta \in X\}$ is consistent,
- (ii) for any $\eta, \nu \in 2^{<\omega}$, if $\eta \not\perp \nu$, then $\{\varphi(x, a_\eta), \varphi(x, a_\nu)\}$ is inconsistent.

We say a theory has ATP if there exists a formula having ATP. If a theory does not have ATP, then we say the theory has NATP.

If a theory has ATP, then the theory always has TP_2 and SOP_1 ([AK20, Proposition 4.4, 4.6]). Thus the class of NTP_2 theories and the class of NSOP_1 theories are subclasses of the class of NATP theories. Therefore we have the following diagram:

$$\begin{array}{ccccccc} \text{NTP}_2 & \longrightarrow & \text{NATP} & & & & \\ \uparrow & & \uparrow & & & & \\ \text{simple} & \longrightarrow & \text{NSOP}_1 & \longrightarrow & \text{NSOP}_2 & \longrightarrow & \dots \end{array}$$

Now we recall notions of forking, dividing, and pre-independence.

Definition 2.5. Let \mathcal{L} be a language, T an \mathcal{L} -theory, \mathbb{M} a monster model of T , $A \subset \mathbb{M}$ a small set, $\varphi(x, y)$ an \mathcal{L} -formula, $p(x)$ a type over A .

- (i) A global extension \tilde{p} of p is said to be *invariant over A* if $\varphi(x, c') \in \tilde{p}$ whenever $\varphi(x, c) \in \tilde{p}$ and $c \equiv_A c'$.
- (ii) A global extension \tilde{p} of p is said to be *Lascar invariant over A* if $\varphi(x, c') \in \tilde{p}$ whenever $\varphi(x, c) \in \tilde{p}$ and $c \equiv_A^L c'$.
- (iii) A global extension \tilde{p} of p is called a *coheir of p (over A)* if for all $\varphi(x, c) \in \tilde{p}$, there exists $m \in M$ such that $\models \varphi(m, c)$.
- (iv) A sequence $(c_i)_{i \in \omega}$ is called an *invariant sequence over A in p* if there exists a global A -invariant extension \tilde{p} of p over A such that $c_i \models \tilde{p}|_{A c < i}$ for all $i \in \omega$.
- (v) A sequence $(c_i)_{i \in \omega}$ is called a *coheir sequence over A in p* if there exists a global coheir extension \tilde{p} of p over A such that $c_i \models \tilde{p}|_{A c < i}$ for all $i \in \omega$.
- (vi) We say $\varphi(x, c)$ *divides over A* if there exists an A -indiscernible sequence $(c_i)_{i \in \omega}$ such that $c_0 = c$ and $\{\varphi(x, c_i)\}_{i \in \omega}$ is inconsistent.
- (vii) We say $\varphi(x, c)$ *forks over A* if there exist $\psi_0(x, c_0), \dots, \psi_n(x, c_n)$ such that $\varphi(x, c) \vdash \bigvee \psi_i(x, c_i)$ and $\psi_i(x, c_i)$ divides over A for each i .
- (viii) We say $p(x)$ *divides over A* if there exists $\varphi(x, c)$ such that $p(x) \vdash \varphi(x, c)$ and $\varphi(x, c)$ divides over A .
- (ix) We say $p(x)$ *forks over A* if there exists $\varphi(x, c)$ such that $p(x) \vdash \varphi(x, c)$ and $\varphi(x, c)$ forks over A .
- (x) We say $\varphi(x, c)$ *Kim-divides over A* if there exists an A -invariant sequence $(c_i)_{i \in \omega}$ in $\text{tp}(c/A)$ such that $\{\varphi(x, c_i)\}_{i \in \omega}$ is inconsistent.
- (xi) We say $\varphi(x, c)$ *coheir-divides over A* if there exists a coheir sequence $(c_i)_{i \in \omega}$ over A in $\text{tp}(c/A)$ such that $\{\varphi(x, c_i)\}_{i \in \omega}$ is inconsistent.
- (xii) By the same manner above, we define Kim-forking, coheir-forking for formulas, and Kim-dividing, Kim-forking, coheir-dividing, coheir-forking for types.

Definition 2.6. [CK12] [Adl08] A ternary relation \perp is called a *pre-independence relation* if it is invariant under automorphisms. For a pre-independence relation \perp , we say

- (i) it satisfies *monotonicity* if $aa' \perp_A bb'$ implies $a \perp_A b$,
- (ii) it satisfies *base monotonicity* if $a \perp_A bc$ implies $a \perp_{Ab} c$,
- (iii) it satisfies *transitivity on the left (over A)* if $a \perp_{Ab} c$ and $b \perp_A c$ implies $ab \perp_A c$,
- (iv) it satisfies *right extension (over A)* if for all a, b, c, A with $a \perp_A b$, there exists $c' \equiv_{Ab} c$ such that $a \perp_A bc'$,

- (v) it satisfies *left extension (over A)* if for all a, b, c, A with $a \downarrow_A b$, there exists $c' \equiv_{Aa} c$ such that $ac' \downarrow_A b$.
- (vi) We say a set A is an *extension base for \downarrow* if $a \downarrow_A A$ for all a .
- (vii) We say \downarrow *preserves indiscernibility over a set A* if I is aA -indiscernible whenever $a \downarrow_A I$ and I is A -indiscernible.

Definition 2.7. [CK12] [Adl08] Let \mathcal{L} be a language, T an \mathcal{L} -theory, \mathbb{M} a monster model of T , $A, B \subset \mathbb{M}$ small sets, and $a \in \mathbb{M}$. Suppose $A \subseteq B$.

- (i) $a \downarrow_A^d B$ if $\text{tp}(a/B)$ does not divide over A .
- (ii) $a \downarrow_A^f B$ if $\text{tp}(a/B)$ does not fork over A .
- (iii) $a \downarrow_A^K B$ if $\text{tp}(a/B)$ does not Kim-fork over A .
- (iv) $a \downarrow_A^i B$ if there exists a global Lascar invariant type over A containing $\text{tp}(a/B)$.
- (v) $a \downarrow_A^u B$ if there exists a global coheir over A containing $\text{tp}(a/B)$.

Fact 2.8. [CK12] [Adl08] For all a, b and a model M $a \downarrow_M^u b \Rightarrow a \downarrow_M^i b \Rightarrow a \downarrow_M^f b \Rightarrow a \downarrow_M^d b$ and $a \downarrow_M^f b \Rightarrow a \downarrow_M^K b$. If M is a model, then \downarrow^u satisfies *monotonicity, base monotonicity, transitivity on the left over M , right extension over M , and left extension over M* .

Definition 2.9. Let $p(x) = \text{tp}(a/M)$.

- (i) A global extension \tilde{p} of p is called a *strong coheir of p (over M)* if \tilde{p} is a coheir over M and for any $A \supseteq M$, if $a' \models \tilde{p}|_A$ then $A \downarrow_M^K a'$.
- (ii) A sequence $(a_i)_{i \in \omega}$ is called a *strong coheir sequence over M in p* if there exists a global strong coheir extension \tilde{p} of p over M such that $a_i \models \tilde{p}|_{Ma < i}$ for all $i \in \omega$.
- (iii) We say a formula $\varphi(x, a)$ *strong coheir-divides over M* if there exists a strong coheir sequence $(a_i)_{i \in \omega}$ over M in p such that $\{\varphi(x, a_i)\}_{i \in \omega}$ is inconsistent.
- (iv) We say that $\text{tp}(a/Bb)$ is *strictly invariant over B* (denoted by $a \downarrow_B^{ist} b$) if there is a global extension p , which is Lascar invariant over B (so $a \downarrow_B^i b$) and for any $C \supseteq Bb$, if $c \models p|_C$ then $C \downarrow_B^f c$.
- (v) We write $a \downarrow_B^{iKst} b$ if there is a global extension p of $\text{tp}(a/Bb)$, which is Lascar invariant over B (so $a \downarrow_B^i b$) and for any $C \supseteq Bb$, if $c \models p|_C$ then $C \downarrow_B^K c$.
- (vi) We write $a \downarrow_B^{uKst} b$ if there is a global strong coheir of $\text{tp}(a/Bb)$.

3. KIM-DIVIDING AND COHEIR-DIVIDING ON NATP THEORIES

In [CK12], Chernikov and Kaplan proved that forking and dividing are equivalent over models, in NTP_2 theories. In other words, they proved that in NTP_2 theories, for a model M , if a formula forks over M , then it divides over M .

The strategy they have taken can be divided into two main parts. First they show that in NTP_2 and over a set B , if a pre-independence relation \downarrow satisfies every property in Definition 2.6, then every formula dividing over B coheir-divides over B . In particular, in NTP_2 theories, for any model M , if a formula divides over M , then it coheir-divides over M . As a corollary they show that in NTP_2 theories, \downarrow^{ist} is an extension base over models.

As the second step, they prove Kim's lemma for NTP_2 theories over models. That is, if a formula $\varphi(x, a)$ divides over a model M , then for any sequence $(a_i)_{i \in \omega}$ such that $a_0 = a$ and $a_i \perp_M^{ist} a_{<i}$ for all $i \in \omega$, the set $\{\varphi(x, a_i)\}_{i \in \omega}$ is inconsistent.

Using these observation, they prove that forking and dividing are equivalent over models in NTP_2 theories. If a formula $\varphi(x, a)$ forks over a model M , then there exist $\psi_0(x, a_0), \dots, \psi_n(x, a_n)$ such that ψ_i divides over M for each $i \leq n$ and $\varphi(x, a) \vdash \bigvee \psi_i(x, a_i)$. Since they prove that \perp^{ist} is an extension base over models, there exists a global extension $p(x, x_0, \dots, x_n)$ of $\text{tp}(a, a_0, \dots, a_n/M)$ satisfying (iv) in Definition 2.9. If we choose any sequence $(a^j, a_0^j, \dots, a_n^j)_{j \in \omega}$ such that $(a^j, a_0^j, \dots, a_n^j) \models p|_M a^{<j}, a_0^{<j}, \dots, a_n^{<j}$ for each $j \in \omega$, then by Kim's lemma, $\{\varphi(x, a^j)\}_{j \in \omega}$ is inconsistent. Thus $\varphi(x, a)$ divides over M .

Following their strategy, we are trying to show that Kim-forking and Kim-dividing are equivalent over models in NATP theories, and obtained a small result which corresponds to their first step. Note that we can not expect to prove that forking and dividing are equivalent over models since they are not equivalent in NSOP₁ theories.

To explain our result, we need a notion of $\mathcal{T}_\alpha^\delta$, which describes trees that grow downward. The notion is originally from [KR20].

Definition 3.1. Suppose α and δ are ordinals. We define $\mathcal{T}_\alpha^\delta$ to be the set of functions η so that

- (i) $\text{dom}(\eta)$ is an end-segment of α of the form $[\beta, \alpha)$ for β equal to 0 or a successor ordinal. If α is a successor or 0, we allow $\beta = \alpha$, i.e. $\text{dom}(\eta) = \emptyset$.
- (ii) $\text{ran}(\eta) \subseteq \delta$
- (iii) finite support: the set $\{\gamma \in \text{dom}(\eta) : \eta(\gamma) \neq 0\}$ is finite.

We interpret $\mathcal{T}_\alpha^\delta$ as an $\mathcal{L}_{s, \alpha}$ -structure by defining

- (iv) $\eta \sqsubseteq \nu$ if and only if $\eta \subseteq \nu$. Write $\eta \perp \nu$ if $\neg(\eta \sqsubseteq \nu)$ and $\neg(\nu \sqsubseteq \eta)$.
- (v) $\eta \wedge \nu = \eta|_{[\beta, \alpha)} = \nu|_{[\beta, \alpha)}$ where $\beta = \min\{\gamma : \eta|_{[\gamma, \alpha)} = \nu|_{[\gamma, \alpha)}\}$, if non-empty (note that β will not be a limit, by finite support). Define $\eta \wedge \nu$ to be the empty function if this set is empty (note that this cannot occur if α is a limit).
- (vi) $\eta <_{lex} \nu$ if and only if $\eta \triangleleft \nu$ or, $\eta \perp \nu$ with $\text{dom}(\eta \wedge \nu) = [\gamma + 1, \alpha)$ and $\eta(\gamma) < \nu(\gamma)$.
- (vii) for all $\beta \in \alpha \setminus \text{lim}(\alpha)$, $P_\beta = \{\eta \in \mathcal{T}_\alpha^\delta : \text{dom}(\eta) = [\beta, \alpha)\}$ (the β -th floor).

We will also need the following notation.

- (viii) (canonical inclusion) For $\alpha' > \alpha$, \mathcal{T}_α^2 can be embedded in $\mathcal{T}_{\alpha'}^2$ with respect to $\mathcal{L}_{s, \alpha'}$ by a map $f_{\alpha, \alpha'} : \mathcal{T}_\alpha^2 \rightarrow \mathcal{T}_{\alpha'}^2 : \eta \mapsto \eta \cup \{(\beta, 0) : \beta \in \alpha' \setminus \alpha\}$. Unless otherwise stated, we regard \mathcal{T}_α^2 as $f_{\alpha, \alpha'}(\mathcal{T}_\alpha^2)$ in $\mathcal{T}_{\alpha'}^2$.
- (ix) $\eta \perp_{lex} \nu$ if and only if $\eta <_{lex} \nu$ and $\eta \not\triangleleft \nu$. For an indexed set $\{a_\eta\}_{\eta \in \mathcal{T}_\alpha^2}$ and $\eta \in \mathcal{T}_\alpha^2$, by $a_{\perp_{lex} \eta}$ we mean the set $\{a_\nu : \nu \perp_{lex} \eta\}$.
- (x) for each $\eta \in \mathcal{T}_\alpha^\delta$, let $h(\eta)$ be an ordinal such that $\text{dom}(\eta) = [h(\eta), \alpha)$.
- (xi) for each $\eta \in \mathcal{T}_\alpha^\delta$, let C_η (the cone on η) be the set of all $\nu \in \mathcal{T}_\alpha^\delta$ such that $\eta \sqsubseteq \nu$.

Remark 3.2. Let α, β be ordinals and suppose $\beta < \alpha$. Let us consider \mathcal{T}_α^2 .

- (i) By finite support, $<_{lex}$ is a well-ordering on P_β .
- (ii) If α is a cardinal, then $|P_\beta| = \alpha$.
- (iii) If α is a cardinal, then $|C_\eta| < \alpha$ for all $\eta \in \mathcal{T}_\alpha^2$.

Lemma 3.3. *Let θ and κ be cardinals such that κ is uncountable, $\theta^+ < \kappa$ and $\text{cf}(\kappa) = \kappa$, where θ^+ is the successor cardinal of θ . Let c be a θ -coloring on \mathcal{T}_κ^2 (a function from \mathcal{T}_κ^2 to θ) which satisfies:*

(*) *for each $\beta < \alpha$ and $\eta, \nu \in P_\beta$, if $\eta <_{lex} \nu$ then $c(\eta) \leq c(\nu)$.*

Then for any infinite cardinal $\lambda < \kappa$, there exists an \mathcal{L} -embedding $f : \mathcal{T}_\lambda^2 \rightarrow \mathcal{T}_\kappa^2$ and $\iota < \theta$ such that $c(\eta) = \iota$ for all $\eta \in f(\mathcal{T}_\lambda^2)$.

Proof. By Remark 3.2 (ii), $|P_\beta| = \kappa$ for each $\beta < \kappa$. Thus for each $\beta < \kappa$, there exists $\iota_\beta < \theta$ such that $|\{\eta \in P_\beta : c(\eta) = \iota_\beta\}| = \kappa$. By Remark 3.2 (i), $\{\eta \in P_\beta : c(\eta) = \iota_\beta\}$ has the least element with respect to $<_{lex}$, say η_β . We show that $c(\eta) = \iota_\beta$ for all $\eta \in P_\beta$ with $\eta_\beta < \eta$. Suppose not. Then there exists $\eta \in P_\beta$ such that $\eta_\beta < \eta$ and $\iota_\beta < c(\eta)$. Let $\nu = \eta_\beta \wedge \eta$. Then $\{\eta \in P_\beta : c(\eta) = \iota_\beta\} \subseteq C_\nu$ by (*). But by Remark 3.2 (iii), $|C_\nu| < \kappa$. It is a contradiction.

Let λ be an infinite cardinal such that $\theta^+ \leq \lambda < \kappa$. For each $\beta < \lambda$, let γ_β be the largest ordinal such that $\eta_\beta(\gamma_\beta) = 1$. We can find such an ordinal by finite support. Since $\text{cf}(\kappa) = \kappa$, we can choose an ordinal $\gamma < \kappa$ such that $\lambda < \gamma$, and $\gamma_\beta < \gamma$ for all $\beta < \lambda$. Let ξ be an element of \mathcal{T}_κ^2 such that $\text{dom}(\xi) = [\gamma, \kappa)$, $\xi(\gamma) = 1$ and $\xi(\gamma') = 0$ for all $\gamma' > \gamma$. Then each floor in C_ξ is monochromatic with respect to c . Since the cone has at least λ -many floors and we have only θ -many colors, we can find λ -many floors in the cone which have the same color. This completes the proof. \square

Proof of the main theorem. Suppose $\varphi(x, a)$ Kim-divides over M . Then there exists a global M -invariant type $p(y)$ containing $\text{tp}(a/M)$ such that for all $(a_i)_{i \in \omega} \models p^{\otimes \omega}|_M$, the set $\{\varphi(x, a_i)\}_{i \in \omega}$ is k -inconsistent for some $k \in \omega$. To get a contradiction, we assume there is no global coheir extension $q(y)$ of $\text{tp}(a/M)$ such that for all $(a_i)_{i \in \omega} \models q^{\otimes \omega}|_M$, the set $\{\varphi(x, a_i)\}_{i \in \omega}$ is inconsistent. Note that the number of all global coheir extensions of $\text{tp}(a/M)$ is bounded. Let $\{q_i\}_{i < \theta}$ be an enumeration of all global coheir extension of $\text{tp}(a/M)$, where θ is a cardinal number. Let κ be a cardinal such that $\theta^+ < \kappa$ and $\text{cf}(\kappa) = \kappa$.

Claim. For any ordinal α and any set $A \supseteq M$, there exists $\{b_\eta\}_{\eta \in \mathcal{T}_\alpha^2}$ which satisfies:

- (i) $b_\eta \models p|_{M b_\eta}$ for each $\eta \in \mathcal{T}_\alpha^2$, and
- (ii) For each $\eta \in \mathcal{T}_\alpha^2$, there exists $i < \theta$ such that $b_\eta \models q_i|_{A b_\perp_{lex} \eta}$ and $b_\eta \not\models q_{i'}|_{A b_\perp_{lex} \eta}$ for all $i' < i$.
- (iii) $A' \downarrow_M^u A$ where $A' = \{b_\eta\}_{\eta \in \mathcal{T}_\alpha^2}$.

Furthermore, for any set $A \supseteq M$, we can construct a sequence $\{\{b_\eta\}_{\eta \in \mathcal{T}_\alpha^2}\}_{\alpha < \kappa}$ whose elements satisfy (i) and (ii) above, and $\{b_\eta\}_{\eta \in \mathcal{T}_\alpha^2} \subseteq \{b_\eta\}_{\eta \in \mathcal{T}_{\alpha'}^2}$ for $\alpha < \alpha' < \kappa$ by canonical inclusion.

Proof of Claim. First we construct $\{b_\eta\}_{\eta \in \mathcal{T}_0^2}$. Choose any $A \supseteq M$. Then there exist $b \models p|_M$ and $b' \models q_0|_A$. Since $b' \downarrow_M^u A$, by left extension, there exists $b^* \equiv_{M b'} b$ such that $b^* b' \downarrow_M^u A$. In particular, $b^* \models p|_M$ and $b^* \downarrow_M^u A$. Thus there exists $i < \theta$ such that $b^* \models q_i|_A$. We may assume i is the smallest one of those indices. Let $b_\emptyset := b^*$. Then $\{b_\eta\}_{\eta \in \mathcal{T}_0^2}$ satisfies (i), (ii), and (iii).

Now we consider the successor case. Assume that the statement is true for some ordinal β . Let $\alpha := \beta + 1$. Choose any $A \supseteq M$ and suppose we have constructed $\{\{b_\eta\}_{\eta \in \mathcal{T}_{\beta'}^2}\}_{\beta' < \beta}$ whose members satisfies (i), (ii), (iii), and $\{b_\eta\}_{\eta \in \mathcal{T}_{\beta'}^2} \subseteq \{b_\eta\}_{\eta \in \mathcal{T}_{\beta''}^2}$ for each $\beta' < \beta'' \leq \beta$ by canonical inclusion. For each $\eta \in \mathcal{T}_\beta^2$, let $b_\eta^0 := b_\eta$. By

the induction hypothesis, we have $\{b_\eta^1\}_{\eta \in \mathcal{T}_\beta^2}$ satisfying (i), (ii), and (iii) over $A \cup A^0$ where $A^0 = \{b_\eta^0\}_{\eta \in \mathcal{T}_\beta^2}$. Let $A^1 = \{b_\eta^1\}_{\eta \in \mathcal{T}_\beta^2}$.

For each $\eta \in \mathcal{T}_\alpha^2 \setminus \{\emptyset\}$, let $b_\eta := b_{\eta|_\beta}^{\eta(\beta)}$. Choose any $b \models p|_{AA^0A^1}$. Note that $A^0A^1 \perp_M^u A$ by left transitivity. Hence we can apply left extension to get $b^* \equiv_{A^0A^1} b$ such that $b^*A^0A^1 \perp_M^u A$. In particular, $b^* \perp_M^u A$. Thus there exists $i < \theta$ such that $b^* \models q_i|_A$. We may assume i is the smallest one of those indices. Let $b_\emptyset := b^*$. Then $\{b_\eta\}_{\eta \in \mathcal{T}_\alpha^2}$ satisfies (i), (ii), and (iii). Clearly $\{b_\eta\}_{\eta \in \mathcal{T}_\beta^2} \subseteq \{b_\eta\}_{\eta \in \mathcal{T}_\alpha^2}$ by canonical inclusion.

Finally, let α be a limit ordinal and assume that the statement is true for all $\beta < \alpha$. Suppose we have constructed a sequence $\{\{b_\eta\}_{\eta \in \mathcal{T}_\beta^2}\}_{\beta < \alpha}$ such that $\{b_\eta\}_{\eta \in \mathcal{T}_\beta^2} \subseteq \{b_\eta\}_{\eta \in \mathcal{T}_{\beta'}^2}$ for all $\beta < \beta' < \alpha$, and $\{b_\eta\}_{\eta \in \mathcal{T}_\beta^2}$ satisfies (i),(ii), and (iii) for each $\beta < \alpha$. Then the union $\{b_\eta\}_{\eta \in \mathcal{T}_\alpha^2} := \bigcup_{\beta < \alpha} \{b_\eta\}_{\eta \in \mathcal{T}_\beta^2}$ satisfies (i), (ii), and (iii). This completes the proof of the claim. \square

Now we have $\{b_\eta\}_{\eta \in \mathcal{T}_\kappa^2}$ satisfies (i) and (ii) over M , by the claim above. Let c be a θ -coloring on \mathcal{T}_κ^2 such that $c(\eta) = i$ if and only if $b_\eta \models q_i|_{Mb_{\perp_{lex}}\eta}$ and $b_\eta \not\models q_{i'}|_{Mb_{\perp_{lex}}\eta}$ for all $i' < i$, as in (ii). By the construction, if η and ν are in the same floor and $\eta <_{lex} \nu$, then $\{b_\xi\}_{\xi \in C_\eta} \equiv_{Mb_{\perp_{lex}}\eta} \{b_\xi\}_{\xi \in C_\nu}$, and hence $c(\eta) \leq c(\nu)$ by the minimality of the index i in (ii). Thus this coloring satisfies the hypothesis in Lemma 3.3, and we can find an \mathcal{L} -embedding $f : \mathcal{T}_\omega^2 \rightarrow \mathcal{T}_\kappa^2$ such that $c(f(\mathcal{T}_\omega^2)) = \iota$ for some $\iota < \theta$. For each $\eta \in \mathcal{T}_\omega^2$, let $c_\eta := b_{f(\eta)}$. Then $\{c_\eta\}_{\eta \in \mathcal{T}_\omega^2}$ satisfies the following conditions:

- (iv) $c_\eta \models p|_{Mc_{>\eta}}$ for each $\eta \in \mathcal{T}_\omega^2$.
- (v) $c_\eta \models q_i|_{Mc_{\perp_{lex}}\eta}$ for each $\eta \in \mathcal{T}_\omega^2$.

In particular, every finite antichain X in \mathcal{T}_ω^2 is an initial segment of some coheir sequence in q_i over M . Thus the set $\{\varphi(x, c_\eta)\}_{\eta \in X}$ is consistent. On the other hand, every finite path X in \mathcal{T}_ω^2 with $|X| \geq k$ is an initial segment of some Morley sequence in p over M . Thus the set $\{\varphi(x, c_\eta)\}_{\eta \in X}$ is inconsistent. Therefore by compactness, we can construct an antichain tree, which yields a contradiction with the assumption that T is NATP. \square

The following two corollaries can be shown by using the same argument in [CK12, Lemma 3.1, Proposition 3.7].

Corollary 3.4. *Suppose that T is NATP. If a formula Kim-forks over a model M , then it quasi divides over M .*

Corollary 3.5. *In NATP theories, every model is an extension base for \perp^{uKst} .*

Thus if we can prove Kim's lemma for Kim-dividing in NATP, we can prove that Kim-forking and Kim-dividing are equivalent in NATP, over models.

Conjecture 3.6. *In NATP theories, if a formula $\varphi(x, a)$ Kim-divides over a model M , then $\{\varphi(x, a_i)\}_{i \in \omega}$ is inconsistent for any strong coheir sequence $(a_i)_{i \in \omega}$ over M .*

REFERENCES

- [Adl08] Hans Adler, An introduction to theories without the independence property, *Archive for Mathematical Logic*, (2008).

- [AK20] JinHoo Ahn and Joonhee Kim, SOP₁, SOP₂, and Antichain Tree Property, *submitted*, 2020
- [Con12] Gabriel Conant, Dividing lines in unstable theories, Unpublished, https://www.dpmms.cam.ac.uk/~gc610/Math/Properties_in_Unstable_Theories.pdf, 2012.
- [CK12] Artem Chernikov and Itay Kaplan, Forking and Dividing in NTP₂ theories, *The Journal of Symbolic Logic*, **77**, (2012).
- [DS04] Mirna Džamonja and Saharon Shelah, On \triangleleft^* -maximality, *Ann. Pure Appl. Logic*, **125** (2004), no. 1-3, 119-158.
- [KK11] Byunghan Kim and Hyeung-Joon Kim, Notions around tree property 1, *Ann. Pure Appl. Logic* **162** (2011).
- [KR20] Itay Kaplan and Nicholas Ramsey, On Kim-Independence, *Journal of the European Mathematical Society*, **22**, (2020).
- [She90] Saharon Shelah, *Classification theory and the number of nonisomorphic models*, Second edition. *Studies in Logic and the Foundations of Mathematics*, **92**. North-Holland Publishing Co., Amsterdam, 1990.

DEPARTMENT OF MATHEMATICS
Yonsei University
Seoul 03722
South Korea
kimjoonhee@yonsei.ac.kr