RIGIDITY AND DEGENERATION OF 3-DIMENSIONAL HYPERBOLIC CONE STRUCTURES

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Abstract. In this note, we survey rigidity of hyperbolic cone structures and give an example of degeneration with decreasing cone angles. This example is constructed by gluing four copies of a certain polyhedron. We can explicitly describe the isometry types of such hyperbolic polyhedra. Furthermore, we introduce a generalization of cone structure to avoid intersection of cone loci.

1. Introduction

The Mostow–Prasad rigidity [15, 16] implies that the isometry type of a finite volume hyperbolic 3-manifold is uniquely determined by its topology. Hence there is no deformation of (complete) finite volume hyperbolic structures on a 3-manifold. Nevertheless, we can obtain deformation via hyperbolic cone structures by allowing cone-type singularities. Deformation via cone structures has two major applications: Dehn fillings [8] and geometrization of orbifolds [2, 3, 6].

Related to the uniqueness of deformation via cone structures, there are two notions: local and global rigidity. Local rigidity asserts that the deformation space is locally parametrized by cone angles. Global rigidity asserts uniqueness for fixed cone angles. Local rigidity for finite volume cone structures holds if the cone angles are at most 2π [7]. On the other hand, global rigidity for finite volume cone structures is known to hold only when the cone angles at most π [10].

Small deformation of cone structures is possible by local rigidity, Sometimes, however, there are degenerations, in which continuous deformation cannot be extended. The following types of degeneration are known:

(1) collapsing (where the volumes decrease to zero),
(2) appearance of an essential Euclidean sub-cone-surface, and
(3) intersection of cone loci.

The first and second types do not occur if the cone angles decrease. Kojima [10] showed that the third type does not occur if the cone angles are less than π. One might expect that cone structures do not degenerate if the cone angles decrease. However, this conjecture fails in an example by the author [21]. Cone loci may intersect even if the cone angles decrease.

Such degeneration is likely to be ordinary, but explicit construction is hard in general. In our single example, we construct cone structures by gluing four copies of a certain polyhedron. Then we are reduced to considering isometry types of such polyhedra.

Furthermore, we introduce the notion of holed cone structures. The author is preparing a paper for a detailed account [20]. The construction of our example is naturally extended to holed cone structures. Then we can avoid intersection of cone loci.

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2. Rigidity of hyperbolic cone structures

In this section, we survey rigidity of hyperbolic cone structures on a 3-manifold. Let X be a 3-manifold, and let Σ be a link in X. Let $\Sigma_1, \dots, \Sigma_n$ denote the components of Σ. Suppose that $X \setminus \Sigma$ admits an incomplete hyperbolic structure, and the completed metric has the form

$$dr^2 + \sinh^2 r d\theta^2 + \cosh^2 r dz^2$$

in cylindrical coordinates around each component Σ_i for $1 \leq i \leq n$, where r is the distance from the singular locus, z is the distance along the singular locus, and θ is the angle measured modulo $\theta_i > 0$. Then the metric on (X, Σ) is called a *(hyperbolic) cone structure*. More precisely, an equivalence class of such cone metrics by isometries isotopic to the identity is a cone structure. The angle θ_i is called the *cone angle* at the cone locus Σ_i. In our definition, the cone loci consist of disjoint closed geodesics.

If $\theta_i = 2\pi$, the metric around Σ_i is smooth. By generalizing the notion, $\theta_i = 0$ means that Σ_i is a cusp. If $\theta_i = 2\pi/n_i$ for $n_i \in \mathbb{N}$, the metric space (X, Σ) can be regarded as a hyperbolic orbifold.

From now on, fix a pair (X, Σ). Let $\mathcal{C}_{[0,\alpha]} = \mathcal{C}_{[0,\alpha]}(X, \Sigma)$ denote the set of cone structures on (X, Σ) such that the cone angles are at most α. We usually consider $\mathcal{C}_{[0,2\pi]}$. The set $\mathcal{C}_{[0,\alpha]}$ admits the pointed Gromov–Hausdorff topology, which is induced by the geometric convergence of metric spaces. The continuous map $\Theta \colon \mathcal{C}_{[0,\alpha]} \to [0, \alpha]^n$ is defined by $\Theta(g) = (\theta_1, \dots, \theta_n)$, where θ_i is the cone angle at Σ_i in the cone-manifold $(X, \Sigma; g)$.

Suppose that the cone structures on (X, Σ) have finite volume. In practice, it is sufficient to suppose only that $X \setminus \Sigma$ admits a cusped hyperbolic structure of finite volume. Then local and global rigidity for cone structures are known as follows.

Theorem 2.1 (The local rigidity by Hodgson and Kerckhoff [7])**.** *The space $\mathcal{C}_{[0,2\pi]}$ is Hausdorff, and the map $\Theta \colon \mathcal{C}_{[0,2\pi]} \to [0, 2\pi]^n$ is a local homeomorphism. In other words, the space $\mathcal{C}_{[0,2\pi]}$ is locally parametrized by the cone angles.*

This result is induced from the infinitesimal rigidity of cone structures, which extends the local rigidity theory of Weil [17] by using Hodge theory. Infinitesimal deformations preserving cone angles are represented by L^2-harmonic forms, which are finally shown to vanish. The possibility of local deformation follows from a calculation of the dimension of representation space. This result cannot be generalized to the case that cone angles exceed 2π. Izmestiev [9] constructed infinitesimally flexible hyperbolic cone-manifolds with cone angles more than 2π.

Theorem 2.2 (The global rigidity by Kojima [10])**.** *The map $\Theta \colon \mathcal{C}_{[0,\pi]} \to [0, \pi]^n$ is injective. In other words, the cone structure is determined by the cone angles if the cone angles do not exceed π.*

Global rigidity is not known when cone angles exceed π. Let g_0 be an element in $\mathcal{C}_{[0,\pi]}$ such that $\Theta(g_0) = (0, \dots, 0)$. The cusped hyperbolic structure g_0 is unique for (X, Σ) by the Mostow–Prasad rigidity [15, 16]. Theorem 2.2 follows from this fact and Theorems 2.1 and 2.3. Cone structures are uniquely deformed from g_0 to $g \in \mathcal{C}_{[0,\pi]}$ with respect to a fixed path of increasing cone angles.

Theorem 2.3 ([10])**.** *Let $g \in \mathcal{C}_{[0,\pi]}$. Suppose that $\Theta(g) = (\theta_1, \dots, \theta_n) \in [0, \pi]^n$. Then there is $A \subset \mathcal{C}_{[0,\pi]}$ such that $g \in A$ and $\Theta|_A \colon A \to [0, \theta_1] \times \cdots \times [0, \theta_n]$ is a homeomorphism. In other words, we can obtain a continuous family of cone structures from g to g_0 by arbitrarily decreasing cone angles.*

A *continuous degenerating family* of cone structures is a continuous map $\gamma \colon [0, 1) \to \mathcal{C}_{[0,2\pi]}$ such that $\lim_{t \to 1} \Theta(\gamma(t)) \in [0, 2\pi]^n$ but $\gamma(t)$ does not converge in $\mathcal{C}_{[0,2\pi]}$ as $t \to 1$. Theorem 2.3 implies that there is no continuous degenerating family of cone structures with decreasing cone angles if the cone angles are at most π. However, Theorem 3.1 implies that Theorem 2.3 cannot be generalized for cone angles less than 2π.

Similar results are known for 3-dimensional hyperbolic cone-manifolds with vertices. Local rigidity for cone angles less than 2π was proved by Mazzeo and Montcouquiol [13], and independently Weiss [19]. Global rigidity for cone angles at most π was proved by Weiss [18]. The methods in the proofs are similar to [7, 10].

A few results are also known for 3-dimensional hyperbolic cone-manifolds of infinite volume. In this case, similarly to hyperbolic 3-manifolds without cone singularity, there is deformation preserving the cone angles. Nevertheless, one may expect that the cone angles and the end invariants determine the isometric type.

Local rigidity for geometrically finite cone-manifolds means that the deformation space is locally parametrized by the conformal invariant at infinity and the cone angles. In the case that there are no rank-one cusps, this local rigidity was proved by Bromberg [4].

It is possible to consider the case that cone loci are open. Global rigidity for quasi-Fuchsian cone-manifolds with cone angles less than π was proved by Lecuire and Schlenker [12], using quite a different argument from the above results. This is an analog of Bers' simultaneous uniformization theorem. Local rigidity for quasi-Fuchsian cone-manifolds are proved only when the cone angles are less than π by Moroianu and Schlenker [14].

3. An alternating link in the thickened torus

We consider a link $L = L_1 \sqcup \cdots \sqcup L_4 \subset T^2 \times I$ as indicated in the left of Figure 1, where I is an open interval. (A fundamental domain of $T^2 \times I$ is drawn.) Let $\mathcal{C} = \mathcal{C}_{[0,2\pi]} = \mathcal{C}_{[0,2\pi]}(T^2 \times I, L)$ denote the space of cone structures on $(T^2 \times I, L)$, where the components of $T^2 \times \partial I$ keep to be two cusps. Note that any of the cone angles cannot be equal to 2π. The map $\Theta\colon \mathcal{C} \to [0, 2\pi)^4$ assigns the cone angles at L_i.

Theorem 3.1. *There is a continuous degenerating family of cone structures on $(T^2 \times I, L)$ with decreasing cone angles. In this degeneration, two of the cone loci L intersect transversally. Two simultaneous intersections may occur.*

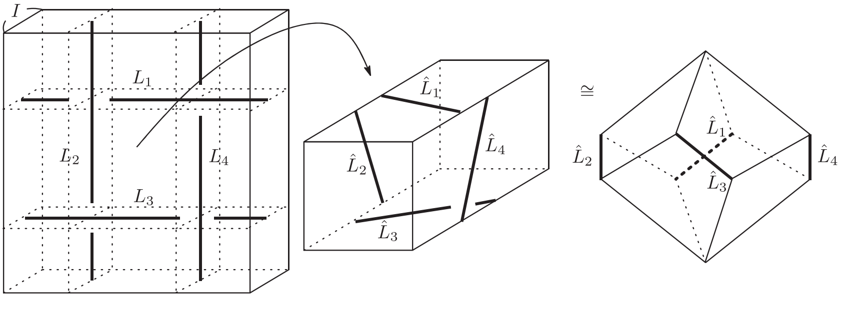

Figure 1. Decomposition of $(T^2 \times I, L)$ into trapezohedra

The space $(T^2 \times I, L)$ is topologically decomposed into four (tetragonal) trapezohedra as indicated in Figure 1. The four trapezohedra correspond to the complementary regions of the diagram of L in T^2.

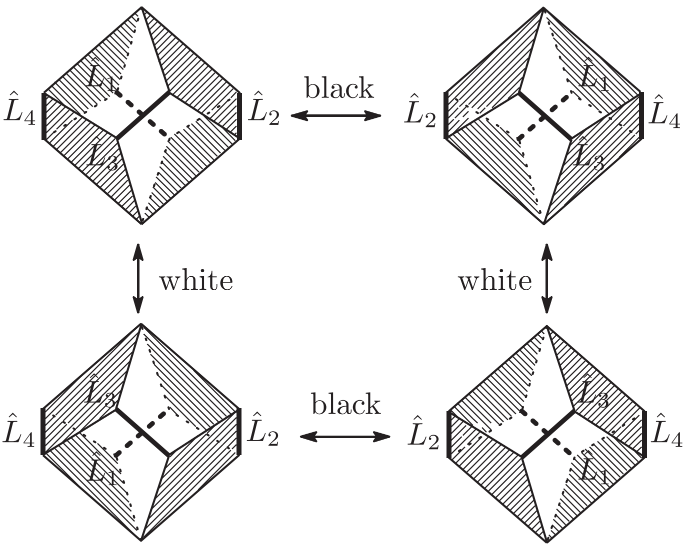

FIGURE 2. Gluing four copies of a trapezohedron

Conversely, we can construct a cone structure on $(T^2 \times I, L)$ by gluing four copies of a hyperbolic trapezohedra with right dihedral angles except at $\hat{L}_i$ as indicated in Figure 2. This construction can be called "double of double". Let $\mathcal{C}_{\mathrm{sym}}$ denote the set of cone structures obtained by this construction. A cone structure in $\mathcal{C}_{\mathrm{sym}}$ is symmetric with respect to an action by $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ on $(T^2 \times I, L)$.

The two ideal vertices disjoint from $\hat{L}_i$ correspond to the components of $T^2 \times \partial I$. The edge $\hat{L}_i$ corresponds to the cone locus L_i. If the dihedral angle at $\hat{L}_i$ is equal to α_i, the cone angle at L_i is equal to $2\alpha_i$. We remark that $\hat{L}_i$ degenerates to an ideal vertex if $\alpha_i = 0$. We use the term "trapezohedron" also for such a degenerated polyhedron.

The local rigidity implies that the set $\mathcal{C}_{\mathrm{sym}}$ is the union of components of $\mathcal{C}$. In fact, the space $\mathcal{C}_{\mathrm{sym}}$ is connected. Though we have no proof, it is very likely that $\mathcal{C}_{\mathrm{sym}} = \mathcal{C}$. Otherwise the global rigidity for $\mathcal{C}$ fails.

The above argument for $g_0 \in \mathcal{C}$ with $\Theta(g_0) = 0$ induces a decomposition of $T^2 \times I \setminus L$ into four regular ideal octahedra. This decomposition was given in [1, 5], and the "double of double" construction was described in detail in [11].

4. Dihedral angles of a tetragonal trapezohedron

By the construction in Section 3, we are reduced to consider hyperbolic trapezohedra (possibly $\hat{L}_i$ degenerates to an ideal vertex) with right dihedral angles except at $\hat{L}_i$. Let $\mathcal{A}$ denote the image of the map $\frac{1}{2}\Theta\colon \mathcal{C}_{\mathrm{sym}} \to [0,\pi)^4$. In other words, $(\alpha_1, \dots, \alpha_4) \in \mathcal{A}$ if and only if there exists a trapezohedron with dihedral angles α_i at $\hat{L}_i$ and $\pi/2$ at the other edges in the hyperbolic space. We explicitly describe $\mathcal{A}$. From now on, the indices $i = 1, \dots, 4$ are regarded modulo 4. See [21] for details.

Theorem 4.1. *The map $\frac{1}{2}\Theta\colon \mathcal{C}_{\mathrm{sym}} \to \mathcal{A}$ is a homeomorphism. In particular, the isometry class of a hyperbolic trapezohedron is determined by the element of $\mathcal{A}$.*

Theorem 4.1 is non-trivial because we do not know whether the global rigidity holds in general.

Theorem 4.2. *For $1 \leq i \leq 4$, let a function Φ_i be defined by*

$$\Phi_i(c_1, \ldots, c_4) = c_i c_{i+1}(c_i c_{i+1} + 1)c_{i+2}c_{i+3} - c_i c_{i+1}(c_i + c_{i+1})(c_{i+2} + c_{i+3}) + (c_i + c_{i+1})^2 - c_i c_{i+1} - 1.$$

Let $\partial\mathcal{A}$ denote the frontier of $\mathcal{A}$ in $[0, \pi)^4$. Then the set $\mathcal{A}$ consists of the elements $(\alpha_1, \ldots, \alpha_4) \in [0, \pi)^4$ satisfying $\Phi_i(\cos\alpha_1, \ldots, \cos\alpha_4) < 0$ or $\cos\alpha_i + \cos\alpha_{i+1} > 0$ for any i. Moreover, it holds that $\partial\mathcal{A} = \bigcup_i \partial_i\mathcal{A}$, where

$$\partial_i\mathcal{A} = \{(\alpha_1, \ldots, \alpha_4) \in [0, \pi)^4 \mid \Phi_i(\cos\alpha_1, \ldots, \cos\alpha_4) = 0,\ \cos\alpha_i + \cos\alpha_{i+1} \leq 0, \cos\alpha_i \leq \cos\alpha_{i+2},\ \cos\alpha_{i+1} \leq \cos\alpha_{i+3}\}.$$

As $(\alpha_1, \ldots, \alpha_4) \in \mathcal{A}$ approaches to $\partial_i\mathcal{A}$, the edge between $\hat{L}_i$ and $\hat{L}_{i+1}$ degenerates.

Theorem 4.2 enables us to see the shape of $\mathcal{A}$ more explicitly.

- $\mathcal{A} \neq [0, \pi)^4$.
- The space $\mathcal{A}$ is connected.
- $(\alpha, \beta, \alpha, \beta) \in \mathcal{A}$ for any $\alpha, \beta \in [0, \pi)$.
- $\partial_i\mathcal{A} \cap \partial_{i+2}\mathcal{A} = \emptyset$.
- $\partial_i\mathcal{A} \cap \partial_{i+1}\mathcal{A} \neq \emptyset$. This corresponds to a degeneration in $\mathcal{C}_{\mathrm{sym}}$ in which two intersections of cone loci occur.
- $[0, \arccos(1 - \sqrt{2}))^4 \subset \mathcal{A}$.
- $(\arccos(1 - \sqrt{2}), \arccos(1 - \sqrt{2}), 0, 0) \notin \mathcal{A}$.

Theorem 3.1 follows from Theorem 4.2. For instance, $(2\pi/3, \ldots, 2\pi/3) \in \mathcal{A}$ and $(2\pi/3, 2\pi/3, 0, 0) \notin \mathcal{A}$. The edge between $\hat{L}_1$ and $\hat{L}_2$ degenerates while decreasing cone angles from $(2\pi/3, \ldots, 2\pi/3)$ to $(2\pi/3, 2\pi/3, 0, 0)$. This corresponds to a degeneration of cone structures in $\mathcal{C}_{\mathrm{sym}}$.

We sketch an outline of the proof. Let us consider a hyperbolic trapezohedron T whose dihedral angles are α_i at $\hat{L}_i$ and $\pi/2$ at the other edges. We use the upper half-space model of hyperbolic 3-space. Regard $\partial\mathbb{H}^3 = \mathbb{R}^2 \cup \{\infty\}$. The trapezohedron T has two ideal vertices disjoint from $\hat{L}_i$. We set them at ∞ and $O = (0, 0)$. We project T to $\mathbb{R}^2 \subset \partial\mathbb{H}^3$ as indicated in Figure 3.

We consider the following points and circles:

- $\widetilde{P}_i$ and $\widetilde{Q}_i$ are the end points of the edge $\hat{L}_i$.
- P_i and Q_i are respectively the images of $\widetilde{P}_i$ and $\widetilde{Q}_i$ by the projection.
- The circle C_i is the boundary of the geodesic plane containing $OQ_{i-1}P_iQ_i$.
- R_i is the center of C_i.
- S_i is the intersectional point of C_i and C_{i+1} other than O.

The dihedral angles α_i are indicated in Figure 4. The point Q_i is the intersection of the segments OS_i and P_iP_{i+1}. Since the other dihedral angles are equal to π, we may assume that

$$P_1 = (p_1, p_2), P_2 = (-p_3, p_2), P_3 = (-p_3, -p_4), P_4 = (p_1, -p_4),$$
$$R_1 = (p_1, tp_1), R_2 = (-tp_2, p_2), R_3 = (-p_3, -tp_3), R_4 = (tp_4, -p_4),$$

for $p_i > 0$ and $t \geq 0$. Let $q_i = \dfrac{p_{i+1}}{p_i}$. Since a positive constant multiple on $\mathbb{R}^2$ extends an isometry of $\mathbb{H}^3$, an isometry type of T determines q_i and t. Moreover, we have $\cos\alpha_i = \dfrac{q_i - t}{\sqrt{1 + t^2}}$.

Conversely, we can construct the above points by $q_i > 0$ and $t \geq 0$. Then the condition for the projection of a trapezohedron is as follows:

- The segments OS_i and P_iP_{i+1} intersect, and
- their intersection Q_i is distinct from P_{i+1}.

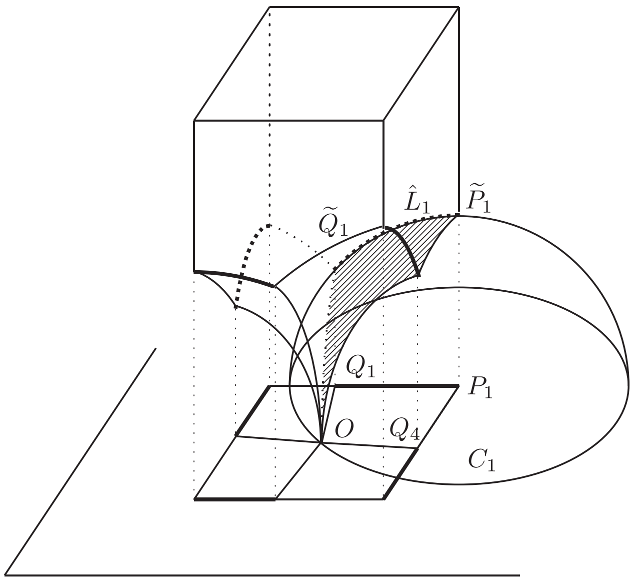

FIGURE 3. Projection of a trapezohedron

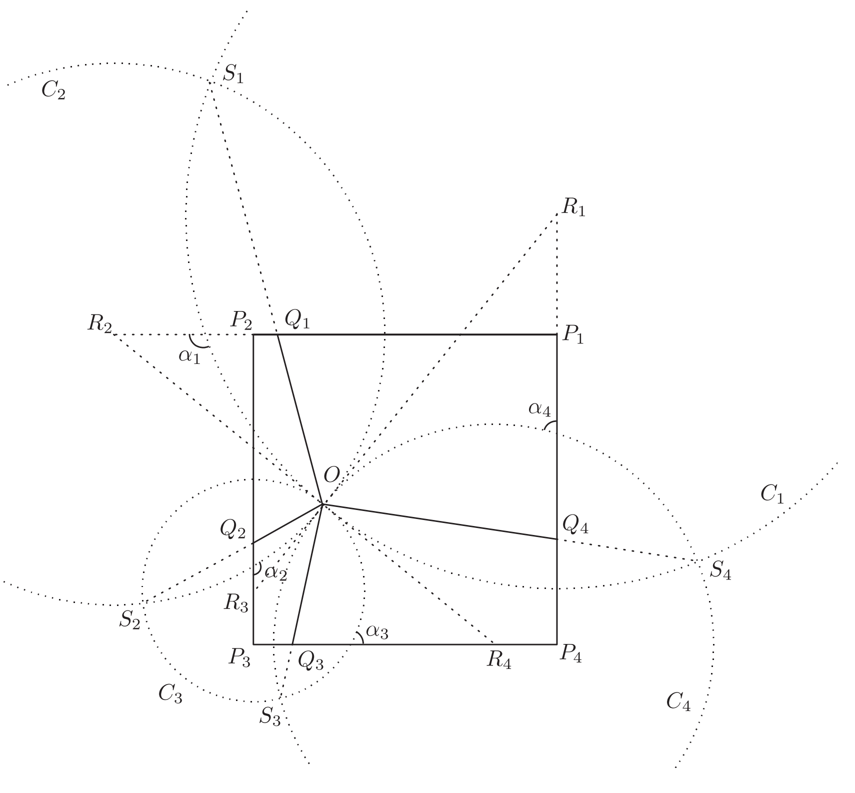

FIGURE 4. Image of the projection

This is equivalent to the following inequalities for each i:

$$t \geq \frac{1}{2}(q_i - q_i^{-1}), \quad (1 - q_i q_{i+1})t < q_i + q_{i+1}.$$

The latter inequality concerns degeneration. If $Q_i = P_{i+1}$, the edge between $\hat{L}_i$ and $\hat{L}_{i+1}$ degenerates, which corresponds to intersection of L_i and L_{i+1} in $T^2 \times I$.

Let

$$\mathcal{B} = \{(q_1, \ldots, q_4, t) \in \mathbb{R}^4_{>0} \times \mathbb{R}_{\geq 0} \mid \prod_{i=1}^{4} q_i = 1, \ t \geq \frac{1}{2}(q_i - q_i^{-1})\},$$

$$\mathcal{B}_0 = \{(q_1, \ldots, q_4, t) \in \mathcal{B} \mid (1 - q_i q_{i+1})t < q_i + q_{i+1}\}.$$

Define $f \colon \mathcal{B} \to \mathbb{R}^4$ by

$$f(q_1, \ldots, q_4, t) = \left(\frac{q_1 - t}{\sqrt{1+t^2}}, \ldots, \frac{q_4 - t}{\sqrt{1+t^2}} \right) = (\cos\alpha_1, \ldots, \cos\alpha_4).$$

Then f is a homeomorphism onto $(-1, 1]^4$. Moreover, we have $f(\mathcal{B}_0) = \cos(\mathcal{A})$, where $\cos(\alpha_1, \ldots, \alpha_4) = (\cos\alpha_1, \ldots, \cos\alpha_4)$. Theorems 4.1 and 4.2 follow from this description. In particular, $\mathcal{B}_0 \neq \mathcal{B}$ implies that $\mathcal{A} \neq [0, \pi)^4$.

The fact that $(\alpha, \ldots, \alpha) \in \mathcal{A}$ for any $\alpha \in [0, \pi)$ is elementarily shown in the left of Figure 5. A degeneration is shown in the right of Figure 5.

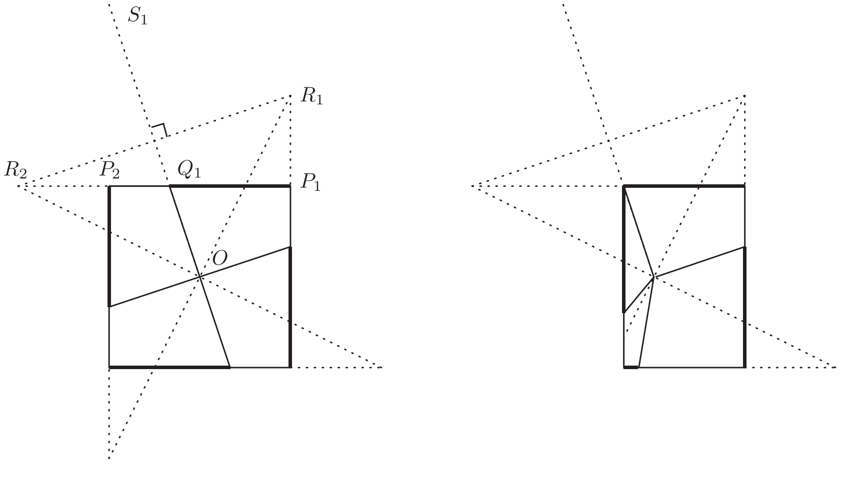

Figure 5. Examples of projections

5. Holed cone structures

We introduce holed cone structures on a 3-manifold as a generalization of cone structures. This enables to avoid intersection of cone loci in deformation, and extend the deformation space. The author is preparing a paper for a detailed account [20].

Definition 5.1. Let X be a 3-manifold, and let L be a link in X. Let B be union of finitely many (possibly empty) disjoint closed 3-balls in $X \setminus \Sigma$. A *(hyperbolic) holed cone metric* on (X, Σ) is a (hyperbolic) cone metric on $(X \setminus \mathrm{int}(B), \Sigma)$ with smooth boundary ∂B. We call each component of B a *hole*.

Definition 5.2. Let g and g' be (hyperbolic) holed cone metrics on (X, Σ) respectively with holes B and B'. The metrics g and g' are equivalent if there are holed cone metrics g_i with holes B_i on (X, Σ) for $0 \leq i \leq n$ such that $g_0 = g$, $g_n = g'$, and for each $0 \leq i \leq n-1$ either

- there is a map $f\colon (X, \Sigma) \to (X, \Sigma)$ isotopic to the identity (preserving Σ) such that the restriction of f to $(X \setminus \mathrm{int}(B_i), \Sigma; g_i)$ is an isometry onto $(X \setminus \mathrm{int}(B_{i+1}), \Sigma; g_{i+1})$,
- $B_i \subset B_{i+1}$, and g_{i+1} is the restriction of g_i to $X \setminus \mathrm{int}(B_{i+1})$, or
- $B_{i+1} \subset B_i$, and g_i is the restriction of g_{i+1} to $X \setminus \mathrm{int}(B_i)$.

We call an equivalent class a *(hyperbolic) holed cone structure*. Let $\mathcal{HC}(X, \Sigma)$ denote the set of holed cone structures on (X, Σ). The deformation space $\mathcal{HC}(X, \Sigma)$ is endowed with the quotient topology induced from C^∞-topology of metrics.

For a holed cone structure, the holonomy representation of $\pi_1(X \setminus \Sigma)$ to $\mathrm{Isom}(\mathbb{H}^3)$ is well-defined.

Our example can be extended to holed cone structures. We construct a holed cone structure on $(T^2 \times I, L)$ by gluing four copies of a "holed trapezohedra" in the same manner as in Section 3. Here a holed trapezohedron is the complement of holes in a trapezohedron endowed with a hyperbolic metric such that each face is totally geodesic. This is isometrically immersed in $\mathbb{H}^3$ as indicated in Figure 6. We suppose that the holes do not intersect $\hat{L}_i$.

The construction in Section 4 is naturally extended to holed trapezohedra (see Figure 7). As a result, every element in $\mathcal{B}$ corresponds to a holed trapezohedron. Hence every quadruple in $[0, 2\pi)^4$ is realized as cone angles of a holed cone structure on $(T^2 \times I, L)$. Let $\mathcal{HC}_{\mathrm{sym}}(T^2 \times I, L)$ denote the set of holed cone structures obtained by this construction. Then $\Theta\colon \mathcal{HC}_{\mathrm{sym}}(T^2 \times I, L) \to [0, 2\pi)^4$ is a homeomorphism.

We do not know whether $\mathcal{HC}_{\mathrm{sym}}(T^2 \times I, L)$ is a component of $\mathcal{HC}(T \times I, L)$. This question concerns local rigidity for holed cone structures. It is doubtful whether local rigidity holds for all the holed cone structures with cone angles at most 2π. However, we may have a chance if we restrict the shape of holes. Deformation via holed cone structures may be effective to consider global rigidity for cone structures.

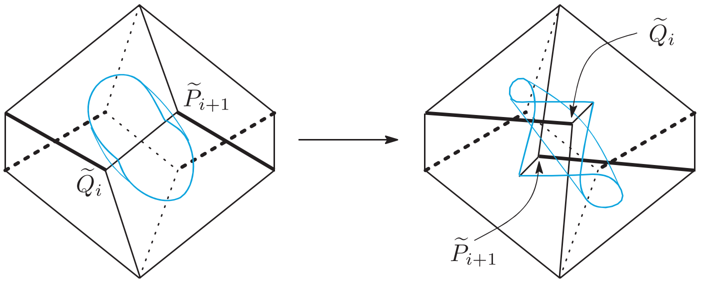

Figure 6. A holed trapezohedron

References

[1] C. Adams, A. Calderon, and N. Mayer, *Generalized bipyramids and hyperbolic volumes of alternating k-uniform tiling links*, Topology Appl., 271 (2020), p. 107045.

[2] M. Boileau, B. Leeb, and J. Porti, *Geometrization of 3-dimensional orbifolds*, Ann. of Math., 162 (2005), pp. 195–290.

[3] M. Boileau, S. Maillot, and J. Porti, *Three-Dimensional Orbifolds and their Geometric Structures*, vol. 15 of Panoramas et Synthèses, Soc. Math. de France, 2003.

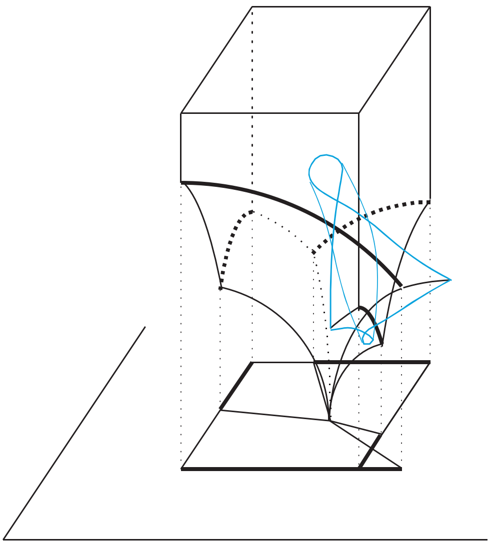

FIGURE 7. Projection of a holed trapezohedron

[4] K. BROMBERG, *Rigidity of geometrically finite hyperbolic cone-manifolds*, Geom. Dedicata, 105 (2004), pp. 143–170.
[5] A. CHAMPANERKAR, I. KOFMAN, AND J. S. PURCELL, *Geometrically and diagrammatically maximal knots*, J. Lond. Math. Soc., 94 (2016), pp. 883–908.
[6] D. COOPER, C. D. HODGSON, AND S. P. KERCKHOFF, *Three-dimensional Orbifolds and Cone-Manifolds*, vol. 5 of MSJ Memoirs, Math. Soc. of Japan, 2000.
[7] C. D. HODGSON AND S. P. KERCKHOFF, *Rigidity of hyperbolic cone-manifolds and hyperbolic Dehn surgery*, J. Differential Geom., 48 (1998), pp. 1–59.
[8] ———, *Universal bounds for hyperbolic dehn surgery*, Ann. of Math., 162 (2005), pp. 367–421.
[9] I. IZMESTIEV, *Examples of infinitesimally flexible 3-dimensional hyperbolic cone-manifolds*, J. Math. Soc. Japan, 63 (2011), pp. 581–598.
[10] S. KOJIMA, *Deformations of hyperbolic 3-cone-manifolds*, J. Differential Geom., 49 (1998), pp. 469–516.
[11] A. KOLPAKOV AND B. MARTELLI, *Hyperbolic four-manifolds with one cusp*, Geom. Funct. Anal., 23 (2013), pp. 1903–1933.
[12] C. LECUIRE AND J.-M. SCHLENKER, *The convex core of quasifuchsian manifolds with particles*, Geom. Topol., 18 (2014), pp. 2309–2373.
[13] R. MAZZEO AND G. MONTCOUQUIOL, *Infinitesimal rigidity of cone-manifolds and the Stoker problem for hyperbolic and Euclidean polyhedra*, J. Differential Geom., 87 (2011), pp. 525–576.
[14] S. MOROIANU AND J.-M. SCHLENKER, *Quasi-Fuchsian manifolds with particles*, J. Differential Geom., 83 (2009), pp. 75–129.
[15] G. D. MOSTOW, *Strong rigidity of locally symmetric spaces*, vol. 78 of Ann. of Math. Studies, Princeton University Press, 1973.
[16] G. PRASAD, *Strong rigidity of $\mathbb{Q}$-rank 1 lattices*, Invent. Math., 21 (1973), pp. 255–286.
[17] A. WEIL, *On discrete subgroups of Lie groups*, Ann. of Math., 72 (1960), pp. 369–384.
[18] H. WEISS, *Global rigidity of 3-dimensional cone-manifolds*, J. Differential Geom., 76 (2007), pp. 495–523.
[19] ———, *The deformation theory of hyperbolic cone-3-manifolds with cone-angles less than 2π*, Geom. Topol., 17 (2013), pp. 329–367.
[20] K. YOSHIDA, *Holed cone structures on 3-manifolds*, in preparation.
[21] ———, *Degeneration of 3-dimensional hyperbolic cone structures with decreasing cone angles*, preprint, arXiv:1909.06622, (2019).

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