A QUICK TOUR TO THE DISTANCE ON TEICHMÜLLER SPACE VIA RENORMALIZED VOLUME

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ABSTRACT. We give a quick introduction to a distance on the Teichmüller space defined via the notion of renormalized volume. The definitions and properties are summarised without proofs.

1. INTRODUCTION

The purpose of this note is to give a quick summery of results in [Masa] which is about a distance and compactification of the Teichmüller space via the renormalized volume.

There have been many attempts to relate the hyperbolic volume of 3-manifolds with quantities which appear in Teichmüller theory. In this note, we discuss closed orientable surface S of genus ≥ 2. One way to associate hyperbolic 3-manifolds via $\mathcal{T}(S)$ is to consider quasi-Fuchsian manifolds. By the Bers simultaneous uniformization [Ber70], the space of quasi-Fuchsian manifolds is parametrized by the product $\mathcal{T}(S) \times \mathcal{T}(S)$. Let $\mathrm{qf}(X, Y)$ denote the quasi-Fuchsian manifold parametrized by $X, Y \in \mathcal{T}(S)$. Although quasi-Fuchsian manifolds are of infinite hyperbolic volumes, there are several natural ways to extract finite "volumes" of them. One standard and classical object is the volume of convex core, which we call the convex core volume of $\mathrm{qf}(X, Y)$ and denote it by $V_C(X, Y)$. Another natural notion is so-called the renormalized volume which is extensively studied by several authors [BBB19, BBB2, BBP, BC17, KM18, KS08, Sch13, Sch19]. Let $V_R(X, Y)$ denote the renormalized volume of $\mathrm{qf}(X, Y)$. In [Masa], we define a distance d_R on $\mathcal{T}(S)$ via the renormalized volume, and demonstrate that the distance d_R is natural to the volume of hyperbolic 3-manifolds.

Let us first summarise known results about distances on $\mathcal{T}(S)$ and volume of hyperbolic 3-manifolds. Brock [Bro03] has shown that the Weil-Petersson distance $d_{\mathrm{wp}}(X, Y)$ on $\mathcal{T}(S)$ is coarsely equal to the convex core volume $V_C(X, Y)$. By [BC17], $V_C(X, Y)$ differs from $V_R(X, Y)$ by a finite amount, Brock's work shows the coarse correspondence between $d_{\mathrm{wp}}(X, Y)$ and $V_R(X, Y)$ as well. The error constants in Brock's result are not explicit, but by using the work of Krasnov-Schlenker[KS08], Kojima-McShane [KM18] showed

$$V_R(X, Y) \leq 3\pi|\chi(S)| d_{\mathcal{T}}(X, Y)$$

where $\chi(S)$ is the Euler characteristics of S. Notice that opposite inequality is impossible as by any Dehn twist τ, $d_{\mathcal{T}}(X, \tau^n(X)) \to \infty$ whereas $V_R(X, \tau^n X)$ is

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bounded as $n \to \infty$. The works of Brock and Kojima-McShane extend to the hyperbolic volume of mapping tori $M(\psi)$ of pseudo-Anosov mapping class ψ. Namely by replacing $V_R(X,Y)$ with $\mathrm{vol}(M(\psi))$ and distances with translation distances of ψ, we get the same estimates.

In [Masa], we introduced a new distance, denoted d_R via the renormalized volume. One important feature of d_R is the following.

Theorem 1.1. *Let $\psi \in \mathrm{MCG}(S)$ be a pseudo-Anosov mapping class and $M(\psi) = S \times I/(x,1) \sim (\psi(x),0)$ denote the mapping torus of ψ. Then the translation distance of ψ with respect to d_R is equal to the hyperbolic volume of $M(\psi)$, that is, for any $X \in \mathcal{T}(S)$, we have*

$$\lim_{n\to\infty} \frac{1}{n} d_R(X, \psi^n X) = \mathrm{vol}(M(\psi)).$$

In this note, we give a quick overview of [Masa] about d_R. We do not give proofs and refer [Masa] for the details.

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2. Renormalized volume, distance and compactification

The idea of the renormalized volume comes from Graham-Witten [GW99] and it is studied by several authors for hyperbolic 3-manifolds (see e.g. [BBB19, BBB2, BBP, BC17, KM18, KS08, Sch13, Sch19]).

For $X, Y \in \mathcal{T}(S)$, let $\mathrm{QD}(X)$ denote the space of quadratic differentials on X, and $q_Y(X) \in \mathrm{QD}(X)$ denote the mapping defined via Bers embedding with base point X. We further let $\mathrm{QD}(S) = \cup_{X \in \mathrm{QD}(X)}$ denote the space of quadratic differentials on S which is a bundle over $\mathcal{T}(S)$. Notice the notation $q_Y(X)$ is not standard for the Bers embedding. For the discussion in this note, we would like to regard q_Y as the function determined by Y. It is known by Nehari's inequality that $q_Y(X)$ is contained in a closed metric ball $\mathrm{QD}_B(X)$ with respect to L^∞-norm.

In this note, we adopt the following variation formula as the definition of V_R.

Theorem 2.1 ([KM18, Lemma 2.4], [Sch19, Corollary 3.13])**.** *For any $Y \in \mathcal{T}(S)$, $V_R(\cdot, Y)$ is differentiable on $\mathcal{T}(S)$. If $\sigma : [-1,1] \to \mathcal{T}(S)$ is a differentiable path,*

$$\left.\frac{d}{dt}\right|_{t=0} V_R(\sigma(t), Y) = -\mathrm{Re}\langle q_Y(\sigma(0)), \dot{\sigma}(0)\rangle.$$

Now we define the space $\mathrm{LQ}(S)$ which is the space of sections of a bundle over $\mathcal{T}(S)$.

Definition 2.2. Let $C := 3\sqrt{\pi(g-1)}$. Then we define

$$\mathrm{LQ}(S) := \prod_{X \in \mathcal{T}(S)} \left\{[-C d_{\mathrm{wp}}(b,X), C d_{\mathrm{wp}}(b,X)] \times \mathrm{QD}_B(X)\right\},$$

(LQ stands for Lipschitz and Quadratic differential). Furthermore by the notation $(\xi, q) \in \mathrm{LQ}(S)$, we mean a point given by $\xi : \mathcal{T}(S) \to \mathbb{R}$ and $q : \mathcal{T}(S) \to \mathrm{QD}(S)$, where $\xi(X) \in [-d_{\mathrm{wp}}(b,X), d_{\mathrm{wp}}(b,X)]$ and $q(X) \in \mathrm{QD}_B(X)$. We equip $\mathrm{LQ}(S)$ with the topology of point-wise convergence, or equivalently the product topology.

It turns out that $\mathrm{LQ}(S)$ is a "nice" space.

Proposition 2.3 ([Masa, Proposition 5.4]). *The space* $\mathrm{LQ}(S)$ *is a compact, Hausdorff, and second countable (hence metrizable) space.*

In [Masa], we constructed a compactification of $\mathcal{T}(S)$ by embedding it in $\mathrm{LQ}(S)$.

2.1. **Lipschitz property of the renormarized volume.** In [Sch19] and [KM18], Schlenker and Kojima-McShane proved that there are explicit upper bounds of the renormalized volume V_R in terms of the WP metric and the Teichmüller distance.

Theorem 2.4 ([Sch19, Theorem 5.4],[KM18]). *Let* $X, Y \in \mathcal{T}(S)$. *Then we have*

(1) $V_R(X,Y) \leq 3\sqrt{\pi(g-1)}d_{\mathrm{wp}}(X,Y)$, *and*
(2) $V_R(X,Y) \leq 3\pi|\chi(S)|d_{\mathcal{T}}(X,Y)$.

The proof of Theorem 2.4 is reduced to the following.

Lemma 2.5 ([Sch13, Proof of Theorem 1.2], [KM18, Proof of Theorem 1.4]). *Let* $Y \in \mathcal{T}(S)$ *and* $\sigma : [0,T] \to \mathcal{T}(S)$ *be a differentiable path.*

(1) *If* σ *is a geodesic with respect to the Teichmüller metric, then*

$$\left|\frac{d}{dt}V_R(\sigma(t),Y)\right| \leq 3\pi|\chi(S)|.$$

(2) *If* σ *is a geodesic with respect to the WP metric, then*

$$\left|\frac{d}{dt}V_R(\sigma(t),Y)\right| \leq 3\sqrt{\pi(g-1)}.$$

Theorem 2.4 is obtained by integrating quantities in Lemma 2.5 along corresponding geodesic segments.

Imitating horofunctions defined with distances, we define a function on $\mathcal{T}(S)$ via the renormalized volume as follows. Let us fix a base point $b \in \mathcal{T}(S)$.

Definition 2.6. Let $Z \in \mathcal{T}(S)$. We define $\nu_Z : \mathcal{T}(S) \to \mathbb{R}$ by

$$\nu_Z(X) := V_R(X,Z) - V_R(b,Z)$$

for $X \in \mathcal{T}(S)$. We call ν_Z a *volume horofunction.*

The variation formula of V_R (Theorem 2.1) gives the following integral expression of ν_Z:

Proposition 2.7 ([Masa, Proposition 6.4]). *Let* $X, Z \in \mathcal{T}(S)$ *and let* $\sigma : [0,T] \to \mathcal{T}(S)$ *be a piecewise differentiable path connecting* X *and* b. *Then*

$$\nu_Z(X) := \int_0^T -\mathrm{Re}\langle q_Z(\sigma(t)), \dot{\sigma}(t)\rangle dt.$$

By Proposition 2.7, we see that the function ν_Z is a Lipschitz map:

Proposition 2.8 ([Masa, Proposition 6.5]). *The function* $\nu_Z : \mathcal{T}(S) \to \mathbb{R}$ *is a Lipschitz map with respect to both the Teichmüller metric and the WP metric, i.e.*

(1) $|\nu_Z(X) - \nu_Z(Y)| \leq 3\sqrt{\pi(g-1)}d_{\mathrm{wp}}(X,Y)$, *and*
(2) $|\nu_Z(X) - \nu_Z(Y)| \leq 3\pi|\chi(S)|d_{\mathcal{T}}(X,Y)$.

Thus we see that ν_Z is a Lipchitz function and vanishes at the base point b. From now on we consider the WP metric on $\mathcal{T}(S)$ and Lipchitz functions with respect to the WP metric. Let $\mathrm{Lip}_b^C\mathcal{T}(S)$ denote the space of C-Lipchitz functions on $\mathcal{T}(S)$ for $C = 3\sqrt{\pi(g-1)}$ which vanishes at b. We have a map

$$\mathcal{V}' : \mathcal{T}(S) \to \mathrm{Lip}_b^C\mathcal{T}(S)$$

defined by $\mathcal{V}'(Z) := \nu_Z$.

Proposition 2.9 ([Masa, Proposition 6.6])**.** *The map $\mathcal{V}' : \mathcal{T}(S) \to \mathrm{Lip}_b^C\mathcal{T}(S)$ is injective and continuous.*

Recall that for $C = 3\sqrt{\pi(g-1)}$,

$$\mathrm{Lip}_b^C\mathcal{T}(S) \subset \prod_{x\in\mathcal{T}(S)} [-C \cdot d_{\mathrm{wp}}(b, X), C \cdot d_{\mathrm{wp}}(b, X)],$$

and $q_X(Y) \in \mathrm{QD}_B(Y)$ (Bers embedding). We are ready to define a function which gives our compactification.

Definition 2.10. We define a map

$$\mathcal{V} : \mathcal{T}(S) \to \mathrm{LQ}(S)$$

by $\mathcal{V}(Z) = (\nu_Z(X), q_Z(X))_{X\in\mathcal{T}(S)}$.

The map $\mathcal{V}$ is an embedding:

Proposition 2.11 ([Masa, Proposition 6.9])**.** *The map $\mathcal{V} : \mathcal{T}(S) \to \mathrm{LQ}(S)$ is a homeomorphism onto its image.*

By Proposition 2.11 and 2.3, the closure $\overline{\mathcal{V}(\mathcal{T}(S))}$ is compact.

Definition 2.12. We denote the closure by $\overline{\mathcal{T}(S)}^{\mathrm{vh}} := \overline{\mathcal{V}(\mathcal{T}(S))}$ (volume and horo) and the boundary by $\partial_{\mathrm{vh}}\mathcal{T}(S) := \overline{\mathcal{V}(\mathcal{T}(S))} \setminus \mathcal{V}(\mathcal{T}(S))$.

The construction of $\overline{\mathcal{T}(S)}^{\mathrm{vh}}$ is compatible with the action of the mapping class group $\mathrm{MCG}(S)$ on $\mathcal{T}(S)$.

Proposition 2.13 ([Masa, Proposition 6.11])**.** *The action of* $\mathrm{MCG}(S)$ *on* $\mathcal{T}(S)$ *extends to a continuous action by homeomorphisms on* $\overline{\mathcal{T}(S)}^{\mathrm{vh}}$ *by*

$$\psi \cdot \nu(X) := \nu(\psi^{-1}X) - \nu(\psi^{-1}b) \text{ for each } X \in \mathcal{T}(S) \tag{2.1}$$
$$\psi \cdot q := \psi^* q$$

and $\psi(\nu, q) = (\psi \cdot \nu, \psi \cdot q)$.

2.2. **Volume of mapping tori.** A mapping class $\psi \in \mathrm{MCG}(S)$ is called *pseudo-Anosov* if ψ has exactly two fixed points $F_+(\psi), F_-(\psi) \in \mathcal{PMF}(S)$ which we may characterize as $\lim_{n\to\infty} \psi^n(X) = F_+(\psi)$ and $\lim_{n\to-\infty} \psi^n(X) = F_-(\psi)$ for any $X \in \mathcal{T}(S)$ in the Thurston compactification. Thurston has shown that the mapping torus

$$M(\psi) := S \times [0, 1]/((\psi(x), 0) \sim (x, 1))$$

admits a complete hyperbolic metric of finite volume. Let $\mathrm{vol}(M(\psi))$ denote the hyperbolic volume of $M(\psi)$. The following proposition follows from [BB16, KM18, Sch13].

Proposition 2.14 (c.f. [Masa, Proposition 6.18])**.** *Let $\psi \in \mathrm{MCG}(S)$ be pseudo-Anosov. Then*

$$\lim_{n\to\infty} \frac{1}{n} V_R(b, \psi^{-n}b) = \mathrm{vol}(M(\psi)).$$

3. A distance on $\mathcal{T}(S)$ via renormalized volume

3.1. **Renormalized volume function V_R is not a distance.** The renormalized volume of quasi-Fuchsian manifolds defines a function

$$V_R : \mathcal{T}(S) \times \mathcal{T}(S) \to \mathbb{R}.$$

with the properties:

- $V_R(X,Y) \geq 0$ and $V_R(X,Y) = 0$ if and only if $X = Y$ ([BBB19, BBP]), and
- $V_R(X,Y) = V_R(Y,X)$ (by definition of quasi-Fuchsian manifolds).

Therefore it is natural to ask if V_R defines a distance on $\mathcal{T}(S)$ (see e.g. [DHM15, Problem 5.7(Agol)]).

In [Masa], we showed that the triangle inequality is not valid.

Theorem 3.1 ([Masa, Theorem 7.2])**.** *The function $V_R : \mathcal{T}(S) \times \mathcal{T}(S) \to \mathbb{R}$ does NOT satisfy the triangle inequality.*

3.2. **A distance via the renormalized volume.** We now define a distance on $\mathcal{T}(S)$.

Definition 3.2. Given $X, Y \in \mathcal{T}(S)$, let

$$d_R(X,Y) := \sup_{(\nu,q)} \nu(X) - \nu(Y),$$

where the supremum is taken over $(\nu, q) \in \overline{\mathcal{T}(S)}^{\mathrm{vh}}$.

Remark 3.3. We remark that as $\overline{\mathcal{T}(S)}^{\mathrm{vh}}$ is compact the supremum is actually attained by some $(\nu, q) \in \overline{\mathcal{T}(S)}^{\mathrm{vh}}$. Hence for any piecewise differentiable path $\sigma : [0,T] \to \mathcal{T}(S)$ connecting X and Y, we have

$$d_R(X,Y) = \int_0^T -\mathrm{Re}\langle q(\sigma(t)), \dot{\sigma}(t)\rangle dt, \tag{3.1}$$

for some $(\nu, q) \in \overline{\mathcal{T}(S)}^{\mathrm{vh}}$. Note also that if one takes the supremum over $\mathcal{T}(S)$ (not $\overline{\mathcal{T}(S)}^{\mathrm{vh}}$), one still gets the same distance as $\mathcal{T}(S) \subset \overline{\mathcal{T}(S)}^{\mathrm{vh}}$ is open dense.

It is also worth mentioning that if one considers the horofunctions with respect to a distance, say d, then the distance defined similarly to the one in Definition 3.2 recovers the original distance d by the triangle inequality. Due to the lack of the triangle inequality for V_R, the function d_R differs from V_R.

The following properties of d_R is easy consequences of above discussions, see [Masa] for details.

Theorem 3.4 ([Masa, Theorem 7.5])**.** *We have the following estimates of d_R in terms of d_{wp}, $d_{\mathcal{T}}$, and V_R.*

1. $d_R(X,Y) \leq 3\sqrt{\pi(g-1)}\, d_{\mathrm{wp}}(X,Y)$,
2. $d_R(X,Y) \leq 3\pi|\chi(S)|\, d_{\mathcal{T}}(X,Y)$,
3. $V_R(X,Y) \leq d_R(X,Y)$.

Theorem 3.5 ([Masa, Theorem 7.6])**.** *The function $d_R : \mathcal{T}(S) \times \mathcal{T}(S) \to \mathbb{R}$ gives a (possibly asymmetric) distance, that is: for any $X, Y, Z \in \mathcal{T}(S)$, we have*

1. $d_R(X,Y) \geq 0$ *and* $d_R(X,Y) = 0 \iff X = Y$.
2. $d_R(X,Y) \leq d_R(X,Z) + d_R(Z,Y)$.

As an immediate corollary of the work of Brock [Bro03], we have:

Theorem 3.6 ([Masa, Theorem 7.9])**.** *The distance d_R is quasi isometric to the WP distance d_{wp}. More precisely, there exists constants $L \geq 1$ and $K \geq 0$ which depends only on S such that*

$$\frac{1}{L} d_{\mathrm{wp}}(X, Y) - K \leq d_R(X, Y) \leq 3\sqrt{\pi(g-1)}\, d_{\mathrm{wp}}(X, Y).$$

Let us re-state the main theorem.

Theorem 3.7. *Let $\psi \in \mathrm{MCG}(S)$ be a pseudo-Anosov mapping class and $M(\psi)$ the mapping torus of ψ. Then the translation length $\tau_v(\psi)$ of ψ with respect to d_R is equal to the hyperbolic volume of the mapping torus $M(\psi)$, i.e. for any $X \in \mathcal{T}(S)$,*

$$\tau_v(\psi) := \lim_{k\to\infty} \frac{d_R(X, \psi^k(X))}{k} = \mathrm{vol}(M(\psi)).$$

In the proof, we utilize some ergodic theory, which is inspired by Karlsson-Ledrappier [KL06, Proof of Theorem 1.1], see [Masa, Theorem 7.10] for the proofs.

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