Local rigidity of right-angled Coxeter groups in hyperbolic 5-space

Tomoshige Yukita

School of Education, Faculty of Education and Integrated Arts and Sciences,
Waseda University

1 Local rigidity of discrete subgroups of Isom ($\mathbb{H}^d$)

Let $\mathbb{H}^d$ be the hyperbolic d-space and Isom ($\mathbb{H}^d$) be the isometry group of $\mathbb{H}^d$. For a discrete subgroup $\Gamma <$ Isom ($\mathbb{H}^d$), we denote the set of all group homomorphisms of Γ to Isom ($\mathbb{H}^d$) by $\mathcal{R}(\Gamma)$, called the *representation space* of Γ. Note that we do not restrict our attention to faithful or discrete representations. We endow $\mathcal{R}(\Gamma)$ with the compact-open topology. The inclusion map $\Gamma \hookrightarrow$ Isom ($\mathbb{H}^d$) is denoted by ρ_0. The isometry group Isom ($\mathbb{H}^d$) acts on $\mathcal{R}(\Gamma)$ by conjugation.

Definition 1.1. A discrete subgroup $\Gamma <$ Isom ($\mathbb{H}^d$) is said to be *locally rigid* if the orbit Isom ($\mathbb{H}^d$)$\cdot$ ρ_0 is open in the representation space $\mathcal{R}(\Gamma)$.

Remark 1.1. Mostow rigidity theorem does not imply the local rigidity. The theorem states that the quotient of the set of all the faithful and discrete representations of a lattice $\Gamma <$ Isom ($\mathbb{H}^d$) to Isom ($\mathbb{H}^d$) by conjugation is the one point set for $d \geq 3$. However, it is possible to exist representations neither faithful nor discrete near the inclusion map ρ_0. We shall give such an example of Γ in Section 3.

We call a continuous path ρ_t in $\mathcal{R}(\Gamma)$ passing through ρ_0 a *deformation* of Γ. A deformation ρ_t is said to be *trivial* if there exists a continuous path $g_t \in$ Isom ($\mathbb{H}^d$) such that $\rho_t = g_t \rho_0 g_t^{-1}$ for any t. The local rigidity of Γ is equivalent to say that any deformation of Γ is trivial. Roughly speaking, a non-trivial deformation of Γ corresponds to a deformation of the hyperbolic orbifold $\mathbb{H}^n/\Gamma$. Consider the following problem.

Problem 1.1. Given a discrete subgroup $\Gamma <$ Isom ($\mathbb{H}^d$), when is Γ locally rigid?

There are well-known results about the local rigidity of lattices in Isom ($\mathbb{H}^d$). Here a discrete subgroup $\Gamma <$ Isom ($\mathbb{H}^d$) is called a *lattice* if the hyperbolic orbifold $\mathbb{H}^d/\Gamma$ has finite volume.

Theorem 1.1 ([9])**.** *If $d \geq 3$ and the hyperbolic orbifold $\mathbb{H}^d/\Gamma$ is compact, then Γ is locally rigid.*

Theorem 1.2 ([2])**.** *If $d \geq 4$ and Γ is a lattice, then Γ is locally rigid.*

In this note, we consider the local rigidity of discrete subgroups of Isom ($\mathbb{H}^d$) for $d \geq 4$. By Theorem 1.2, we focus on discrete subgroups which are not lattices.

2 Hyperbolic Coxeter groups

Definition 2.1. A closed subset $P \subset \mathbb{H}^d$ is called a *hyperbolic d-polytope* if the interior of P in $\mathbb{H}^d$ is not empty and there exists a finite family of closed half-spaces $H_1^-, \ldots, H_N^-$ such that $P = \bigcap_{i=1}^{N} H_i^-$, where H_i^- is the closed half-space bounded by the hyperplane H_i. We assume that such a finite family of closed half-spaces is minimal, that is, $P \subsetneq \bigcap_{j \neq i} H_j^-$ for any i. We call $P \subset \mathbb{H}^d$ a *hyperbolic Coxeter d-polytope* if dihedral angles of P are of the form $\frac{\pi}{m}$ for $2 \leq m \leq \infty$. Moreover, if all dihedral angles of P are $\frac{\pi}{2}$, we call P a *right-angled* hyperbolic Coxeter polytope.

For a hyperbolic Coxeter d-polytope $P \subset \mathbb{H}^d$, we denote the subgroup of $\mathrm{Isom}(\mathbb{H}^d)$ generated by the reflections in the bounding hyperplanes of P by Γ_P, called the *hyperbolic Coxeter group* associated with P. If P is right-angled, then the group Γ_P is callled the *right-angled* hyperbolic Coxeter group.

Theorem 2.1 ([8, Lemma 1.3. pp.199, Proposition 1.4. pp.200])**.** (1) *If P is a hyperbolic Coxeter d-polytope, then the subgroup $\Gamma_P < Isom(\mathbb{H}^d)$ is discrete.* (2) *For any discrete subgroup $\Gamma < Isom(\mathbb{H}^d)$ generated by finitely many reflections, there exists a hyperbolic Coxeter d-polytope P such that $\Gamma = \Gamma_P$.*

In this note, if a hyperbolic Coxeter polytope P has a hyperplane intersecting with all bounding hyperplanes of P perpendicularly (see Figure 1), the hyperbolic Coxeter group Γ_P is said to be *Fuchsian*. If Γ_P is Fuchsian, then Γ_P can be viewed as a discrete subgroup of $\mathrm{Isom}(\mathbb{H}^{d-1})$ by considering the intersection of P and a hyperplane intersecting with bounding hyperplanes of P.

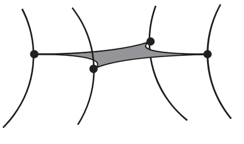

Figure 1: Schematic picture of a hyperplane intersecting with bounding hyperplanes of P perpendicularly

Definition 2.2. For a hyperbolic d-polytope $P = \bigcap_{i=1}^{N} H_i^-$, the intersection $H_i \cap P$ is called a *facet* of P.

For two facets F_i and F_j of P, the mutual positions of the facets are divided into three cases; intersecting, parallel, and ultraparallel (see Figure 2). F_i and F_j are *parallel* (resp. *ultraparallel*) if they meet at a point at infinity (resp. do not intersect in $\overline{\mathbb{H}^d}$).

It is known that the hyperbolic Coxeter group Γ_P has the following group presentation.

$$\Gamma_P = \left\langle s_1, \cdots, s_N \middle| s_1^2 = \cdots = s_N^2 = 1,\ (s_i s_j)^{m_{ij}} = 1 \right\rangle, \tag{2.1}$$

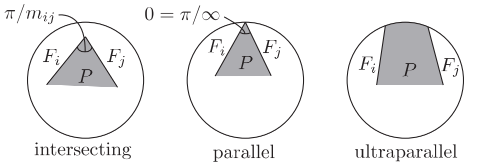

Figure 2: Schematic pictures of the mutual positions of two facets F_i and F_j

where s_i is the reflection in the hyperplane H_i and m_{ij} means that the dihedral angle between F_i and F_j is equal to $\dfrac{\pi}{m_{ij}}$. This group presentation enable us to describe a neighborhood of $\rho_0 \in \mathcal{R}(\Gamma_P)$ in terms of the outer normal vectors of the bounding hyperplanes of P.

3 Representation spaces of hyperbolic Coxeter groups

We recall the hyperboloid model of $\mathbb{H}^d$ (for details see [7]). Let $\mathbb{R}^{d,1}$ be the $(d+1)$-Lorentzian space with the following Lorentzian inner product.

$$\langle x, y\rangle := x_1y_1 + \cdots + x_dy_d - x_{d+1}y_{d+1}.$$

A vector $v \in \mathbb{R}^{d,1}$ is called a *space-like, light-like*, and *time-like* if $\langle v, v\rangle > 0, \langle v, v\rangle = 0$,and $\langle v, v\rangle < 0$, respectively. For a nonzero vector $v \in \mathbb{R}^{d,1}$, the orthogonal complement $v^\perp$ is said to be *space-like, light-like*, and *time-like* if v is time-like, light-like, and space-like, respectively. The set of all the time-like vectors whose the last coordinates are positive is denoted by H^d, and is called the *hyperboloid model* of $\mathbb{H}^d$, that is,

$$H^d = \{\, x = (x_1, \cdots, x_{d+1}) \mid \langle x, x\rangle = -1,\ x_{d+1} > 0 \,\}.$$

For any hyperplane $H \subset H^d$, there exists a space-like unit vector $v \in \mathbb{R}^{d,1}$ such that $H = H^d \cap v^\perp$. Note that the vector v is unique up to the sign. We call such space-like unit vectors *outer normal vectors* of a hyperplane H. Here and in the sequel we write H_v for a pair (H, v) of a hyperplane H and its outer normal vector v. Then, the closed half-space H_v^- bounded by a hyperplane H_v is written as follows.

$$H_v^- = \left\{ x \in H^d \;\middle|\; \langle x, v\rangle \leq 0 \right\}.$$

The reflection $s_v \in \mathrm{Isom}(\mathbb{H}^d)$ in a hyperplane H_v is defined by $s_v(x) = x - 2\langle x, v\rangle v$. Let dS^d be the set of space-like unit vectors and Ref be the set of reflections. We have the following covering map,

$$dS^d \to \mathrm{Ref}; v \mapsto s_v.$$

Lemma 3.1 ([1, Lemma 2.3])**.** *For any reflection $s \in Isom(\mathbb{H}^d)$, there exists a neighborhood $U \subset Isom(\mathbb{H}^d)$ of s such that if $s' \in U$ and $s'^2 = 1$, then s' is also a reflection.*

Consider a hyperbolic Coxeter group Γ_P, where $P = \displaystyle\bigcap_{i=1}^{N} H_{v_i}^-$. For simplicity, we write H_i^- and H_i for $H_{v_i}^-$ and H_{v_i}. By Lemma 3.1, we can find an open neighborhood U of $\rho_0 \in \mathcal{R}(\Gamma_P)$ such that for any $\rho \in U$, the isometries $\rho(s_1), \ldots, \rho(s_N)$ are reflections. Therefore for the study of the local rigidity of hyperbolic Coxeter groups, it is sufficient to consider representations such that the generators are reflections. By considering the covering map $dS^d \to \mathrm{Ref}$, we obtain the following injective continuous map $\Phi : U \to (\mathbb{R}^{d+1})^N$ defined on a sufficiently small neighborhood U of ρ_0.

$$\Phi : U \to \mathrm{Ref}^N \to (\mathbb{R}^{d+1})^N;\ \rho \mapsto (\rho(s_1), \cdots, \rho(s_N)) \mapsto (v_1(\rho), \cdots, v_N(\rho)).$$

By the group presentation (2.1), a map $\rho : \{s_1, \ldots, s_N\} \to \mathrm{Ref}^N$ extends to a group homomorphism $\rho : \Gamma_P \to \mathrm{Isom}\,(\mathbb{H}^d)$ if and only if $(\rho(s_i)\rho(s_j))^{m_{ij}} = 1$. Therefore we see that

$$\Phi(U) \subset \left\{ (v_1, \cdots, v_N) \in (\mathbb{R}^{d+1})^N \,\middle|\, \langle v_i, v_i \rangle = 1,\ \langle v_i, v_j \rangle = -\cos\frac{\pi}{m_{ij}} \right\}.$$

We give an example of a non-trivial deformation of a 3-dimensional right-angled Coxeter group Γ_P. Note that if $P \subset \mathbb{H}^3$ is compact, then Γ_P is locally rigid by Theorem 1.1. Consider the regular ideal octahedron O satisfying the following properties (see Figure 3).

(*i*) O is combinatorially equivalent to Euclidean regular octahedron.

(*ii*) All dihedral angles of O are $\pi/2$.

(*iii*) All vertices of O are points on the boundary at infinity.

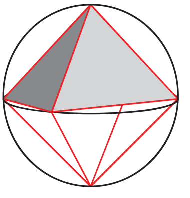

Figure 3: Depicted in the projective ball model

Definition 3.1. An edge e of hyperbolic 3-polytope is said to be *contractible* if the two faces containing e have at least four edges and the dihedral angels at the edges incident to e are $\frac{\pi}{2}$ (see Figure 4).

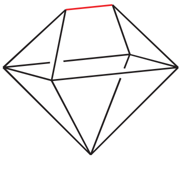

Figure 4: The four edges incident to the red edge have dihedral angles $\frac{\pi}{2}$, so that the red edge is contractible.

Theorem 3.1 ([5, Proposition 1, 2.]). *Let P be a hyperbolic 3-polytope of finite volume whose dihedral angles are at most $\frac{\pi}{2}$.*

(1) *Suppose that P has a contractible edge e and the dihedral angle at e is θ_0. For any $0 < \theta \leq \theta_0$, there exists a hyperbolic 3-polytope P_θ of finite volume satisfying the following properties: P_θ and P are combinatorially equivalent. The dihedral angles of P_θ at the edges other than e are the same as that of P. The dihedral angle of P_θ at the edge e is θ.*

(2) *Suppose that P has a contractible edge e. There exists a hyperbolic 3-polytope P_0 of finite volume satisfying the following properties (see Figure 5): P_0 is combinatorially equivalent to the 3-polytope obtained by contracting the edge e of P to a vertex of valency 4. The dihedral angles of P_0 at the edges are the same as that of P.*

(3) *Suppose that P has a vertex of valency 4. Then there exists $0 < \theta \leq \pi/2$ and a hyperbolic 3-polytope P_θ of finite volume satisfying the following properties (see Figure 5): P_θ is combinatorially equivalent to the 3-polytope obtained by inserting an edge e into a vertex of P of valency 4. The dihedral angles of P_θ at the edges other than e are the same as that of P. The dihedral angle of P_θ at the edge e is θ.*

By Theorem 3.1, we obtain an deformation of O as in the Figure 5

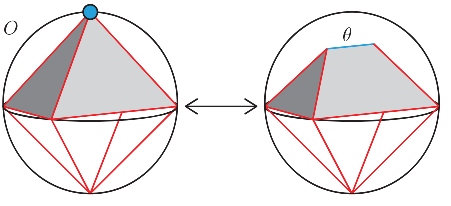

Figure 5: The dihedral angle at the blue edge is θ and the other dihedral angles are $\pi/2$ in both left and right polytopes.

The deformation of O as in the Figure 5 gives rise to a non-trivial deformation of Γ_O as follows. We denote the 3-polytope in the right of Figure 5 by O_θ. Label the faces of O by $F_1, \ldots, F_8$ and denote the outer normal vectors of F_i by v_i. We adopt the labels on the faces of O_θ as same as $\mathcal{O}$ and the outer normal vectors are denoted by $v_1(\theta), \ldots, v_8(\theta)$. Then

$$O_\theta = \bigcap_{i=1}^{8} H_i^-(\theta), \quad H_i^-(\theta) = \left\{ x \in H^3 \;\middle|\; \langle x, v_i(\theta) \rangle \leq 0 \right\}.$$

We define a map $\rho_\theta : \{s_1, \ldots, s_8\} \to \mathrm{Isom}\,(\mathbb{H}^3)$ $(0 \leq \theta \leq \pi/2)$ by

$$s_i(\theta) = x - 2\,\langle x, v_i(\theta)\rangle\, v_i(\theta).$$

We can see that ρ_θ defines a group homomorphism $\rho_\theta \in \mathcal{R}(\Gamma_O)$ as follows. The right-angled hyperbolic Coxeter group Γ_O has the following group presentation.

$$\Gamma_O = \left\langle s_1, \cdots, s_8 \middle| s_1^2 = \cdots = s_8^2 = 1,\ (s_i s_j)^2 = 1 \text{ if faces } F_i, F_j \text{ are intersecting} \right\rangle.$$

It is trivial that ρ_θ preserves the relations of the form $s_i^2 = 1$. Since the dihedral angles at the red colored edges of $\mathcal{O}_\theta$ are the same as that of $\mathcal{O}$, ρ_θ preserves the relations of the form $(s_i s_j)^2 = 1$. Therefore we obtain a non-trivial deformation of Γ_O. In particular, this example tells us that Mostow rigidity theorem does not imply the local rigidity.

4 Local rigidity of hyperbolic Coxeter groups with Fuchsian ends

By Theorem 1.2, lattices in $\mathrm{Isom}\,(\mathbb{H}^d)$ are locally rigid for $d \geq 4$, so that we consider the following problem.

Problem 4.1. Let $\Gamma < \mathrm{Isom}\,(\mathbb{H}^d)$ be a discrete subgroup. Suppose that $d \geq 4$ and the hyperbolic orbifold $\mathbb{H}^d/\Gamma$ has infinite volume. When is Γ locally rigid?

For the study of the geometry of $\mathbb{H}^d/\Gamma$ of infinite volume, the notion of the convex core is important. Here the *convex core* $\mathrm{core}\,(\Gamma)$ of Γ is the quotient of the convex hull of the limit set of Γ by Γ.

Definition 4.1. Suppose that Γ is finitely generated.

(1) Γ is said to be *geometrically finite* if $\mathrm{core}\,(\Gamma)$ has finite volume and is not compact.

(2) Γ is said to be *convex cocompact* if $\mathrm{core}\,(\Gamma)$ is compact.

(3) Γ *has totally geodesic boundary* if every component of the boundary of $\mathrm{core}\,(\Gamma)$ is totally geodesic.

It is known that convex cocompact hyperbolic 3-manifolds are topologically the interiors of 3-manifolds with boundary consisting of closed surfaces of genus bigger than 1. Moreover, such hyperbolic 3-manifolds have non-trivial deformations parametrized by the deformation of conformal structure on the boundary surfaces. However, we can not hope to find the same phenomenon in higher dimensions.

Theorem 4.1 ([4, Theorem 1.1., p. 758.])**.** *Let M^d be a compact hyperbolic d-manifold with boundary. Suppose that $d \geq 4$ and every component of the boundary of M^d is totally geodesic. For any holonomy representation $\rho_0 : \pi_1(M) \to \mathrm{Isom}\,(\mathbb{H}^d)$, the discrete group $\rho_0(\pi_1(M))$ is locally rigid.*

In contrast to Theorem 4.1, there are a few examples of non-trivial deformations of geometrically finite discrete subgroups of $\mathrm{Isom}\,(\mathbb{H}^4)$ having totally geodesic boundary. The first example of such a non-trivial deformation is due to Kerckhoff and Storm [3]. They considered a right-angled hyperbolic 4-polytope P_{22} obtained by removing two *disjoint* facets from the regular ideal 24-cell. Here we do not explain what P_{24} is in detail, but an important property of P_{24} is that all facets of P_{24} are the regular ideal octahedra. Since the removed facets are disjoint, P_{22} has two ends each isometric to the regular ideal octahedron times $[0, \infty)$. We denote the right-angled hyperbolic Coxeter group associated with P_{22} by Γ_{22}. By cutting the ends of P_{22}, we see that the convex core $\mathrm{core}\,(\Gamma_{22})$ is isometric to P_{24}. In $\mathrm{core}\,(\Gamma_{22})$, we view the removed two facets as totally geodesic boundary. Then, Kerckhoff and Storm constructed a non-trivial deformation of Γ_{22} by deforming the two boundary regular ideal octahedra (see Figure 6).

Conjecture 4.1 ([4, Conjecture 6.2., p. 782]). Let M^d be a hyperbolic d-manifold of *finite volume* with boundary. Suppose that $d \geq 5$ and every component of the boundary of M^d is totally geodesic. For any holonomy representation $\rho_0 : \pi_1(M) \to \mathrm{Isom}\,(\mathbb{H}^d)$, the discrete group $\rho_0(\pi_1(M))$ is locally rigid.

In general, it is not easy to construct explicitly the hyperbolic manifolds satisfying the given conditions. Concerning the Conjecture 4.1, the coloring technique due to Kolpakov and Slavich is a way to construct hyperbolic manifolds of finite volume with geodesic boundary from right-angled hyperbolic polytopes(see [6] for details).

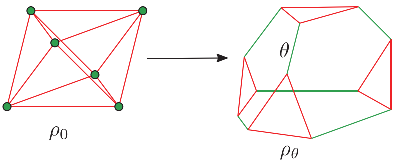

Figure 6: Deformation of the boundary regular ideal octahedron

Let P be a right-angled hyperbolic d-polytope. Suppose that facets $F_1, \ldots, F_k$ of P are mutually disjoint. Then, the polytope P' obtained from P by removing the facets $F_1, \ldots, F_k$ has k ends each isometric to $F_i \times [0, \infty)$, and hence the group $\Gamma_{P'}$ is geometrically finite and has totally geodesic boundary.

Definition 4.2. [1]. Let P be a right-angled hyperbolic d-polytope. The right-angled hyperbolic Coxeter group Γ_P has *Fuchsian end* if there exists finitely many hyperplanes $H_1, \ldots, H_k$ such that the polytope $\widetilde{P} = P \cap \bigcap_{i=1}^{k} H_i^-$ is right-angled hyperbolic d-polytope of finite volume and the facets $F_1 = H_1 \cap \widetilde{P}, \ldots, F_k = H_k \cap \widetilde{P}$ are mutually disjoint.

By applying the coloring technique to right-angled Coxeter groups with Fuchsian ends, we can construct various hyperbolic manifolds satisfying the conditions in the Conjecture 4.1. The following is a first example of locally rigid 5-dimensional right-angled Coxeter groups with Fuchsian ends.

Theorem 4.2 ([10])**.** *There exists a right-angled hyperbolic* 5*-polytope of finite volume such that for any family* $F_1, \ldots, F_k$ *of mutually disjoint facets, the right-angled Coxeter group* $\Gamma_{P'}$ *with Fuchsian ends associated with the polytope* P' *obtained by removing the facets* $F_1, \ldots, F_k$ *from* P *is locally rigid.*

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