On model companions of some classes of groups

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Abstract

We investigate some classes of groups in point of view of the existence of model companion and amalgamation. Especially, we prove that torsion free groups has no model companion

1 Introduction

Model companion is the one of the most basic concepts in model theory.

Definition 1. Let T_i $(i < 2)$ and T be L-theories.

1. T_0 is said to be a companion of T_1 if for each $i < 1$, every model $M_i \models T_i$ has an extension $M_{1-i} \supset M_i$ such that $M_{1-i} \models T_{1-i}$.
2. T^* is said to be a model companion of T if T^* is a companion of T and model complete.

If a model companion exists, then it is unique. Well known examples of model companions in algebraic structures are the following: many important algebraic theory has a model companion.

- ACF is the model companion of the theory of fields.

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- RCF is the model companion of the theory of ordered fields.
- DCF is the model companion of the differential fields.
- The theory of divisible torsion free Abelian groups is the model companion of the theory of torsion free Abelian groups.

On the other hand, there are natural example without a model companion. The most famous one is the theory of groups.

Fact 2. The theory of groups has no model companion.

(One can find several other examples which have no model companion in [1].) To show above fact, several textbooks refer a group theoretic lemma. Akito Tsuboi, in a personal communication, asked that there is an elementary proof of the above fact without such lemma. He also showed using some randomization technique that the theory of torsion-free groups has no model companion, inspired by Saracino's following results [2][3]:

Proposition 3. The following theories has no model companion.

1. The theory of n-step nilpotent groups ($n > 1$).
2. The theory of n-step torsion-free nilpotent groups ($n > 1$).

In this article, we give another proof showing that both the theory of groups and the theory of torsion-free groups has no model companion, using a kind of free product of groups. Free product amalgamating isomorphic subgroups can be considered as an amalgamation in model theoretic meaning.

It is also very interesting that discussing the existence of model companion of exponent n groups with the connection to the Burnside problem, however, this topic will appear in another opportunity in the future work.

2 Non-existence of model companions

First we recall a basic fact about model companions.

Definition 4. Let T be an L-theory and $M \models T$. M is said to be an existentially closed model of T if for every extension $M \subset N \models T$, $\bar{a} \in M$, and a quantifier free formula $\varphi(x, \bar{y})$, $N \models \exists x \varphi(x, \bar{a})$ implies $M \models \exists x \varphi(x, \bar{a})$.

Fact 5. Suppose that T is a $\forall\exists$-theory. Then T has a model companion if and only if the class of existentially closed model of T is an elementary class.

Torsion-free grous are universal theories, so we can use the above fact freely.

Definition 6. Let G be a group.

1. G is said to be torsion-free if for all $a \in G$, $a^n \neq e \to a = e$ for every $n > 0$.

We start recalling the definition of free product of two groups.

Definition 7. Let G_i $(i < 2)$ be groups. The free product $G_0 * G_1$ of G_0 and G_1 is a unique group H with group morphisms $\sigma_i : G_i \to H$ $(i < 2)$ satisfying the following universal property: For every group X and morphisms $\tau_i : G_i \to X$ $(i < 2)$ there are unique morphisms $\eta : H \to X$ such that $\eta \circ \tau_i = \sigma_i$ $(i < 2)$.

$G_0 * G_1$ can be seen as a common extension of G_0 and G_1, since σ_i must be injective, and it looks like a "free" group outside G_i. Especially, every elements in G_0 and G_1 except e don't commute as follows.

Fact 8. Let canonical morphisms $\sigma_i : G_i \to G_0 * G_1$ be identity maps, i.e. $G_i \leq G_0 * G_1$. Then for any $a_i \in G_i$ except e, $a_0 a_1 \neq a_1 a_0$.

Now we introduce a generalization of the free product.

Definition 9. Let G_0 and G_1 have a common subgroup A. We say a group H with group morphisms $\sigma_i : G_i \to H$ $(i < 2)$ such that $\sigma_0 | A = \sigma_1 | A$ is said to be a free product of G_0 and G_1 with A amalgamated if it satisfies the following universal property: For every group X and morphisms $\tau_i : G_i \to X$ $(i < 2)$ there are unique morphisms $\eta : H \to X$ such that $\eta \circ \tau_i = \sigma_i$ $(i < 2)$.

In this article, free product with amalgamation is denoted by $G_0 *_A G_1$, and we simply call it a free prodcut over A. We don't recall the proof of the existence of $G_0 *_A G_1$ but it can be obtain from $G_0 * G_1$ as a natural quotient group. It is not trivial that the canonical morphisms σ_i are injective, so first we introduce a basic structure theorem, called normal form theorem, for $G_0 *_A G_1$ without proof.

In what follows, for each $i < 2$, we fix a transversal set T_i for the family of cosets of A in G_i, for each i. In other words, $T_i \cap Ag$ has exactly one element for each $g \in G_i \setminus A$. In the following definition, we omit the subscription of σ_i for the simplicity.

Definition 10. For an element $x \in G_0 *_A G_1$, we say an equation $x = \sigma(a)\sigma(r_0)\cdots\sigma(r_{n-1})$ is a normal expression of x if

1. $a \in A$,
2. $r_j \in T_0 \cup T_1$ for all $j < n$,
3. $r_j \in T_i \Rightarrow r_{j+1} \in T_{1-i}$ for all $j < n$.

Remark 11. We allow normal expressions with $n = 0$. In this case, expressions has a form $x = \sigma(a)$.

The following is an elementary result on free product with amalgamation.

Theorem 12. Every $x \in G_0 *_A G_1$ has a unique normal expression.

Corollary 13.
1. The canonical morphisms $\sigma_i : G_i \to G_0 *_A G_1$ is injective.
2. $\sigma_0(G_0) \cap \sigma_1(G_1) = A$.
3. For every $g_i \in G_i \setminus A$, $g_0 g_1 \neq g_1 g_0$ in $G_0 *_A G_1$. (Here, we consider G_i as a subgroup of $G_0 *_A G_1$ by (1).)

Proof. (1): Let $e \neq g \in G_i$ and suppose $\sigma_i(g) = \sigma_i(e) = e$ in the product. Take $r \in T_i$ and $a \in A$ such that $ar = g$. Then e has two normal expression $\sigma_i(e)$ and one of $\sigma_i(a)\sigma_i(r)$, $\sigma(a)$, or $\sigma(r)$.

(2): Similar to (1).

(3): We can find $a \in A$ and $r_i \in T_i$ such that $g_0 g_1 = a r_0 r_1$. Also, there are $a' \in A$ and $r'_i \in T_i$ such that $g_1 g_0 = a' r'_1 r'_0$. If $g_0 g_1 = g_1 g_0$ in $G_0 *_A G_1$, then we can find two different normal expression for it in the same manner as (1). □

From now on, we always assume $G_i \leq G_0 *_A G_1$ and σ_i is the identity map (hence we don't write it in equations).

Definition 14. Let $x \in G_0 *_A G_1$ and $x = a r_0 \cdots r_{n-1}$ the normal expression of x.

1. The expression length $len(x)$ of x is n.
2. The expression type $etp(x)$ of x is the pair $(i, j) \in \{0, 1\}^2$ where $r_0 \in G_i$ and $r_{n-1} \in G_j$. If $len(x) = 0$, then we define $etp(x)$ as the empty set.

Remark 15.
1. If $a \in A$ then x and xa have the same expression length and type.

2. If $etp(x) = (i, j)$, $etp(y) = (k, l)$ and $j \neq k$, then $len(xy) = len(x) + len(y)$ and $etp(xy) = (i, l)$.
3. If $etp(x) = (i, j)$, $etp(y) = (k, l)$ and $i \neq l$, then $len(xy) \geq 1$. In addition, if $len(xy) \geq 2$, then $etp(xy) = (i, 1 - i)$.
4. $len(xy) \geq |len(x) - len(y)|$.

Proposition 16. Let A be a common subgroup of G_i $(i < 2)$. If G_0 and G_1 are torsion-free, then so is $G_0 *_A G_1$.

Proof. Suppose that $x \in G_0 *_A G_1$ and $x^m = e$ for some $m > 1$. Let $x = ar_0 \cdots r_{n-1}$ be the normal expression of x.

(Case 1): Suppose $len(x) = n \leq 1$, i.e. $x \in G_i$ for some $i < 2$. In this case, $x = e$ because G_i is torsion-free.

(Case 2): Suppose that $len(x) = n$ is odd. Then r_0 and r_{n-1} belong to the same G_i. Let $y = (ar_0)^{-1} x a r_0$. Then $y^m = e$ and $len(y) \leq n - 1$. Hence we can deduce Case 2 to Case 1 or Case 3 below.

(Case 3): Suppose that $len(x) = n = 2k > 0$. In this case, $etp(x) = (i, 1 - i)$ for some i. Hence $len(x^m) = len(x)m > 0$, which contradicts that $x^m = e$. □

Now we can discuss the existence of model companion.

Theorem 17. Let T be a $\forall\exists$-theory of a class of groups such that

1. there is a finitely generated infinite group $A \models T$.
2. the class of models of T is closed under taking:
   (a) group product ($G_0 \times G_1$ for $G_i \models T$), and
   (b) free product over A ($G_0 *_A G_1$ for $G_i \models T$),

Then T has no model companion. Especially, the theory of groups and the theory of torsion-free groups have no model companion.

Proof. Suppose that T has a model companion. Then the class of existentially closed models are elementary, so we can find ω-saturated existentially closed $G_0 \models T$. We can see $A \leq G_0$. Let $\{a_0, \cdots, a_{n-1}\}$ be generators of A Consider a set $\Sigma(x) = \{x \neq a \mid a \in A\} \cup \{\forall y((\bigwedge_{i<n} ya_i = a_i y) \to xy = yx)\}$ of $L(A)$-formulas. Since A is infinite, $\Sigma(x)$ is finitely satisfiable. Take $g \models \Sigma(x)$ in G_0. Let $G_1 = A \times H$ for some nontrivial $H \models T$ with $H \ni h \neq e$. Consider the free product $G_0 *_A G_1$. In the product, $gh \neq hg$ holds though $ga_i = a_i g$. Since $G_0 \leq G_0 *_A G_1 \models T$ and G_0 is existentially closed, we can find $h' \in G_0$ such that $gh' \neq h'g$ and $h'a_i = a_i h'$ for all i. This contradicts to the choice of g. □

References

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