Homotopy motions of surfaces in 3-manifolds –Résumé –

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This article is a résumé of the paper [6] developed from the research announcement [5]. In the paper, we introduced the concept of a homotopy motion of a subset in a manifold and gave a systematic study of homotopy motions of surfaces in closed oriented 3-manifolds. This notion arises from various natural problems in 3-manifold theory such as domination of manifold pairs, homotopical behavior of simple loops on a Heegaard surface, and monodromies of virtual branched covering surface bundles associated to a Heegaard splitting (see [6, Section 0.2]).

1. The homotopy motion groups

A *homotopy motion* of a subspace Σ in a manifold M is a homotopy $F = \{f_t\}_{t\in I} : \Sigma \times I \to M$, such that the initial end f_0 is the inclusion map $j : \Sigma \to M$ and the terminal end f_1 is an embedding with image Σ, where $f_t : \Sigma \to M$ ($t \in I = [0,1]$) is the continuous map from Σ to M defined by $f_t(x) = F(x,t)$. Roughly speaking, the *homotopy motion group* $\Pi(M,\Sigma)$ is the group of equivalence classes of homotopy motions of Σ in M, where the product is defined by concatenation of homotopies.

Example 1.1. Let φ be an element of the mapping class group $\mathrm{MCG}(\Sigma)$ of Σ. Consider the 3-manifold $M := \Sigma \times \mathbb{R}/(x,t) \sim (\varphi(x), t+1)$, which is the Σ-bundle over S^1 with monodromy φ. We denote the image of $\Sigma \times \{0\}$ in M by the same symbol Σ and call it a *fiber surface*. Then we have a natural homotopy motion $\lambda = \{f_t\}$ of Σ in M defined by $f_t(x) = [x,t]$, where $[x,t]$ is the element of M represented by (x,t) (see Figure 1(i)). Its terminal end is equal to φ^{-1}, because $f_1(x) = [x,1] = [\varphi^{-1}(x), 0] = \varphi^{-1}(x)$.

Example 1.2. Let h be an orientation-reversing free involution of a closed, orientable surface Σ. Consider the 3-manifold $N := \Sigma\times[0,1]/(x,t) \sim (h(x), 1-t)$, which is the orientable twisted I-bundle over the closed, non-orientable surface Σ/h. The boundary ∂N is identified with Σ by the homeomorphism

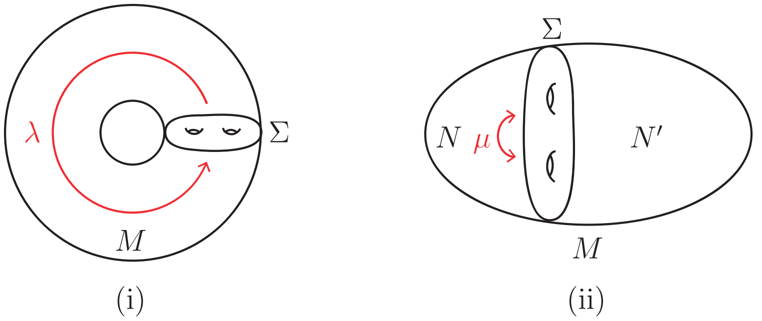

FIGURE 1. (i) The homotopy motion λ. (ii) The homotopy motion μ.

$\Sigma \to \partial N$ mapping x to $[x, 0]$, where $[x, t]$ denotes the element of N represented by (x, t). Then we have a natural homotopy motion $\mu = \{f_t\}_{t \in I}$ of $\Sigma = \partial N$ in N, defined by $f_t(x) = [x, t]$. Its terminal end is equal to h, because $f_1(x) = [x, 1] = [h(x), 0] = h(x)$ for every $x \in \Sigma = \partial N$. Let N' be any compact, orientable 3-manifold whose boundary is identified with Σ, i.e., a homeomorphism $\partial N' \cong \Sigma$ is fixed, and let $M = N \cup N'$ be the closed, orientable 3-manifold obtained by gluing N and N' along the common boundary Σ. Then the homotopy motion $\mu = \{f_t\}_{t \in I}$ of Σ in N defined as above can be regarded as that of Σ in M (see Figure 1(ii)). If N' is also a twisted I-bundle associated with an orientation-reversing involution h' of Σ, then we have another homotopy motion μ' of Σ in N' with terminal end h'.

We now describe key examples that arise from open book decompositions. Recall that an *open book decomposition* of a closed, orientable 3-manifold M is defined to be the pair (L, π), where

(1) L is a (fibered) link in M; and
(2) $\pi : M - L \to S^1$ is a fibration such that $\pi^{-1}(\theta)$ is the interior of a Seifert surface Σ_θ of L for each $\theta \in S^1$.

We call L the *binding* and Σ_θ a *page* of the open book decomposition (L, π). The monodromy of the fibration π is called the *monodromy* of (L, π). We think of the monodromy φ of (L, π) as an element of $\mathrm{MCG}(\Sigma_0, \mathrm{rel}\, \partial\Sigma_0)$, the mapping class group of Σ_0 relative to $\partial\Sigma_0$, i.e., the group of self-homeomorphisms of Σ_0 that fix $\partial\Sigma_0$, modulo isotopy fixing $\partial\Sigma_0$. The pair (M, L), as well as the projection π, is then recovered from Σ_0 and φ. Indeed, we have

$$(M, L) \cong (\Sigma_0 \times \mathbb{R}, \partial\Sigma_0 \times \mathbb{R}) / \sim,$$

where $\sim$ is defined by $(x, s) \sim (\varphi(x), s+1)$ for $x \in \Sigma_0$ and $s \in \mathbb{R}$, and $(y, 0) \sim (y, s)$ for $y \in \partial\Sigma_0$ and any $s \in \mathbb{R}$. So, we occasionally denote the open book decomposition (L, π) by (Σ_0, φ). Under this identification, the Seifert surface Σ_θ is identified with the image $\Sigma \times \{\theta\}$. We define an $\mathbb{R}$-action $\{r_t\}_{t\in\mathbb{R}}$ on M, called a *book rotation*, by $r_t([x, s]) = [x, s+t]$, where $[x, s]$ denotes the element of M represented by (x, s).

Given an open book decomposition (L, π) of M, we obtain a Heegaard splitting $M = V_1 \cup_\Sigma V_2$, where

$$V_1 = \mathrm{cl}(\pi^{-1}([0, 1/2])) = \pi^{-1}([0, 1/2]) \cup L = \cup_{0\le\theta\le1/2}\Sigma_\theta,$$
$$V_2 = \mathrm{cl}(\pi^{-1}([1/2, 1])) = \pi^{-1}([1/2, 1]) \cup L = \cup_{1/2\le\theta\le1}\Sigma_\theta,$$
$$\Sigma = \Sigma_0 \cup \Sigma_{1/2}.$$

We call this the Heegaard splitting of M *induced from* the open book decomposition (L, π).

Example 1.3. Under the above setting, we define two particular homotopy motions in M. The first one, $\rho = \rho_{(L,\pi)} = \rho_{(\Sigma_0,\varphi)}$, is defined by restricting the book rotation, with time parameter rescaled by the factor 1/2, to the Heegaard surface Σ, namely $\rho(t) = r_{t/2}|_\Sigma$, see Figure 2. The second one, $\sigma = \sigma_{(L,\pi)} = \sigma_{(\Sigma_0,\varphi)}$, is defined by

$$\sigma(t)(x) = \begin{cases} r_t(x) & (x \in \Sigma_0) \\ x & (x \in \Sigma_{1/2}), \end{cases}$$

see Figure 3. We call ρ and σ, respectively, the *half book rotation* and the *unilateral book rotation* associated with the open book decomposition (L, π) (or (Σ_0, φ)).

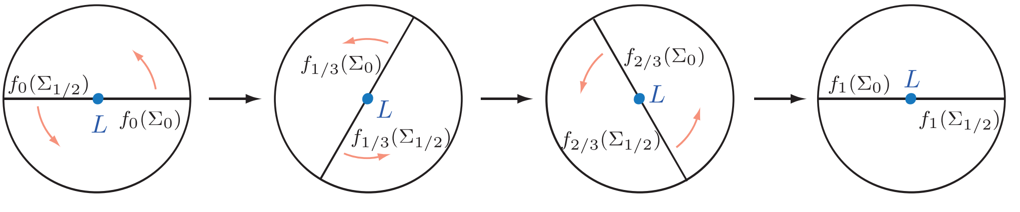

FIGURE 2. The homotopy motion $\rho = \{f_t\}_{t\in I}$.

We now give a formal definition of homotopy motion groups. Let Σ be a subspace of a manifold M, and $j : \Sigma \to M$ the inclusion map. Let $C(\Sigma, M)$ be

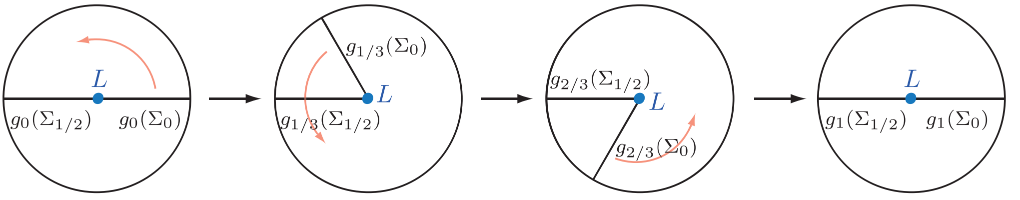

FIGURE 3. The homotopy motion $\sigma = \{g_t\}_{t\in I}$.

the space of continuous maps from Σ to M with the compact-open topology, and $J(\Sigma, M)$ the subspace of $C(\Sigma, M)$ consisting of embeddings of Σ into M with image $j(\Sigma)$. We call a path

$$\alpha : (I, \{1\}, \{0\}) \to (C(\Sigma, M), J(\Sigma, M), \{j\})$$

a *homotopy motion* of Σ in M. We call the maps $\alpha(0)$ and $\alpha(1)$ from Σ to M the *initial end* and the *terminal end*, respectively, of the homotopy motion. Two homotopy motions $(I, \{1\}, \{0\}) \to (C(\Sigma, M), J(\Sigma, M), \{j\})$ are said to be *equivalent* if they are homotopic via a homotopy through maps of the same form. We define

$$\Pi(M, \Sigma) := \pi_1(C(\Sigma, M), J(\Sigma, M), j)$$

to be the set of equivalence classes of homotopy motions, as usual in the definition of relative homotopy groups $\pi_n(X, A, x_0)$ for $x_0 \in A \subset X$, where X is a topological space. We equip $\Pi(M, \Sigma)$ with a group structure as in the following way. Let α and β be homotopy motions. Then the *concatenation*

$$\alpha \cdot \beta : (I, \{1\}, \{0\}) \to (C(\Sigma, M), J(\Sigma, M), \{j\})$$

of them is defined by

$$\alpha \cdot \beta(t) = \begin{cases} \alpha(2t) & (0 \leq t \leq 1/2) \\ \beta(2t-1) \circ \alpha(1) & (1/2 \leq t \leq 1). \end{cases}$$

We can easily check that the concatenation naturally induces a product of elements of $\pi_1(C(\Sigma, M), J(\Sigma, M))$. The *identity motion* $e : (I, \{1\}, \{0\}) \to (C(\Sigma, M), J(\Sigma, M), \{j\})$ defined by $e(t) = j$ $(t \in I)$ represents the identity element of $\Pi(M, \Sigma)$. The *inverse* $\bar{\alpha}$ of a homotopy motion α is defined by

$$\bar{\alpha}(t) = \alpha(1-t) \circ \alpha(1)^{-1},$$

where we regard $\alpha(1)$ as a self-homeomorphism of Σ, and $\alpha(1)^{-1}$ denotes its inverse. Then the inverse of $[\alpha]$ in the group $\pi_1(C(\Sigma, M), J(\Sigma, M))$ is given by $[\bar{\alpha}]$.

Definition 1.4. We call the group $\Pi(M, \Sigma)$ the *homotopy motion group* of Σ in M.

Since the inclusion map j is nothing but the identity if we think of the codomain of j as Σ, $J(\Sigma, M)$ can be canonically identified with $\mathrm{Homeo}(\Sigma)$. Thus, the terminal end $\alpha(1) = f_1$ of a homotopy motion $\alpha = \{f_t\}_{t\in I}$ can be regarded as an element of $\mathrm{Homeo}(\Sigma)$. Therefore, we obtain a map

$$\partial_+ : \Pi(M, \Sigma) \to \mathrm{MCG}(\Sigma)$$

by taking the equivalence class of a homotopy motion $\alpha = \{f_t\}_{t\in I}$ to the mapping class of $\alpha(1) = f_1 \in \mathrm{Homeo}(\Sigma)$. Here $\mathrm{MCG}(\Sigma) = \pi_0(\mathrm{Homeo}(\Sigma))$ is the mapping class group of Σ. Clearly, this map is a homomorphism. (To be precise, this holds when we think of $\mathrm{Homeo}(\Sigma)$ as acting on X from the right: under the usual convention where $\mathrm{Homeo}(\Sigma)$ acts on X from the left, which we employ in this paper, the map ∂_+ is actually an anti-homomorphism.)

Definition 1.5. We denote the image of ∂_+ by $\Gamma(M, \Sigma)$. Namely, $\Gamma(M, \Sigma)$ is the subgroup of the mapping class group $\mathrm{MCG}(\Sigma)$ defined by

$$\begin{aligned}\Gamma(M, \Sigma) &= \{[f] \in \mathrm{MCG}(\Sigma) \mid \exists \text{ homotopy motion } \{f_t\}_{t\in I} \text{ such that } f = f_1.\} \\ &= \{[f] \in \mathrm{MCG}(\Sigma) \mid j \circ f : \Sigma \to M \text{ is homotopic to } j : \Sigma \to M.\}.\end{aligned}$$

The kernel of ∂_+ is denoted by $\mathcal{K}(M, \Sigma)$: thus we have the following sequence.

$$1 \longrightarrow \mathcal{K}(M, \Sigma) \longrightarrow \Pi(M, \Sigma) \xrightarrow{\partial_+} \Gamma(M, \Sigma) \longrightarrow 1$$

For an element $[\alpha]$ of $\mathcal{K}(M, \Sigma)$, we may choose its representative α so that $\alpha(1) = j$. Then α induces a continuous map $\hat{\alpha} : \Sigma \times S^1 \to M$ that sends $(x, t) \in \Sigma \times S^1$ to $\alpha(t)(x) = \alpha(x, t)$, where we identify S^1 with $\mathbb{R}/\mathbb{Z}$. Then we can construct two homomorphisms deg and Φ defined on $\mathcal{K}(M, \Sigma)$ as follows. (See [6, Section 2] for well-definedness.)

Definition 1.6. (1) We denote by $\deg : \mathcal{K}(M, \Sigma) \to \mathbb{Z}$ the homomorphism defined by

$$\deg([\alpha]) = \deg(\hat{\alpha} : \Sigma \times S^1 \to M).$$

We call $\deg([\alpha])$ the *degree* of the element $[\alpha] \in \mathcal{K}(M, \Sigma)$.

(2) Suppose the genus $g(\Sigma) \geq 2$. We denote by Φ the homomorphism

$$\Phi : \mathcal{K}(M, \Sigma) \to Z(j_*(\pi_1(\Sigma, x_0)), \pi_1(M, x_0)), \ \Phi([\alpha]) = [u],$$

where $u : (I, \partial I) \to (M, \{x_0\})$, $u(t) = \alpha(t)(x_0)$. Here $Z(j_*(\pi_1(\Sigma, x_0)), \pi_1(M, x_0))$ denotes the centralizer of $j_*(\pi_1(\Sigma, x_0))$ in $\pi_1(M, x_0)$.

The homomorphism $\deg : \mathcal{K}(M, \Sigma) \to \mathbb{Z}$ does not vanish if and only if (M, Σ) is *dominated by* $\Sigma \times S^1$, namely, there exists a map $\phi : \Sigma \times S^1 \to M$ such that $\phi|_{\Sigma \times \{0\}}$ is an embedding with image $\Sigma \subset M$ and that the degree of ϕ is non-zero.

2. The homotopy motion groups of surfaces in 3-manifolds - two extreme cases -

In this section, we describe the homotopy motion groups $\Gamma(M, \Sigma)$ of surfaces in 3-manifolds for the two extreme cases: the case where Σ is incompressible and the case where Σ is *homotopically trivial* in the sense that the inclusion map $j : \Sigma \to M$ is homotopic to a constant map.

Theorem 2.1. *Let M be a closed, orientable Haken manifold, and suppose that Σ is a closed, orientable, incompressible surface in M. Then the following hold.*

(1) *If M is a Σ-bundle over S^1 with monodromy φ and Σ is a fiber surface, then $\Pi(M, \Sigma)$ is the infinite cyclic group generated by the homotopy motion λ described in Example* 1.1.
(2) *If Σ separates M into two submanifolds, M_1 and M_2, precisely one of which is a twisted I-bundle, then $\Pi(M, \Sigma)$ is the order-*2 *cyclic group generated by the homotopy motion μ described in Example* 1.2.
(3) *If Σ separates M into two submanifolds, M_1 and M_2, both of which are twisted I-bundles, then $\Pi(M, \Sigma)$ is the infinite dihedral group generated by the homotopy motions μ and μ' described in Example* 1.2.
(4) *Otherwise, $\Pi(M, \Sigma)$ is the trivial group.*

This theorem is proved by using the positive solution of Simon's conjecture [13] concerning manifold compactifications of covering spaces, with finitely generated fundamental groups, of compact 3-manifolds. A proof of Simon's conjecture can be found in Canary's expository article [3, Theorem 9.2], where he attributes it to Long and Reid.

Theorem 2.2. *Let Σ be a closed, orientable surface embedded in a closed, orientable 3-manifold M. Then the following hold.*

(1) *If Σ is homotopically trivial and if M is aspherical, then $\Pi(M, \Sigma) \cong \pi_1(M) \times \mathrm{MCG}(\Sigma)$. To be more precise, $\Gamma(M, \Sigma) = \mathrm{MCG}(\Sigma)$, and $\mathcal{K}(M, \Sigma)$ is identified with the factor $\pi_1(M)$. Moreover, the homomorphism $\deg : \mathcal{K}(M, \Sigma) \to \mathbb{Z}$ vanishes.*

(2) *Conversely, if* $\Gamma(M,\Sigma) = \mathrm{MCG}(\Sigma)$ *and if* M *is irreducible, then* Σ *is homotopically trivial.*

3. The homotopy motion groups of Heegaard surfaces closed orientable 3-manifolds

In this section, we study the homotopy motion groups of Heegaard surfaces of 3-manifolds. Throughout this section, $M = V_1 \cup_\Sigma V_2$ denotes a Heegaard splitting of a closed, orientable 3-manifold.

3.1. The group $\mathcal{K}(M,\Sigma)$ for Heegaard surfaces of closed orientable 3-manifolds

For irreducible 3-manifolds, we obtain the following complete determination of the group $\mathcal{K}(M,\Sigma)$.

Theorem 3.1. *Let* M *be a closed, orientable, irreducible* 3*-manifold and* Σ *a Heegaard surface of* M*.*

(1) *Suppose that* M *is aspherical. Then* (M,Σ) *is not dominated by* $\Sigma\times S^1$*. To be precise,* Φ *gives an isomorphism* $\mathcal{K}(M,\Sigma) \cong Z(\pi_1(M))$*, and the homomorphism* $\deg : \mathcal{K}(M,\Sigma) \to \mathbb{Z}$ *vanishes. Thus if* M *is a Seifert fibered space with orientable base orbifold, then* $\mathcal{K}(M,\Sigma)$ *is isomorphic to* $\mathbb{Z}^3$ *or* $\mathbb{Z}$ *according to whether* M *is the* 3*-torus* T^3 *or not; otherwise,* $\mathcal{K}(M,\Sigma)$ *is the trivial group.*

(2) *Suppose that* M *is non-aspherical, namely* M *has the geometry of* S^3*. Then* (M,Σ) *is dominated by* $\Sigma\times S^1$*. To be precise, the following holds.*
   (i) *If* $g(\Sigma) \geq 2$*, then the product homomorphism* $\Phi\times\deg$ *induces an isomorphism* $\mathcal{K}(M,\Sigma) \cong Z(\pi_1(M)) \times |\pi_1(M)|\cdot\mathbb{Z}$*.*
   (ii) *If* $g(\Sigma) \leq 1$*, then the homomorphism* $\deg$ *induces an isomorphism* $\mathcal{K}(M,\Sigma) \cong |\pi_1(M)|\cdot\mathbb{Z}$*.*

For 3-manifolds which are not necessarily irreducible, we obtain the following partial result.

Theorem 3.2. *Let* M *be a closed, orientable* 3*-manifold and* Σ *a Heegaard surface of* M*.*

(1) *If* M *contains an aspherical prime summand, then* (M,Σ) *is not dominated by* $\Sigma\times S^1$*.*

(2) *If* $M = \#_g(S^2\times S^1)$ *for some* $g \geq 1$*, then* (M,Σ) *is dominated by* $\Sigma\times S^1$*. To be precise,* $\deg(\mathcal{K}(M,\Sigma)) = \mathbb{Z}$*.*

(3) *If* $M = \mathbb{RP}^3\#\mathbb{RP}^3$*, then* (M,Σ) *is dominated by* $\Sigma\times S^1$*. To be precise,* $\deg(\mathcal{K}(M,\Sigma)) = 2\mathbb{Z}$*.*

By the geometrization theorem established by Perelman, we obtain the following corollary.

Corollary 3.3. *Let M be a closed, orientable, 3-manifold which is either prime or geometric, and let Σ a Heegaard surface of M. Then (M, Σ) is dominated by $\Sigma \times S^1$ if and only if M is non-aspherical, namely M admits the geometry of S^3 or $S^2 \times \mathbb{R}$.*

We remark that Theorems 3.1 and 3.2 are intimately related with the result of Kotschick-Neofytidis [7, Theorem 1], which says that a closed, orientable 3-manifold M is dominated by a product $\Sigma \times S^1$ for some closed, orientable surface Σ if and only if M is finitely covered by either a product $F \times S^1$, for some aspherical surface F, or a connected sum $\#_g(S^2 \times S^1)$ for some non-negative integer g.

3.2. Gap between $\Gamma(M, \Sigma)$ and its natural subgroup $\langle \Gamma(V_1), \Gamma(V_2) \rangle$

For a Heegaard splitting $M = V_1 \cup_\Sigma V_2$, let $\Gamma(V_i)$ be the kernel of the homomorphism $\mathrm{MCG}(V_i) \to \mathrm{Out}(\pi_1(V_i))$ $(i = 1, 2)$. Since $\mathrm{MCG}(V_i)$ is regarded as a subgroup of $\mathrm{MCG}(\Sigma)$, the group $\Gamma(V_i)$ is regarded as a subgroup of $\mathrm{MCG}(\Sigma)$. In [4, Question 5.4], Minsky raised a question concerning the subgroup $\langle \Gamma(V_1), \Gamma(V_2) \rangle$ generated by $\Gamma(V_1)$ and $\Gamma(V_2)$. The corresponding question for 2-bridge spheres for 2-bridge links were completely solved by Lee-Sakuma [9, 11], and applied the study of epimorphisms among 2-bridge knot groups [1, Theorem 8.1]) and variations of McShane's identity [10] (see [8] for summary).

Now observe that the group $\langle \Gamma(V_1), \Gamma(V_2) \rangle$ is contained in the group $\Gamma(M, \Sigma)$. The above results show that it is more natural to work with the group $\Gamma(M, \Sigma)$ for [4, Question 5.4], and the following questions naturally arise.

Question 3.4. (1) When is the group $\langle \Gamma(V_1), \Gamma(V_2) \rangle$ equal to $\Gamma(M, \Sigma)$?

(2) When is the group $\langle \Gamma(V_1), \Gamma(V_2) \rangle$ equal to the free product $\Gamma(V_1) * \Gamma(V_2)$?

A partial answer to the second question was given by Bowditch-Ohshika-Sakuma in [12, Theorem B] (cf. Bestvina-Fujiwara [2, Section 3]), which says that if the Hempel distance is large enough, then the orientation-preserving subgroup $\langle \Gamma^+(V_1), \Gamma^+(V_2) \rangle$ is equal to the free product $\Gamma^+(V_1) * \Gamma^+(V_2)$. A main purpose of [6] is to give the following partial answer to Question 3.4(1).

Theorem 3.5. *Let $M = V_1 \cup_\Sigma V_2$ be a Heegaard splitting of a closed, orientable 3-manifold M induced from an open book decomposition. If M has an aspherical prime summmand, then we have $\langle \Gamma(V_1), \Gamma(V_2) \rangle \lneq \Gamma(M, \Sigma)$.*

In fact, it is proved that neither the half book rotation ρ nor the unilateral book rotation σ, defined in Example 1.3, is not contained in $\langle \Gamma(V_1), \Gamma(V_2) \rangle$. This theorem is proved by using a $\mathbb{Z}^2$-valued invariant for elements of $\Gamma(M, \Sigma)$, which in turn is constructed by using Theorem 3.1.

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