Some remarks on locally o-minimal structures

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概要

abstract Locally o-minimal structures are some local adaptation from o-minimal structures. They were treated, e.g. in [1], [2]. We try to characterize types of definably complete locally o-minimal structures. And we argue about the dp-rank of them.

1. Introduction

We recall some definitions and fundamental facts at first.

Definition 1. Let M be a densely linearly ordered structure without endpoints.

M is $o-minimal$ if every definable subset of M^1 is a finite union of points and intervals.

M is $locally\ o-minimal$ if for any element $a \in M$ and any definable subset $X \subset M^1$, there is an open interval $I \subset M$ such that $I \ni a$ and $I \cap X$ is a finite union of points and intervals.

M is $definably\ complete$ if any definable subset X of M^1 has the supremum and infimum in $M \cup \{\pm\infty\}$.

Here we consider densely linearly ordered structures only.

Example 2. [1], [2]

$(\mathbf{R}, +, <, \mathbf{Z})$ where $\mathbf{Z}$ is the interpretation of a unary predicate, and $(\mathbf{R}, +, <, \sin)$ are definably complete locally o-minimal structures.

Fact 3. [1] *Definably complete local o-minimality is preserved under elementary equivalence.*

Thus we argue in a sufficiently large saturated model $\mathcal{M}$ as usual.

O-minimal structures are characterized by means of behavior of $1-$types. They consider two kinds of $1-$types by the way to cut linear orders of parameter sets, e.g. in [5]. Here we consider nonisolated types only.

Definition 4. Let M be a densely linearly ordered structure and $A \subset M$.

And let $p(x) \in S_1^{or}(A)$, that is, $p(x)$ is complete over A w.r.t. the order relation.

We say that $p(x)$ is *cut over* A if for any $a \in A$, if $a < x \in p(x)$, then there is $b \in A$ such that $a < b < x \in p(x)$, and similarly if $x < a \in p(x)$, then there is $c \in A$ such that $x < c < a \in p(x)$.

We say that $q(x) \in S_1^{or}(A)$ is *noncut over* A if $q(x)$ is not a cut type.

And sometimes we call $q(x) \in S_1(A)$ *cut* (*noncut*) over A if $q(x)$ contains a cut (noncut) $p(x) \in S_1^{or}(A)$.

Remark 5. *Let M be a densely linearly ordered structure and $A \subset M$. And let $p(x) \in S_1^{or}(A)$ be noncut.*

There are four kinds of noncut types.

$p(x) = \{b < x < a : b < a \in A\}$ for some fixed a, or $\{a < x < b : a < b \in A\}$ for some fixed a.

Here we call these types bounded noncut types.

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2. Characterization of definably complete locally o-minimal structures

In o-minimal structures, types of the order relation are complete. Similar argument hold in definably complete locally o-minimal structures to some extent.

Lemma 6. *Let M be a definably complete locally o-minimal structure and $A \subset M$.*

Then any bounded noncut type $p(x) \in S_1^{or}(dcl(A))$ is complete over $dcl(A)$.

Proof ;

Let $p(x) \in S_1^{or}(dcl(A))$ be bounded noncut, that is, $p(x) = \{c < x < a : c < a \in dcl(A)\}$ for some fixed $a \in dcl(A)$ (Another case is proved similarly). By local o-minimality, for any formula $\varphi(x, \overline{b})$ over $dcl(A)$, there is an interval $I \subset M$ such that $a \in I$ and $I \cap \varphi(M, \overline{b})$ is a union of finite points and intervals. Thus there is $c \in I$ such that either for any $d \in I$ with $c < d < a$, $M \models \varphi(d, \overline{b})$, or for any $d \in I$ with $c < d < a$, $M \models \neg\varphi(d, \overline{b})$. Now we assume that for any $d \in I$ with $c < d < a$, $M \models \varphi(d, \overline{b})$. We consider the formula $\forall y(x < y < a \longrightarrow \varphi(y, \overline{b}))$. By definably completeness, there is the infimum $e \in M$ such that for any f with $e < f < a$, $M \models \varphi(f, \overline{b})$ (If $e = -\infty$, then "$x < a$" implies $\varphi(x, \overline{b})$). And $e \in dcl(A)$. ∎

Notation 7. *In the lemma above, for $p(x) = \{c < x < a : c < a \in dcl(A)\}$ for some* *fixed $a \in dcl(A)$, there is the infimum $e \in M$ such that for any f with $e < f < a$ satisfies the formula $\varphi(y, \bar{b})$. We denote "b_φ" such boundary point in the following.*

Next we characterize about the definability of types. We recall the definition.

Definition 8. Let M be a structure.

A type $p(\overline{x}) \in S_n(M)$ is $definable$ if for any $L-$formula $\varphi(\overline{x}, \overline{y})$, there is an $L(M)-$formula $d\varphi(\overline{y})$ such that for all $\overline{a} \in M$, $\varphi(\overline{x}, \overline{a}) \in p(\overline{x})$ if and only if $M \models d\varphi(\overline{a})$.

We can prove the next fact.

Fact 9. *Let M be a definably complete locally o-minimal structure and let $p(x) \in S_1(M)$. Then $p(x)$ is definable if and only if $p(x)$ is noncut.*

We can generalize the fact above for $n-$types to a certain extent.

Definition 10. [6] Let $\mathcal{M}$ be a sufficiently large saturated densely linearly ordered structure and $A \subset \mathcal{M}$.

We say that $p(\overline{x}) \in S_n(A)$ is *noncut over A* if ;

1) $n = 1$, we define it by the same way as above, and

2) $n \geq 2$, let $\overline{x} = (x_1, x_2, \cdots, x_n)$, we define inductively,

$q(\overline{x}') := tp(x_1 \cdots x_{n-1}/A)$ is noncut over A and $tp(x_n/Aa_1 \cdots a_{n-1})$ is noncut over $A \cup \{a_1, \cdots, a_{n-1}\}$ for any realizations $a_1 \cdots a_{n-1}$ of $q(\overline{x}')$.

Fact 11. *Let M be a locally o-minimal structure and let $p(\overline{x}) \in S_n(M)$ be noncut. Then $p(\overline{x})$ is definable.*

Next we characterize definably complete locally o-minimal structures by the notion of forking. We recall some definitions.

Definition 12. A formula $\varphi(\bar{x}, \bar{a})$ *divides* over a set A if there is a sequence $\{\bar{a}_i : i \in \omega\}$ with $tp(\bar{a}_i/A) = tp(\bar{a}/A)$ such that $\{\varphi(\bar{x}, \bar{a}_i) : i \in \omega\}$ is k-inconsistent for some $k \in \omega$.

A formula $\phi(\bar{x}, \bar{a})$ $forks$ over A if $\phi(\bar{x}, \bar{a}) \vdash \bigvee_{i<n} \psi_i(\bar{x}, \bar{b}_i)$ and each $i < n$, $\psi_i(\bar{x}, \bar{b}_i)$ divides over A.

In some papers, they consider the notion of dimension for (definably complete) locally o-minimal structures, e.g. in [3].

Definition 13. Let M be a densely linearly ordered structure without endpoints and let $X \subset M^n$ be a nonempty definable subset.

The dimension $dim(X)$ of X is the maximal nonnegative integer d such that $\pi(X)$ has a nonempty interior for some coodinate projection $\pi : M^n \longrightarrow M^d$.

According to the argument in [7], we recall some lemmas.

Lemma 14. *Let $\mathcal{M}$ be a sufficiently large saturated definably complete locally o-minimal structure and $A \subset \mathcal{M}$. And let $p(x), q(x) \in S_1(A)$ (with $dim(p) = dim(q) = 1$).*

Then either

(a) (i) all $A-$definable $f : p(\mathcal{M}) \longrightarrow q(\mathcal{M})$ are increasing, or

(ii) all $A-$definable $f : p(\mathcal{M}) \longrightarrow q(\mathcal{M})$ are decreasing.

(b) In case (i), whenever $B \supset A$, $a \in p(\mathcal{M})$ and $a > dcl(B) \cap p(\mathcal{M})$, then $dcl(aA) \cap q(\mathcal{M}) > dcl(B) \cap q(\mathcal{M})$,

In case (ii), whenever $B \supset A$, $a \in p(\mathcal{M})$ and $a < dcl(B) \cap p(\mathcal{M})$, then $dcl(aA) \cap q(\mathcal{M}) > dcl(B) \cap q(\mathcal{M})$.

In the lemma above, we just say that if there is a function f between $p(\mathcal{M})$ and $q(\mathcal{M})$, then f has these properties. There is no definable function between a cut type and a noncut type. By this lemma, they consider characteristic extensions of complete types in o-minimal structures. Here we adapt the argument for noncut types.

Definition 15. Let $p(x_1, \cdots, x_n) \in S_n(A)$ of dimension n and $A \subset B$.

Fix some sequence $\eta = (\eta(1), \cdots, \eta(n))$ where each $\eta(i)$ is 1 or 0.

For $1 \leq i \leq n$, let $p_i(x_1, \cdots, x_i)$ be the restriction of p to the variables $x_1, \cdots, x_i$.

We define an extension $p_B^\eta \in S_n(B)$ of p. Choose a realization $(b_1, \cdots, b_n)$ of p_B^η inductively as follows ;

$b_1 \in p_1(\mathcal{M})$ and if $\eta(1) = 1$, then $b_1 > dcl(B) \cap p_1(\mathcal{M})$, while if $\eta(1) = 0$, then $b_1 < dcl(B) \cap p_1(\mathcal{M})$.

For some realization $b_1, \cdots, b_i$ of $p_i(x_1, \cdots, x_i)$, let b_{i+1} be a realization of $p_{i+1}(b_1, \cdots, b_i, x_{i+1})$ such that :

if $\eta(i+1) = 1$, then $b_{i+1} > dcl(B, b_1, \cdots, b_i) \cap p_{i+1}(b_1, \cdots b_i, \mathcal{M})$ and

if $\eta(i+1) = 0$, then $b_{i+1} < dcl(B, b_1, \cdots, b_i) \cap p_{i+1}(b_1, \cdots b_i, \mathcal{M})$.

Lemma 16. [7] *Let $p(\bar{x}) \in S_n(A)$ of dimension n and let $q(y) \in S_1(A)$ of dimension 1.*

Then there is $\eta \in {}^n2$ as in the definition above such that ;

for any $B \supset A$ and any realization $\bar{a}$ of $p_B^\eta(\overline{x})$, $dcl(\bar{a}A) \cap q(\mathcal{M}) > dcl(B) \cap q(\mathcal{M})$.

By the lemmas above, we can prove the next fact.

Proposition 17. *Let $\mathcal{M}$ be a sufficiently large saturated definably complete locally o-minimal structure. And let $c \in \mathcal{M}$ and $A \subset \mathcal{M}$ with $dim(\mathrm{tp}(c/A)) = 1$, and $A \cup \{c\} \subset B$.*

Moreover let $r(x) \in S_1(B)$ be a bounded noncut type of c over $dcl(B)$ satisfying $r \restriction Ac \nvdash r$.

Then $r(x)$ divides over A.

Sketch of proof ;

Let $r(x) := \{ c < x < d : d \in dcl(B) \}$ *(Another case is proved similarly). As* $r \upharpoonright Ac \nvdash r$, *there is an* $L(B)-$*formula* $\varphi(x, \bar{b}) \in r(x)$ *such that for any* $L(Ac)-$*formula* $\psi(x) \in r(x)$, $b_\varphi < b_\psi$.

Case 1. $c \equiv_A b_\varphi$.

We consider an automorphism $\sigma \in Aut_A(\mathcal{M})$ *such that* $\sigma(c) = b_\varphi$. *And let* ${b_\varphi}' = \sigma(b_\varphi)$. *Thus* $c < b_\varphi < {b_\varphi}'$.

Case 2. $c \not\equiv_A b_\varphi$.

Now there is an $L(A)-$*formula* $\psi(x)$ *such that* $\neg\psi(x) \in \mathrm{tp}(c/A)$ *and* $\psi(x) \in \mathrm{tp}(b_\varphi/A)$. *As for any* $L(A)-$*formula* $\psi(x) \in r(x)$, $b_\varphi < b_\psi$, *so for any* d *with* $c < d < b_\varphi$, $\models \psi(d)$.

In the lemmas above, let $p(\bar{x}) = \mathrm{tp}(cb_\varphi/A)$, $q(x) = r(x) \upharpoonright A$, *and the noncut extension* $p'(\bar{x}) \in S^\eta(\bar{b}cA)$ *of* p *such that for any* $c'{b_\varphi}' \models p'(\bar{x})$, $dcl(c'{b_\varphi}'A) \cap q(\mathcal{M}) > dcl(\bar{b}cA) \cap q(\mathcal{M})$. *If* $c < c' \leq b_\varphi$, *then* $\models \psi(c')$, *a contradiction. Thus* $c < b_\varphi < c' < {b_\varphi}'$.

We iterate this construction infinitely many times and prove that the formula $c < x < b_\varphi$ *divides over* A. ∎

Corollary 18. *Let* $\mathcal{M}$ *be a sufficiently large saturated definably complete locally o-minimal structure. And let* $A \subset B \subset C \subset \mathcal{M}$ *and* $p(x) \in S_1(C)$ *be a cut type over* C.

Suppose that there is $d \in B$ *with* $dim(\mathrm{tp}(d/A)) = 1$ *and a bounded noncut type* $q(x) \in S_1(dcl(B))$ *of* d *such that* $q \upharpoonright Ad \nvdash q$ *and* $p \vdash q$.

Then $p(x)$ *divides over* A.

3. Dp-rank of locally o-minimal structures

We recall some definitions.

Definition 19. An *independent partition pattern* of a partial type $p(\bar{x})$ is a sequence of formulas $(\varphi^\alpha(\bar{x}, \bar{y}^\alpha))_{\alpha<\kappa}$ and tuples $\bar{b}_i^\alpha$ for $a < \kappa$ and $i < \omega$ satisfying that ;

for any $\alpha < \kappa$, $\{ \varphi^\alpha(\bar{x}, \bar{b}_i^\alpha) \mid i < \omega \}$ is $k^\alpha-$inconsistent for some $k^\alpha < \omega$, and for any $\eta \in \omega^\kappa$, $\{ \varphi^\alpha(\bar{x}, \bar{b}_{\eta(\alpha)}^\alpha) \mid \alpha < \kappa \}$ is consistent with $p(\bar{x})$.

For a theory T, the invariant $\kappa_{inp}^n(T)$ is the smallest infinite cardinal κ such that no $n-$type has an $inp-$pattern of cardinality κ.

A formula $\varphi(\bar{x}, \bar{y})$ has the *independence property* if there are sequences $(\bar{a}_i : i < \omega)$ and $(\bar{b}_I : I \subset \omega)$ such that $\models \varphi(\bar{a}_i, \bar{b}_I)$ if and only if $i \in I$.

A formula $\varphi(\bar{x}, \bar{y})$ has the *tree property of the second kind* (TP_2) if there are tuples

$(\bar{b}_i^\alpha)_{\alpha,i<\omega}$ such that $\{\varphi(\bar{x},\bar{b}_i^\alpha) \mid i<\omega\}$ is 2−inconsistent for any $\alpha<\omega$, and for any $\eta\in\omega^\omega$, $\{\varphi(\bar{x},\bar{b}_{\eta(\alpha)}^\alpha) \mid \alpha<\omega\}$ is consistent.

We call the *burden* of $p(\bar{x})$ the supremum of the cardinalities κ of all *inp*−patterns for p. It is known that the burden of p is equal to the (classical) weight of p in simple theories.

Theorem 20. *e.g.* [15]
For any theory T,
any formula $\varphi(\bar{x},\bar{y})$ with $|\bar{x}|=n$ is NTP_2 if and only if $\kappa_{inp}^n(T)\le|T|^+$ $(<\infty)$.

Definition 21. An *independent contradictory types pattern* is a sequence of formulas $(\varphi^\alpha(\bar{x},\bar{y}^\alpha))_{\alpha<\kappa}$ and tuples $\bar{b}_i^\alpha$ for $\alpha<\kappa$ and $i<\omega$ satisfying that ;
for any $\eta\in\omega^\kappa$, the following set of formulas is consistent,
$\Gamma_\eta(\bar{x}) := \{\varphi^\alpha(\bar{x},\bar{b}_i^\alpha) \mid \alpha<\kappa,\, i<\omega,\, \eta(\alpha)=i\} \cup \{\neg\varphi^\alpha(\bar{x},\bar{b}_i^\alpha) \mid \alpha<\kappa,\, i<\omega,\, \eta(\alpha)\ne i\}$.

Theorem 22. *e.g.* [11]
If T is NIP, then $\kappa_{ict}(T)=\kappa_{inp}(T)$. Otherwise, $\kappa_{ict}(T)=\infty$.

And we recall the definition of dp-rank from [16].

Definition 23. Let $p(\bar{x})$ be a partial type over a set $A\subset\mathcal{M}$. We define the $dp-rank$ of $p(\bar{x})$ as follows.
The dp-rank of $p(\bar{x})$ is always greater than or equal to 0. Let μ be a cardinal.
We say that $p(\bar{x})$ has dp-rank $\le\mu$ if given any realization a of p and any $1+\mu$ mutually A−indiscernible sequences, at least one of them is indiscernible over Aa.
And we say that p has dp-rank $=\mu$ if it has dp-rank $\le\mu$, but it is not the case that it has dp-rank $\le\lambda$ for any $\lambda<\mu$.
We call p $dp-minimal$ if it has dp-rank 1, we denote rk-dp$(p)=1$.
We call p *dependent* if it has an ordinal dp-rank, that is, rk-dp$(p)<\infty$.
And we call p *strongly dependent* if rk-dp$(p)\le\omega$.

There exists many examples whose theories are dp-minimal. For example, structures of superstable with U-rank = 1, C-minimal, p-adics, ordered set with finite width, tree, and so on. Here we recall the next fact. It is proved by the argument about the notion of Vapnik-Chervonenkis density (or VC-minimality).

Theorem 24. [14], [17]
Weakly o-minimal theories are dp-minimal.

But there are examples of locally o-minimal structures whose theories have the independence property.

Many results are proved under the strong assumption that structures $M \prec \mathcal{M}$ are dp-minimal.

Proposition 25. [13]

Let M be an inp-minimal linearly ordered structure without endpoints and $A \subset M$. And let M be $|A|^+$-saturated and $p(x) \in S_1(M)$.

Then the following are equivalent ;

1. $p(x)$ divides over A.

2. There exist $a, b \in M$ such that $p \vdash a < x < b$ and $a \equiv_A b$.

We can prove the next fact under the same assumption.

Fact 26. *Let $\mathcal{M}$ be a sufficiently large saturated definably complete locally o-minimal structure and $A \subset \mathcal{M}$. And let $Th(\mathcal{M})$ be inp-minimal.*

Moreover let $p(x) \in S_1(dcl(A))$ be an unbounded noncut type.

Then $p(x)$ does not fork over $\emptyset$.

4. Further problems

For definably complete locally o-minimal structures, we can prove the next fact easily.

Lemma 27. *Let M be a definably complete locally o-minimal structure and $A \subset M$ with $dcl(A) \neq \emptyset$.*

Then the isolated $1-$types of $Th(M, a)_{a \in A}$ are dense.

Thus I will try to characterize definably complete locally o-minimal structures by means of prime models.

And I will try to characterize locally o-minimal structures satisfying some additional conditions of their theories. The additional conditions are ; definably complete, dp-minimal, strongly dependent, NIP or NTP_2.

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